Unit-1

QUADRATIC EQUATIONS

In this unit, students will learn how to

- *define quadratic equation.*
- *solve a quadratic equation in one variable by factorization.*
- *solve a quadratic equation in one variable by completing square.*
- *derive quadratic formula by using method of completing square.*
- *solve a quadratic equation by using quadratic formula.*
- *solve the equations of the type $ax^4 + bx^2 + c = 0$ by reducing it to the quadratic form.*
- *solve the equations of the type $a\,p(x) + \frac{b}{p(x)} = c$.*
- *solve reciprocal equations of the type $a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0$.*
- *solve exponential equations involving variables in exponents.*
- *solve equations of the type $(x + a)(x + b)(x + c)(x + d) = k$ where $a + b = c + d$.*
- *solve radical equations of the types*
  (i) $\sqrt{ax + b} = cx + d$,
  (ii) $\sqrt{x + a} + \sqrt{x + b} = \sqrt{x + c}$,
  (iii) $\sqrt{x^2 + px + m} + \sqrt{x^2 + px + n} = q$.

1.1. Quadratic Equation

An equation, which contains the square of the unknown (variable) quantity, but no higher power, is called a **quadratic equation** or an equation of the **second degree**.

A second degree equation in one variable x of the form

$ax^2 + bx + c = 0$, where $a \neq 0$ and a, b, c are real numbers,

is called the **general or standard form** of a quadratic equation.

Here a is the co-efficient of x^2, b is the co-efficient of x and constant term is c.

The equations $x^2 - 7x + 6 = 0$ and $3x^2 + 4x = 5$ are the examples of quadratic equations.

$x^2 - 7x + 6 = 0$ is in standard form but

$3x^2 + 4x = 5$ is not in standard form.

Remember that: If $a = 0$ in $ax^2 + bx + c = 0$, then it reduces to a linear equation $bx + c = 0$.

If $b = 0$ in a quadratic equation $ax^2 + bx + c = 0$, then it is called a **pure quadratic** equation. For example $x^2 - 16 = 0$ and $4x^2 = 7$ are the pure quadratic equations.

Activity: Write two pure quadratic equations.

1.2 Solution of quadratic equations

To find solution set of a quadratic equation, following methods are used:

(i) factorization (ii) completing square

1.2(i) Solution by factorization:

In this method, write the quadratic equation in the standard form as

$ax^2 + bx + c = 0$ (i)

If two numbers r and s can be found for equation (i) such that $r + s = b$ and $rs = ac$, then $ax^2 + bx + c$ can be factorized into two linear factors.

The procedure is explained in the following examples.

Example 1: Solve the quadratic equation $3x^2 - 6x = x + 20$ by factorization.

Solution: $3x^2 - 6x = x + 20$ (i)

The standard form of (i) is $3x^2 - 7x - 20 = 0$ (ii)

Here $a = 3$, $b = -7$, $c = -20$ and $ac = 3 \times -20 = -60$

As $-12 + 5 = -7$ and $-12 \times 5 = -60$, so

the equation (ii) can be written as

$3x^2 - 12x + 5x - 20 = 0$

or $3x(x - 4) + 5(x - 4) = 0$

$\Rightarrow$ $(x - 4)(3x + 5) = 0$

Either $x - 4 = 0$ or $3x + 5 = 0$, that is,

Activity: Factorize $x^2 - x - 2 = 0$.

$x = 4$ or $3x = -5 \Rightarrow x = -\dfrac{5}{3}$

$\therefore$ $x = -\dfrac{5}{3}$, 4 are the solutions of the given equation.

Thus, the solution set is $\left\{-\dfrac{5}{3}, 4\right\}$.

Example 2: Solve $5x^2 = 30x$ by factorization.

Solution: $5x^2 = 30x$

$5x^2 - 30x = 0$ which is factorized as

$5x(x - 6) = 0$

> **Remember that:** Cancelling of x on both sides of $5x^2 = 30x$ means the loss of one root *i.e.*, $x = 0$

Either $5x = 0$ or $x - 6 = 0$ $\Rightarrow x = 0$ or $x = 6$

$\therefore$ $x = 0, 6$ are the roots of the given equation.

Thus, the solution set is $\{0, 6\}$.

1.2(ii) Solution by completing square:

To solve a quadratic equation by the method of completing square is illustrated through the following examples.

Example 1: Solve the equation $x^2 - 3x - 4 = 0$ by completing square.

Solution: $x^2 - 3x - 4 = 0$ (i)

Shifting constant term -4 to the right, we have

$x^2 - 3x = 4$ (ii)

Adding the square of $\dfrac{1}{2} \times$ coefficient of x, that is,

$\left(-\dfrac{3}{2}\right)^2$ on both sides of equation (ii), we get

$$x^2 - 3x + \left(-\frac{3}{2}\right)^2 = 4 + \left(-\frac{3}{2}\right)^2$$

$$\left(x - \frac{3}{2}\right)^2 = 4 + \frac{9}{4} = \frac{16 + 9}{4}$$

or $$\left(x - \frac{3}{2}\right)^2 = \frac{25}{4}$$

Taking square root of both sides of the above equation, we have

$$\sqrt{\left(x - \frac{3}{2}\right)^2} = \pm\sqrt{\frac{25}{4}}$$

$$\Rightarrow \quad x - \frac{3}{2} = \pm\frac{5}{2} \text{ or } x = \frac{3}{2} \pm \frac{5}{2}$$

Either $x = \dfrac{3}{2} + \dfrac{5}{2} = \dfrac{3 + 5}{2} = \dfrac{8}{2} = 4$ or $x = \dfrac{3}{2} - \dfrac{5}{2} = \dfrac{3 - 5}{2} = \dfrac{-2}{2} = -1$

$\therefore$ 4 and -1 are the roots of the given equation.

Thus, the solution set is $\{-1, 4\}$.

Example 2: Solve the equation $2x^2 - 5x - 3 = 0$ by completing square.

Solution: $2x^2 - 5x - 3 = 0$

Dividing each term by 2, to make coefficient of x^2 equal to 1.

$$x^2 - \frac{5}{2}x - \frac{3}{2} = 0$$

Remember that: For our convenience, we make the co-efficient of x^2 equal to 1 in the method of completing square.

Shifting constant term $-\frac{3}{2}$ to the right

$$x^2 - \frac{5}{2}x = \frac{3}{2} \quad \text{(i)}$$

Multiply co-efficient of x with $\frac{1}{2}$ *i.e.*, $\frac{1}{2}\left(-\frac{5}{2}\right) = -\frac{5}{4}$

Now adding $\left(-\frac{5}{4}\right)^2$ on both sides of the equation (i), we have

$$x^2 - \frac{5}{2}x + \left(-\frac{5}{4}\right)^2 = \frac{3}{2} + \left(-\frac{5}{4}\right)^2, \text{ that is,}$$

$$\left(x - \frac{5}{4}\right)^2 = \frac{3}{2} + \frac{25}{16} = \frac{24 + 25}{16}$$

or $$\left(x - \frac{5}{4}\right)^2 = \frac{49}{16}$$

Taking square root of both sides of the above equation, we have

$$\sqrt{\left(x - \frac{5}{4}\right)^2} = \pm\sqrt{\frac{49}{16}}$$

$$\Rightarrow \quad x - \frac{5}{4} = \pm\frac{7}{4}$$

Either $x - \frac{5}{4} = \frac{7}{4}$ or $x - \frac{5}{4} = -\frac{7}{4}$, that is,

$$x = \frac{7}{4} + \frac{5}{4} \quad \text{or} \quad x = -\frac{7}{4} + \frac{5}{4}$$

$$= \frac{7+5}{4} = \frac{12}{4} = 3 \quad \text{or} \quad = \frac{-7+5}{4} = \frac{-2}{4} = \frac{-1}{2}$$

$\therefore$ $x = -\frac{1}{2}, 3$ are the roots of the given equation.

Thus, the solution set is $\left\{-\frac{1}{2}, 3\right\}$.

EXERCISE 1.1

1. Write the following quadratic equations in the standard form and point out pure quadratic equations.

(i). $(x + 7)(x - 3) = -7$ (ii) $\frac{x^2+4}{3} - \frac{x}{7} = 1$

(iii) $\frac{x}{x+1} + \frac{x+1}{x} = 6$ (iv) $\frac{x+4}{x-2} - \frac{x-2}{x} + 4 = 0$

(v) $\frac{x+3}{x+4} - \frac{x-5}{x} = 1$ (vi) $\frac{x+1}{x+2} + \frac{x+2}{x+3} = \frac{25}{12}$

2. Solve by factorization:

(i) $x^2 - x - 20 = 0$ (ii) $3y^2 = y(y - 5)$

(iii) $4 - 32x = 17x^2$ (iv) $x^2 - 11x = 152$

(v) $\frac{x+1}{x} + \frac{x}{x+1} = \frac{25}{12}$ (vi) $\frac{2}{x-9} = \frac{1}{x-3} - \frac{1}{x-4}$

3. Solve the following equations by completing square:

(i) $7x^2 + 2x - 1 = 0$ (ii) $ax^2 + 4x - a = 0\ ,\ a \neq 0$

(iii) $11x^2 - 34x + 3 = 0$ (iv) $lx^2 + mx + n = 0,\ l \neq 0$

(v) $3x^2 + 7x = 0$ (vi) $x^2 - 2x - 195 = 0$

(vii) $-x^2 + \frac{15}{2} = \frac{7}{2}x$ (viii) $x^2 + 17x + \frac{33}{4} = 0$

(ix) $4 - \frac{8}{3x+1} = \frac{3x^2+5}{3x+1}$ (x) $7(x + 2a)^2 + 3a^2 = 5a(7x + 23a)$

1.3 Quadratic Formula:

1.3. (i) Derivation of quadratic formula by using completing square method.

The quadratic equation in standard form is

$$ax^2 + bx + c = 0 \quad , \quad a \neq 0$$

Dividing each term of the equation by a, we get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Shifting constant term $\frac{c}{a}$ to the right, we have

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Adding $\left(\frac{b}{2a}\right)^2$ on both sides, we obtain

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}=\frac{b^2}{4a^2}-\frac{c}{a}$$

or $$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

Taking square root of both sides, we get

$$\sqrt{\left(x+\frac{b}{2a}\right)^2}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

or $$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$\Rightarrow$ $$x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Thus, $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$, $a\neq 0$ is known as **"quadratic formula"**.

1.3 (ii) Use of quadratic formula:

The quadratic formula is a useful tool for solving all those equations which can or can not be factorized. The method to solve the quadratic equation by using quadratic formula is illustrated through the following examples.

Example 1: Solve the quadratic equation $2+9x=5x^2$ by using quadratic formula.

Solution: $2+9x=5x^2$

The given equation in standard form can be written as

$$5x^2-9x-2=0$$

Comparing with the standard quadratic equation $ax^2+bx+c=0$, we observe that

$$a=5\ ,\ b=-9\ ,\ c=-2$$

Putting the values of a, b and c in quadratic formula

Activity: Using quadratic formula, write the solution set of $x^2+x-2=0$.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$, we have

$$x=\frac{-(-9)\pm\sqrt{(-9)^2-4(5)(-2)}}{2(5)}$$

or $$x=\frac{9\pm\sqrt{81+40}}{10}=\frac{9\pm\sqrt{121}}{10}=\frac{9\pm 11}{10}$$

Either $x=\dfrac{9+11}{10}$ or $x=\dfrac{9-11}{10}$, that is,

$x=\dfrac{20}{10}=2$ or $x=\dfrac{-2}{10}=-\dfrac{1}{5}$

$\therefore$ $2, -\frac{1}{5}$ are the roots of the given equation.

Thus, the solution set is $\left\{-\frac{1}{5}, 2\right\}$.

Example 2: Using quadratic formula, solve the equation $\frac{2x+1}{x+2} - \frac{x-2}{x+4} = 0$.

Solution: $\frac{2x+1}{x+2} - \frac{x-2}{x+4} = 0$

Simplifying and writing in the standard form

$(2x+1)(x+4) - (x-2)(x+2) = 0$

$2x^2 + 8x + x + 4 - (x^2 - 4) = 0$

$2x^2 + 9x + 4 - x^2 + 4 = 0$

or $x^2 + 9x + 8 = 0$

Here $a = 1$, $b = 9$, $c = 8$

Using quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we have

$$x = \frac{-9 \pm \sqrt{(9)^2 - 4 \times 1 \times 8}}{2 \times 1}$$

$$= \frac{-9 \pm \sqrt{81 - 32}}{2} = \frac{-9 \pm \sqrt{49}}{2} = \frac{-9 \pm 7}{2}$$

$\Rightarrow$ $x = \frac{-9+7}{2} = \frac{-2}{2} = -1$

or $x = \frac{-9-7}{2} = \frac{-16}{2} = -8$

$\therefore$ $-1, -8$ are the roots of the given equation. Thus, the solution set is $\{-8, -1\}$.

EXERCISE 1.2

1. Solve the following equations using quadratic formula:

(i) $2 - x^2 = 7x$

(ii) $5x^2 + 8x + 1 = 0$

(iii) $\sqrt{3}\,x^2 + x = 4\sqrt{3}$

(iv) $4x^2 - 14 = 3x$

(v) $6x^2 - 3 - 7x = 0$

(vi) $3x^2 + 8x + 2 = 0$

(vii) $\frac{3}{x-6} - \frac{4}{x-5} = 1$

(viii) $\frac{x+2}{x-1} - \frac{4-x}{2x} = 2\frac{1}{3}$

(ix) $\frac{a}{x-b} + \frac{b}{x-a} = 2$

(x) $-(l+m) - lx^2 + (2l+m)x = 0, \ l \neq 0$

1.4 Equations reducible to quadratic form

We now discuss different types of equations, which can be reduced to a quadratic equation by some proper substitution.

Type (i) The equations of the type $ax^4 + bx^2 + c = 0$

Replacing $x^2 = y$ in equation $ax^4 + bx^2 + c = 0$,

we get a quadratic equation in y.

Example 1: Solve the equation $x^4 - 13x^2 + 36 = 0$.

Solution: $x^4 - 13x^2 + 36 = 0$ (i)

Let $x^2 = y$. Then $x^4 = y^2$

Equation (i) becomes

$y^2 - 13y + 36 = 0$ which can be factorized as

$y^2 - 9y - 4y + 36 = 0$

$y(y - 9) - 4(y - 9) = 0$

$(y - 9)(y - 4) = 0$

Either $y - 9 = 0$ or $y - 4 = 0$, that is,

$y = 9$ or $y = 4$

Put $y = x^2$

$x^2 = 9$ or $x^2 = 4$

$\Rightarrow$ $x = \pm 3$ or $x = \pm 2$

$\therefore$ The solution set is $\{\pm 2, \pm 3\}$

Type (ii) The equations of the type $ap(x) + \dfrac{b}{p(x)} = c$

Example 2: Solve the equation $2(2x - 1) + \dfrac{3}{2x - 1} = 5$.

Solution: Given that $2(2x - 1) + \dfrac{3}{2x - 1} = 5$ (i)

Let $2x - 1 = y$.

Then the equation (i) becomes

$$2y + \frac{3}{y} = 5 \quad \text{or} \quad 2y^2 + 3 = 5y$$

$\Rightarrow$ $2y^2 - 5y + 3 = 0$

Using quadratic formula

$$y = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 2 \times 3}}{2 \times 2}$$

$$= \frac{5 \pm \sqrt{25-24}}{4} = \frac{5 \pm \sqrt{1}}{4} = \frac{5 \pm 1}{4}$$

We have $y = \frac{5+1}{4} = \frac{6}{4} = \frac{3}{2}$ or $y = \frac{5-1}{4} = \frac{4}{4} = 1$

When $y = \frac{3}{2}$,

$$2x - 1 = \frac{3}{2} \qquad (\because \; y = 2x - 1)$$

$$\Rightarrow \quad 2x = \frac{3}{2} + 1 = \frac{5}{2} \quad \Rightarrow \; x = \frac{5}{4}$$

When $y = 1$,

$$2x - 1 = 1 \qquad (\because \; y = 2x - 1)$$

$$\Rightarrow \quad 2x = 1 + 1 = 2 \quad \Rightarrow \; x = 1.$$

Thus, the solution set is $\left\{1, \frac{5}{4}\right\}$.

Type (iii) Reciprocal equations of the type:

$$a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0 \;\text{ or }\; ax^4 + bx^3 + cx^2 + bx + a = 0$$

An equation is said to be a **reciprocal equation**, if it remains unchanged, when x is replaced by $\frac{1}{x}$.

Replacing x by $\frac{1}{x}$ in $ax^4 - bx^3 + cx^2 - bx + a = 0$, we have

$$a\left(\frac{1}{x}\right)^4 - b\left(\frac{1}{x}\right)^3 + c\left(\frac{1}{x}\right)^2 - b\left(\frac{1}{x}\right) + a = 0 \text{ which is simplified as}$$

$a - bx + cx^2 - bx^3 + ax^4 = 0$. We get the same equation.

Thus $ax^4 - bx^3 + cx^2 - bx + a = 0$ is a reciprocal equation.

The method for solving reciprocal equation is illustrated through an example.

Example 3: Solve the equation $2x^4 - 5x^3 - 14x^2 - 5x + 2 = 0$.

Solution: $2x^4 - 5x^3 - 14x^2 - 5x + 2 = 0$

Dividing each term by x^2

$$\frac{2x^4}{x^2} - \frac{5x^3}{x^2} - \frac{14x^2}{x^2} - \frac{5x}{x^2} + \frac{2}{x^2} = 0$$

$$2x^2 - 5x - 14 - \frac{5}{x} + \frac{2}{x^2} = 0$$

$$2\left(x^2 + \frac{1}{x^2}\right) - 5\left(x + \frac{1}{x}\right) - 14 = 0 \qquad \text{(i)}$$

Let $x + \frac{1}{x} = y$. Then $x^2 + \frac{1}{x^2} = y^2 - 2$

So equation (i) becomes $2(y^2 - 2) - 5y - 14 = 0$ or $2y^2 - 4 - 5y - 14 = 0$

$2y^2 - 5y - 18 = 0$

$2y^2 - 9y + 4y - 18 = 0$ or $y(2y - 9) + 2(2y - 9) = 0$

$\Rightarrow (2y - 9)(y + 2) = 0$

Either $2y - 9 = 0$ or $y + 2 = 0$

As $y = x + \frac{1}{x}$, so we have

$2\left(x + \frac{1}{x}\right) - 9 = 0$ or $x + \frac{1}{x} + 2 = 0$

$2x^2 - 9x + 2 = 0$ or $x^2 + 2x + 1 = 0$

By quadratic formula, we get

$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4 \times 2 \times 2}}{2 \times 2}$ or $x = \frac{-2 \pm \sqrt{(2)^2 - 4 \times 1 \times 1}}{2 \times 1}$

$\Rightarrow x = \frac{9 \pm \sqrt{81 - 16}}{4}$ or $x = \frac{-2 \pm \sqrt{4 - 4}}{2}$

$= \frac{9 \pm \sqrt{65}}{4}$ or $x = \frac{-2 \pm 0}{2} \Rightarrow x = -1, -1$

Thus, the solution set is $\left\{-1, \frac{9 - \sqrt{65}}{4}, \frac{9 + \sqrt{65}}{4}\right\}$.

Type (iv) Exponential equations:

In **exponential equations,** variable occurs in exponent.

The method of solving such equations is illustrated through an example.

Example 4: Solve the equation $5^{1+x} + 5^{1-x} = 26$.

Solution: $5^{1+x} + 5^{1-x} = 26$

$5^1 . 5^x + 5^1 . 5^{-x} = 26$ or $5 . 5^x + \frac{5}{5^x} - 26 = 0$ (i)

Let $5^x = y$. Then equation (i) becomes

$5y + \frac{5}{y} - 26 = 0$

$5y^2 + 5 - 26y = 0$ or $5y^2 - 26y + 5 = 0$

$5y^2 - 25y - y + 5 = 0$

$5y(y - 5) - 1(y - 5) = 0$

$(y - 5)(5y - 1) = 0$

Either $y - 5 = 0$ or $5y - 1 = 0$, that is,

$$y = 5 \quad \text{or} \quad 5y = 1 \Rightarrow y = \frac{1}{5}$$

Put $\quad y = 5^x$

$$5^x = 5^1 \quad \text{or} \quad 5^x = 5^{-1} \Rightarrow x = 1 \quad \text{or} \quad x = -1$$

$\therefore$ The solution set is $\{\pm 1\}$.

Type (v) The equations of the type:

$$(x + a)(x + b)(x + c)(x + d) = k, \text{ where } a + b = c + d$$

Example 5: Solve the equation $(x - 1)(x + 2)(x + 8)(x + 5) = 19$.

Solution: $(x - 1)(x + 2)(x + 8)(x + 5) = 19$

or $\quad [(x - 1)(x + 8)][(x + 2)(x + 5)] - 19 = 0 \qquad (\because -1 + 8 = 2 + 5)$

$$(x^2 + 7x - 8)(x^2 + 7x + 10) - 19 = 0 \qquad \text{(i)}$$

Let $\quad x^2 + 7x = y$

Then eq. (i) becomes

$$(y - 8)(y + 10) - 19 = 0$$
$$y^2 + 2y - 80 - 19 = 0$$
$$y^2 + 2y - 99 = 0$$
$$y^2 + 11y - 9y - 99 = 0$$
$$y(y + 11) - 9(y + 11) = 0$$
$$(y + 11)(y - 9) = 0$$

Either $\quad y + 11 = 0 \quad$ or $\quad y - 9 = 0$

Put $\quad y = x^2 + 7x$, so

$x^2 + 7x + 11 = 0 \quad$ or $\quad x^2 + 7x - 9 = 0$

Solving by quadratic formula, we have

$$x = \frac{-7 \pm \sqrt{(7)^2 - 4(1)(11)}}{2(1)} \quad \text{or} \quad x = \frac{-7 \pm \sqrt{(7)^2 - 4(1)(-9)}}{2(1)}$$

$$= \frac{-7 \pm \sqrt{49 - 44}}{2} = = \frac{-7 \pm \sqrt{5}}{2} \qquad = \frac{-7 \pm \sqrt{49 + 36}}{2} = \frac{-7 \pm \sqrt{85}}{2}$$

$\therefore$ The solution set is $\left\{\frac{-7 \pm \sqrt{5}}{2}, \frac{-7 \pm \sqrt{85}}{2}\right\}$.

EXERCISE 1.3

Solve the following equations.

1. $2x^4 - 11x^2 + 5 = 0$
2. $2x^4 = 9x^2 - 4$
3. $5x^{1/2} = 7x^{1/4} - 2$
4. $x^{2/3} + 54 = 15x^{1/3}$
5. $3x^{-2} + 5 = 8x^{-1}$
6. $(2x^2 + 1) + \frac{3}{2x^2 + 1} = 4$
7. $\frac{x}{x - 3} + 4\left(\frac{x - 3}{x}\right) = 4$
8. $\frac{4x + 1}{4x - 1} + \frac{4x - 1}{4x + 1} = 2\frac{1}{6}$
9. $\frac{x - a}{x + a} - \frac{x + a}{x - a} = \frac{7}{12}$
10. $x^4 - 2x^3 - 2x^2 + 2x + 1 = 0$

11. $2x^4 + x^3 - 6x^2 + x + 2 = 0$

12. $4.2^{2x+1} - 9.2^x + 1 = 0$

13. $3^{2x+2} = 12.3^x - 3$

14. $2^x + 64.2^{-x} - 20 = 0$

15. $(x + 1)(x + 3)(x - 5)(x - 7) = 192$

16. $(x - 1)(x - 2)(x - 8)(x + 5) + 360 = 0$

1.5 Radical equations

An equation involving expression under the radical sign is called a **radical equation**.

e.g., $\sqrt{x+3} = x + 1$ and $\sqrt{x-1} = \sqrt{x-2} + 1$

1.5 (i) Equations of the type: $\sqrt{ax+b} = cx + d$

Example 1: Solve the equation $\sqrt{3x+7} = 2x + 3$.

Solution: $\sqrt{3x+7} = 2x + 3$ (i)

Squaring both sides of the equation (i), we get

$$(\sqrt{3x+7})^2 = (2x+3)^2$$

or $3x + 7 = 4x^2 + 12x + 9$

Simplifying the above equation, we have

$$4x^2 + 9x + 2 = 0$$

Applying quadratic formula,

$$x = \frac{-9 \pm \sqrt{(9)^2 - 4 \times 4 \times 2}}{2 \times 4}$$

$$= \frac{-9 \pm \sqrt{81 - 32}}{8} = \frac{-9 \pm \sqrt{49}}{8} = \frac{-9 \pm 7}{8}$$

Note: Extraneous root is introduced either by squaring the given equation or clearing it of fractions.

$\therefore$ $x = \frac{-9+7}{8} = \frac{-2}{8} = \frac{-1}{4}$

or $x = \frac{-9-7}{8} = \frac{-16}{8} = -2$

Checking:

Putting $x = -\frac{1}{4}$ in the equation (i), we have

$$\sqrt{3\left(-\frac{1}{4}\right) + 7} = 2\left(-\frac{1}{4}\right) + 3 \Rightarrow \sqrt{\frac{-3+28}{4}} = -\frac{1}{2} + 3 \Rightarrow \sqrt{\frac{25}{4}} = \frac{5}{2} \text{ which is true.}$$

Putting $x = -2$ in the equation (i), we get

$$\sqrt{3(-2)+7} = 2(-2)+3 \quad \Rightarrow \sqrt{1} = -1 \text{ which is not true.}$$

On checking, we find that $x = -2$ does not satisfy the equation (i), so it is an extraneous root. Thus the solution set is $\left\{-\frac{1}{4}\right\}$.

1.5 (ii) Equations of the type $\sqrt{x+a} + \sqrt{x+b} = \sqrt{x+c}$

Example 2: Solve the equation $\sqrt{x+3} + \sqrt{x+6} = \sqrt{x+11}$.

Solution:

$$\sqrt{x+3} + \sqrt{x+6} = \sqrt{x+11} \qquad \text{(i)}$$

Squaring both sides of the equation (i), we have

$$x+3+x+6+2\left(\sqrt{x+3}\right)\left(\sqrt{x+6}\right) = x+11$$

or $$2\sqrt{x^2+9x+18} = -x+2 \qquad \text{(ii)}$$

Squaring both sides of the equation (ii), we get

$$4\left(x^2+9x+18\right) = x^2-4x+4$$

or $$3x^2+40x+68 = 0$$

Applying quadratic formula, we get

$$x = \frac{-40 \pm \sqrt{(40)^2 - 4\times 3\times 68}}{2\times 3} = \frac{-40 \pm \sqrt{1600-816}}{6}$$

$$= \frac{-40 \pm \sqrt{784}}{6} = \frac{-40 \pm 28}{6}$$

We have $x = \frac{-40+28}{6} = \frac{-12}{6} = -2$ or $x = \frac{-40-28}{6} = \frac{-68}{6} = \frac{-34}{3}$

Checking: Putting $x = \frac{-34}{3}$ in the equation (i), we have

$$\sqrt{\frac{-34}{3}+3} + \sqrt{\frac{-34}{3}+6} = \sqrt{\frac{-34}{3}+11}$$

or $$\sqrt{\frac{-34+9}{3}} + \sqrt{\frac{-34+18}{3}} = \sqrt{\frac{-34+33}{3}}$$

$\Rightarrow$ $$\sqrt{\frac{25}{3}\times(-1)} + \sqrt{\frac{16}{3}\times(-1)} = \sqrt{\frac{1}{3}\times(-1)}$$

$$\Rightarrow \frac{5}{\sqrt{3}}i + \frac{4}{\sqrt{3}}i = \frac{1}{\sqrt{3}}i \text{ which is not true.}$$

As $\frac{-34}{3}$ is extraneous root, so the solution set is $\{-2\}$.

1.5(iii) Equations of the type: $\sqrt{x^2+px+m}+\sqrt{x^2+px+n}=q$

Example 3: Solve the equation $\sqrt{x^2-3x+36}-\sqrt{x^2-3x+9}=3$.

Solution: $\sqrt{x^2-3x+36}-\sqrt{x^2-3x+9}=3$

Let $x^2-3x=y$

Then $\sqrt{y+36}-\sqrt{y+9}=3$

Squaring both sides, we get

$y+36+y+9-2\left(\sqrt{y+36}\right)\left(\sqrt{y+9}\right)=9$

$2y+45-2\sqrt{(y+36)\,(y+9)}=9$

$-2\sqrt{y^2+45y+324}=-2y-36$ or $-2\sqrt{y^2+45y+324}=-2(y+18)$

$\Rightarrow \sqrt{y^2+45y+324}=y+18$

Again squaring both sides, we get

$y^2+45y+324=y^2+36y+324$

$9y=0 \quad \Rightarrow \quad y=0$

As $x^2-3x=y$, so $x^2-3x=0$

$\Rightarrow \quad x(x-3)=0$

Either $x=0$ or $x-3=0 \Rightarrow x=3$

$\therefore$ $x=0\,,3$ are the roots of the equation.

Thus, the solution set is $\{0, 3\}$.

EXERCISE 1.4

Solve the following equations.

1. $2x+5=\sqrt{7x+16}$
2. $\sqrt{x+3}=3x-1$
3. $4x=\sqrt{13x+14}-3$
4. $\sqrt{3x+100}-x=4$
5. $\sqrt{x+5}+\sqrt{x+21}=\sqrt{x+60}$
6. $\sqrt{x+1}+\sqrt{x-2}=\sqrt{x+6}$
7. $\sqrt{11-x}-\sqrt{6-x}=\sqrt{27-x}$
8. $\sqrt{4a+x}-\sqrt{a-x}=\sqrt{a}$
9. $\sqrt{x^2+x+1}-\sqrt{x^2+x-1}=1$
10. $\sqrt{x^2+3x+8}+\sqrt{x^2+3x+2}=3$
11. $\sqrt{x^2+3x+9}+\sqrt{x^2+3x+4}=5$

MISCELLANEOUS EXERCISE - 1

1. Multiple Choice Questions

Four possible answers are given for the following questions. Tick (✓) the correct answer.

(i) Standard form of quadratic equation is

(a) $bx + c = 0$, $b \neq 0$ (b) $ax^2 + bx + c = 0$, $a \neq 0$

(c) $ax^2 = bx$, $a \neq 0$ (d) $ax^2 = 0$, $a \neq 0$

(ii) The number of terms in a standard quadratic equation $ax^2 + bx + c = 0$ is

(a) 1 (b) 2 (c) 3 (d) 4

(iii) The number of methods to solve a quadratic equation is

(a) 1 (b) 2 (c) 3 (d) 4

(iv) The quadratic formula is

(a) $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (b) $x = \dfrac{b \pm \sqrt{b^2 - 4ac}}{2a}$

(c) $x = \dfrac{-b \pm \sqrt{b^2 + 4ac}}{2a}$ (d) $x = \dfrac{b \pm \sqrt{b^2 + 4ac}}{2a}$

(v) Two linear factors of $x^2 - 15x + 56$ are

(a) $(x - 7)$ and $(x + 8)$ (b) $(x + 7)$ and $(x - 8)$

(c) $(x - 7)$ and $(x - 8)$ (d) $(x + 7)$ and $(x + 8)$

(vi) An equation, which remains unchanged when x is replaced by $\dfrac{1}{x}$ is called a/an

(a) Exponential equation (b) Reciprocal equation

(c) Radical equation (d) None of these

(vii) An equation of the type $3^x + 3^{2-x} + 6 = 0$ is a/an

(a) Exponential equation (b) Radical equation

(c) Reciprocal equation (d) None of these

(viii) The solution set of equation $4x^2 - 16 = 0$ is

(a) $\{\pm 4\}$ (b) $\{4\}$ (c) $\{\pm 2\}$ (d) ± 2

(ix) An equation of the form $2x^4 - 3x^3 + 7x^2 - 3x + 2 = 0$ is called a/an

(a) Reciprocal equation (b) Radical equation

(c) Exponential equation (d) None of these

2. Write short answers of the following questions.

(i) Solve $x^2 + 2x - 2 = 0$

(ii) Solve by factorization $5x^2 = 15x$

(iii) Write in standard form $\dfrac{1}{x+4} + \dfrac{1}{x-4} = 3$

(iv) Write the names of the methods for solving a quadratic equation.

(v) Solve $\left(2x - \dfrac{1}{2}\right)^2 = \dfrac{9}{4}$ (vi) Solve $\sqrt{3x + 18} = x$

(vii) Define quadratic equation.

(viii) Define reciprocal equation.

(ix) Define exponential equation.

(x) Define radical equation.

3. Fill in the blanks

(i) The standard form of the quadratic equation is________.

(ii) The number of methods to solve a quadratic equation are ________.

(iii) The name of the method to derive a quadratic formula is ________.

(iv) The solution of the equation $ax^2 + bx + c = 0$, $a \neq 0$ is ________.

(v) The solution set of $25x^2 - 1 = 0$ is ________.

(vi) An equation of the form $2^{2x} - 3.2^x + 5 = 0$ is called a/an ________ equation.

(vii) The solution set of the equation $x^2 - 9 = 0$ is ________.

(viii) An equation of the type $x^4 + x^3 + x^2 + x + 1 = 0$ called a/an ________ equation.

(ix) A root of an equation, which do not satisfy the equation is called ________ root.

(x) An equation involving impression of the variable under ________ is called radical equation.

SUMMARY

- An equation which contains the square of the unknown (variable) quantity, but no higher power, is called a **quadratic equation** or an equation of the **second degree.**
- A **second degree equation** in one variable x, $ax^2 + bx + c = 0$
  where $a \neq 0$ and a, b, c are real numbers, is called the **general or standard form** of a quadratic equation.
- An equation is said to be a **reciprocal equation,** if it remains unchanged, when x is replaced by $\frac{1}{x}$.
- In **exponential equations,** variables occur in exponents.
- An equation involving expression under the **radical sign** is called a **radical equation.**
- Quadratic formula for $ax^2 + bx + c = 0$, $a \neq 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Any quadratic equation is solved by the following three methods.
  (i) Factorization (ii) Completing square
  (iii) Quadratic formula

Unit-2

THEORY OF QUADRATIC EQUATIONS

In this unit, students will learn how to

- *define discriminant ($b^2 - 4ac$) of the quadratic expression $ax^2 + bx + c$.*
- *find discriminant of a given quadratic equation.*
- *discuss the nature of roots of a quadratic equation through discriminant.*
- *determine the nature of roots of a given quadratic equation and verify the result by solving the equation.*
- *determine the value of an unknown involved in a given quadratic equation when the nature of its roots is given.*
- *find cube roots of unity.*
- *recognize complex cube roots of unity as ω and ω^2.*
- *prove the properties of cube roots of unity.*
- *use properties of cube roots of unity to solve appropriate problems.*
- *find the relation between the roots and the coefficients of a quadratic equation.*
- *find the sum and product of roots of a given quadratic equation without solving it.*
- *find the value(s) of unknown(s) involved in a given quadratic equation when*
  - *sum of roots is equal to a multiple of the product of roots,*
  - *sum of the squares of roots is equal to a given number,*
  - *roots differ by a given number,*
  - *roots satisfy a given relation (e.g., the relation $2\alpha + 5\beta = 7$ where α and β are the roots of given equation),*
  - *both sum and product of roots are equal to a given number.*
- *define symmetric functions of roots of a quadratic equation.*
- *evaluate a symmetric function of the roots of a quadratic equation in terms of its coefficients.*

- *establish the formula,*
  x^2 – (sum of roots) x + (product of roots) = 0,
  to find a quadratic equation from the given roots.
- *form the quadratic equation whose roots, for example, are of the type:*
  - $2\alpha + 1, 2\beta + 1,$
  - $\alpha^2, \beta^2,$
  - $\frac{1}{\alpha}, \frac{1}{\beta},$
  - $\frac{\alpha}{\beta}, \frac{\beta}{\alpha},$
  - $\alpha + \beta, \frac{1}{\alpha} + \frac{1}{\beta},$

  where α, β are the roots of a given quadratic equation.
- *describe the method of synthetic division.*
- *use synthetic division to*
  - *find quotient and remainder when a given polynomial is divided by a linear polynomial,*
  - *find the value(s) of unknown(s) if the zeros of a polynomial are given,*
  - *find the value(s) of unknown(s) if the factors of a polynomial are given,*
  - *solve a cubic equation if one root of the equation is given,*
  - *solve a biquadratic (quartic) equation if two of the real roots of the equation are given.*
- *solve a system of two equations in two variables when*
  - *one equation is linear and the other is quadratic,*
  - *both the equations are quadratic.*
- *solve the real life problems leading to quadratic equations.*

2.1 Nature of the roots of a quadratic equation

On solving quadratic equations, we get different kinds of roots. Now we will discuss the nature or characteristics of the roots of the quadratic equation without actually solving it.

2.1.1 Discriminant ($b^2 - 4ac$) of the quadratic expression $ax^2 + bx + c$.

We know that two roots of the equation $ax^2 + bx + c = 0$, $a \neq 0$ (i)

are $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

The nature of these roots depends on the value of the expression "$b^2 - 4ac$" which is called the **"discriminant"** of the quadratic equation (i) or the quadratic expression $ax^2 + bx + c$.

2.1.2 To find the discriminant of a given quadratic equation.

We explain the procedure to find the discriminant of a given quadratic equation through the following example:

Example 1: Find the discriminant of the following equations.

(a) $2x^2 - 7x + 1 = 0$ (b) $x^2 - 3x + 3 = 0$

Solution:

(a) $2x^2 - 7x + 1 = 0$

Here $a = 2$, $b = -7$, $c = 1$

Disc. $= b^2 - 4ac$

$= (-7)^2 - 4(2)(1)$

$= 49 - 8 = 41$

(b) $x^2 - 3x + 3 = 0$

Here $a = 1$, $b = -3$, $c = 3$

Disc. $= b^2 - 4ac$

$= (-3)^2 - 4(1)(3)$

$= 9 - 12 = -3$

2.1.3 Nature of the roots of a quadratic equation through discriminant.

The roots of the quadratic equation $ax^2 + bx + c = 0$, $(a \neq 0)$ are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and its discriminant is $b^2 - 4ac$.

When a, b and c are rational numbers.

(i) If $b^2 - 4ac > 0$ and is a perfect square, then the roots are rational (real) and unequal.

(ii) If $b^2 - 4ac > 0$ and is not a perfect square, then the roots are irrational (real) and unequal.

(iii) If $b^2 - 4ac = 0$, then the roots are rational (real) and equal.

(iv) If $b^2 - 4ac < 0$, then the roots are imaginary (complex conjugates).

2.1.4 Determine the nature of the roots of a given quadratic equation and verify the result by solving the equation.

We illustrate the procedure through the following examples:

Example 2: Using discriminant, find the nature of the roots of the following equations and verify the results by solving the equations.

(a) $x^2 - 5x + 5 = 0$ (b) $2x^2 - x + 1 = 0$

(c) $x^2 + 8x + 16 = 0$ (d) $7x^2 + 8x + 1 = 0$

Solution: **(a)** $x^2 - 5x + 5 = 0$

Compare with the standard quadratic equation

$$ax^2 + bx + c = 0$$

Here $a = 1, b = -5$ and $c = 5$

$$\text{Disc.} = b^2 - 4ac$$

$$= (-5)^2 - 4(1)(5) = 25 - 20 = 5 > 0$$

As the Disc. is positive and is not a perfect square.

Therefore, the roots are irrational (real) and unequal.

Now solving the equation $x^2 - 5x + 5 = 0$ by quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(5)}}{2(1)} = \frac{5 \pm \sqrt{25 - 20}}{2} = \frac{5 \pm \sqrt{5}}{2}$$

Evidently, the roots are irrational (real) and unequal.

(b) $2x^2 - x + 1 = 0$

Here $a = 2$, $b = -1$ and $c = 1$

$$\text{Disc.} = b^2 - 4ac$$

$$= (-1)^2 - 4(2)(1) = 1 - 8 = -7 < 0$$

As the Disc. is negative,

therefore, the roots of the equation are imaginary and unequal.

Verification by solving the equation.

$$2x^2 - x + 1 = 0$$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(1)}}{2(2)}$$

$$= \frac{1 \pm \sqrt{1 - 8}}{4} = \frac{1 \pm \sqrt{-7}}{4}$$

Evidently, the roots are imaginary and unequal.

(c) $x^2 + 8x + 16 = 0$

Here $a = 1, b = 8$ and $c = 16$

$$\text{Disc.} = b^2 - 4ac$$

$$= (8)^2 - 4(1)(16)$$

$$= 64 - 64 = 0$$

As the discriminant is zero, therefore

the roots are rational (real) and equal.

Verification by solving the equation.

$$x^2 + 8x + 16 = 0$$

$$(x + 4)^2 = 0$$

$$\Rightarrow \quad x = -4, -4$$

So the roots are rational (real) and equal.

(d) $7x^2 + 8x + 1 = 0$

Here $a = 7$, $b = 8$ and $c = 1$

$$\text{Disc.} = b^2 - 4ac$$

$$= (8)^2 - 4(7)(1)$$

$$= 64 - 28 = 36 = (6)^2$$

which is positive and perfect square.

$\therefore$ The roots are rational (real) and unequal.

Now solving the equation by factors, we get

$$7x^2 + 8x + 1 = 0$$

$$7x^2 + 7x + x + 1 = 0$$

$$7x(x + 1) + 1(x + 1) = 0$$

$$(x + 1)(7x + 1) = 0$$

Either $x + 1 = 0$ or $7x + 1 = 0$, that is

$$x = -1 \quad \text{or} \quad 7x = -1 \Rightarrow x = -\frac{1}{7}$$

Thus, the roots are (real) rational and unequal.

2.1.5 To determine the value of an unknown involved in a given quadratic equation when nature of its roots is given.

We illustrate the procedure through the following example:

Examples 3: Find k, if the roots of the equation

$(k + 3)x^2 - 2(k + 1)x - (k + 1) = 0$ are equal, if $k \neq -3$.

Solution: $(k + 3)x^2 - 2(k + 1)x - (k + 1) = 0$

Here $a = k + 3$, $b = -2(k + 1)$ and $c = -(k + 1)$

$\because$ As roots are equal, so Disc. = 0, that is,

$\therefore$ $b^2 - 4ac = 0$

$$[-2(k + 1)]^2 - 4(k + 3)[-(k + 1)] = 0$$

$$4[k + 1]^2 + 4(k + 3)(k + 1) = 0 \quad \text{or} \quad 4(k + 1)(k + 1 + k + 3) = 0$$

$$\Rightarrow \quad 4(k + 1)(2k + 4) = 0 \quad \text{or} \quad 8(k + 1)(k + 2) = 0$$

$$\Rightarrow \quad k + 1 = 0 \quad \text{or} \quad k + 2 = 0$$

$$\Rightarrow \quad k = -1 \quad \text{or} \quad k = -2$$

Thus, roots will be equal if $k = -2, -1$.

EXERCISE 2.1

1. Find the discriminant of the following given quadratic equations:
   (i) $2x^2 + 3x - 1 = 0$ (ii) $6x^2 - 8x + 3 = 0$
   (iii) $9x^2 - 30x + 25 = 0$ (iv) $4x^2 - 7x - 2 = 0$
2. Find the nature of the roots of the following given quadratic equations and verify the result by solving the equations:
   (i) $x^2 - 23x + 120 = 0$ (ii) $2x^2 + 3x + 7 = 0$
   (iii) $16x^2 - 24x + 9 = 0$ (iv) $3x^2 + 7x - 13 = 0$
3. For what value of k, the expression
   $k^2x^2 + 2(k + 1)x + 4$ is perfect square.
4. Find the value of k, if the roots of the following equations are equal.
   (i) $(2k - 1)x^2 + 3kx + 3 = 0$
   (ii) $x^2 + 2(k + 2)x + (3k + 4) = 0$
   (iii) $(3k + 2)x^2 - 5(k + 1)x + (2k + 3) = 0$
5. Show that the equation $x^2 + (mx + c)^2 = a^2$ has equal roots,
   if $c^2 = a^2(1 + m^2)$
6. Find the condition that the roots of the equation $(mx + c)^2 - 4ax = 0$ are equal.
7. If the roots of the equation $(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$ are equal, then $a = 0$ or $a^3 + b^3 + c^3 = 3abc$.
8. Show that the roots of the following equations are rational.
   (i) $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$
   (ii) $(a + 2b)x^2 + 2(a + b + c)x + (a + 2c) = 0$
9. For all values of k, prove that the roots of the equation
   $x^2 - 2\left(k + \frac{1}{k}\right)x + 3 = 0$, $k \neq 0$ are real.
10. Show that the roots of the equation
    $(b - c)x^2 + (c - a)x + (a - b) = 0$ are real.

2.2 Cube Roots of Unity and Their Properties.

2.2.1 The cube roots of unity.

Let a number x be the cube root of unity,

i.e., $x = (1)^{1/3}$

or $x^3 = 1$

$\Rightarrow$ $x^3 - 1 = 0$

$(x^3) - (1)^3 = 0$

$(x - 1)(x^2 + x + 1) = 0$ [using $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$]

Either $x - 1 = 0$ or $x^2 + x + 1 = 0$

$\Rightarrow \quad x = 1 \quad$ or $\quad x = \dfrac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)}$

$$= \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

$\therefore$ Three **cube roots** of unity are

$1, \dfrac{-1+i\sqrt{3}}{2}$ and $\dfrac{-1-i\sqrt{3}}{2}$, where $i = \sqrt{-1}$.

2.2.2 Recognise complex cube roots of unity as ω and ω².

The two complex cube roots of unity are $\dfrac{-1+\sqrt{-3}}{2}$ and $\dfrac{-1-\sqrt{-3}}{2}$.

If we name anyone of these as ω (pronounced as omega), then the other is ω^2. We shall prove this statement in the next article.

2.2.3 Properties of cube roots of unity.

(a) Prove that each of the complex cube roots of unity is the square of the other.

Proof: The complex cube roots of unity are $\dfrac{-1+\sqrt{-3}}{2}$ and $\dfrac{-1-\sqrt{-3}}{2}$.

We prove that

$$\left(\frac{-1+\sqrt{-3}}{2}\right)^2 = \frac{-1-\sqrt{-3}}{2} \quad \text{and} \quad \left(\frac{-1-\sqrt{-3}}{2}\right)^2 = \frac{-1+\sqrt{-3}}{2}$$

$$\left(\frac{-1+\sqrt{-3}}{2}\right)^2 = \frac{1+(-3)-2\sqrt{-3}}{4} = \frac{-2-2\sqrt{-3}}{4} = \frac{2(-1-\sqrt{-3})}{4} = \frac{-1-\sqrt{-3}}{2}$$

$$\left(\frac{-1-\sqrt{-3}}{2}\right)^2 = \frac{1+(-3)+2\sqrt{-3}}{4} = \frac{-2+2\sqrt{-3}}{4} = \frac{2(-1+\sqrt{-3})}{4} = \frac{-1+\sqrt{-3}}{2}$$

Thus, each of the complex cube root of unity is the square of the other, that is,

if $\omega = \dfrac{-1+\sqrt{-3}}{2}$, then $\omega^2 = \dfrac{-1-\sqrt{-3}}{2}$ and if $\omega = \dfrac{-1-\sqrt{-3}}{2}$, then $\omega^2 = \dfrac{-1+\sqrt{-3}}{2}$.

(b) Prove that the product of three cube roots of unity is one.

Proof: Three cube roots of unity are

$1, \dfrac{-1+\sqrt{-3}}{2}$ and $\dfrac{-1-\sqrt{-3}}{2}$

The **product** of cube roots of unity $= (1)\left(\frac{-1+\sqrt{-3}}{2}\right)\left(\frac{-1-\sqrt{-3}}{2}\right)$

$$= \frac{(-1)^2 - (\sqrt{-3})^2}{4} = \frac{1-(-3)}{4} = \frac{1+3}{4} = \frac{4}{4} = 1$$

i.e., $(1)(\omega)(\omega^2) = 1$ or $\omega^3 = 1$

Remember that:
$\omega^4 = \omega^3 . \omega = 1 . \omega = \omega$

(c) Prove that each complex cube root of unity is reciprocal of the other.

Proof: We know that $\omega^3 = 1 \Rightarrow \omega . \omega^2 = 1$, so

$$\omega = \frac{1}{\omega^2} \quad \text{or} \quad \omega^2 = \frac{1}{\omega}$$

Thus, each complex cube root of unity is **reciprocal** of the other.

(d) Prove that the sum of all the cube roots of unity is zero.

i.e., $\mathbf{1 + \omega + \omega^2 = 0}$

Proof: The cube roots of unity are

$$1, \frac{-1+\sqrt{-3}}{2} \text{ and } \frac{-1-\sqrt{-3}}{2}.$$

If $\omega = \frac{-1+\sqrt{-3}}{2}$, then $\omega^2 = \frac{-1-\sqrt{-3}}{2}$

The **sum** of all the roots $= 1 + \omega + \omega^2$

$$= 1 + \frac{-1+\sqrt{-3}}{2} + \frac{-1-\sqrt{-3}}{2}$$

$$= \frac{2-1+\sqrt{-3}-1-\sqrt{-3}}{2} = \frac{0}{2} = 0$$

Thus, $1 + \omega + \omega^2 = 0$

We can easily deduce the following results, that is,

(i) $1 + \omega^2 = -\omega$ (ii) $1 + \omega = -\omega^2$ (iii) $\omega + \omega^2 = -1$

2.2.4 Use of properties of cube roots of unity to solve appropriate problems.

We can reduce the higher powers of ω into 1, ω and ω^2.

e.g., $\omega^7 = (\omega^3)^2 . \omega = (1)^2 . \omega = \omega$

$\omega^{23} = (\omega^3)^7 . \omega^2 = (1)^7 . \omega^2 = \omega^2$

$\omega^{63} = (\omega^3)^{21} = (1)^{21} = 1$

$$\omega^{-5} = \frac{1}{\omega^5} = \frac{1}{\omega^3 . \omega^2} = \frac{1}{1 . \omega^2} = \frac{\omega^3}{\omega^2} = \omega$$

$$\omega^{-16} = \frac{1}{\omega^{16}} = \frac{1}{(\omega^3)^5 . \omega}$$

$$= \frac{1}{(1)^5 . \omega} = \frac{\omega^3}{\omega} = \omega^2$$

$\omega^{-27} = \dfrac{1}{\omega^{27}} = \dfrac{1}{(\omega^3)^9} = \dfrac{1}{(1)^9} = 1$

Example 1: Evaluate $(-1+\sqrt{-3})^8 + (-1-\sqrt{-3})^8$.

Solution: $(-1+\sqrt{-3})^8 + (-1-\sqrt{-3})^8$

$= \left[2\left(\dfrac{-1+\sqrt{-3}}{2}\right)\right]^8 + \left[2\left(\dfrac{-1-\sqrt{-3}}{2}\right)\right]^8$

$= (2\omega)^8 + (2\omega^2)^8$

$= 256\ \omega^8 + 256\ \omega^{16}$

$= 256\ [\omega^8 + \omega^{16}]$

$= 256\ [(\omega^3)^2 . \omega^2 + (\omega^3)^5 . \omega]$ $\quad (\because \omega^3 = 1)$

$= 256\ [\omega^2 + \omega]$ $\quad (\omega + \omega^2 = -1)$

$= 256\ (-1) = -256$

Example 2: Prove that $x^3 - y^3 = (x-y)(x-\omega y)(x-\omega^2 y)$.

Solution: $x^3 - y^3 = (x-y)(x-\omega y)(x-\omega^2 y)$

R.H.S $= (x-y)(x-\omega y)(x-\omega^2 y)$

$= (x-y)[x^2 - \omega^2 xy - \omega xy + \omega^3 y^2]$

$= (x-y)[x^2 - xy(\omega^2+\omega) + (1)y^2]$

$= (x-y)[x^2 - xy(-1) + y^2]$

$= (x-y)[x^2 + xy + y^2]$

$= x^3 - y^3 =$ L.H.S

EXERCISE 2.2

1. Find the cube roots of $-1, 8, -27, 64$.
2. Evaluate

   (i) $(1-\omega-\omega^2)^7$ (ii) $(1-3\omega-3\omega^2)^5$

   (iii) $(9+4\omega+4\omega^2)^3$ (iv) $(2+2\omega-2\omega^2)(3-3\omega+3\omega^2)$

   (v) $(-1+\sqrt{-3})^6 + (-1-\sqrt{-3})^6$ (vi) $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^9 + \left(\dfrac{-1-\sqrt{-3}}{2}\right)^9$

   (vii) $\omega^{37} + \omega^{38} - 5$ (viii) $\omega^{-13} + \omega^{-17}$

3. Prove that $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$.
4. Prove that $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)$ ..... $2n$ factors $= 1$.

2.3 Roots and co-efficients of a quadratic equation.

We know that $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$ are roots of the equation $ax^2+bx+c=0$ where a, b are coefficients of x^2 and x respectively. While c is the constant term.

2.3.1 Relation between roots and co-efficients of a quadratic equation.

If $\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$,

then we can find the **sum** and the **product** of the roots as follows.

Sum of the roots $= \alpha + \beta$

$$= \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}$$

Product of the roots $= \alpha\beta$

$$= \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$= \frac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2} = \frac{b^2-(b^2-4ac)}{4a^2}$$

$$= \frac{b^2-b^2+4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

If we denote the sum of roots and product of roots by S and P respectively, then

$$S = -\frac{b}{a} = -\frac{\text{Co-efficient of } x}{\text{Co-efficient of } x^2}$$

and $$P = \frac{c}{a} = \frac{\text{Constant term}}{\text{Co-efficient of } x^2}.$$

2.3.2 The sum and the product of the roots of a given quadratic equation without solving it.

We illustrate the method through the following example.

Example 1: Without solving, find the sum and product of the roots of the equations.

(a) $3x^2-5x+7=0$ (b) $x^2+4x-9=0$

Solution: **(a)** Let α and β be the roots of the equation

$3x^2-5x+7=0$

Then sum of roots $= \alpha+\beta = -\frac{b}{a} = -\left(\frac{-5}{3}\right) = \frac{5}{3}$

and product of roots $= \alpha\beta = \frac{c}{a} = \frac{7}{3}$

(b) Let α and β be the roots of the equation $x^2+4x-9=0$

Then $\alpha + \beta = -\frac{b}{a} = -\frac{4}{1} = -4$

and $\alpha\beta = \frac{c}{a} = \frac{-9}{1} = -9$

2.3.3 To find unknown values involved in a given quadratic equation.

The procedure is illustrated through the following examples.

(a) Sum of the roots is equal to a multiple of the product of the roots.

Example 1: Find the value of h, if the sum of the roots is equal to 3-times the product of the roots of the equation $3x^2 + (9-6h)x + 5h = 0$.

Solution: Let α, β be the roots of the equation

$$3x^2 + (9-6h)x + 5h = 0$$

Then $\alpha + \beta = -\frac{b}{a} = -\left(\frac{9-6h}{3}\right) = \frac{6h-9}{3}$

$$\alpha\beta = \frac{c}{a} = \frac{5h}{3}$$

Since $\alpha + \beta = 3(\alpha\beta)$

$$\frac{6h-9}{3} = 3\left(\frac{5h}{3}\right) \quad \text{or} \quad \frac{3(2h-3)}{3} = 5h$$

$$2h - 3 = 5h \Rightarrow 2h - 5h = 3$$

$$-3h = 3 \Rightarrow h = \frac{3}{-3} = -1$$

(b) Sum of the squares of the roots is equal to a given number.

Example 2: Find p, if the sum of the squares of the roots of the equation $4x^2 + 3px + p^2 = 0$ is unity.

Solution: If α, β are the roots of $4x^2 + 3px + p^2 = 0$,

then $\alpha + \beta = -\frac{b}{a} = -\frac{3p}{4}$

and $\alpha\beta = \frac{c}{a} = \frac{p^2}{4}$

Since $\alpha^2 + \beta^2 = 1$ (Given)

$$(\alpha + \beta)^2 - 2\alpha\beta = 1$$

$$\Rightarrow \left(\frac{-3p}{4}\right)^2 - 2\left(\frac{p^2}{4}\right) = 1 \quad \text{or} \quad \frac{9p^2}{16} - \frac{p^2}{2} = 1$$

$$\Rightarrow 9p^2 - 8p^2 = 16 \Rightarrow p^2 = 16 \Rightarrow p = \pm 4$$

(c) Two roots differ by a given number.

Example 3: Find h, if the roots of the equation $x^2 - hx + 10 = 0$ differ by 3.

Solution: Let α and $\alpha - 3$ be the roots of $x^2 - hx + 10 = 0$.

Then $\alpha + \alpha - 3 = -\frac{b}{a} = -\left(\frac{-h}{1}\right) = h$

$2\alpha - 3 = h \Rightarrow 2\alpha = h + 3 \Rightarrow \alpha = \frac{h+3}{2}$ (i)

and $\alpha(\alpha - 3) = \frac{c}{a} = \frac{10}{1} = 10$ or $\alpha(\alpha - 3) = 10$ (ii)

Putting value of α from equation (i) in equation (ii), we get

$$\left(\frac{h+3}{2}\right)\left(\frac{h+3}{2} - 3\right) = 10 \Rightarrow \left(\frac{h+3}{2}\right)\left(\frac{h+3-6}{2}\right) = 10$$

$$\left(\frac{h+3}{2}\right)\left(\frac{h-3}{2}\right) = 10 \Rightarrow h^2 - 9 = 40, \text{ that is,}$$

$$h^2 = 49 \Rightarrow h = \pm 7$$

(d) Roots satisfy a given relation
(e.g. $2\alpha + 5\beta = 7$, where α, β are the roots of a given equation).

Example 4: Find p, if the roots α, β of the equation $x^2 - 5x + p = 0$, satisfy the relation $2\alpha + 5\beta = 7$.

Solution: If α, β are the roots of the equation $x^2 - 5x + p = 0$.

Then $\alpha + \beta = -\frac{b}{a} = -\left(\frac{-5}{1}\right) = 5$

$\alpha + \beta = 5 \Rightarrow \beta = 5 - \alpha$ (i)

and $\alpha\beta = \frac{c}{a} = \frac{p}{1} = p \Rightarrow \alpha\beta = p$ (ii)

Since $2\alpha + 5\beta = 7$ (Given) (iii)

Put the value of β from equation (i) in equation (iii)

$$2\alpha + 5(5 - \alpha) = 7$$

$2\alpha + 25 - 5\alpha = 7$ or $-3\alpha = 7 - 25$, that is

$-3\alpha = -18 \Rightarrow \alpha = 6$ (iv)

$\beta = 5 - 6 = -1$ Use (i) and (iv)

Put the values of α and β in eq. (ii)

$$6(-1) = p \Rightarrow p = -6$$

(e) Both sum and product of the roots are equal to a given number.

Example 5: Find m, if sum and product of the roots of the equation $5x^2 + (7 - 2m)x + 3 = 0$ is equal to a given number, say λ.

Solution: Let α, β be the roots of the equation

$$5x^2 + (7 - 2m)x + 3 = 0$$

Then $\alpha + \beta = -\frac{b}{a} = -\frac{7-2m}{5} = \frac{2m-7}{5}$

and $\alpha\beta = \frac{c}{a} = \frac{3}{5}$

Let $\alpha + \beta = \lambda$ (i) and $\alpha\beta = \lambda$ (ii)

Then from (i) and (ii) $\alpha + \beta = \alpha\beta$, that is,

$$\therefore \quad \frac{2m-7}{5} = \frac{3}{5} \Rightarrow 2m - 7 = 3 \Rightarrow 2m = 10 \Rightarrow m = 5$$

EXERCISE 2.3

1. Without solving, find the sum and the product of the roots of the following quadratic equations.

(i)	$x^2 - 5x + 3 = 0$	(ii)	$3x^2 + 7x - 11 = 0$
(iii)	$px^2 - qx + r = 0$	(iv)	$(a + b)x^2 - ax + b = 0$
(v)	$(l + m)x^2 + (m + n)x + n - l = 0$	(vi)	$7x^2 - 5mx + 9n = 0$

2. Find the value of k, if
   (i) sum of the roots of the equation $2kx^2 - 3x + 4k = 0$ is twice the product of the roots.
   (ii) sum of the roots of the equation $x^2 + (3k - 7)x + 5k = 0$ is $\frac{3}{2}$ times the product of the roots.
3. Find k, if
   (i) sum of the squares of the roots of the equation
   $4kx^2 + 3kx - 8 = 0$ is 2.
   (ii) sum of the squares of the roots of the equation
   $x^2 - 2kx + (2k + 1) = 0$ is 6.
4. Find p, if
   (i) the roots of the equation $x^2 - x + p^2 = 0$ differ by unity.
   (ii) the roots of the equation $x^2 + 3x + p - 2 = 0$ differ by 2.
5. Find m, if
   (i) the roots of the equation $x^2 - 7x + 3m - 5 = 0$ satisfy the relation $3\alpha + 2\beta = 4$
   (ii) the roots of the equation $x^2 + 7x + 3m - 5 = 0$ satisfy the relation $3\alpha - 2\beta = 4$
   (iii) the roots of the equation $3x^2 - 2x + 7m + 2 = 0$ satisfy the relation $7\alpha - 3\beta = 18$
6. Find m, if sum and product of the roots of the following equations is equal to a given number λ.
   (i) $(2m + 3)x^2 + (7m - 5)x + (3m - 10) = 0$
   (ii) $4x^2 - (3 + 5m)x - (9m - 17) = 0$

2.4 Symmetric functions of the roots of a quadratic equation.

2.4.1 Define symmetric functions of the roots of a quadratic equation

Definition:

Symmetric functions are those functions in which the roots involved are such that the value of the expressions involving them remain unaltered, when roots are interchanged. For example, if

$$f(\alpha, \beta) = \alpha^2 + \beta^2, \text{ then}$$

$$f(\beta, \alpha) = \beta^2 + \alpha^2 = \alpha^2 + \beta^2 \qquad (\because\ \beta^2 + \alpha^2 = \alpha^2 + \beta^2)$$

$$= f(\alpha, \beta)$$

Example: Find the value of $\alpha^3 + \beta^3 + 3\alpha\beta$, if $\alpha = 2$, $\beta = 1$. Also find the value of $\alpha^3 + \beta^3 + 3\alpha\beta$ if $\alpha = 1$, $\beta = 2$.

Solution: When $\alpha = 2$ and $\beta = 1$,

$$\alpha^3 + \beta^3 + 3\alpha\beta = (2)^3 + (1)^3 + 3(2)(1)$$
$$= 8 + 1 + 6 = 15$$

When $\alpha = 1$ and $\beta = 2$,

$$\alpha^3 + \beta^3 + 3\alpha\beta = (1)^3 + (2)^3 + 3(1)(2)$$
$$= 1 + 8 + 6 = 15$$

The expression $\alpha^3 + \beta^3 + 3\alpha\beta$ represents a symmetric function of α and β.

2.4.2. Evaluate a symmetric function of roots of a quadratic equation in terms of its co-efficients

If α, β are the roots of the quadratic equation

$$ax^2 + bx + c = 0, \ (a \neq 0) \qquad \text{(i)}$$

Then $\alpha + \beta = -\dfrac{b}{a}$ (ii)

and $\alpha\beta = \dfrac{c}{a}$ (iii)

The functions given in equations (ii) and (iii) are the symmetric functions for the quadratic equation (i).

Some more symmetric functions of two variables α, β are given below:

(i) $\alpha^2 + \beta^2$ (ii) $\alpha^3 + \beta^3$

(iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ (iv) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$

Example 1: If α, β are the roots of the quadratic equation

$$px^2 + qx + r = 0 \quad , \quad (p \neq 0)$$

then evaluate $\alpha^2\beta + \alpha\beta^2$

Solution: Since α, β are the roots of $px^2 + qx + r = 0$, therefore,

$$\alpha+\beta=-\frac{q}{p} \quad \text{and} \quad \alpha\beta=\frac{r}{p}$$

$$\alpha^2\beta+\alpha\beta^2=\alpha\beta(\alpha+\beta)$$

$$=\frac{r}{p}\left(-\frac{q}{p}\right)=\frac{-qr}{p^2}$$

Example 2: If α, β are the roots of the equation $2x^2+3x+4=0$, then find the value of (i) $\alpha^2+\beta^2$ (ii) $\frac{1}{\alpha}+\frac{1}{\beta}$

Solution: Since α, β are the roots of the equation $2x^2+3x+4=0$, therefore,

$$\alpha+\beta=-\frac{3}{2} \quad \text{and} \quad \alpha\beta=\frac{4}{2}=2$$

(i) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\left(\frac{-3}{2}\right)^2-2\,(2)$

$$=\frac{9}{4}-4=\frac{9-16}{4}=-\frac{7}{4}$$

(ii) $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\beta+\alpha}{\alpha\beta}=\frac{\alpha+\beta}{\alpha\beta}=(\alpha+\beta)\frac{1}{\alpha\beta}$

$$=\left(-\frac{3}{2}\right)\left(\frac{1}{2}\right)=\frac{-3}{4}$$

EXERCISE 2.4

1. If α, β are the roots of the equation $x^2+px+q=0$, then evaluate
   (i) $\alpha^2+\beta^2$ (ii) $\alpha^3\beta+\alpha\beta^3$
   (iii) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$
2. If α, β are the roots of the equation $4x^2-5x+6=0$, then find the values of
   (i) $\frac{1}{\alpha}+\frac{1}{\beta}$ (ii) $\alpha^2\beta^2$
   (iii) $\frac{1}{\alpha^2\beta}+\frac{1}{\alpha\beta^2}$ (iv) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$
3. If α, β are the roots of the equation $lx^2+mx+n=0$ $(l\neq 0)$, then find the values of
   (i) $\alpha^3\beta^2+\alpha^2\beta^3$ (ii) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$

2.5 Formation of a quadratic equation.

If α and β are the roots of the required quadratic equation.

Let $\quad x = \alpha \quad$ and $\quad x = \beta$

i.e., $\quad x - \alpha = 0 \quad , \quad x - \beta = 0$

and $\quad (x - \alpha)(x - \beta) = 0$

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

which is the required quadratic equation in the standard form.

2.5.1 Find a quadratic equation from given roots and establish the formula x^2 – (sum of the roots) x + product of the roots = 0.

Let α, β be the roots of the quadratic equation

$$ax^2 + bx + c = 0 \quad , \quad (a \neq 0) \qquad \text{(i)}$$

Then $\quad \alpha + \beta = -\dfrac{b}{a} \quad$ and $\quad \alpha\beta = \dfrac{c}{a}$

Rewrite eq. (i) as $\quad x^2 + \dfrac{b}{a}x + \dfrac{c}{a} = 0$

or $\quad x^2 - \left(-\dfrac{b}{a}\right)x + \dfrac{c}{a} = 0$

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

or $\quad x^2$ – (sum of roots) x + product of roots = 0 , that is,

$x^2 - Sx + P = 0$ where $S = \alpha + \beta$ and $P = \alpha\beta$

Example 1: Form a quadratic equation with roots 3 and 4.

Solution: Since 3 and 4 are the roots of the required quadratic equation, therefore,

S = Sum of the roots = 3 + 4 = 7

P = Product of the roots = (3) (4) = 12

As $\quad x^2 - Sx + P = 0$, so the required quadratic equation is $x^2 - 7x + 12 = 0$

2.5.2 Form quadratic equations whose roots are of the type

(i) $2\alpha + 1 , 2\beta + 1$ $\quad$ (ii) α^2, β^2 $\quad$ (iii) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$ $\quad$ (iv) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$

(v) $\alpha + \beta, \dfrac{1}{\alpha} + \dfrac{1}{\beta}$ where α, β are the roots of a given quadratic equation.

Example 2: If α, β are the roots of the equation $2x^2 - 3x - 5 = 0$, form quadratic equations having roots

(i) $2\alpha + 1 , 2\beta + 1$ $\quad$ (ii) α^2, β^2 $\quad$ (iii) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$

(iv) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$ $\quad$ (v) $\alpha + \beta, \dfrac{1}{\alpha} + \dfrac{1}{\beta}$

Solution: As α, β are the roots of the equation

$$2x^2 - 3x - 5 = 0,$$

therefore, $\alpha + \beta = -\left(\frac{-3}{2}\right) = \frac{3}{2}$ and $\alpha\beta = \frac{-5}{2} = -\frac{5}{2}$

(i) S = Sum of the roots = $2\alpha + 1 + 2\beta + 1$

$$= 2(\alpha + \beta) + 2 = 2\left(\frac{3}{2}\right) + 2 = 5$$

P = Product of the roots = $(2\alpha + 1)(2\beta + 1)$

$$= 4\alpha\beta + 2(\alpha + \beta) + 1$$

$$= 4\left(-\frac{5}{2}\right) + 2\left(\frac{3}{2}\right) + 1$$

$$= -10 + 3 + 1 = -6$$

Using $x^2 - Sx + P = 0$, we have

$$x^2 - 5x - 6 = 0$$

(ii) S = Sum of the roots = $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$$= \left(\frac{3}{2}\right)^2 - 2\left(-\frac{5}{2}\right) = \frac{9}{4} + 5 = \frac{29}{4}$$

P = Product of the roots = $\alpha^2 \,.\, \beta^2 = (\alpha\beta)^2 = \left(-\frac{5}{2}\right)^2 = \frac{25}{4}$

Using $x^2 - Sx + P = 0$, we have

$$x^2 - \frac{29}{4}x + \frac{25}{4} = 0 \quad \Rightarrow \quad 4x^2 - 29x + 25 = 0$$

(iii) S = Sum of the roots = $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = (\alpha + \beta) \cdot \frac{1}{\alpha\beta}$

$$= \frac{3}{2} \cdot \left(-\frac{2}{5}\right) \qquad (\because \ \alpha\beta = -\frac{5}{2})$$

$$= -\frac{3}{5}$$

P = Product of the roots = $\frac{1}{\alpha} \cdot \frac{1}{\beta} = \frac{1}{\alpha\beta} = -\frac{2}{5}$ $\qquad (\because \ \alpha\beta = -\frac{5}{2})$

Using $x^2 - Sx + P = 0$, we have

$$x^2 - \left(-\frac{3}{5}\right)x + \left(-\frac{2}{5}\right) = 0 \quad \Rightarrow \quad 5x^2 + 3x - 2 = 0$$

(iv) S = Sum of the roots = $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = [(\alpha + \beta)^2 - 2\alpha\beta] \cdot \frac{1}{\alpha\beta}$

$$= \left[\left(\frac{3}{2}\right)^2 - 2\left(-\frac{5}{2}\right)\right] \times \left(-\frac{2}{5}\right) \qquad (\because \ \alpha\beta = -\frac{5}{2})$$

$$= \left(\frac{9}{4}+5\right)\times\left(-\frac{2}{5}\right) = \frac{29}{4}\times\left(-\frac{2}{5}\right) = -\frac{29}{10}$$

P = Product of the roots $= \frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha} = 1$

Using $x^2 - Sx + P = 0$, we have

$$x^2 - \left(-\frac{29}{10}\right)x + 1 = 0 \quad \Rightarrow \; 10x^2 + 29x + 10 = 0$$

(v) S = Sum of the roots $= \alpha + \beta + \frac{1}{\alpha} + \frac{1}{\beta} = \alpha + \beta + \frac{\beta + \alpha}{\alpha\beta}$

$$= (\alpha + \beta)\left(1 + \frac{1}{\alpha\beta}\right) = \frac{3}{2}\left(1 - \frac{2}{5}\right) = \frac{3}{2}\times\frac{3}{5}$$

$$= \frac{9}{10}$$

P = Product of the roots $= (\alpha + \beta)\,.\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = (\alpha + \beta)\left(\frac{\beta + \alpha}{\alpha\beta}\right)$

$$= (\alpha + \beta)^2 \times \frac{1}{\alpha\beta} = \left(\frac{3}{2}\right)^2 \times \left(-\frac{2}{5}\right)$$

$$= \frac{9}{4}\times\left(-\frac{2}{5}\right) = -\frac{9}{10}$$

Using $x^2 - Sx + P = 0$, we have

$$x^2 - \frac{9}{10}x + \left(-\frac{9}{10}\right) = 0 \quad \Rightarrow \; 10x^2 - 9x - 9 = 0$$

Example 3: If α, β are the roots of the equation $x^2 - 7x + 9 = 0$, then form an equation whose roots are 2α and 2β.

Solution: Since α, β are the roots of the equation $x^2 - 7x + 9 = 0$, therefore,

$$\alpha + \beta = -\frac{b}{a} = -\left(\frac{-7}{1}\right) = 7$$

and $\quad \alpha\beta = \frac{c}{a} = \frac{9}{1} = 9$

The roots of the required equation are $2\alpha, 2\beta$

S = Sum of roots $= 2\alpha + 2\beta = 2(\alpha + \beta) = 2(7) = 14$

P = Product of roots $= (2\alpha)\,(2\beta) = 4\alpha\beta = 4(9) = 36$

Thus the required quadratic equation will be

$x^2 - Sx + P = 0$, that is,

$x^2 - 14x + 36 = 0$

EXERCISE 2.5

1. Write the quadratic equations having following roots.

(a)	1, 5	(b)	4, 9	(c)	−2, 3
(d)	0, −3	(e)	2, −6	(f)	−1, −7
(g)	$1 + i, 1 - i$	(h)	$3 + \sqrt{2}, 3 - \sqrt{2}$		

2. If α, β are the roots of the equation $x^2 - 3x + 6 = 0$.
Form equations whose roots are

(a) $2\alpha + 1 , 2\beta + 1$ (b) α^2 , β^2 (c) $\frac{1}{\alpha}, \frac{1}{\beta}$

(d) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ (e) $\alpha + \beta, \frac{1}{\alpha} + \frac{1}{\beta}$

3. If α, β are the roots of the equation $x^2 + px + q = 0$.
Form equations whose roots are

(a) α^2, β^2 (b) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

2.6 Synthetic Division

Synthetic division is the process of finding the quotient and remainder, when a polynomial is divided by a linear polynomial. In fact synthetic division is simply a shortcut of long division method.

2.6.1 Describe the synthetic division method.

The method of synthetic division is described through the following example.

Example 1: Using synthetic division, divide the polynomial $P(x) = 5x^4 + x^3 - 3x$ by $x - 2$.

Solution: $(5x^4 + x^3 - 3x) \div (x - 2)$

From divisor, $x - a$, here $a = 2$

Now write the co-efficients of the dividend in a row with zero as the co-efficient of the missing powers of x in the descending order as shown below.

Dividend $P(x) = 5x^4 + 1 \times x^3 + 0 \times x^2 - 3x + 0 \times x^0$

Now write the co-efficients of x from dividend in a row and $a = 2$ on the left side.

	5	1	0	−3	0
2	↓	10	22	44	82
	5	11	22	41	82

(i) Write 5 the first co-efficient as it is in the row under the horizontal line.

(ii) Multiply 5 with 2 and write the result 10 under 1. write the sum of 1 + 10 = 11 under the line.

(iii) Multiply 11 with 2 and place the result 22 under 0. Add 0 and 22 and write the result 22 under the line.

(iv) Multiply 22 with 2, place the result 44 under −3. Write 41 as the sum of 44 and −3 under the line.

(v) Multiply 41 with 2 and put the result 82 under 0. The sum of 0 and 82 is 82.

In the resulting row 82 separated by the vertical line segment is the remainder and 5, 11, 22, 41 are the co-efficients of the quotient.

As the highest power of x in dividend is 4, therefore the highest power of x in quotient will be $4 - 1 = 3$.

Thus **Quotient** $= Q(x) = 5x^3 + 11x^2 + 22x + 41$ and the **Remainder** $= \boldsymbol{R} = 82$

2.6.2 Use synthetic division to

(a) find quotient and remainder, when a given polynomial is divided by a linear polynomial.

Example 2: Using synthetic division, divide $P(x) = x^4 - x^2 + 15$ by $x + 1$

Solution: $(x^4 - x^2 + 15) \div (x + 1)$

As $x + 1 = x - (-1)$, so $a = -1$

Now write the co-efficients of dividend in a row and $a = -1$ on the left side.

	1	0	−1	0	15
−1	↓	−1	1	0	0
	1	−1	0	0	15

$\therefore$ Quotient $= Q(x) = x^3 - x^2 + 0.\, x + 0 = x^3 - x^2$

and Remainder = 15

(b) find the value (s) of unknown (s), if the zeros of a polynomial are given.

Example 3: Using synthetic division, find the value of h. If the zero of polynomial

$P(x) = 3x^2 + 4x - 7h$ is 1.

Solution: $P(x) = 3x^2 + 4x - 7h$ and its zero is 1.

Then by synthetic division.

	3	4	$-7h$
1	↓	3	7
	3	7	$7 - 7h$

Remainder $= 7 - 7h$

Since 1 is the zero of the polynomial, therefore,

Remainder = 0, that is,

$7 - 7h = 0$

$7 = 7h \Rightarrow h = 1$

c) find the value (s) of unknown (s), if the factors of a polynomial are given.

Example 4: Using synthetic division, find the values of l and m, if $x - 1$ and $x + 1$ are the factors of the polynomial $P(x) = x^3 + 3lx^2 + mx - 1$

Solution: Since $x - 1$ and $x + 1$ are the factors of $P(x) = x^3 + 3lx^2 + mx - 1$.

therefore, 1 and −1 are zeros of polynomial $P(x)$.

Now by synthetic division

	1	$3l$	m	-1
1	$\downarrow$	1	$3l+1$	$3l+m+1$
	1	$3l+1$	$3l+m+1$	$3l+m$

Since 1 is the zero of polynomial, therefore remainder is zero, that is,

$3l + m = 0$ (i)

and

	1	$3l$	m	-1
-1	$\downarrow$	-1	$-3l+1$	$3l-m-1$
	1	$3l-1$	$-3l+m+1$	$3l-m-2$

Since -1 is the zero of polynomial, therefore, remainder is zero, that is,

$3l - m - 2 = 0$ (ii)

Adding eqs. (i) and (ii), we get

$$6l - 2 = 0$$

$$6l = 2 \quad \Rightarrow \quad l = \frac{2}{6} = \frac{1}{3}$$

Put the value of l in eq. (*i*) $3\left(\frac{1}{3}\right) + m = 0$ or $1 + m = 0 \Rightarrow m = -1$

Thus $l = \frac{1}{3}$ and $m = -1$

(d) solve a cubic equation, if one root of the equation is given.

Example 5: Using synthetic division, solve the equation $3x^3 - 11x^2 + 5x + 3 = 0$ when 3 is the root of the equation.

Solution: Since 3 is the root of the equation $3x^3 - 11x^2 + 5x + 3 = 0$.

Then by synthetic division, we get

	3	-11	5	3
3	$\downarrow$	9	-6	-3
	3	-2	-1	0

The depressed equation is $3x^2 - 2x - 1 = 0$

$$3x^2 - 3x + x - 1 = 0$$

$$3x(x - 1) + 1(x - 1) = 0$$

$$(x - 1)(3x + 1) = 0$$

Either $x - 1 = 0$ or $3x + 1 = 0$, that is,

$$x = 1 \qquad \text{or} \qquad 3x = -1 \;\Rightarrow\; x = -\frac{1}{3}$$

Hence 3, 1 and $-\frac{1}{3}$ are the roots of the given equation.

(e) solve a biquadratic (quartic) equation, if two of the real roots of the equation are given.

Example 6: By synthetic division, solve the equation

$x^4 - 49x^2 + 36x + 252 = 0$ having roots -2 and 6.

Solution: Since -2 and 6 are the roots of the given equation $x^4 - 49x^2 + 36x + 252 = 0$.

Then by synthetic division, we get

	1	0	–49	36	252
–2	↓	–2	4	90	–252
	1	–2	–45	126	0
6		6	24	–126	
	1	4	–21	0	

∴ The depressed equation is $x^2 + 4x - 21 = 0$

$x^2 + 7x - 3x - 21 = 0$

$x(x + 7) - 3(x + 7) = 0$

$(x + 7)(x - 3) = 0$

Either $x + 7 = 0$ or $x - 3 = 0$

$x = -7$ or $x = 3$

Thus $-2, 6, -7$ and 3 are the roots of the given equation.

EXERCISE 2.6

1. Use synthetic division to find the quotient and the remainder, when
   (i) $(x^2 + 7x - 1) \div (x + 1)$ (ii) $(4x^3 - 5x + 15) \div (x + 3)$
   (iii) $(x^3 + x^2 - 3x + 2) \div (x - 2)$
2. Find the value of h using synthetic division, if
   (i) 3 is the zero of the polynomial $2x^3 - 3hx^2 + 9$
   (ii) 1 is the zero of the polynomial $x^3 - 2hx^2 + 11$
   (iii) -1 is the zero of the polynomial $2x^3 + 5hx - 23$
3. Use synthetic division to find the values of l and m, if
   (i) $(x + 3)$ and $(x - 2)$ are the factors of the polynomial $x^3 + 4x^2 + 2lx + m$
   (ii) $(x - 1)$ and $(x + 1)$ are the factors of the polynomial $x^3 - 3lx^2 + 2mx + 6$

4. Solve by using synthetic division, if

(i) 2 is the root of the equation $x^3 - 28x + 48 = 0$

(ii) 3 is the root of the equation $2x^3 - 3x^2 - 11x + 6 = 0$

(iii) -1 is the root of the equation $4x^3 - x^2 - 11x - 6 = 0$

5. Solve by using synthetic division, if

(i) 1 and 3 are the roots of the equation $x^4 - 10x^2 + 9 = 0$

(ii) 3 and -4 are the roots of the equation $x^4 + 2x^3 - 13x^2 - 14x + 24 = 0$

2.7 Simultaneous equations

A system of equations having a common solution is called a system of **simultaneous equations**.

The set of all the ordered pairs (x, y), which satisfies the system of equations is called the **solution set** of the system.

2.7(i) Solve a system of two equations in two variables,

(*a*) when one equation is linear and the other is quadratic.

To solve a system of equations in two variables x and y. Find 'y' in terms of x from the given linear equation. Substitute the value of y in the quadratic equation, we get an other quadratic equation in one variable x. Solve this equation for x and then find the values of y.

The values of x and y provide the solution set of the system of equations.

Example 1: Solve the system of equations

$3x + y = 4$ and $3x^2 + y^2 = 52.$

Solution: The given equations are

$3x + y = 4$ (i)

and $3x^2 + y^2 = 52$ (ii)

From eq. (i) $y = 4 - 3x$ (iii)

Put value of y in eq. (ii)

$3x^2 + (4 - 3x)^2 = 52$

$3x^2 + 16 - 24x + 9x^2 - 52 = 0$

$12x^2 - 24x - 36 = 0$ or $x^2 - 2x - 3 = 0$

By factorization

$x^2 - 3x + x - 3 = 0$

$x(x - 3) + 1(x - 3) = 0$

$\Rightarrow$ $(x - 3)(x + 1) = 0$

Either $x - 3 = 0$ or $x + 1 = 0$, that is,

$x = 3$ or $x = -1$

Put the values of x in eq. (iii)

When $x = 3$ When $x = -1$

In an ordered pair (x, y), x always occupies first place and y second place.

$y = 4 - 3x$
$y = 4 - 3\,(3) = 4 - 9$
$y = -5$

$y = 4 - 3x$
$y = 4 - 3(-1) = 4 + 3$
$y = 7$

$\therefore$ The ordered pairs are $(3, -5)$ and $(-1, 7)$

Thus, the solution set is $\{(3, -5), (-1, 7)\}$

(*b*) **when both the equations are quadratic.**

The method to solve the equations is illustrated through the following examples.

Example 2: Solve the equations

$$x^2 + y^2 + 2x = 8 \quad \text{and} \quad (x-1)^2 + (y+1)^2 = 8$$

Solution: The given equations are

$$x^2 + y^2 + 2x = 8 \qquad \text{(i)}$$
$$(x-1)^2 + (y+1)^2 = 8 \qquad \text{(ii)}$$

From equation (ii), we get

$$x^2 - 2x + 1 + y^2 + 2y + 1 = 8$$

or $x^2 + y^2 - 2x + 2y = 6$ (iii)

Subtracting eq. (iii) from eq. (i), we have

$$4x - 2y = 2 \quad \text{or} \quad 2x - y = 1$$

$\Rightarrow$ $y = 2x - 1$ (iv)

Put the value of y in eq. (ii)

$$(x-1)^2 + (2x - 1 + 1)^2 = 8$$
$$x^2 - 2x + 1 + 4x^2 - 8 = 0$$
$$5x^2 - 2x - 7 = 0$$
$$5x^2 - 7x + 5x - 7 = 0 \quad \text{or} \quad x(5x-7) + 1\,(5x-7) = 0$$

$\Rightarrow$ $(5x-7)(x+1) = 0$

Either $5x - 7 = 0$ or $x + 1 = 0$, that is,

$$5x = 7 \Rightarrow x = \frac{7}{5} \quad \text{or} \quad x = -1$$

Now putting the values of x in eq. (iv), we have

When $x = \frac{7}{5}$

$$y = 2\left(\frac{7}{5}\right) - 1$$
$$y = \frac{14}{5} - 1 = \frac{14-5}{5} = \frac{9}{5}$$

When $x = -1$

$$y = 2(-1) - 1$$
$$y = -3$$

Thus, the solution set is $\left\{(-1, -3), \left(\frac{7}{5}, \frac{9}{5}\right)\right\}$.

Example 3: Solve the equations

$x^2 + y^2 = 7$ and $2x^2 + 3y^2 = 18.$

Solution: Given equations are

$$x^2 + y^2 = 7 \qquad \text{(i)}$$

$$2x^2 + 3y^2 = 18 \qquad \text{(ii)}$$

Multiply equation (i) with 3

$$3x^2 + 3y^2 = 21 \qquad \text{(iii)}$$

Subtracting equations (ii) from (iii)

$$x^2 = 3 \quad \Rightarrow \quad x = \pm\sqrt{3}$$

When $x = \sqrt{3}$, then from equation (i)

$$x^2 + y^2 = 7 \quad \text{or} \quad 3 + y^2 = 7 \quad \Rightarrow \ y^2 = 4 \quad \Rightarrow \ y = \pm 2$$

When $x = -\sqrt{3}$, then $y = \pm 2$

Thus, the required solution set is $\{(\pm\sqrt{3}, \pm 2)\}$.

Example 4: Solve the equations

$x^2 + y^2 = 20$ and $6x^2 + xy - y^2 = 0$

Solution: Given equations are

$$x^2 + y^2 = 20 \qquad \text{(i)}$$

$$6x^2 + xy - y^2 = 0 \qquad \text{(ii)}$$

The equation (ii) can be written as

$$y^2 - xy - 6x^2 = 0$$

$$\Rightarrow \qquad y = \frac{-(-x) \pm \sqrt{(-x)^2 - 4 \times 1 \times (-6x^2)}}{2 \times 1}$$

$$= \frac{x \pm \sqrt{x^2 + 24x^2}}{2} = \frac{x \pm \sqrt{25x^2}}{2}$$

$$= \frac{x \pm 5x}{2}$$

We have $y = \frac{x + 5x}{2} = \frac{6x}{2} = 3x$ or $y = \frac{x - 5x}{2} = \frac{-4x}{2} = -2x$

Substituting $y = 3x$ in the equation (i), we get

$$x^2 + (3x)^2 = 20 \quad \text{or} \quad x^2 + 9x^2 = 20$$

$$\Rightarrow \qquad 10x^2 = 20 \quad \Rightarrow \ x^2 = 2 \quad \Rightarrow \ x = \pm\sqrt{2}$$

When $x = \sqrt{2}$, $y = 3(\sqrt{2}) = 3\sqrt{2}$ and when $x = -\sqrt{2}$, $y = 3(-\sqrt{2}) = -3\sqrt{2}$

Substituting $y = -2x$ in the equation (i), we have

$$x^2 + (-2x)^2 = 20 \quad \text{or} \quad x^2 + 4x^2 = 20$$

$$\Rightarrow \qquad 5x^2 = 20 \quad \Rightarrow \ x^2 = 4 \quad \Rightarrow \ x = \pm 2$$

When $x = 2$, $y = -2(2) = -4$ and when $x = -2$, $y = -2(-2) = 4$

Thus, the solution is $\{(\sqrt{2}, 3\sqrt{2}), (-\sqrt{2}, -3\sqrt{2}), (2, -4), (-2, 4)\}$.

Example 5: Solve the equations

$x^2 + y^2 = 40$ and $3x^2 - 2xy - y^2 = 80.$

Solution: Given equations are

$$x^2 + y^2 = 40 \qquad \text{(i)}$$

$$3x^2 - 2xy - y^2 = 80 \qquad \text{(ii)}$$

Multiplying equation (i) by 2, we have

$$2x^2 + 2y^2 = 80 \qquad \text{(iii)}$$

Subtracting the equation (iii) from equation (ii), we get

$$x^2 - 2xy - 3y^2 = 0 \qquad \text{(iv)}$$

The equation (iv) can be written as

$$x^2 - 3xy + xy - 3y^2 = 0$$

or $\quad x(x - 3y) + y(x - 3y) = 0$

$\Rightarrow \quad (x - 3y)(x + y) = 0$

Either $\quad x - 3y = 0 \quad$ or $\quad x + y = 0$

$\qquad x = 3y \quad$ or $\quad x = -y$

Put in eq. (i),

$(3y)^2 + y^2 = 40$	$(-y)^2 + y^2 = 40$
$10y^2 = 40$	$2y^2 = 40$
$y^2 = 4$	$y^2 = 20$
$y = \pm 2$	$y = \pm 2\sqrt{5}$

$y = 2$	$y = -2$	$y = 2\sqrt{5}$	$y = -2\sqrt{5}$
$x = 3y$	$x = 3y$	$x = -y$	$x = -y$
$x = 3(2)$	$x = 3(-2)$	$x = -\left(2\sqrt{5}\right)$	$x = -\left(-2\sqrt{5}\right)$
$x = 6$	$x = -6$	$x = -2\sqrt{5}$	$= 2\sqrt{5}$

$\therefore$ The solution set is $\left\{(6, 2), (-6, -2), \left(2\sqrt{5}, -2\sqrt{5}\right), \left(-2\sqrt{5}, 2\sqrt{5}\right)\right\}$

EXERCISE 2.7

Solve the following simultaneous equations.

1.	$x + y = 5$	;	$x^2 - 2y - 14 = 0$
2.	$3x - 2y = 1$	;	$x^2 + xy - y^2 = 1$
3.	$x - y = 7$	;	$\frac{2}{x} - \frac{5}{y} = 2$
4.	$x + y = a - b$	;	$\frac{a}{x} - \frac{b}{y} = 2$
5.	$x^2 + (y - 1)^2 = 10$	;	$x^2 + y^2 + 4x = 1$
6.	$(x + 1)^2 + (y + 1)^2 = 5$	;	$(x + 2)^2 + y^2 = 5$
7.	$x^2 + 2y^2 = 22$	;	$5x^2 + y^2 = 29$
8.	$4x^2 - 5y^2 = 6$	;	$3x^2 + y^2 = 14$
9.	$7x^2 - 3y^2 = 4$	;	$2x^2 + 5y^2 = 7$
10.	$x^2 + 2y^2 = 3$	;	$x^2 + 4xy - 5y^2 = 0$
11.	$3x^2 - y^2 = 26$	;	$3x^2 - 5xy - 12y^2 = 0$
12.	$x^2 + xy = 5$	;	$y^2 + xy = 3$
13.	$x^2 - 2xy = 7$	;	$xy + 3y^2 = 2$

2.7(ii) Solving Real Life Problems with Quadratic Equations

There are many problems which lead to quadratic equations. To form an equation, we use symbols for unknown quantities in the problems. Then roots of the equation may provide the answer to these problems.

The procedure to solve these problems is explained in the following examples.

Example 1: Three less than a certain number multiplied by 9 less than twice the number is 104. Find the number.

Solution: Let the required number be x. Then

three less than the number = $x - 3$

and 9 less than twice the number = $2x - 9$

According to the given condition, we have

$$(x - 3)(2x - 9) = 104$$

$$2x^2 - 15x + 27 = 104$$

$$2x^2 - 15x - 77 = 0$$

Factorizing, we get

$$(2x + 7)(x - 11) = 0 \quad \Rightarrow \quad x = -\frac{7}{2}, \quad x = 11$$

i.e., $x = -\frac{7}{2}$ and 11 are the required numbers.

Example 2: The length of a rectangle is 4cm more than its breadth. If the area of the rectangle is 45cm^2. Find its sides.

Solution: Let the breadth in cm be x.

Then the length in cm will be $x + 4$.

By the given condition rectangular area = 45 cm^2, that is,

$$x(x + 4) = 45$$
$$x^2 + 4x - 45 = 0$$
$$(x + 9)(x - 5) = 0$$
$$x + 9 = 0 \quad \text{or} \quad x - 5 = 0$$
$$x = -9 \quad \text{or} \quad x = 5$$

If $x = 5$, then $x + 4 = 5 + 4 = 9$ (neglecting –ve value)

Thus the breadth is 5cm and length is 9cm.

Example 3: The sum of the co-ordinates of a point is 6 and the sum of their squares is 20. Find the co-ordinates of the point.

Solution: Let (x, y) be the co-ordinates of the point. Then by the given conditions, we have

$$x + y = 6 \qquad \text{(i)}$$
$$x^2 + y^2 = 20 \qquad \text{(ii)}$$

From eq. (i) $\quad y = 6 - x \qquad$ (iii)

Putting $y = 6 - x$ in eq. (ii), we get

$$x^2 + (6 - x)^2 = 20$$
$$x^2 + 36 + x^2 - 12x - 20 = 0$$
$$2x^2 - 12x + 16 = 0 \quad \text{or} \quad x^2 - 6x + 8 = 0$$

Factorizing, we get

$$(x - 4)(x - 2) = 0 \quad \Rightarrow \quad x = 4 \quad \text{or} \quad x = 2$$

using eq. (iii), $\quad y = 6 - 4 = 2 \quad$ or $\quad y = 6 - 2 = 4$

$\therefore$ the co-ordinates of the point are (4, 2) or (2, 4)

EXERCISE 2.8

1. The product of two positive consecutive numbers is 182. Find the numbers.
2. The sum of the squares of three positive consecutive numbers is 77. Find them.
3. The sum of five times a number and the square of the number is 204. Find the number.
4. The product of five less than three times a certain number and one less than four times the number is 7. Find the number.
5. The difference of a number and its reciprocal is $\frac{15}{4}$. Find the number.
6. The sum of the squares of two digits of a positive integral number is 65 and the number is 9 times the sum of its digits. Find the number.

7. The sum of the co-ordinates of a point is 9 and sum of their squares is 45. Find the co-ordinates of the point.

8. Find two integers whose sum is 9 and the difference of their squares is also 9.

9. Find two integers whose difference is 4 and whose squares differ by 72.

10. Find the dimensions of a rectangle, whose perimeter is 80cm and its area is $375cm^2$.

MISCELLANEOUS EXERCISE - 2

1. Multiple Choice Questions

Four possible answers are given for the following questions. Tick (✓) the correct answer.

(i) If α, β are the roots of $3x^2 + 5x - 2 = 0$, then $\alpha + \beta$ is

(a) $\frac{5}{3}$ (b) $\frac{3}{5}$ (c) $\frac{-5}{3}$ (d) $\frac{-2}{3}$

(ii) If α, β are the roots of $7x^2 - x + 4 = 0$, then $\alpha\beta$ is

(a) $\frac{-1}{7}$ (b) $\frac{4}{7}$ (c) $\frac{7}{4}$ (d) $\frac{-4}{7}$

(iii) Roots of the equation $4x^2 - 5x + 2 = 0$ are

(a) irrational (b) imaginary (c) rational (d) none of these

(iv) Cube roots of -1 are

(a) $-1, -\omega, -\omega^2$ (b) $-1, \omega, -\omega^2$ (c) $-1, -\omega, \omega^2$ (d) $1, -\omega, -\omega^2$

(v) Sum of the cube roots of unity is

(a) 0 (b) 1 (c) -1 (d) 3

(vi) Product of cube roots of unity is

(a) 0 (b) 1 (c) -1 (d) 3

(vii) If $b^2 - 4ac < 0$, then the roots of $ax^2 + bx + c = 0$ are

(a) irrational (b) rational (c) imaginary (d) none of these

(viii) If $b^2 - 4ac > 0$, but not a perfect square then roots of $ax^2 + bx + c = 0$ are

(a) imaginary (b) rational (c) irrational (d) none of these

(ix) $\frac{1}{\alpha} + \frac{1}{\beta}$ is equal to

(a) $\frac{1}{\alpha}$ (b) $\frac{1}{\alpha} - \frac{1}{\beta}$ (c) $\frac{\alpha - \beta}{\alpha\beta}$ (d) $\frac{\alpha + \beta}{\alpha\beta}$

(x) $\alpha^2 + \beta^2$ is equal to

(a) $\alpha^2 - \beta^2$ (b) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

(c) $(\alpha + \beta)^2 - 2\alpha\beta$ (d) $\alpha + \beta$

(xi) Two square roots of unity are

(a) 1, −1 (b) 1, ω (c) 1, −ω (d) ω, ω^2

(xii) Roots of the equation $4x^2 - 4x + 1 = 0$ are

(a) real, equal (b) real, unequal (c) imaginary (d) irrational

(xiii) If α, β are the roots of $px^2 + qx + r = 0$, then sum of the roots 2α and 2β is

(a) $\frac{-q}{p}$ (b) $\frac{r}{p}$ (c) $\frac{-2q}{p}$ (d) $-\frac{q}{2p}$

(xiv) If α, β are the roots of $x^2 - x - 1 = 0$, then product of the roots 2α and 2β is

(a) −2 (b) 2 (c) 4 (d) −4

(xv) The nature of the roots of equation $ax^2 + bx + c = 0$ is determined by

(a) sum of the roots (b) product of the roots

(c) synthetic division (d) discriminant

(xvi) The discriminant of $ax^2 + bx + c = 0$ is

(a) $b^2 - 4ac$ (b) $b^2 + 4ac$ (c) $-b^2 + 4ac$ (d) $-b^2 - 4ac$

2. Write short answers of the following questions.

(i) Discuss the nature of the roots of the following equations.

(a) $x^2 + 3x + 5 = 0$ (b) $2x^2 - 7x + 3 = 0$

(c) $x^2 + 6x - 1 = 0$ (d) $16x^2 - 8x + 1 = 0$

(ii) Find ω^2, if $\omega = \frac{-1 + \sqrt{-3}}{2}$

(iii) Prove that the sum of the all cube roots of unity is zero.

(iv) Find the product of complex cube roots of unity.

(v) Show that $x^3 + y^3 = (x + y)(x + \omega y)(x + \omega^2 y)$

(vi) Evaluate $\omega^{37} + \omega^{38} + 1$

(vii) Evaluate $(1 - \omega + \omega^2)^6$

(viii) If ω is cube root of unity, form an equation whose roots are 3ω and $3\omega^2$.

(ix) Using synthetic division, find the remainder and quotient when $(x^3 + 3x^2 + 2) \div (x - 2)$

(x) Using synthetic division, show that $x - 2$ is the factor of $x^3 + x^2 - 7x + 2$.

(xi) Find the sum and product of the roots of the equation $2px^2 + 3qx - 4r = 0$.

(xii) Find $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ of the roots of the equation $x^2 - 4x + 3 = 0$

(xiii) If α, β are the roots of $4x^2 - 3x + 6 = 0$, find

(a) $\alpha^2 + \beta^2$ (b) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ (c) $\alpha - \beta$

(xiv) If α, β are the roots of $x^2 - 5x + 7 = 0$, find an equation whose roots are

(a) $-\alpha, -\beta$ (b) $2\alpha, 2\beta$.

3. Fill in the blanks

(i) The discriminant of $ax^2 + bx + c = 0$ is ________.

(ii) If $b^2 - 4ac = 0$, then roots of $ax^2 + bx + c = 0$ are ________.

(iii) If $b^2 - 4ac > 0$, then the roots of $ax^2 + bx + c = 0$ are ________.

(iv) If $b^2 - 4ac < 0$, then the root of $ax^2 + bx + c = 0$ are ________.

(v) If $b^2 - 4ac > 0$ and perfect square, then the roots of $ax^2 + bx + c = 0$ are ________.

(vi) If $b^2 - 4ac > 0$ and not a perfect square, then roots of $ax^2 + bx + c = 0$ are ________.

(vii) If α, β are the roots of $ax^2 + bx + c = 0$, then sum of the roots is ________.

(viii) If α, β are the roots of $ax^2 + bx + c = 0$, then product of the roots is ________.

(ix) If α, β are the roots of $7x^2 - 5x + 3 = 0$, then the sum of the roots is ________.

(x) If α, β are the roots of $5x^2 + 3x - 9 = 0$, then product of the roots is ________.

(xi) For a quadratic equation $ax^2 + bx + c = 0$, $\frac{1}{\alpha\beta}$ is equal to ________.

(xii) Cube roots of unity are ________.

(xiii) Under usual notation sum of the cube roots of unity is ________.

(xiv) If 1, ω, ω^2 are the cube roots of unity, then ω^{-7} is equal to ________.

(xv) If α, β are the roots of the quadratic equation, then the quadratic equation is written as ________.

(xvi) If 2ω and $2\omega^2$ are the roots of an equation, then equation is ________.

SUMMARY

- **Discriminant** of the quadratic expression $ax^2 + bx + c$ is "$b^2 - 4ac$".
- The **cube roots** of unity are 1, $\frac{-1+\sqrt{-3}}{2}$ and $\frac{-1-\sqrt{-3}}{2}$.
- **Complex cube roots** of unity are ω and ω^2.
- **Properties of cube roots of unity.**

(a) The **product** of three cube roots of unity is one. *i.e.*, $(1)(\omega)(\omega^2) = \omega^3 = 1$

(b) Each of the complex cube roots of unity is **reciprocal** of the other.

(c) Each of the complex cube roots of unity is the **square** of the other.

(d) The **sum** of all the cube roots of unity is zero, *i.e.*, $1 + \omega + \omega^2 = 0$

- The **roots** of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ are
  $$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$
- The **sum** and the **product** of the roots of $ax^2 + bx + c = 0$, $a \neq 0$ are
  $$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a} \text{ respectively.}$$
- **Symmetric functions** of the roots of a quadratic equation are those functions in which the roots involved are such that the values of the expressions remain unaltered, when roots are interchanged.
- Formation of a quadratic equation if its roots are given;
  $$x^2 - (\text{sum of the roots})\, x + \text{product of the roots} = 0$$
  $$\Rightarrow \quad \mathbf{x^2 - (\alpha + \beta)\, x + \alpha\beta = 0.}$$
- **Synthetic division** is the process of finding the quotient and remainder, when a polynomial is divided by a linear polynomial.
- A system of equations having a common solution is called a system of **simultaneous equations**.

Unit-3

VARIATIONS

In this unit, students will learn how to

- *define ratio, proportions and variations (direct and inverse).*
- *find 3^{rd}, 4^{th}, mean and continued proportion.*
- *apply theorems of invertendo, alternendo, componendo, dividendo and componendo & dividendo to find proportions.*
- *define joint variation.*
- *solve problems related to joint variation.*
- *use k-method to prove conditional equalities involving proportions.*
- *solve real life problems based on variations.*

3.1 Ratio, Proportions and Variations

3.1(i) Define (*a*) ratio, (*b*) proportion and (*c*) variations (direct and inverse).

(a) Ratio

A relation between two quantities of the same kind (measured in same unit) is called **ratio**. If a and b are two quantities of the same kind and b is not zero, then the ratio of a and b is written as $a : b$ or in fraction $\frac{a}{b}$

e.g., if a hockey team wins 4 games and loses 5, then the ratio of the games won to games lost is 4 : 5 or in fraction $\frac{4}{5}$

Remember that:

(*i*) The **order** of the elements in a ratio is important.

(*ii*) In ratio $a : b$, the first term a is called **antecedent** and the second term b is called **consequent**.

(*iii*) A ratio has no **units**.

Example 1: Find the ratio of

(*i*) 200gm to 700 gm (*ii*) 1km to 600m

Solution: (*i*) Ratio of 200gm to 700 gm

$$200 : 700 = \frac{200}{700} = \frac{2}{7} = 2 : 7$$

Where 2 : 7 is the simplest (lowest) form of the ratio 200 : 700.

(*ii*) Ratio of 1km to 600m

Since 1km = 1000m

then $1000 : 600 = \frac{1000}{600} = \frac{10}{6} = \frac{5}{3} = 5 : 3$

or 1km : 600m = 1000:600

$$= \frac{1000}{100} : \frac{600}{100} = 10:6 = 5:3$$

Example 2: Find a, if the ratios $a + 3 : 7 + a$ and 4 : 5 are equal.

Solution: Since the ratios $a + 3 : 7 + a$ and 4 : 5 are equal.

$\therefore$ in fraction form

$$\frac{a+3}{7+a} = \frac{4}{5}$$

$$5(a + 3) = 4(7 + a)$$

$$5a + 15 = 28 + 4a$$

$$5a - 4a = 28 - 15$$

$$a = 13$$

Thus the given ratios will be equal if $a = 13$.

Example 3: If 2 is added in each number of the ratio 3 : 4, we get a new ratio 5 : 6. Find the numbers.

Solution: Because the ratio of two numbers is 3 : 4.

Multiply each number of the ratio with x. Then the numbers be $3x$, $4x$ and the ratio becomes $3x:4x$. Now according to the given condition

$$\frac{3x+2}{4x+2}=\frac{5}{6}$$

$$6(3x+2)=5(4x+2) \Rightarrow 18x+12=20x+10$$

$$18x-20x=10-12 \Rightarrow -2x=-2 \Rightarrow x=1$$

Thus the required numbers are

$$3x=3(1)=3$$

and $$4x=4(1)=4.$$

Example 4: Find the ratio $3a+4b : 5a+7b$ if $a : b = 5 : 8$.

Solution: Given that $a : b = 5 : 8$ or $\frac{a}{b}=\frac{5}{8}$

Now $$3a+4b : 5a+7b=\frac{3a+4b}{5a+7b}$$

$$=\frac{\frac{3a+4b}{b}}{\frac{5a+7b}{b}}=\frac{3\left(\frac{a}{b}\right)+4\left(\frac{b}{b}\right)}{5\left(\frac{a}{b}\right)+7\left(\frac{b}{b}\right)} \quad \text{(Dividing numerator and denominator by } b\text{)}$$

$$=\frac{3\left(\frac{5}{8}\right)+4(1)}{5\left(\frac{5}{8}\right)+7(1)} \qquad \left(\because \frac{a}{b}=\frac{5}{8}\right)$$

$$=\frac{\frac{15}{8}+4}{\frac{25}{8}+7}=\frac{\frac{15+32}{8}}{\frac{25+56}{8}}=\frac{47}{81}$$

Hence, $3a+4b : 5a+7b = 47 : 81$.

(b) Proportion

A **proportion** is a statement, which is expressed as an equivalence of two ratios.

If two ratios $a : b$ and $c : d$ are equal, then we can write $a : b = c : d$

Where quantities a, d are called **extremes**, while b, c are called **means**.

Symbolically the proportion of a, b, c and d is written as

$$a : b :: c : d$$

or $$a : b = c : d$$

or $\frac{a}{b} = \frac{c}{d}$

i.e., $ad = bc$

This shows that, **Product of extremes = Product of means.**

Example 5: Find x, if 60m : 90m :: 20kg : x kg

Solution: Given that 60m : 90m :: 20kg : x kg

$60 : 90 = 20 : x$

∵ Product of extremes = Product of means

∴ $60x = 90 \times 20$

$x = \frac{90 \times 20}{60} = 30$ *i.e.,* x is 30 kg

Example 6: Find the cost of 15kg of sugar, if 7 kg of sugar costs 560 rupees.

Solution: Let the cost of 15kg of sugar be x-rupees.

Then in proportion form

15kg : 7kg :: Rs. x : Rs. 560

$15 : 7 = x : 560$

∵ Product of extremes = Product of means

∴ $15 \times 560 = 7x$

$7x = 15 \times 560$

$x = \frac{15 \times 560}{7} = 15(80) = 1200$

Thus, x = Rs. 1200.

EXERCISE 3.1

1. Express the following as a ratio $a : b$ and as a fraction in its simplest (lowest) form.
   (*i*) Rs. 750 , Rs. 1250 (*ii*) 450cm , 3 m
   (*iii*) 4kg , 2kg 750gm (*iv*) 27min. 30 sec, 1 hour
   (*v*) 75°, 225°
2. In a class of 60 students, 25 students are girls and remaining students are boys. Compute the ratio of
   (*i*) boys to total students (*ii*) boys to girls
3. If $3(4x - 5y) = 2x - 7y$, find the ratio $x : y$.
4. Find the value of p, if the ratios $2p + 5 : 3p + 4$ and $3 : 4$ are equal.
5. If the ratios $3x + 1 : 6 + 4x$ and $2 : 5$ are equal. Find the value of x.

6. Two numbers are in the ratio 5 : 8. If 9 is added to each number, we get a new ratio 8 : 11. Find the numbers.

7. If 10 is added in each number of the ratio 4 :13, we get a new ratio 1 : 2. What are the numbers?

8. Find the cost of 8kg of mangoes, if 5kg of mangoes cost Rs. 250.

9. If $a : b = 7 : 6$, find the value of $3a + 5b : 7b - 5a$.

10. Complete the following:

(*i*) If $\frac{24}{7} = \frac{6}{x}$, then $4x =$ __________

(*ii*) If $\frac{5a}{3x} = \frac{15b}{y}$, then $ay =$ __________

(*iii*) If $\frac{9pq}{2lm} = \frac{18p}{5m}$, then $5q =$ __________

11. Find x in the following proportions.

(*i*) $3x - 2 : 4 :: 2x + 3 : 7$

(*ii*) $\frac{3x-1}{7} : \frac{3}{5} :: \frac{2x}{3} : \frac{7}{5}$

(*iii*) $\frac{x-3}{2} : \frac{5}{x-1} :: \frac{x-1}{3} : \frac{4}{x+4}$

(*iv*) $p^2 + pq + q^2 : x :: \frac{p^3 - q^3}{p+q} : (p-q)^2$

(*v*) $8 - x : 11 - x :: 16 - x : 25 - x$

(c) Variation:

The word variation is frequently used in all sciences. There are two types of variations: (i) Direct variation (ii) Inverse variation.

(i) Direct Variation

If two quantities are related in such a way that increase (decrease) in one quantity causes increase (decrease) in the other quantity, then this variation is called **direct variation**.

In otherwords, if a quantity y varies directly with regard to a quantity x. We say that y is **directly proportional** to x and is written as $y \propto x$ or $y = kx$. *i.e.*, $\frac{y}{x} = k$, $k \neq 0$.

The sign $\propto$ read as "varies as" is called the sign of proportionality or variation, while $k \neq 0$ is known as constant of variation.

e.g., (*i*) Faster the speed of a car, longer the distance it covers.

(*ii*) The smaller the radius of the circle, smaller the circumference is.

Example 1: Find the relation between distance d of a body falling from rest varies directly as the square of the time t, neglecting air resistance. Find k, if $d = 16$ feet for $t = 1$ sec. Also derive a relation between d and t.

Solution: Since d is the distance of the body falling from rest in time t.

Then under the given condition

$$d \propto t^2$$

i.e., $d = kt^2$ (i)

Since $d = 16$ feet and $t = 1$ sec

Then equation (i) becomes

$16 = k(1)^2$

i.e., $k = 16$

put in eq. (i) $d = 16t^2$

Which is a relationship between the distance d and time t.

Activity:

From the above example:

(*i*) Find time t, when $d = 64$ feet

(*ii*) Find distance d, when $t = 3$ sec

Example 2: If y varies directly as x, find

(*a*) the equation connecting x and y.

(*b*) the constant of variation k and the relation between x and y, when $x = 7$ and $y = 6$

(*c*) the value of y, when $x = 21$.

Solution: (*a*) Given that y varies directly as x.

Therefore $y \propto x$, *i.e.,* $y = kx$, where k is constant of variation.

(*b*) Putting $x = 7$ and $y = 6$ in equation

$y = kx$ (i)

We get $6 = 7k \Rightarrow k = \frac{6}{7}$

Put in eq. (i) $y = \frac{6}{7}x$ (ii)

(*c*) Now put $x = 21$, in equation (ii)

Then $y = \frac{6}{7}(21) = 18$

Example 3: Given that A varies directly as the square of r and $A = \frac{1782}{7}$ cm^2, when $r = 9$cm. If $r = 14$cm, then find A.

Solution: Since A varies directly as square of r

$\therefore$ $A \propto r^2$

or $A = kr^2$ (i)

$\frac{1782}{7} = k(9)^2$

$\frac{1782}{7 \times 81} = k$ or $k = \frac{22}{7}$

Put $k = \frac{22}{7}$ and $r = 14\text{cm}$ in eq. (i)

$$A = \frac{22}{7}(14)^2 = \frac{22}{7} \times 14 \times 14 = 616 \text{ cm}^2$$

Example 4: If y varies directly as cube of x and $y = 81$ when $x = 3$, so evaluate y when $x = 5$.

Solution: Given that y varies directly as cube of x.

i.e., $y \propto x^3$ or $y = kx^3$ (i) (where k is constant)

Put $y = 81$ and $x = 3$ in (i)

$$81 = k(3)^3 \Rightarrow 27k = 81 \Rightarrow k = 3$$

Now put $k = 3$ and $x = 5$ in eq. (i)

$$y = 3(5)^3 = 375$$

(ii) Inverse Variation

If two quantities are related in such a way that when one quantity increases, the other decreases is called **inverse variation**.

In otherwords, if a quantity y varies inversely with regard to quantity x. We say that y is inversely proportional to x or y varies inversely as x and is written as $y \propto \frac{1}{x}$ or $y = \frac{k}{x}$.

i.e., $xy = k$, where $k \neq 0$ is the constant of variation.

Example 1: If y varies inversely as x and $y = 8$, when $x = 4$. Find y, when $x = 16$.

Solution: Since y varies inversely as x, therefore

$$y \propto \frac{1}{x} \quad \text{or} \quad y = \frac{k}{x} \quad \text{(i)}$$

$$\Rightarrow \quad xy = k \quad \text{(ii)}$$

Putting $y = 8$ and $x = 4$ in (ii)

$$k = (x)(y)$$
$$= (4)(8) = 32$$

Now put $k = 32$ and $x = 16$ in (i) $\Rightarrow y = \frac{32}{16} = 2$

Example 2: If y varies inversely as x^2 and $y = 16$, when $x = 5$, so find x, when $y = 100$.

Solution: Since y varies inversely as x^2, therefore

$$y \propto \frac{1}{x^2} \quad \text{or} \quad y = \frac{k}{x^2}$$

$$k = x^2 y \quad \text{(i)}$$

Put $x = 5$ and $y = 16$ in (i)

$$k = (5)^2 \times 16$$
$$k = 400$$

Now put $k = 400$ and $y = 100$ in (i)

$$400 = 100x^2 \quad \text{or} \quad x^2 = \frac{400}{100} = 4$$

$$x = \pm 2$$

EXERCISE 3.2

1. If y varies directly as x, and $y = 8$ when $x = 2$, find
   (*i*) y in terms of x (*ii*) y when $x = 5$
   (*iii*) x when $y = 28$
2. If $y \propto x$, and $y = 7$ when $x = 3$ find
   (*i*) y in terms of x (*ii*) x when $y = 35$ and y when $x = 18$
3. If $R \propto T$ and $R = 5$ when $T = 8$, find the equation connecting R and T. Also find R when $T = 64$ and T when $R = 20$.
4. If $R \propto T^2$ and $R = 8$ when $T = 3$, find R when $T = 6$.
5. If $V \propto R^3$ and $V = 5$ when $R = 3$, find R when $V = 625$.
6. If w varies directly as u^3 and $w = 81$ when $u = 3$. Find w when $u = 5$.
7. If y varies inversely as x and $y = 7$ when $x = 2$, find y when $x = 126$.
8. If $y \propto \frac{1}{x}$ and $y = 4$ when $x = 3$, find x when $y = 24$.
9. If $w \propto \frac{1}{z}$ and $w = 5$ when $z = 7$, find w when $z = \frac{175}{4}$.
10. $A \propto \frac{1}{r^2}$ and $A = 2$ when $r = 3$, find r when $A = 72$.
11. $a \propto \frac{1}{b^2}$ and $a = 3$ when $b = 4$, find a when $b = 8$.
12. $V \propto \frac{1}{r^3}$ and $V = 5$ when $r = 3$, find V when $r = 6$ and r when $V = 320$.
13. $m \propto \frac{1}{n^3}$ and $m = 2$ when $n = 4$, find m when $n = 6$ and n when $m = 432$.

3.1(ii) Find 3rd, 4th, mean and continued proportion:

We are already familiar with proportions that if quantities a, b, c and d are in proportion, then $a : b :: c : d$

i.e., product of extremes = product of means

Third Proportional

If three quantities a, b and c are related as $a : b :: b : c$,

then c is called the third proportion.

Example 1: Find a third proportional of $x + y$ and $x^2 - y^2$.

Solution: Let c be the third proportional,

then $x + y : x^2 - y^2 :: x^2 - y^2 : c$

$$c\,(x + y) = (x^2 - y^2)\,(x^2 - y^2)$$

$$c = \frac{(x^2 - y^2)(x^2 - y^2)}{x + y} = \frac{(x^2 - y^2)(x - y)(x + y)}{(x + y)}$$

$$c = (x^2 - y^2)(x - y) = (x + y)(x - y)^2$$

Fourth Proportional

If four quantities a, b, c and d are related as

$$a : b :: c : d$$

Then d is called the fourth proportional.

Example 2: Find fourth proportional of $a^3 - b^3$, $a + b$ and $a^2 + ab + b^2$

Solution: Let x be the fourth proportional,

then $(a^3 - b^3) : (a + b) :: (a^2 + ab + b^2) : x$

i.e., $x(a^3 - b^3) = (a + b)(a^2 + ab + b^2)$

$$x = \frac{(a + b)(a^2 + ab + b^2)}{a^3 - b^3} = \frac{(a + b)(a^2 + ab + b^2)}{(a - b)(a^2 + ab + b^2)}$$

$$x = \frac{a + b}{a - b}$$

Mean Proportional

If three quantities a, b and c are related as $a : b :: b : c$,

then b is called the mean proportional.

Example 3: Find the mean proportional of $9p^6q^4$ and r^8.

Solution: Let m be the mean proportional,

then $9p^6q^4 : m :: m : r^8$

or $m \cdot m = 9p^6q^4 (r^8)$

$$m^2 = 9p^6q^4r^8$$

$$m = \pm\sqrt{9p^6q^4r^8} = \pm 3p^3q^2r^4$$

Continued Proportion

If three quantities a, b and c are related as

$$a : b :: b : c \quad ,$$

where a is first, b is the mean and c is the third proportional, then a, b and c are in continued proportion.

Example 4: Find p, if 12, p and 3 are in continued proportion.

Solution: Since 12, p and 3 are in continued proportion.

$\therefore$ $12 : p :: p : 3$ *i.e.,* $p \cdot p = (12)(3) \Rightarrow p^2 = 36$

Thus, $p = \pm 6$

EXERCISE 3.3

1. Find a third proportional to

(*i*)	6, 12	(*ii*)	$a^3, 3a^2$
(*iii*)	$a^2 - b^2, a - b$	(*iv*)	$(x - y)^2, x^3 - y^3$
(*v*)	$(x + y)^2, x^2 - xy - 2y^2$	(*vi*)	$\frac{p^2 - q^2}{p^3 + q^3}, \frac{p - q}{p^2 - pq + q^2}$

2. Find a fourth proportional to

(*i*)	5, 8, 15	(*ii*)	$4x^4, 2x^3, 18x^5$
(*iii*)	$15a^5b^6, 10a^2b^5, 21a^3b^3$	(*iv*)	$x^2 - 11x + 24, (x - 3), 5x^4 - 40x^3$

(*v*) $p^3 + q^3, p^2 - q^2, p^2 - pq + q^2$

(*vi*) $(p^2 - q^2)(p^2 + pq + q^2), p^3 + q^3, p^3 - q^3$

3. Find a mean proportional between

(*i*)	20, 45	(*ii*)	$20x^3y^5, 5x^7y$
(*iii*)	$15p^4qr^3, 135q^5r^7$	(*iv*)	$x^2 - y^2, \frac{x - y}{x + y}$

4. Find the values of the letter involved in the following continued proportions.

(*i*)	5, p, 45	(*ii*)	8, x, 18
(*iii*)	12, $3p - 6$, 27	(*iv*)	7, $m - 3$, 28

3.2 Theorems on Proportions

If four quantities a, b, c and d form a proportion, then many other useful properties may be deduced by the properties of fractions.

(1) Theorem of Invertendo

If $a : b = c : d$, then $b : a = d : c$

Example 1: If $3m : 2n = p : 2q$, then

$$2n : 3m = 2q : p$$

Solution: Since $3m : 2n = p : 2q$

$$\therefore \quad \frac{3m}{2n} = \frac{p}{2q}$$

By invertendo theorem

$$\frac{2n}{3m} = \frac{2q}{p}$$

i.e., $2n : 3m = 2q : p$

(2) Theorem of Alternando

If $a : b = c : d$, then $a : c = b : d$

Example 2: If $3p + 1 : 2q = 5r : 7s$, then prove that $3p + 1 : 5r = 2q : 7s$

Solution: Given that $3p + 1 : 2q = 5r : 7s$

Then $\frac{3p+1}{2q} = \frac{5r}{7s}$

By alternando theorem

$$\frac{3p+1}{5r} = \frac{2q}{7s}$$

Thus, $3p + 1 : 5r = 2q : 7s$

(3) Theorem of Componendo

If $a : b = c : d$, then

(*i*) $a + b : b = c + d : d$

and (*ii*) $a : a + b = c : c + d$

Example 3: If $m + 3 : n = p : q - 2$, then

$$m + n + 3 : n = p + q - 2 : q - 2$$

Solution: Since $m + 3 : n = p : q - 2$

$$\therefore \quad \frac{m+3}{n} = \frac{p}{q-2}$$

By componendo theorem

$$\frac{(m+3)+n}{n} = \frac{p+(q-2)}{q-2}$$

or $\frac{m+n+3}{n} = \frac{p+q-2}{q-2}$

Thus $m + n + 3 : n = p + q - 2 : q - 2$

(4) Theorem of Dividendo

If $a : b = c : d$, then

(*i*) $a - b : b = c - d : d$

and (*ii*) $a : a - b = c : c - d$

Example 4: If $m + 1 : n - 2 = 2p + 3 : 3q + 1$.

Then $m - n + 3 : n - 2 = 2p - 3q + 2 : 3q + 1$

Solution: Given that $m + 1 : n - 2 = 2p + 3 : 3q + 1$

Then $\frac{m+1}{n-2} = \frac{2p+3}{3q+1}$

By dividendo theorem

$$\frac{m-n+3}{n-2} = \frac{2p-3q+2}{3q+1}$$

Thus $m - n + 3 : n - 2 = 2p - 3q + 2 : 3q + 1$

(5) Theorem of Componendo-dividendo

If $a : b = c : d$, then

(i) $a + b : a - b = c + d : c - d$

and (ii) $a - b : a + b = c - d : c + d$

Example 5: If $m : n = p : q$.

Then prove that $3m + 7n : 3m - 7n = 3p + 7q : 3p - 7q$

Solution: Since $m : n = p : q$

or $\frac{m}{n} = \frac{p}{q}$

Multiplying both sides by $\frac{3}{7}$, we get

$$\frac{3m}{7n} = \frac{3p}{7q}$$

Then using componendo-dividendo theorem

$$\frac{3m + 7n}{3m - 7n} = \frac{3p + 7q}{3p - 7q}$$

Thus $3m + 7n : 3m - 7n = 3p + 7q : 3p - 7q$

Example 6: If $5m + 3n : 5m - 3n = 5p + 3q : 5p - 3q$,

then show that $m : n = p : q$

Solution: Given that $5m + 3n : 5m - 3n = 5p + 3q : 5p - 3q$

or $\frac{5m + 3n}{5m - 3n} = \frac{5p + 3q}{5p - 3q}$

By componendo-dividendo theorem

$$\frac{(5m + 3n) + (5m - 3n)}{(5m + 3n) - (5m - 3n)} = \frac{(5p + 3q) + (5p - 3q)}{(5p + 3q) - (5p - 3q)}$$

$$\frac{5m + 3n + 5m - 3n}{5m + 3n - 5m + 3n} = \frac{5p + 3q + 5p - 3q}{5p + 3q - 5p + 3q}$$

$$\frac{10m}{6n} = \frac{10p}{6q}$$

Multiplying both sides by $\frac{6}{10}$.

$$\frac{m}{n} = \frac{p}{q}$$

i.e., $m : n = p : q$

Example 7: Using theorem of componendo-dividendo, find the value of

$$\frac{m + 3p}{m - 3p} + \frac{m + 2q}{m - 2q}, \quad \text{if} \quad m = \frac{6pq}{p + q}.$$

Solution: Since $m = \frac{6pq}{p+q}$ or $m = \frac{(3p)(2q)}{p+q}$ (i)

$$\therefore \quad \frac{m}{3p} = \frac{2q}{p+q}$$

By componendo-dividendo theorem

$$\frac{m+3p}{m-3p} = \frac{2q+(p+q)}{2q-(p+q)} = \frac{2q+p+q}{2q-p-q}$$

$$\frac{m+3p}{m-3p} = \frac{p+3q}{q-p} \quad \text{(ii)}$$

Again from eq. (i), we have

$$\frac{m}{2q} = \frac{3p}{p+q}$$

By componendo-dividendo theorem

$$\frac{m+2q}{m-2q} = \frac{3p+(p+q)}{3p-(p+q)} = \frac{3p+p+q}{3p-p-q}$$

$$\frac{m+2q}{m-2q} = \frac{4p+q}{2p-q} \quad \text{(iii)}$$

Adding (ii) and (iii)

$$\frac{m+3p}{m-3p} + \frac{m+2q}{m-2q} = \frac{p+3q}{q-p} + \frac{4p+q}{2p-q} = -\frac{p+3q}{p-q} + \frac{4p+q}{2p-q}$$

$$= \frac{-(p+3q)(2p-q)+(p-q)(4p+q)}{(p-q)(2p-q)}$$

$$= \frac{-2p^2-5pq+3q^2+4p^2-3pq-q^2}{(p-q)(2p-q)}$$

$$= \frac{2p^2-8pq+2q^2}{(p-q)(2p-q)} = \frac{2(p^2-4pq+q^2)}{(p-q)(2p-q)}$$

Example 8: Using theorem of componendo-dividendo, solve the equation

$$\frac{\sqrt{x+3}+\sqrt{x-3}}{\sqrt{x+3}-\sqrt{x-3}} = \frac{4}{3}.$$

Solution: Given equation is $\frac{\sqrt{x+3}+\sqrt{x-3}}{\sqrt{x+3}-\sqrt{x-3}} = \frac{4}{3}$

By componendo-dividendo theorem

$$\frac{\sqrt{x+3}+\sqrt{x-3}+\sqrt{x+3}-\sqrt{x-3}}{\sqrt{x+3}+\sqrt{x-3}-\sqrt{x+3}+\sqrt{x-3}} = \frac{4+3}{4-3}$$

$$\frac{\cancel{2}\sqrt{x+3}}{\cancel{2}\sqrt{x-3}} = \frac{7}{1} \Rightarrow \sqrt{\frac{x+3}{x-3}} = 7$$

Squaring both sides

$$\frac{x+3}{x-3} = 49$$

$$x + 3 = 49\,(x - 3) \quad \Rightarrow x + 3 = 49x - 147 \quad \Rightarrow x - 49x = -147 - 3$$

$$-48x = -150 \quad \Rightarrow 48x = 150 \quad \Rightarrow x = \frac{150}{48} = \frac{25}{8}$$

Example 9: Using componendo-dividendo theorem, solve the equation $\frac{(x+3)^2 - (x-5)^2}{(x+3)^2 + (x-5)^2} = \frac{4}{5}$.

Solution: Given equation is $\frac{(x+3)^2 - (x-5)^2}{(x+3)^2 + (x-5)^2} = \frac{4}{5}$

By componendo-dividendo theorem

$$\frac{(x+3)^2 - (x-5)^2 + (x+3)^2 + (x-5)^2}{(x+3)^2 - (x-5)^2 - (x+3)^2 - (x-5)^2} = \frac{4+5}{4-5}$$

$$\frac{2(x+3)^2}{-2(x-5)^2} = \frac{9}{-1} \quad \Rightarrow \left(\frac{x+3}{x-5}\right)^2 = (3)^2$$

Taking square root $\frac{x+3}{x-5} = \pm 3$

$\frac{x+3}{x-5} = 3$ or $\frac{x+3}{x-5} = -3$

$x + 3 = 3(x - 5)$ | $x + 3 = -3(x - 5)$

$x + 3 = 3x - 15$ | $x + 3 = -3x + 15$

$-2x = -18$ | $4x = 12$

$x = 9$ | $x = 3$

∴ The solution set is $\{3, 9\}$

EXERCISE 3.4

1. Prove that $a : b = c : d$, if

(i) $\frac{4a+5b}{4a-5b} = \frac{4c+5d}{4c-5d}$ (ii) $\frac{2a+9b}{2a-9b} = \frac{2c+9d}{2c-9d}$

(iii) $\frac{ac^2+bd^2}{ac^2-bd^2} = \frac{c^3+d^3}{c^3-d^3}$ (iv) $\frac{a^2c+b^2d}{a^2c-b^2d} = \frac{ac^2+bd^2}{ac^2-bd^2}$

(v) $pa + qb : pa - qb = pc + qd : pc - qd$

(vi) $\frac{a+b+c+d}{a+b-c-d} = \frac{a-b+c-d}{a-b-c+d}$

(vii) $\dfrac{2a+3b+2c+3d}{2a+3b-2c-3d}=\dfrac{2a-3b+2c-3d}{2a-3b-2c+3d}$

(viii) $\dfrac{a^2+b^2}{a^2-b^2}=\dfrac{ac+bd}{ac-bd}$

2. Using theorem of componendo-dividendo

(i) Find the value of $\dfrac{x+2y}{x-2y}+\dfrac{x+2z}{x-2z}$, if $x=\dfrac{4yz}{y+z}$

(ii) Find the value of $\dfrac{m+5n}{m-5n}+\dfrac{m+5p}{m-5p}$, if $m=\dfrac{10np}{n+p}$

(iii) Find the value of $\dfrac{x-6a}{x+6a}-\dfrac{x+6b}{x-6b}$, if $x=\dfrac{12ab}{a-b}$

(iv) Find the value of $\dfrac{x-3y}{x+3y}-\dfrac{x+3z}{x-3z}$, if $x=\dfrac{3yz}{y-z}$

(v) Find the value of $\dfrac{s-3p}{s+3p}+\dfrac{s+3q}{s-3q}$, if $s=\dfrac{6pq}{p-q}$

(vi) Solve $\dfrac{(x-2)^2-(x-4)^2}{(x-2)^2+(x-4)^2}=\dfrac{12}{13}$

(vii) Solve $\dfrac{\sqrt{x^2+2}+\sqrt{x^2-2}}{\sqrt{x^2+2}-\sqrt{x^2-2}}=2$

(viii) Solve $\dfrac{\sqrt{x^2+8p^2}-\sqrt{x^2-p^2}}{\sqrt{x^2+8p^2}+\sqrt{x^2-p^2}}=\dfrac{1}{3}$

(ix) Solve $\dfrac{(x+5)^3-(x-3)^3}{(x+5)^3+(x-3)^3}=\dfrac{13}{14}$

3.3.(i) Joint variation

A combination of direct and inverse variations of one or more than one variables forms **joint variation.**

If a variable y varies directly as x and varies inversely as z.

Then $y \propto x$ and $y \propto \dfrac{1}{z}$

In joint variation, we write it as

$$y \propto \frac{x}{z}$$

i.e., $y = k\dfrac{x}{z}$

Where $k \neq 0$ is the constant of variation.

For example, by Newton's law of gravitation, if one body attracts another with a force (G), that varies directly as the product of their masses (m_1), (m_2) and inversely as the square of the distance (d) between them.

i.e., $G \propto \frac{m_1 m_2}{d^2}$ or $G = k\frac{m_1 m_2}{d^2}$, where $k \neq 0$ is the constant

3.3.(*ii*) Problems related to joint variation.

Procedure to solve the problems related to joint variation is explained through examples.

Example 1: If y varies jointly as x^2 and z and $y = 6$ when $x = 4$, $z = 9$. Write y as a function of x and z and determine the value of y, when $x = -8$ and $z = 12$.

Solution: Since y varies jointly as x^2 and z, therefore

$$y \propto x^2 z$$

i.e., $y = kx^2 z$ (i)

Put $y = 6, x = 4, z = 9$

$$6 = k(4)^2(9)$$

$$\frac{6}{16 \times 9} = k \quad \Rightarrow \quad k = \frac{1}{24}$$

Put $k = \frac{1}{24}$ in eq.(i), $y = \frac{1}{24}x^2 z$

Now put $x = -8$, $z = 12$ in the above equation,

$$y = \frac{1}{24}(-8)^2(12) = 32$$

Example 2: p varies jointly as q and r^2 and inversely as s and t^2, $p = 40$, when $q = 8$, $r = 5$, $s = 3$, $t = 2$. Find p in terms of q, r, s and t. Also find the value of p when $q = -2$, $r = 4$, $s = 3$ and $t = -1$.

Solution: Given that $p \propto \frac{qr^2}{st^2}$

$$p = k\frac{qr^2}{st^2} \qquad \text{(i)}$$

Put $p = 40$, $q = 8$, $r = 5$, $s = 3$ and $t = 2$

$$40 = k\frac{(8)(5)^2}{3(2)^2}$$

$$\frac{40 \times 3 \times 4}{8 \times 25} = k$$

$$k = \frac{12}{5}$$

Then eq. (i) becomes

$$p = \frac{12}{5}\,\frac{qr^2}{st^2}$$

Now for $q = -2$, $r = 4$, $s = 3$ and $t = -1$, we have

$$p = \frac{12}{5}\,\frac{(-2)\,(4)^2}{(3)\,(-1)^2} = -\frac{128}{5}$$

EXERCISE 3.5

1. If s varies directly as u^2 and inversely as v and $s = 7$ when $u = 3$, $v = 2$. Find the value of s when $u = 6$ and $v = 10$.

2. If w varies jointly as x, y^2 and z and $w = 5$ when $x = 2$, $y = 3$, $z = 10$. Find w when $x = 4$, $y = 7$ and $z = 3$.

3. If y varies directly as x^3 and inversely as z^2 and t, and $y = 16$ when $x = 4$, $z = 2$, $t = 3$. Find the value of y when $x = 2$, $z = 3$ and $t = 4$.

4. If u varies directly as x^2 and inversely as the product yz^3, and $u = 2$ when $x = 8$, $y = 7$, $z = 2$. Find the value of u when $x = 6$, $y = 3$, $z = 2$.

5. If v varies directly as the product xy^3 and inversely as z^2 and $v = 27$ when $x = 7$, $y = 6$, $z = 7$. Find the value of v when $x = 6$, $y = 2$, $z = 3$.

6. If w varies inversely as the cube of u, and $w = 5$ when $u = 3$. Find w when $u = 6$.

3.4. *K*-Method

3.4(i) Use *k*-method to prove conditional equalities involving proportions.

If $a : b :: c : d$ is a proportion, then putting each ratio equal to k

i.e., $\frac{a}{b} = \frac{c}{d} = k$

$\frac{a}{b} = k$ and $\frac{c}{d} = k$

$a = bk$ and $c = dk$

Using the above equations, we can solve certain problems relating to proportions more easily. This method is known as *k*-method. We illustrate the *k*-method through the following examples.

Example 1: If $a : b = c : d$, then show that

$$\frac{3a + 2b}{3a - 2b} = \frac{3c + 2d}{3c - 2d}$$

Solution: $a : b = c : d$

Let $\frac{a}{b} = \frac{c}{d} = k$

Then $a = bk$ and $c = dk$

To prove $\frac{3a + 2b}{3a - 2b} = \frac{3c + 2d}{3c - 2d}$

Now L.H.S $= \dfrac{3a+2b}{3a-2b} = \dfrac{3kb+2b}{3kb-2b} = \dfrac{b(3k+2)}{b(3k-2)}$

$= \dfrac{3k+2}{3k-2}$ (i)

Also R.H.S $= \dfrac{3c+2d}{3c-2d} = \dfrac{3kd+2d}{3kd-2d} = \dfrac{d(3k+2)}{d(3k-2)}$

$= \dfrac{3k+2}{3k-2}$ (ii)

$\therefore$ L.H.S = R.H.S

i.e., $\dfrac{3a+2b}{3a-2b} = \dfrac{3c+2d}{3c-2d}$

Example 2: If $a : b = c : d$, then show that

$pa + qb : ma - nb = pc + qd : mc - nd$

Solution: Let $\dfrac{a}{b} = \dfrac{c}{d} = k$, then $a = bk$ and $c = dk$

L.H.S $= pa + qb : ma - nb = \dfrac{pa+qb}{ma-nb} = \dfrac{pkb+qb}{mkb-nb}$

$= \dfrac{b(pk+q)}{b(mk-n)} = \dfrac{pk+q}{mk-n}$

R.H.S $= pc + qd : mc - nd = \dfrac{pc+qd}{mc-nd} = \dfrac{pkd+qd}{mkd-nd}$ $(c = kd)$

$= \dfrac{d(pk+q)}{d(mk-n)} = \dfrac{pk+q}{mk-n}$

i.e., $pa + qb : ma - nb = pc + qd : mc - nd$

Example 3: If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$, then show that $\dfrac{a^3+c^3+e^3}{b^3+d^3+f^3} = \dfrac{ace}{bdf}$

Solution: Let $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = k$

Then $\dfrac{a}{b} = k$, $\dfrac{c}{d} = k$ and $\dfrac{e}{f} = k$

i.e., $a = bk$, $c = dk$ and $e = fk$

To prove $\dfrac{a^3+c^3+e^3}{b^3+d^3+f^3} = \dfrac{ace}{bdf}$

Now L.H.S $= \dfrac{a^3+c^3+e^3}{a^3+d^3+f^3} = \dfrac{(bk)^3+(dk)^3+(fk)^3}{b^3+d^3+f^3}$

$= \dfrac{b^3k^3+d^3k^3+f^3k^3}{b^3+d^3+f^3} = k^3\left(\dfrac{b^3+d^3+f^3}{b^3+d^3+f^3}\right) = k^3$

Also R.H.S $= \dfrac{ace}{bdf} = \dfrac{(bk)\,(dk)\,(fk)}{bdf} = k^3\,\dfrac{bdf}{bdf} = k^3$

$\therefore$ L.H.S = R.H.S

i.e., $\dfrac{a^3 + c^3 + e^3}{b^3 + d^3 + f^3} = \dfrac{ace}{bdf}$

Example 4: If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$, then show that $\dfrac{a^2b + c^2d + e^2f}{ab^2 + cd^2 + ef^2} = \dfrac{a + c + e}{b + d + f}$

Solution: Let $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = k$

$a = bk,\ c = dk,\ e = fk$

To prove $\dfrac{a^2b + c^2d + e^2f}{ab^2 + cd^2 + ef^2} = \dfrac{a + c + e}{b + d + f}$

L.H.S. $= \dfrac{a^2b + c^2d + e^2f}{ab^2 + cd^2 + ef^2}$

$= \dfrac{(bk)^2b + (dk)^2d + (fk)^2f}{(bk)b^2 + (dk)d^2 + (fk)f^2} = \dfrac{k^2b^3 + k^2d^3 + k^2f^3}{kb^3 + kd^3 + kf^3}$

$= \dfrac{k^2\,(b^3 + d^3 + f^3)}{k\,(b^3 + d^3 + f^3)} = k$

R.H.S. $= \dfrac{a + c + e}{b + d + f} = \dfrac{bk + dk + fk}{b + d + f}$

$= \dfrac{k\,(b + d + f)}{b + d + f} = k$

L.H.S. = R.H.S.

Thus, $\dfrac{a^2b + c^2d + e^2f}{ab^2 + cd^2 + ef^2} = \dfrac{a + c + e}{b + d + f}$

EXERCISE 3.6

1. If $a : b = c : d$, $(a, b, c, d \neq 0)$, then show that

(i) $\dfrac{4a - 9b}{4a + 9b} = \dfrac{4c - 9d}{4c + 9d}$ (ii) $\dfrac{6a - 5b}{6a + 5b} = \dfrac{6c - 5d}{6c + 5d}$

(iii) $\dfrac{a}{b} = \sqrt{\dfrac{a^2 + c^2}{b^2 + d^2}}$ (iv) $a^6 + c^6 : b^6 + d^6 = a^3c^3 : b^3d^3$

(v) $p(a + b) + qb : p(c + d) + qd = a : c$

(vi) $a^2 + b^2 : \dfrac{a^3}{a + b} = c^2 + d^2 : \dfrac{c^3}{c + d}$

(vii) $\dfrac{a}{a - b} : \dfrac{a + b}{b} = \dfrac{c}{c - d} : \dfrac{c + d}{d}$

2. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$ $(a, b, c, d, e, f \neq 0)$, then show that

(i) $\frac{a}{b}=\sqrt{\frac{a^2+c^2+e^2}{b^2+d^2+f^2}}$ (ii) $\frac{ac+ce+ea}{bd+df+fb}=\left[\frac{ace}{bdf}\right]^{2/3}$

(iii) $\frac{ac}{bd}+\frac{ce}{df}+\frac{ea}{fb}=\frac{a^2}{b^2}+\frac{c^2}{d^2}+\frac{e^2}{f^2}$

3.4(ii) Real life problems based on variation

Example 1: The strength "s" of a rectangular beam varies directly as the breadth b and the square of the depth d. If a beam 9cm wide and 12cm deep will support 1200 lb. What weight a beam of 12cm wide and 9cm deep will support?

Solution: By the joint variation, we have $s \propto bd^2$

i.e., $s = kbd^2$ (i)

Put $s = 1200$, $b = 9$ and $d = 12$

$k(9)(12)^2 = 1200$

$k = \frac{1200}{9 \times 144} = \frac{25}{27}$

Put in eq. (i) $s = \frac{25}{27}bd^2$

Now for $b = 12$ and $d = 9$

$s = \frac{25}{27}(12)(9)^2 = \frac{25(12)(9)(9)}{27} = 900$ lb

Example 2: The current in a wire varies directly as the electromotive force E and inversely as the resistance R. If $I = 32$ amperes, when $E = 128$ volts and $R = 8$ ohms. Find I, when $E = 150$ volts and $R = 18$ ohms.

Solution: In joint variation, we have $I \propto \frac{E}{R}$, *i.e.,* $I = \frac{kE}{R}$ (i)

For $I = 32$, $E = 128$ and $R = 8$,

$32 = \frac{k(128)}{8} \Rightarrow \frac{32 \times 8}{128} = k \Rightarrow k = 2$

Put in eq. (i) $I = \frac{2E}{R}$.

Now for $E = 150$ and $R = 18$

$I = \frac{2(150)}{18} = \frac{50}{3}$ amp.

EXERCISE 3.7

1. The surface area A of a cube varies directly as the square of the length l of an edge and $A = 27$ square units when $l = 3$ units.

   Find (i) A when $l = 4$ units (ii) l when $A = 12$ sq. units.

2. The surface area S of the sphere varies directly as the square of radius r,

   and $S = 16\pi$ when $r = 2$. Find r when $S = 36\pi$.

3. In Hook's law the force F applied to stretch a spring varies directly as the amount of elongation S and $F = 32lb$ when $S = 1.6$ in. Find (i) S when $F = 50$ lb (ii) F when $S = 0.8$in.

4. The intensity I of light from a given source varies inversely as the square of the distance d from it. If the intensity is 20 candlepower at a distance of 12ft. from the source, find the intensity at a point 8ft. from the source.

5. The pressure P in a body of fluid varies directly as the depth d. If the pressure exerted on the bottom of a tank by a column of fluid 5ft. high is 2.25 lb/sq. in, how deep must the fluid be to exert a pressure of 9 lb/sq. in?

6. Labour costs c varies jointly as the number of workers n and the average number of days d. If the cost of 800 workers for 13 days is Rs. 286000, then find the labour cost of 600 workers for 18 days.

7. The supporting load c of a pillar varies as the fourth power of its diameter d and inversely as the square of its length l. A pillar of diameter 6 inch and of height 30 feet will support a load of 63 tons. How high a 4 inch pillar must be to support a load of 28 tons?

8. The time T required for an elevator to lift a weight varies jointly as the weight w and the lifting depth d varies inversely as the power p of the motor. If 25 sec. are required for a 4-hp motor to lift 500 lb through 40 ft, what power is required to lift 800 lb, through 120 ft in 40 sec.?

9. The kinetic energy (K.E.) of a body varies jointly as the mass "m" of the body and the square of its velocity "v". If the kinetic energy is 4320 ft/lb when the mass is 45 lb and the velocity is 24 ft/sec, determine the kinetic energy of a 3000 lb automobile travelling 44 ft/sec.

MISCELLANEOUS EXERCISE - 3

1. Multiple Choice Questions

Four possible answers are given for the following questions. Tick (✓) the correct answer.

(i) In a ratio $a : b$, a is called

(a) relation (b) antecedent

(c) consequent (d) None of these

(ii) In a ratio $x : y$, y is called

(a) relation (b) antecedent

(c) consequent (d) None of these

(iii) In a proportion $a : b : : c : d$, a and d are called,

(a) means (b) extremes

(c) third proportional (d) None of these

(iv) In a proportion $a : b : : c : d$, b and c are called

(a) means (b) extremes

(c) fourth proportional (d) None of these

(v) In continued proportion $a : b = b : c$, $ac = b^2$, b is said to be _______ proportional between a and c.

(a) third (b) fourth

(c) means (d) None of these

(vi) In continued proportion $a : b = b : c$, c is said to be ______ proportional to a and b.

(a) third (b) fourth

(c) means (d) None of these

(vii) Find x in proportion $4 : x : : 5 : 15$

(a) $\frac{75}{4}$ (b) $\frac{4}{3}$

(c) $\frac{3}{4}$ (d) 12

(viii) If $u \propto v^2$, then

(a) $u = v^2$ (b) $u = kv^2$

(c) $uv^2 = k$ (d) $uv^2 = 1$

(ix) If $y^2 \propto \frac{1}{x^3}$, then

(a) $y^2 = \frac{k}{x^3}$ (b) $y^2 = \frac{1}{x^3}$

(c) $y^2 = x^2$ (d) $y^2 = kx^3$

(x) If $\frac{u}{v} = \frac{v}{w} = k$, then

(a) $u = wk^2$ (b) $u = vk^2$

(c) $u = w^2k$ (d) $u = v^2k$

(xi) The third proportional of x^2 and y^2 is

(a) $\frac{y^2}{x^2}$ (b) x^2y^2

(c) $\frac{y^4}{x^2}$ (d) $\frac{y^2}{x^4}$

(xii) The fourth proportional w of $x : y : : v : w$ is

(a) $\frac{xy}{v}$ (b) $\frac{vy}{x}$

(c) xyv (d) $\frac{x}{vy}$

(xiii) If $a : b = x : y$, then alternando property is

(a) $\frac{a}{x} = \frac{b}{y}$ (b) $\frac{a}{b} = \frac{x}{y}$

(c) $\frac{a+b}{b} = \frac{x+y}{y}$ (d) $\frac{a-b}{x} = \frac{x-y}{y}$

(xiv) If $a : b = x : y$, then invertendo property is

(a) $\frac{a}{x} = \frac{b}{y}$ (b) $\frac{a}{a-b} = \frac{x}{x-y}$

(c) $\frac{a+b}{b} = \frac{x+y}{y}$ (d) $\frac{b}{a} = \frac{y}{x}$

(xv) If $\frac{a}{b} = \frac{c}{d}$, then componendo property is

(a) $\frac{a}{a+b} = \frac{c}{c+d}$ (b) $\frac{a}{a-b} = \frac{c}{c-d}$

(c) $\frac{ad}{bc}$ (d) $\frac{a-b}{b} = \frac{c-d}{d}$

2. Write short answers of the following questions.

(i) Define ratio and give one example.

(ii) Define proportion.

(iii) Define direct variation.

(iv) Define inverse variation.

(v) State theorem of componendo-dividendo.

(vi) Find x, if $6 : x : : 3 : 5$.

(vii) If x and y^2 varies directly, and $x = 27$ when $y = 4$. Find the value of y when $x = 3$.

(viii) If u and v varies inversely, and $u = 8$, when $v = 3$. Find v when $u = 12$.

(ix) Find the fourth proportional to 8, 7, 6.

(x) Find a mean proportional to 16 and 49.

(xi) Find a third proportional to 28 and 4.

(xii) If $y \propto \frac{x^2}{z}$ and $y = 28$ when $x = 7$, $z = 2$, then find y.

(xiii) If $z \propto xy$ and $z = 36$ when $x = 2$, $y = 3$, then find z.

(xiv) If $w \propto \frac{1}{v^2}$ and $w = 2$ when $v = 3$, then find w.

3. Fill in the blanks

(i) The simplest form of the ratio $\frac{(x+y)(x^2+xy+y^2)}{x^3-y^3}$ is ________.

(ii) In a ratio $x : y$; x is called ________.

(iii) In a ratio $a : b$; b is called ________.

(iv) In a proportion $a : b :: x : y$; a and y are called ________.

(v) In a proportion $p : q :: m : n$; q and m are called ________.

(vi) In proportion $7 : 4 :: p : 8$, $p =$ ________.

(vii) If $6 : m :: 9 : 12$, then $m =$ ________.

(viii) If x and y varies directly, then $x =$ ________.

(ix) If v varies directly as u^3, then $u^3 =$ ________.

(x) If w varies inversely as p^2, then $k =$ ________.

(xi) A third proportional of 12 and 4, is ________.

(xii) The fourth proportional of 15, 6, 5 is ________.

(xiii) The mean proportional of $4m^2n^4$ and p^6 is ________.

(xiv) The continued proportion of 4, m and 9 is ________.

SUMMARY

- A relation between two quantities of the same kind is called **ratio.**
- A **proportion** is a statement, which is expressed as equivalence of two ratios.

  If two ratios $a : b$ and $c : d$ are equal, then we can write $a : b = c : d$

- If two quantities are related in such a way that increase (decrease) in one quantity causes increase (decrease) in the other quantity is called **direct variation.**
- If two quantities are related in such a way that when one quantity increases, the other decreases is called **inverse variation.**
- Theorem on proportions:

  (1) **Theorem of Invertendo**

  If $a : b = c : d$, then $b : a = d : c$

  (2) **Theorem of Alternando**

  If $a : b = c : d$, then $a : c = b : d$

  (3) **Theorem of Componendo**

  If $a : b = c : d$, then

  (*i*) $a + b : b = c + d : d$

  and (*ii*) $a : a + b = c : c + d$

(4) **Theorem of Dividendo**

If $a : b = c : d$, then

(*i*) $a - b : b = c - d : d$

(*ii*) $a : a - b = c : c - d$

(5) **Theorem of Componendo-dividendo**

If $a : b = c : d$, then

$a + b : a - b = c + d : c - d$

- A combination of direct and inverse variations of one or more than one variable forms **joint variation**.
- ***K*-Method,**

(a) If $\frac{a}{b} = \frac{c}{d}$,

then $\frac{a}{b} = \frac{c}{d} = k$ or $a = bk$ and $c = dk$

(b) If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$, then $a = bk$, $c = dk$ and $e = fk$

Unit-4

PARTIAL FRACTIONS

In this unit, students will learn how to

- *define proper, improper and rational fraction.*
- *resolve an algebraic fraction into partial fractions when its denominator consists of*
  - *non-repeated linear factors,*
  - *repeated linear factors,*
  - *non-repeated quadratic factors,*
  - *repeated quadratic factors.*

4.1. Fraction

The quotient of two numbers or algebraic expressions is called a **fraction.** The quotient is indicated by a bar (—). We write, the dividend above the bar and the divisor below the bar. For example, $\frac{x^2+2}{x-2}$ is a fraction with $x - 2 \neq 0$. If $x - 2 = 0$, then the fraction is not defined because $x - 2 = 0 \Rightarrow x = 2$ which makes the denominator of the fraction zero.

4.1.1 Rational Fraction

An expression of the form $\frac{N(x)}{D(x)}$, where $N(x)$ and $D(x)$ are polynomials in x with real coefficients and $D(x) \neq 0$, is called a **rational fraction.**

For example, $\frac{x^2+3}{(x+1)^2(x+2)}$ and $\frac{2x}{(x-1)(x+2)}$ are rational fractions.

4.1.2 Proper Fraction:

A rational fraction $\frac{N(x)}{D(x)}$, with $D(x) \neq 0$ is called a proper fraction if degree of the polynomial $N(x)$ in the numerator is less than the degree of the polynomial $D(x)$ in the denominator. For example, $\frac{2}{x+1}, \frac{2x-3}{x^2+4}$ and $\frac{3x^2}{x^3+1}$ are proper fractions.

4.1.3 Improper Fraction:

A rational fraction $\frac{N(x)}{D(x)}$, with $D(x) \neq 0$ is called an **improper fraction** if degree of the polynomial $N(x)$ is greater or equal to the degree of the polynomial $D(x)$.

e.g., $\frac{5x}{x+2}, \frac{3x^2+2}{x^2+7x+12}, \frac{6x^4}{x^3+1}$ are improper fractions.

Every improper fraction can be reduced by division to the sum of a polynomial and a proper fraction. This means that if degree of the numerator is greater or equal to the degree of the denominator, then we can divide $N(x)$ by $D(x)$ obtaining a quotient polynomial $Q(x)$ and a remainder polynomial $R(x)$, whose degree is less than the degree of $D(x)$.

Thus $\frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$, with $D(x) \neq 0$. Where $Q(x)$ is quotient polynomial and $\frac{R(x)}{D(x)}$ is a proper fraction. For example, $\frac{x^2+1}{x+1}$ is an improper fraction.

$\therefore \quad \frac{x^2+1}{x+1} = (x-1) + \frac{2}{x+1}$ *i.e.,* an improper fraction $\frac{x^2+1}{x+1}$ has been resolved to a quotient polynomial $Q(x) = x - 1$ and a proper fraction $\frac{2}{x+1}$

Example 1: Resolve the fraction $\frac{x^3-x^2+x+1}{x^2+5}$ into proper fraction.

Solution: Let $N(x) = x^3 - x^2 + x + 1$ and $D(x) = x^2 + 5$

By long division, we have

$$\begin{array}{r|l} & x - 1 \\ x^2 + 5 & x^3 - x^2 + \ \ x + 1 \\ & \underline{-x^3 \qquad \pm 5x} \\ & \quad - x^2 - 4x + 1 \\ & \quad \underline{\mp x^2 \qquad \mp 5} \\ & \qquad - 4x + 6 \end{array}$$

$$\frac{x^3 - x^2 + x + 1}{x^2 + 5} = (x - 1) + \frac{-4x + 6}{x^2 + 5}$$

Activity: Separate proper and improper fractions

(i) $\frac{x^2 + x + 1}{x^2 + 2}$ (ii) $\frac{2x + 5}{(x + 1)(x + 2)}$ (iii) $\frac{x^3 + x^2 + 1}{x^3 - 1}$ (iv) $\frac{2x}{(x - 1)(x - 2)}$

Activity: Convert the following improper fractions into proper fractions.

(i) $\frac{3x^2 - 2x - 1}{x^2 - x + 1}$ (ii) $\frac{6x^3 + 5x^2 - 6}{2x^2 - x - 1}$

4.2 Resolution of Fraction into Partial Fractions

Consider $\frac{1}{x - 1}$, $\frac{-2}{x + 1}$, $\frac{4}{x}$, a set of three fractions each of which is prefixed by a positive or negative sign. It is easy to find a single fraction, which is equal to the sum of these fractions.

Thus $$\frac{1}{x - 1} - \frac{2}{x + 1} + \frac{4}{x} = \frac{x(x + 1) - 2x(x - 1) + 4(x - 1)(x + 1)}{x(x - 1)(x + 1)}$$

$$= \frac{x^2 + x - 2x^2 + 2x + 4x^2 - 4}{x(x - 1)(x + 1)}$$

$$= \frac{3x^2 + 3x - 4}{x(x - 1)(x + 1)}$$

The single fraction $\frac{3x^2 + 3x - 4}{x(x - 1)(x + 1)}$ is the simplified form of the given fractions and is known as **resultant fraction**. The given fractions $\frac{1}{x - 1}$, $\frac{-2}{x + 1}$ and $\frac{4}{x}$ are called components or **partial fractions**. In this chapter, we shall be given a rational fraction (or resultant fraction) and required to find its partial fractions.

Every proper fraction $\frac{N(x)}{D(x)}$, with $D(x) \neq 0$ can be resolved into an algebraic sum of partial fractions as follows:

4.2.1 Resolution of an algebraic fraction into partial fractions, when $D(x)$ consists of non-repeated linear factors.

Rule I: If linear factor $(ax + b)$ occurs as a factor of $D(x)$, then there is a partial fraction of the form $\frac{A}{ax+b}$, where A is a constant to be found.

In $\frac{N(x)}{D(x)}$, the polynomial $D(x)$ may be written as,

$D(x) = (a_1x + b_1)(a_2x + b_2) \ldots\ldots (a_nx + b_n)$ with all factors distinct.

We have, $\frac{N(x)}{D(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \frac{A_3}{a_3x+b_3} + \ldots\ldots + \frac{A_n}{a_nx+b_n}$,

where $A_1\ A_2 \ldots A_n$ are constants to be determined. The following examples illustrate how we can find these constants:

Example 1: Resolve $\frac{5x+4}{(x-4)(x+2)}$ into partial fractions.

Solution: Let $\frac{5x+4}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$ (i)

Multiplying throughout by $(x-4)(x+2)$, we get

$5x + 4 = A(x + 2) + B(x - 4)$ (ii)

Equation (ii) is an identity, which holds good for all values of x and hence for

$x = 4$ and $x = -2$.

Put $x - 4 = 0$ *i.e.*, $x = 4$ (factor corresponding to A) on both sides of the equation (ii),

we get $5(4) + 4 = A(4 + 2) \Rightarrow \boxed{A = 4}$

Put $x + 2 = 0$ *i.e.*, $x = -2$ (factor corresponding to B), we get

$5(-2) + 4 = B(-2 - 4) \Rightarrow -6B = -6 \Rightarrow \boxed{B = 1}$

Thus required partial fractions are $\frac{4}{x-4} + \frac{1}{x+2}$

Hence, $\frac{5x+4}{(x-4)(x+2)} = \frac{4}{x-4} + \frac{1}{x+2}$

This method is called the **zero's method**. This method is especially useful with linear factors in the denominator $D(x)$.

Identity: An identity is an equation, which is satisfied by all the values of the variables involved. For example, $2(x + 1) = 2x + 2$ and $\frac{2x^2}{x} = 2x$ are identities, as these equations are satisfied by all values of x.

Example 2: Resolve $\dfrac{1}{3+x-2x^2}$ into partial fractions.

Solution: $\dfrac{1}{3+x-2x^2}$ can be written as for convenience $\dfrac{-1}{2x^2-x-2}$

The denominator $D(x) = 2x^2 - x - 3 = 2x^2 - 3x + 2x - 3$

$= x(2x-3) + 1(2x-3) = (x+1)(2x-3)$

Let, $\dfrac{-1}{2x^2-x-3} = \dfrac{-1}{(x+1)(2x-3)} = \dfrac{A}{x+1} + \dfrac{B}{2x-3}$

Multiplying both the sides by $(x+1)(2x-3)$, we get

$-1 = A(2x-3) + B(x+1)$

Equating coefficients of x and constants on both sides, we get

$2A + B = 0$ (i) $\quad -3A + B = -1$ (ii)

Solving (i) and (ii), we get $A = \dfrac{1}{5}$ and $B = \dfrac{-2}{5}$

Thus, $\dfrac{1}{3+x-2x^2} = \dfrac{1}{5(x+1)} - \dfrac{2}{5(2x-3)}$

Note: General method applicable to resolve all rational fractions of the form $\dfrac{N(x)}{D(x)}$ is as follows:

(i) The numerator $N(x)$ must be of lower degree than the denominator $D(x)$.

(ii) If degree of $N(x)$ is greater than the degree of $D(x)$, then division is used and the remainder fraction $R(x)$ can be broken into partial fractions.

(iii) Make substitution of constants accordingly.

(iv) Multiply both the sides by L.C.M.

(v) Arrange the terms on both sides in descending order.

(vi) Equate the coefficients of like powers of x on both sides, we get as many as equations as there are constants in assumption.

(vii) Solving these equations, we can find the values of constants.

EXERCISE 4.1

Resolve into partial fractions.

1. $\dfrac{7x-9}{(x+1)(x-3)}$
2. $\dfrac{x-11}{(x-4)(x+3)}$
3. $\dfrac{3x-1}{x^2-1}$
4. $\dfrac{x-5}{x^2+2x-3}$
5. $\dfrac{3x+3}{(x-1)(x+2)}$
6. $\dfrac{7x-25}{(x-4)(x-3)}$
7. $\dfrac{x^2+2x+1}{(x-2)(x+3)}$
8. $\dfrac{6x^3+5x^2-7}{3x^2-2x-1}$

4.2.2 Resolution of a fraction when $D(x)$ consists of repeated linear factors.

Rule II: If a linear factor $(ax + b)$ occurs n times as a factor of $D(x)$, then there are n partial fractions of the form.

$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$, where $A_1, A_2, \cdots, A_n$ are constants and $n \geq 2$ is a positive integer.

$$\therefore \quad \frac{N(x)}{D(x)} = \frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}.$$

The method of finding constants and resolving into partial fractions is explained by the following example.

Example: Resolve $\frac{1}{(x-1)^2(x-2)}$ into partial fractions.

Solution: Let, $\frac{1}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$

Multiplying both sides by $(x-1)^2(x-2)$, we get

$$1 = A(x-1)(x-2) + B(x-2) + C(x-1)^2$$

$$\Rightarrow \quad A(x^2 - 3x + 2) + B(x-2) + C(x^2 - 2x + 1) = 1 \qquad \text{(i)}$$

Since (i) is an identity and is true for all values of x

Put $x - 1 = 0$ or $x = 1$ in (i), we get

$$B(1-2) = 1 \quad \Rightarrow \quad -B = 1 \quad \text{or} \quad B = -1$$

Put $x - 2 = 0$ or $x = 2$ in (i), we get

$$C(2-1)^2 = 1 \quad \Rightarrow \quad C = 1$$

Equating coefficients of x^2 on both the sides of (i)

$$A + C = 0 \quad \Rightarrow \quad A = -C \text{ so } A = -1$$

Hence required partial fractions are

$$\frac{-1}{x-1} - \frac{1}{(x-1)^2} + \frac{1}{(x-2)}$$

Thus, $\frac{1}{(x-1)^2(x-2)} = \frac{1}{x+2} - \frac{1}{(x-1)} - \frac{1}{(x-1)^2}$

EXERCISE 4.2

Resolve into partial fractions.

1. $\frac{x^2 - 3x + 1}{(x-1)^2(x-2)}$
2. $\frac{x^2 + 7x + 11}{(x+2)^2(x+3)}$
3. $\frac{9}{(x-1)(x+2)^2}$
4. $\frac{x^4 + 1}{x^2(x-1)}$
5. $\frac{7x + 4}{(3x+2)(x+1)^2}$
6. $\frac{1}{(x-1)^2(x+1)}$
7. $\frac{3x^2 + 15x + 16}{(x+2)^2}$
8. $\frac{1}{(x^2-1)(x+1)}$

4.2.3 Resolution of fraction when $D(x)$ consists of non-repeated irreducible quadratic factors.

Rule III: If a quadratic factor $(ax^2 + bx + c)$ with $a \neq 0$ occurs once as a factor of $D(x)$, the partial fraction is of the form $\frac{Ax + B}{(ax^2 + bx + c)}$, where A and B are constants to be found.

Example: Resolve $\frac{11x + 3}{(x-3)(x^2+9)}$ into partial fractions.

Solution: Let $\frac{11x+3}{(x-3)(x^2+9)} = \frac{A}{(x-3)} + \frac{Bx+C}{x^2+9}$

Multiplying both the sides by $(x-3)(x^2+9)$

$\Rightarrow \quad 11x + 3 = A(x^2+9) + (Bx+C)(x-3)$

$\Rightarrow \quad 11x + 3 = A(x^2+9) + B(x^2-3x) + C(x-3)$ (i)

Since (i) is an identity, we have on substituting $x = 3$

$33 + 3 = A(9+9) \quad \Rightarrow \quad 18A = 36 \quad \Rightarrow \quad A = 2$

Comparing the coefficients of x^2 and x on both the sides of (i), we get.

$A + B = 0 \quad \Rightarrow \quad B = -2$

$-3B + C = 11 \quad \Rightarrow \quad -3(-2) + C = 11 \quad \Rightarrow \quad C = 5$

Therefore, the partial fractions are $\frac{2}{x-3} + \frac{-2x+5}{x^2+9}$

Thus, $\frac{11x+3}{(x-3)(x^2+9)} = \frac{2}{x-3} + \frac{-2x+5}{x^2+9}$

EXERCISE 4.3

Resolve into partial fractions.

1. $\frac{3x-11}{(x+3)(x^2+1)}$
2. $\frac{3x+7}{(x^2+1)(x+3)}$
3. $\frac{1}{(x+1)(x^2+1)}$
4. $\frac{9x-7}{(x+3)(x^2+1)}$
5. $\frac{3x+7}{(x+3)(x^2+4)}$
6. $\frac{x^2}{(x+2)(x^2+4)}$
7. $\frac{1}{x^3+1}$ $\left[\text{Hint: } \frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)}\right]$
8. $\frac{x^2+1}{x^3+1}$

4.2.4 Resolution of a fraction when $D(x)$ has repeated irreducible quadratic factors.

Rule IV: If a quadratic factor $(ax^2 + bx + c)$ with $a \neq 0$, occurs twice in the denominator, the corresponding partial fractions are

$$\frac{Ax+B}{(ax^2+bx+c)} + \frac{Cx+D}{(ax^2+bx+c)^2}$$

The constants A, B, C and D are found in the usual way.

Example 1: Resolve $\dfrac{x^3 - 2x^2 - 2}{(x^2+1)^2}$ into partial fractions.

Solution: $\dfrac{x^3 - 2x^2 - 2}{(x^2+1)^2}$ is a proper fraction as degree of numerator is less than the degree of denominator.

Let $\dfrac{x^3 - 2x^2 - 2}{(x^2+1)^2} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{(x^2+1)^2}$

Multiplying both the sides by $(x^2 + 1)^2$, we have

$$x^3 - 2x^2 - 2 = (Ax + B)(x^2 + 1) + Cx + D$$

$$x^3 - 2x^2 - 2 = A(x^3 + x) + B(x^2 + 1) + Cx + D \qquad \text{(i)}$$

Equating the coefficients of x^3, x^2, x and constant on both the sides of (i).

Coefficients of x^3: $A = 1$

Coefficients of x^2: $B = -2$

Coefficients of x: $A + C = 0 \quad \Rightarrow \quad C = -1$

Constants: $B + D = -2$

$D = -2 - B = -2 - (-2) = -2 + 2 = 0 \quad \Rightarrow \quad D = 0$

Thus $\dfrac{x^3 - 2x^2 - 2}{(x^2+1)^2} = \dfrac{x-2}{x^2+1} + \dfrac{-x+0}{(x^2+1)^2} = \dfrac{x-2}{x^2+1} - \dfrac{x}{(x^2+1)^2}$

Example 2: Resolve $\dfrac{2x+1}{(x-1)(x^2+1)^2}$ into partial fractions.

Solution: Assume that $\dfrac{2x+1}{(x-1)(x^2+1)^2} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2}$

Multiplying both the sides by $(x - 1)(x^2 + 1)^2$

$$2x + 1 = A(x^2 + 1)^2 + (Bx + C)(x - 1)(x^2 + 1) + (Dx + E)(x - 1) \qquad \text{(i)}$$

Now we use zeros' method. Put $x - 1 = 0$ or $x = 1$ in (i), we get

$$3 = A(1 + 1)^2 \quad \Rightarrow \quad A = \frac{3}{4}$$

Now writing terms of (i) in descending order.

$2x + 1 = A(x^4 + 2x^2 + 1) + Bx(x^3 - x^2 + x - 1) + C(x^3 - x^2 + x - 1) + D(x^2 - x) + E(x-1)$

or $2x + 1 = A(x^4+2x^2+1) + B(x^4 - x^3 + x^2 - x) + C(x^3 - x^2 + x - 1) + D(x^2 - x) + E(x-1)$

Equating coefficients of x^4, x^3, x^2, and x on both the sides.

Coefficients of x^4: $A + B = 0 \quad \Rightarrow \quad B = \dfrac{-3}{4}$

Coefficients of x^3: $-B + C = 0 \quad \Rightarrow \quad C = \dfrac{-3}{4}$

Coefficients of x^2: $2A + B - C + D = 0 \quad \Rightarrow \quad D = \dfrac{-3}{2}$

Coefficients of x: $-B + C - D + E = 2$

$\dfrac{3}{4} - \dfrac{3}{4} + \dfrac{3}{2} + E = 2 \quad \Rightarrow \quad E = 2 - \dfrac{3}{2} = \dfrac{1}{2}$

Thus required partial fractions are $\dfrac{3}{4(x-1)} + \dfrac{\frac{-3}{4}x - \frac{3}{4}}{x^2+1} + \dfrac{\frac{-3}{2}x + \frac{1}{2}}{(x^2+1)^2}$

$$\therefore \quad \frac{2x+1}{(x-1)(x^2+1)^2} = \frac{3}{4(x-1)} - \frac{3(x+1)}{4(x^2+1)} - \frac{(3x-1)}{2(x^2+1)^2}$$

EXERCISE 4.4

Resolve into partial fractions.

1. $\dfrac{x^3}{(x^2+4)^2}$
2. $\dfrac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2}$
3. $\dfrac{x^2}{(x+1)(x^2+1)^2}$
4. $\dfrac{x^2}{(x-1)(x^2+1)^2}$
5. $\dfrac{x^4}{(x^2+2)^2}$
6. $\dfrac{x^5}{(x^2+1)^2}$

MISCELLANEOUS EXERCISE - 4

1. Multiple Choice Questions

Four possible answers are given for the following questions. Tick (✓) the correct answer.

(i) The identity $(5x+4)^2 = 25x^2 + 40x + 16$ is true for

(a) one value of x (b) two values of x
(c) all values of x (d) none of these

(ii) A function of the form $f(x) = \dfrac{N(x)}{D(x)}$, with $D(x) \neq 0$, where $N(x)$ and $D(x)$ are polynomials in x is called

(a) an identity (b) an equation
(c) a fraction (d) none of these

(iii) A fraction in which the degree of the numerator is greater or equal to the degree of denominator is called

(a) a proper fraction (b) an improper fraction
(c) an equation (d) algebraic relation

(iv) A fraction in which the degree of numerator is less than the degree of the denominator is called

(a) an equation (b) an improper fraction
(c) an identity (d) a proper fraction

(v) $\dfrac{2x+1}{(x+1)(x-1)}$ is:

(a) an improper fraction (b) an equation
(c) a proper fraction (d) none of these

(vi) $(x+3)^2 = x^2 + 6x + 9$ is

(*a*) a linear equation (*b*) an equation

(*c*) an identity (*d*) none of these

(vii) $\frac{x^3+1}{(x-1)(x+2)}$ is

(*a*) a proper fraction (*b*) an improper fraction

(*c*) an identity (*d*) a constant term

(viii) Partial fractions of $\frac{x-2}{(x-1)(x+2)}$ are of the form

(*a*) $\frac{A}{x-1} + \frac{B}{x+2}$ (*b*) $\frac{Ax}{x-1} + \frac{B}{x+2}$

(*c*) $\frac{A}{x-1} + \frac{Bx+C}{x+2}$ (*d*) $\frac{Ax+B}{x-1} + \frac{C}{x+2}$

(ix) Partial fractions of $\frac{x+2}{(x+1)(x^2+2)}$ are of the form

(*a*) $\frac{A}{x+1} + \frac{B}{x^2+2}$ (*b*) $\frac{A}{x+1} + \frac{Bx+C}{x^2+2}$

(*c*) $\frac{Ax+B}{x+1} + \frac{C}{x^2+2}$ (*d*) $\frac{A}{x+1} + \frac{Bx}{x^2+2}$

(x) Partial fractions of $\frac{x^2+1}{(x+1)(x-1)}$ are of the form

(*a*) $\frac{A}{x+1} + \frac{B}{x-1}$ (*b*) $1 + \frac{A}{x+1} + \frac{Bx+C}{x-1}$

(*c*) $1 + \frac{A}{x+1} + \frac{B}{x-1}$ (*d*) $\frac{Ax+B}{(x+1)} + \frac{C}{x-1}$

2. Write short answers of the following questions.

(i) Define a rational fraction.

(ii) What is a proper fraction?

(iii) What is an improper fraction?

(iv) What are partial fractions?

(v) How can we make partial fractions of $\frac{x-2}{(x+2)\,(x+3)}$?

(vi) Resolve $\frac{1}{x^2-1}$ into partial fractions.

(vii) Find partial fractions of $\frac{3}{(x+1)(x-1)}$.

(viii) Resolve $\frac{x}{(x-3)^2}$ into partial fractions.

(ix) How we can make the partial fractions of $\frac{x}{(x+a)(x-a)}$?

(x) Whether $(x+3)^2 = x^2 + 6x + 9$ is an identity?

SUMMARY

- A **fraction** is an indicated quotient of two numbers or algebraic expressions.
- An expression of the form $\frac{N(x)}{D(x)}$, with $D(x) \neq 0$ and $N(x)$ and $D(x)$ are polynomials in x with real coefficients, is called a **rational fraction.** Every fractional expression can be expressed as a quotient of two polynomials.
- A rational fraction $\frac{N(x)}{D(x)}$, with $D(x) \neq 0$ is called a **proper fraction** if degree of the polynomial $N(x)$, in the numerator is less than the degree of the polynomial $D(x)$, in the denominator.
- A rational fraction $\frac{N(x)}{D(x)}$, with $D(x) \neq 0$ is called an **improper fraction** if degree of the polynomial $N(x)$ is greater or equal to the degree of the polynomial $D(x)$.
- **Partial fractions:** Decomposition of resultant fraction $\frac{N(x)}{D(x)}$, with $D(x) \neq 0$, when

  (a) $D(x)$ consists of non-repeated linear factors.
  (b) $D(x)$ consists of repeated linear factors.
  (c) $D(x)$ consists of non-repeated irreducible quadratic factors.
  (d) $D(x)$ consists of repeated irreducible quadratic factors.

Unit-5

SETS AND FUNCTIONS

In this unit, students will learn how to

- *sets*
- *recall the sets denoted by N, W, Z, E, O, P and Q.*
- *recognize operation on sets ($\cup$, $\cap$, \, ...)*
- *perform operations on sets union, intersection, difference, complement.*
- *give formal proofs of the following fundamental properties of union and intersection of two or three sets.*
  - *commutative property of union,*
  - *commutative property of intersection,*
  - *associative property of union,*
  - *associative property of intersection,*
  - *distributive property of union over intersection,*
  - *distributive property of intersection over union,*
  - *De Morgan's laws.*
- *verify the fundamental properties for given sets.*
- *use Venn diagram to represent*
  - *union and intersection of sets,*
  - *complement of a set.*
- *use Venn diagram to verify*
  - *commutative law for union and intersection of sets,*
  - *De Morgan's laws,*
  - *associative laws,*
  - *distributive laws.*
- *recognize ordered pairs and cartesian product.*
- *define binary relation and identify its domain and range.*
- *define function and identify its domain, co-domain and range.*
- *demonstrate the following*
  - *into function,*
  - *one-one function,*
  - *into and one-one function (injective function),*
  - *onto function (surjective function),*
  - *one-one and onto function (bijective function).*
- *examine whether a given relation is a function or not.*
- *differentiate between one-one correspondence and one-one function.*
- *include sufficient exercises to classify/differentiate between the above concepts.*

5.1 SETS

A set is a well-defined collection of objects and it is denoted by capital letters A, B, C etc.

5.1.1(i) Some Important Sets:

In set theory, we usually deal with the following sets of numbers denoted by standard symbols:

N = The set of natural numbers = $\{1, 2, 3, 4, \cdots\}$

W = The set of whole numbers = $\{0, 1, 2, 3, 4, \cdots\}$

Z = The set of all integers = $\{0, \pm 1, \pm 2, \pm 3, \cdots\}$

E = The set of all even integers = $\{0, \pm 2, \pm 4, \cdots\}$

O = The set of all odd integers = $\{\pm 1, \pm 3, \pm 5, \cdots\}$

P = The set of prime numbers = $\{2, 3, 5, 7, 11, 13, 17, \cdots\}$

Q = The set of all rational numbers = $\{x \mid x = \frac{m}{n}$, where $m, n \in Z$ and $n \neq 0\}$

Q' = The set of all irrational numbers = $\{x \mid x \neq \frac{m}{n}$, where $m, n \in Z$ and $n \neq 0\}$

R = The set of all real numbers = $Q \cup Q'$.

5.1.1(ii) Recognize operations on sets ($\cup$, $\cap$, \, ...):

(a) Union of sets

The union of two sets A and B written as $A \cup B$ (read as A union B) is the set consisting of all the elements which are either in A or in B or in both. Thus

$$A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ or } x \in A \text{ and } B \text{ both}\}.$$

For example, if $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6, 7\}$, then $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$

(b) Intersection of sets

The intersection of two sets A and B, written as $A \cap B$ (read as 'A intersection B') is the set consisting of all the common elements of A and B. Thus

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

Clearly $x \in A \cap B \Rightarrow x \in A$ and $x \in B$

For example, if $A = \{a, b, c, d\}$ and $B = \{c, d, e, f\}$, then

$$A \cap B = \{c, d\}$$

(c) Difference of sets

If A and B are two sets, then their difference $A - B$ or $A \setminus B$ is defined as:

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

Similarly $B - A = \{x \mid x \in B \text{ and } x \notin A\}$.

For example, if $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 5, 6, 8\}$, then

$$A - B = \{1, 2, 3, 4, 5\} - \{2, 4, 5, 6, 8\} = \{1, 3\}$$

Also $B - A = \{2, 4, 5, 6, 8\} - \{1, 2, 3, 4, 5\} = \{6, 8\}$.

(d) Complement of a set

If U is a universal set and A is a subset of U, then the complement of A is the set of those elements of U, which are not contained in A and is denoted by A' or A^c.

$\therefore$ $A' = U - A = \{x | x \in U \text{ and } x \notin A\}$.

For example, if $U = \{1, 2, 3, ..., 10\}$ and $A = \{2, 4, 6, 8\}$, then

$A' = U - A$

$= \{1, 2, 3, ..., 10\} - \{2, 4, 6, 8\}$

$= \{1, 3, 5, 7, 9, 10\}$

5.1.1(iii) Perform operations on sets:

Example: If $U = \{1, 2, 3, ..., 10\}$, $A = \{2, 3, 5, 7\}$, $B = \{3, 5, 8\}$, then

find (i) $A \cup B$ (ii) $A \cap B$ (iii) $A - B$

(iv) A' and B'

Solution: (i) $A \cup B = \{2, 3, 5, 7\} \cup \{3, 5, 8\}$

$= \{2, 3, 5, 7, 8\}$

(ii) $A \cap B = \{2, 3, 5, 7\} \cap \{3, 5, 8\}$

$= \{3, 5\}$

(iii) $A \backslash B = \{2, 3, 5, 7\} \backslash \{3, 5, 8\}$

$= \{2, 7\}$

(iv) $A' = U - A = \{1, 2, 3, ..., 10\} - \{2, 3, 5, 7\}$

$= \{1, 4, 6, 8, 9, 10\}$

$B' = U - B = \{1, 2, 3, ..., 10\} - \{3, 5, 8\}$

$= \{1, 2, 4, 6, 7, 9, 10\}$

EXERCISE 5.1

1. If $X = \{1, 4, 7, 9\}$ and $Y = \{2, 4, 5, 9\}$

   Then find:

   (i) $X \cup Y$ (ii) $X \cap Y$

   (iii) $Y \cup X$ (iv) $Y \cap X$

2. If X = Set of prime numbers less than or equal to 17

   and Y = Set of first 12 natural numbers, then find the following

   (i) $X \cup Y$ (ii) $Y \cup X$ (iii) $X \cap Y$ (iv) $Y \cap X$

3. If $X = \phi$, $Y = Z^+$, $T = O^+$, then

   find: (i) $X \cup Y$ (ii) $X \cup T$ (iii) $Y \cup T$

   (iv) $X \cap Y$ (v) $X \cap T$ (vi) $Y \cap T$

4. If $U = \{x \mid x \in N \wedge 3 < x \leq 25\}$, $X = \{x \mid x \text{ is prime} \wedge 8 < x < 25\}$

   and $Y = \{x \mid x \in W \wedge 4 \leq x \leq 17\}$.

Find the value of:

(i) $(X \cup Y)'$ (ii) $X' \cap Y'$

(iii) $(X \cap Y)'$ (iv) $X' \cup Y'$

5. If $X = \{2, 4, 6, \cdots, 20\}$ and $Y = \{4, 8, 12, \cdots, 24\}$, then find the following:

(i) $X - Y$ (ii) $Y - X$

6. If $A = N$ and $B = W$, then find the value of

(i) $A - B$ (ii) $B - A$

5.1.2(iv) Properties of Union and Intersection:

(a) Commutative property of union.

For any two sets A and B, prove that $A \cup B = B \cup A$.

Proof:

Let $x \in A \cup B$

$\Rightarrow$ $x \in A$ or $x \in B$ (by definition of union of sets)

$\Rightarrow$ $x \in B$ or $x \in A$

$\Rightarrow$ $x \in B \cup A$

$\Rightarrow$ $A \cup B \subseteq B \cup A$ (i)

Now let $y \in B \cup A$

$\Rightarrow$ $y \in B$ or $y \in A$ (by definition of union of sets)

$\Rightarrow$ $y \in A$ or $y \in B$

$\Rightarrow$ $y \in A \cup B$

$\Rightarrow$ $B \cup A \subseteq A \cup B$ (ii)

From (i) and (ii), we have $A \cup B = B \cup A$. (by definition of equal sets)

(b) Commutative property of intersection

For any two sets A and B, prove that $A \cap B = B \cap A$

Proof: Let $x \in A \cap B$

$\Rightarrow$ $x \in A$ and $x \in B$ (by definition of intersection of sets)

$\Rightarrow$ $x \in B$ and $x \in A$

$\Rightarrow$ $x \in B \cap A$

$\therefore$ $A \cap B \subseteq B \cap A$ (i)

Now let $y \in B \cap A$

$\Rightarrow$ $y \in B$ and $y \in A$ (by definition of intersection of sets)

$\Rightarrow$ $y \in A$ and $y \in B$

$\Rightarrow$ $y \in A \cap B$

Therefore, $B \cap A \subseteq A \cap B$ (ii)

From (i) and (ii), we have $A \cap B = B \cap A$ (by definition of equal sets)

(c) Associative property of union

For any three sets A, B and C, prove that $(A \cup B) \cup C = A \cup (B \cup C)$

Proof: Let $x \in (A \cup B) \cup C$

$\Rightarrow$ $x \in (A \cup B)$ or $x \in C$

$\Rightarrow$ $x \in A$ or $x \in B$ or $x \in C$

$\Rightarrow$ $x \in A$ or $x \in B \cup C$

$\Rightarrow$ $x \in A \cup (B \cup C)$

$\Rightarrow$ $(A \cup B) \cup C \subseteq A \cup (B \cup C)$ (i)

Similarly $A \cup (B \cup C) \subseteq (A \cup B) \cup C$ (ii)

From (i) and (ii), we have

$(A \cup B) \cup C = A \cup (B \cup C)$

(d) Associative property of intersection

For any three sets A, B and C, prove that $(A \cap B) \cap C = A \cap (B \cap C)$

Proof: Let $x \in (A \cap B) \cap C$

$\Rightarrow$ $x \in (A \cap B)$ and $x \in C$

$\Rightarrow$ $(x \in A$ and $x \in B)$ and $x \in C$

$\Rightarrow$ $x \in A$ and $(x \in B$ and $\in C)$

$\Rightarrow$ $x \in A$ and $x \in B \cap C$

$\Rightarrow$ $x \in A \cap (B \cap C)$

$\therefore$ $(A \cap B) \cap C \subseteq A \cap (B \cap C)$ (i)

Similarly $A \cap (B \cap C) \subseteq (A \cap B) \cap C$ (ii)

From (i) and (ii), we have

$(A \cap B) \cap C = A \cap (B \cap C)$

(e) Distributive property of union over intersection

For any three sets A, B and C, prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof: Let $x \in A \cup (B \cap C)$

$\Rightarrow$ $x \in A$ or $x \in B \cap C$

$\Rightarrow$ $x \in A$ or $(x \in B$ and $x \in C)$

$\Rightarrow$ $(x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$

$\Rightarrow$ $x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow$ $x \in (A \cup B) \cap (A \cup C)$

Therefore $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ (i)

Similarly, now let $y \in (A \cup B) \cap (A \cup C)$

$\Rightarrow$ $y \in (A \cup B)$ and $y \in (A \cup C)$

$\Rightarrow$ $(y \in A$ or $y \in B)$ and $(y \in A$ or $y \in C)$

$\Rightarrow$ $y \in A$ or $(y \in B$ and $y \in C)$

$\Rightarrow$ $y \in A$ or $y \in B \cap C$

$\Rightarrow$ $y \in A \cup (B \cap C)$

$\Rightarrow$ $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ (ii)

From (i) and (ii), we have $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(f) Distributive property of intersection over union

For any three sets A, B and C, prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof: Let $x \in A \cap (B \cup C)$

$\Rightarrow$ $x \in A$ and $x \in B \cup C$

$\Rightarrow$ $x \in A$ and $(x \in B$ or $x \in C)$

$\Rightarrow$ $(x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

$\Rightarrow$ $(x \in A \cap B)$ or $(x \in A \cap C)$

$\Rightarrow$ $x \in (A \cap B) \cup (A \cap C)$

$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ (i)

Similarly $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ (ii)

From (i) and (ii), we have $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(g) De-Morgan's laws

For any two sets A and B

(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

Proof: Let $x \in (A \cup B)'$

$\Rightarrow$ $x \notin A \cup B$ (by definition of complement of set)

$\Rightarrow$ $x \notin A$ and $x \notin B$

$\Rightarrow$ $x \in A'$ and $x \in B'$

$\Rightarrow$ $x \in A' \cap B'$ (by definition of intersection of sets)

$\Rightarrow$ $(A \cup B)' \subseteq A' \cap B'$ (i)

Similarly $A' \cap B' \subseteq (A \cup B)'$ (ii)

Using (i) and (ii), we have $(A \cup B)' = A' \cap B'$

(ii) Let $x \in (A \cap B)'$

$\Rightarrow$ $x \notin A \cap B$

$\Rightarrow$ $x \notin A$ or $x \notin B$

$\Rightarrow$ $x \in A'$ or $x \in B'$

$\Rightarrow$ $x \in A' \cup B'$

$\Rightarrow$ $(A \cap B)' \subseteq A' \cup B'$ (i)

Let $y \in A' \cup B'$

$\Rightarrow$ $y \in A'$ or $y \in B'$

$\Rightarrow$ $y \notin A$ or $y \notin B$

$\Rightarrow$ $y \notin A \cap B$

$\Rightarrow$ $y \in (A \cap B)'$

$\Rightarrow$ $A' \cup B' \subseteq (A \cap B)'$ (ii)

From (i) and (ii), we have proved that

$(A \cap B)' = A' \cup B'$

EXERCISE 5.2

1. If $X = \{1, 3, 5, 7, \cdots, 19\}$, $Y = \{0, 2, 4, 6, 8, \cdots, 20\}$
   and $Z = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$,
   then find the following:

   (i) $X \cup (Y \cup Z)$ (ii) $(X \cup Y) \cup Z$
   (iii) $X \cap (Y \cap Z)$ (iv) $(X \cap Y) \cap Z$
   (v) $X \cup (Y \cap Z)$ (vi) $(X \cup Y) \cap (X \cup Z)$
   (vii) $X \cap (Y \cup Z)$ (viii) $(X \cap Y) \cup (X \cap Z)$

2. If $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 8\}$, $C = \{1, 4, 8\}$.
   Prove the following identities:

   (i) $A \cap B = B \cap A$ (ii) $A \cup B = B \cup A$
   (iii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   (iv) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

3. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
   $A = \{1, 3, 5, 7, 9\}$, $B = \{2, 3, 5, 7\}$,
   then verify the De-Morgan's Laws
   i.e., $(A \cap B)' = A' \cup B'$ and $(A \cup B)' = A' \cap B'$

4. If $U = \{1, 2, 3, \cdots, 20)$, $X = \{1, 3, 7, 9, 15, 18, 20\}$
   and $Y = \{1, 3, 5, \cdots, 17\}$, then show that

   (i) $X - Y = X \cap Y'$ (ii) $Y - X = Y \cap X'$

5.1.2(v) Verify the fundamental properties for given sets:

(a) A and B are any two subsets of U, then $A \cup B = B \cup A$ (commutative law).

For example $A = \{1, 3, 5, 7\}$ and $B = \{2, 3, 5, 7\}$

then $A \cup B = \{1, 3, 5, 7\} \cup \{2, 3, 5, 7\} = \{1, 2, 3, 5, 7\}$

and $B \cup A = \{2, 3, 5, 7\} \cup \{1, 3, 5, 7\} = \{1, 2, 3, 5, 7\}$

Hence, verified that $A \cup B = B \cup A$.

(b) Commutative property of intersection

For example $A = \{1, 3, 5, 7\}$ and $B = [2, 3, 5, 7\}$

Then $A \cap B = \{1, 3, 5, 7\} \cap \{2, 3, 5, 7\} = \{3, 5, 7\}$

and $B \cap A = \{2, 3, 5, 7\} \cap \{1, 3, 5, 7\} = \{3, 5, 7\}$

Hence, verified that $A \cap B = B \cap A$.

(c) If A, B and C are the subsets of U, then $(A \cup B) \cup C = A \cup (B \cup C)$.

(Associative law)

Suppose $A = \{1, 2, 4, 8\}$; $B = \{2, 4, 6\}$

and $C = \{3, 4, 5, 6\}$

then L.H.S $= (A \cup B) \cup C$

$= (\{1, 2, 4, 8\} \cup \{2, 4, 6\}) \cup \{3, 4, 5, 6\}$

$= \{1, 2, 4, 6, 8\} \cup \{3, 4, 5, 6\}$

$= \{1, 2, 3, 4, 5, 6, 8\}$

and R.H.S. $= A \cup (B \cup C)$

$= \{1, 2, 4, 8\} \cup (\{2, 4, 6\} \cup \{3, 4, 5, 6\})$

$= \{1, 2, 4, 8\} \cup \{2, 3, 4, 5, 6\}$

$= \{1, 2, 3, 4, 5, 6, 8\}$

L.H.S. = R.H.S.

Hence, union of sets is associative.

(d) If A, B and C are the subsets of U, then $(A \cap B) \cap C = A \cap (B \cap C)$

(Associative Law).

Suppose $A = \{1, 2, 4, 8\}$; $B = \{2, 4, 6\}$ and $C = \{3, 4, 5, 6\}$

then L.H.S. $= (A \cap B) \cap C$

$= (\{1, 2, 4, 8\} \cap \{2, 4, 6\}) \cap \{3, 4, 5, 6\}$

$= \{2, 4\} \cap \{3, 4, 5, 6\} = \{4\}$

and R.H.S. $= A \cap (B \cap C)$

$= \{1, 2, 4, 8\} \cap (\{2, 4, 6\} \cap \{3, 4, 5, 6\})$

$= \{1, 2, 4, 8\} \cap \{4, 6\} = \{4\}$

L.H.S. = R.H.S.

Hence, intersection of sets is associative.

Distributive laws

(e) Union is distributive over intersection of sets

If A, B and C are the subsets of universal set U, then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Solution: Suppose $A = \{1, 2, 4, 8\}$, $B = \{2, 4, 6\}$ and $C = \{3, 4, 5, 6\}$

then L.H.S $= A \cup (B \cap C)$

$= \{1, 2, 4, 8\} \cup (\{2, 4, 6\} \cap \{3, 4, 5, 6\})$

$= \{1, 2, 4, 8\} \cup \{4, 6\} = \{1, 2, 4, 6, 8\}$

and R.H.S $= (A \cup B) \cap (A \cup C)$

$= (\{1, 2, 4, 8\} \cup \{2, 4, 6\}) \cap (\{1, 2, 4, 8\} \cup \{3, 4, 5, 6\})$

$= \{1, 2, 4, 6, 8\} \cap \{1, 2, 3, 4, 5, 6, 8\}$

$= \{1, 2, 4, 6, 8\}$

L.H.S = R.H.S

(f) Intersection is distributive over union of sets

To prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Suppose $A = \{1, 2, 3, 4, 5, \cdots, 20\}$

$B = \{5, 10, 15, 20, 25, 30\}$

$C = \{3, 9, 15, 21, 27, 33\}$

L.H.S. $= A \cap (B \cup C)$

$= \{1, 2, 3, 4, 5, \cdots, 20\} \cap (\{5, 10, 15, 20, 25, 30\} \cup \{3, 9, 15, 21, 27, 33\})$

$= \{1,2,3,4,5, \cdots, 20\} \cap \{3, 5, 9, 10, 15, 20, 21, 25, 27, 30, 33\}$

$= \{3, 5, 9, 10, 15, 20\}$

R.H.S. $= (A \cap B) \cup (A \cap C)$

$= (\{1, 2, 3, 4, \cdots, 20\} \cap \{5, 10, 15, 20, 25, 30\})$

$\cup (\{1, 2, 3, 4, 5, \cdots, 20\} \cap \{3, 9, 15, 21, 27, 33\})$

$= \{5, 10, 15, 20\} \cup \{3, 9, 15\} = \{3, 5, 9, 10, 15, 20\}$

L.H.S. = R.H.S.

(g) De Morgan's Laws $(A \cap B)' = A' \cup B'$ and $(A \cup B)' = A' \cap B'$

Suppose $U = \{1, 2, 3, 4, \cdots, 10\}$

$A = \{2, 4, 6, 8, 10\} \Rightarrow A' = \{1, 3, 5, 7, 9\}$

$B = \{1, 2, 3, 4, 5, 6\} \Rightarrow B' = \{7, 8, 9, 10\}$

Now consider $A \cap B = \{2, 4, 6, 8, 10\} \cap \{1, 2, 3, 4, 5, 6\}$

$= \{2, 4, 6\}$

Then L.H.S. $= (A \cap B)' = U - (A \cap B)$

$= \{1, 2, 3, 4, \cdots, 10\} - \{2, 4, 6\}$

$= \{1, 3, 5, 7, 8, 9, 10\}$

and R.H.S. $= A' \cup B'$

$= \{1, 3, 5, 7, 9\} \cup \{7, 8, 9, 10\}$

$= \{1, 3, 5, 7, 8, 9, 10\}$

L.H.S. = R.H.S.

$(A \cup B)' = A' \cap B'$

Suppose $U = \{1, 2, 3, 4, \cdots, 10\}$

$A = \{2, 4, 6, 8, 10\} \Rightarrow A' = \{1, 3, 5, 7, 9\}$

$B = \{1, 2, 3, 4, 5, 6\} \Rightarrow B' = \{7, 8, 9, 10\}$

Now consider $A \cup B = \{2, 4, 6, 8, 10\} \cup \{1, 2, 3, 4, 5, 6\}$

$= \{1, 2, 3, 4, 5, 6, 8, 10\}$

L.H.S. $= (A \cup B)' = U - (A \cup B)$

$= \{1, 2, 3, 4, \cdots, 10\} - \{1, 2, 3, 4, 5, 6, 8, 10\}$

$= \{7, 9\}$

and R.H.S $= A' \cap B' = \{1, 3, 5, 7, 9\} \cap \{7, 8, 9, 10\}$

$= \{7, 9\}$

L.H.S. = R.H.S.

5.1.3 VENN DIAGRAM

British mathematician John Venn (1834 – 1923) introduced rectangle for a universal set U and its subsets A and B as closed figures inside this rectangle.

5.1.3(vi) Use Venn diagrams to represent:

(a) Union and intersection of sets

Disjoint sets | Overlapping sets | A B

A B

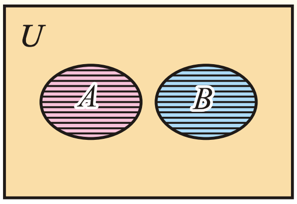


Fig. 1

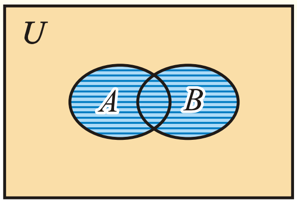


Fig. 2

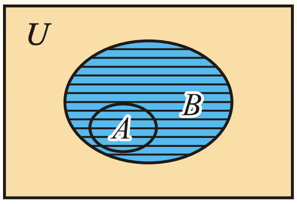


Fig. 3

A B

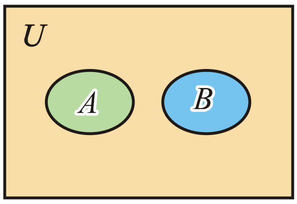


Fig. 4

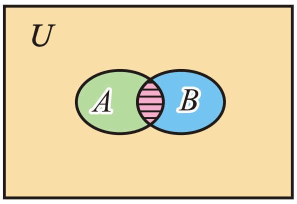


Fig. 5

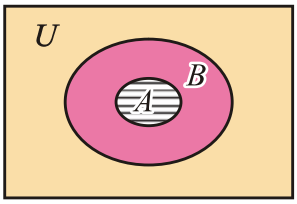
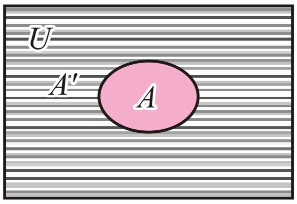


Fig. 6

(Regions shown by horizontal line segments in figures 1 to 6.)

(b) Complement of a set

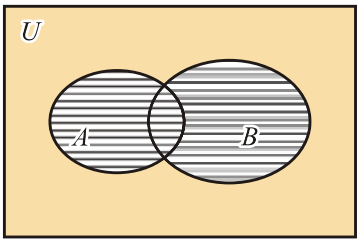
$U - A = A'$ is shown by horizontal line segments.



5.1.3 (vii) Use Venn diagram to verify:

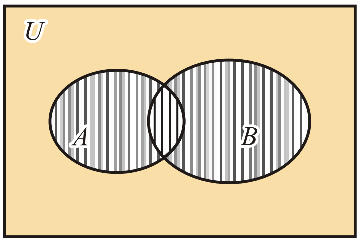
(a) Commutative law for union and intersection of sets

A B



$A \cup B$ is shown by horizontal line segments.

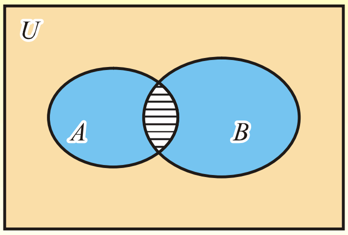
B A



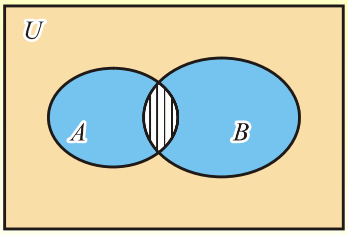
$B \cup A$ is shown by vertical line segments.

The regions shown in both cases are equal. Thus $A \cup B = B \cup A$.

$A \quad B$



$B \quad A$



$A \cap B$ is shown by horizontal line segments. $B \cap A$ is shown by vertical line segments.

The regions shown in both cases are equal. Thus $A \cap B = B \cap A$.

(b) De Morgan's laws

(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

(i) $(A \cup B)' = A' \cap B'$

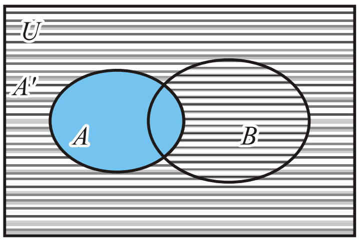


Fig. 1: A' is shown by horizontal line segments

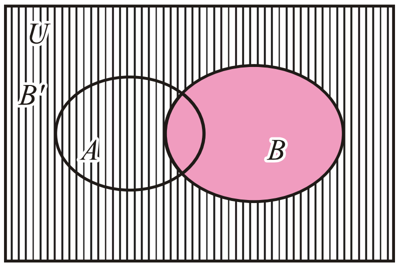


Fig. 2: B' is shown by vertical line segments

$A' \quad B'$

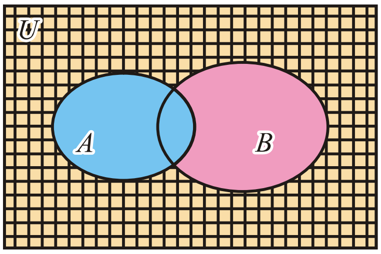


Fig. 3: $A' \cap B'$ is shown by squares

$(A \quad B)'$

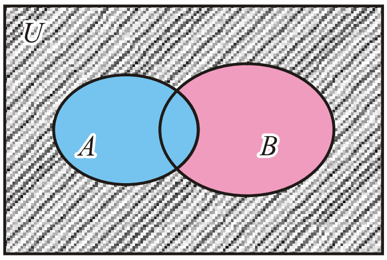


Fig. 4: $(A \cup B)'$ is shown by slanting line segments

Regions shown in Fig. 3 and Fig. 4 are equal.

Thus $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

$A' \quad B'$

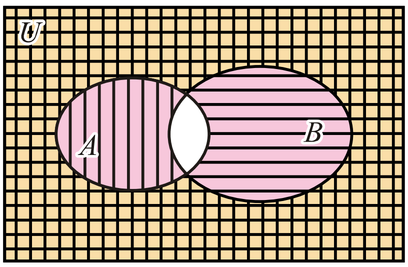


Fig. 5: $A' \cup B'$ is shown by squares, horizontal and vertical line segments.

$(A \cap B)'$

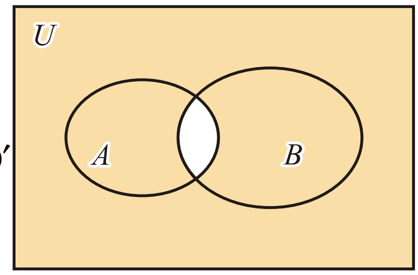


Fig. 6: $U - (A \cap B) = (A \cap B)'$ is shown by shading.

Regions shown in Fig. 5 and Fig. 6 are equal.

Thus $(A \cap B)' = A' \cup B'$

(c) Associative law:

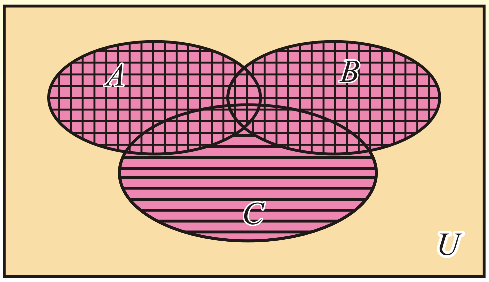


Fig. 1

$(A \cup B) \cup C$ is shown in the above figure.

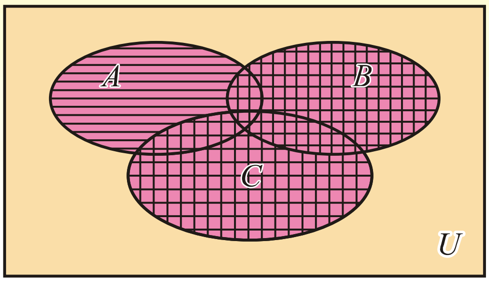


Fig. 2

$A \cup (B \cup C)$ is shown in the above figure.

Regions shown in fig. 1 and fig. 2 by different ways are equal.

Thus $(A \cup B) \cup C = A \cup (B \cup C)$

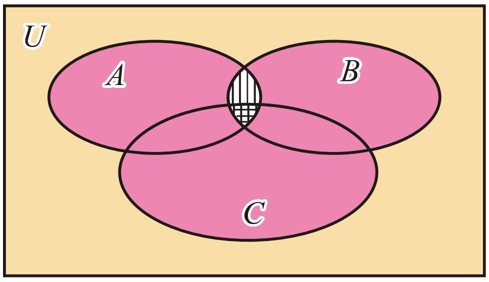


Fig. 3

$(A \cap B) \cap C$ is shown in figure 3 by double crossing line segments

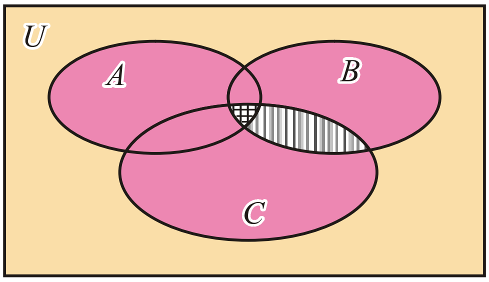


Fig. 4

$A \cap (B \cap C)$ is shown in figure 4 by double crossing line segments

Regions shown in Fig. 3 and fig. 4 are equal.

Thus $(A \cap B) \cap C = A \cap (B \cap C)$

(d) Distributive law:

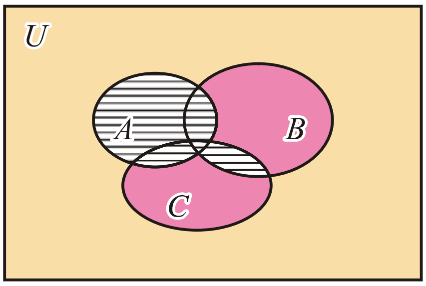


Fig. 1: $A \cup (B \cap C)$ is shown by horizontal line segments in the above figure.

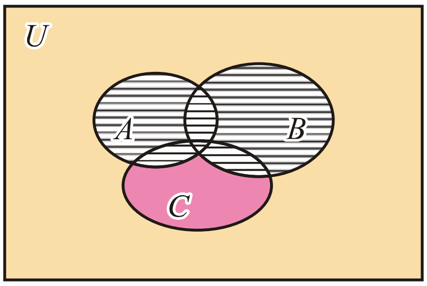


Fig. 2: $A \cup B$ is shown by horizontal line segments in the above figure.

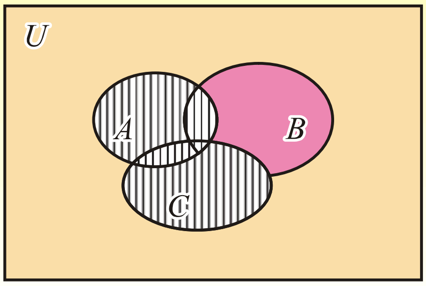


Fig. 3: $A \cup C$ is shown by vertical line segments in Fig. 3.

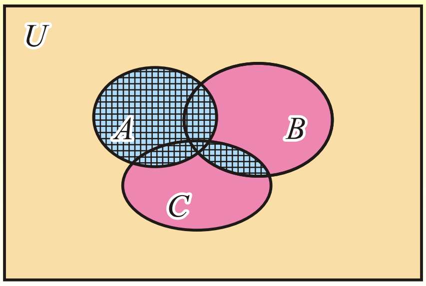


Fig. 4: $(A \cup B) \cap (A \cup C)$ is shown by double crossing line segments in Fig. 4.

Regions shown in Fig. 1 and Fig. 4 are equal.

Thus $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

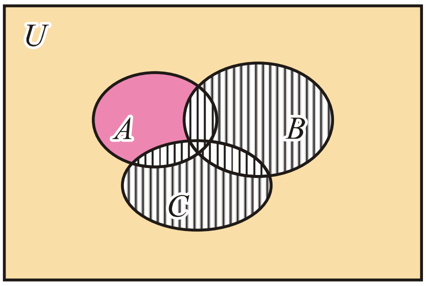


Fig. 5: $B \cup C$ is shown by vertical line segments in Fig. 5.

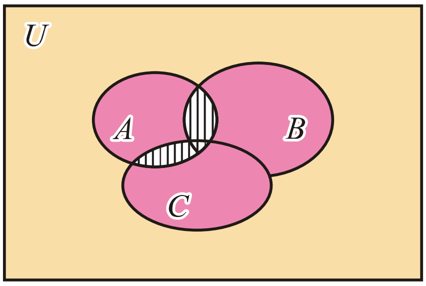


Fig. 6: $A \cap (B \cup C)$ is shown in Fig. 6 by vertical line segments.

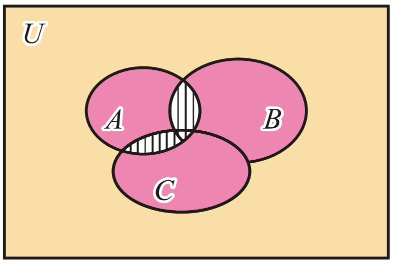


Fig. 7: $(A \cap B) \cup (A \cap C)$ is shown in Fig. 7 by slanting line segments.

Regions displayed in Fig. 6 and Fig. 7 are equal.

Thus $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

EXERCISE 5.3

1. If $U = \{1, 2, 3, 4, \cdots, 10\}$
$A = \{1, 3, 5, 7, 9\}$
$B = \{1, 4, 7, 10\}$,
then verify the following questions.
(i) $A - B = A \cap B'$ (ii) $B - A = B \cap A'$
(iii) $(A \cup B)' = A' \cap B'$ (iv) $(A \cap B)' = A' \cup B'$
(v) $(A - B)' = A' \cup B$ (vi) $(B - A)' = B' \cup A$

2. If $U = \{1, 2, 3, 4, \cdots, 10\}$
$A = \{1, 3, 5, 7, 9\}$; $B = \{1, 4, 7, 10\}$; $C = \{1, 5, 8, 10\}$
then verify the following:
(i) $(A \cup B) \cup C = A \cup (B \cup C)$
(ii) $(A \cap B) \cap C = A \cap (B \cap C)$
(iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(iv) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

3. If $U = N$; then verify De-Morgan's laws by using $A = \phi$ and $B = P$.

4. If $U = \{1, 2, 3, 4, \cdots, 10\}$, $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 3, 4, 5, 8\}$, then prove the following questions by Venn diagram:
(i) $A - B = A \cap B'$ (ii) $B - A = B \cap A'$
(iii) $(A \cup B)' = A' \cap B'$ (iv) $(A \cap B)' = A' \cup B'$
(v) $(A - B)' = A' \cup B$ (vi) $(B - A)' = B' \cup A$

5.1.4 (viii) Ordered pairs and Cartesian product:

5.1.4(a) Ordered pairs:

Any two numbers x and y, written in the form (x, y) is called an ordered pair. In an ordered pair (x, y), the order of numbers is important, that is, x is the first co-ordinate and y is the second co-ordinate. For example, (3, 2) is different from (2, 3).

It is obvious that $(x, y) \neq (y, x)$ unless $x = y$.

Note that $(x, y) = (s, t)$, iff $x = s$ and $y = t$

5.1.4(b) Cartesian product:

Cartesian product of two non-empty sets A and B denoted by $A \times B$ consists of all ordered pairs (x, y) such that $x \in A$ and $y \in B$.

Example: If $A = \{1, 2, 3\}$ and $B = \{2, 5\}$, then find $A \times B$ and $B \times A$.

Solution: $A \times B = \{(1, 2), (1, 5), (2, 2), (2, 5), (3, 2), (3, 5)\}$

Since set A has 3 elements and set B has 2 elements.

Hence product set $A \times B$ has $3 \times 2 = 6$ ordered pairs.

We note that $B \times A = \{(2, 1), (2, 2), (2, 3), (5, 1), (5, 2), (5, 3)\}$

Evidently $A \times B \neq B \times A$.

EXERCISE 5.4

1. If $A = \{a, b\}$ and $B = \{c, d\}$, then find $A \times B$ and $B \times A$.
2. If $A = \{0, 2, 4\}$, $B = \{-1, 3\}$, then find $A \times B$, $B \times A$, $A \times A$, $B \times B$
3. Find a and b, if
   (i) $(a - 4, b - 2) = (2, 1)$ (ii) $(2a + 5, 3) = (7, b - 4)$
   (iii) $(3 - 2a, b - 1) = (a - 7, 2b + 5)$
4. Find the sets X and Y, if $X \times Y = \{(a, a), (b, a), (c, a), (d, a)\}$
5. If $X = \{a, b, c\}$ and $Y = \{d, e\}$, then find the number of elements in
   (i) $X \times Y$ (ii) $Y \times X$ (iii) $X \times X$

5.2 Binary relation

If A and B are any two non-empty sets, then a subset $R \subseteq A \times B$ is called **binary relation** from set A into set B, because there exists some relationship between first and second element of each ordered pair in R.

Domain of relation denoted by *Dom R* is the set consisting of all the first elements of each ordered pair in the relation.

Range of relation denoted by *Rang R* is the set consisting of all the second elements of each ordered pair in the relation.

Example 1: Suppose $A = \{2, 3, 4, 5\}$ and $B = \{2, 4, 6, 8\}$

Form a relation $R : A \rightarrow B = \{x\, R\, y$ such that $y = 2x$ for $x \in A, y \in B\}$

$\Rightarrow \quad R = \{(2, 4), (3, 6), (4, 8)\}$

Dom $R = \{2, 3, 4\} \subseteq A$ and Rang $R = \{4, 6, 8\} \subseteq B$.

Example 2: Suppose $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 5\}$

Form a relation $R : A \rightarrow B = \{x\, R\, y$ such that $x + y = 6$ for $x \in A, y \in B\}$

$\Rightarrow \quad R = \{(1, 5), (3, 3), (4, 2)\}$

Dom $R = \{1, 3, 4\} \subseteq A$ and Rang $R = \{2, 3, 5\} \subseteq B$

5.3 Function or Mapping:

5.3. (i) Suppose A and B are two non-empty sets, then relation $f : A \rightarrow B$ is called a function if (i) Dom $f = A$ (ii) every $x \in A$ appears in one and only one ordered pair in f.

Alternate Definition:

Suppose A and B are two non-empty sets, then relation $f : A \rightarrow B$ is called a function if (i) Dom $f = A$ (ii) $\forall\, x \in A$ we can associate some unique image element $y = f(x) \in B$.

Domain, Co-domain and Range of Function:

If $f : A \rightarrow B$ is a function, then A is called the domain of f and B is called the co-domain of f.

Domain f is the set consisting of all first elements of each ordered pair in f and range f is the set consisting of all second elements of each ordered pair in f.

Example: Suppose $A = \{0, 1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$

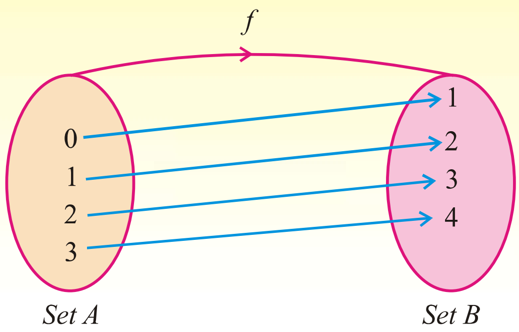
Define a function $f : A \to B$

$f = \{(x, y) \mid y = x + 1 \ \forall\, x \in A, y \in B\}$

$f = \{(0, 1), (1, 2), (2, 3), (3, 4)\}$

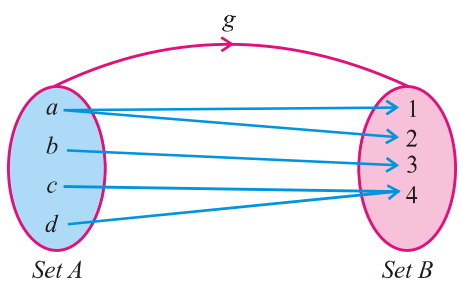
Dom $f = \{0, 1, 2, 3\} = A$

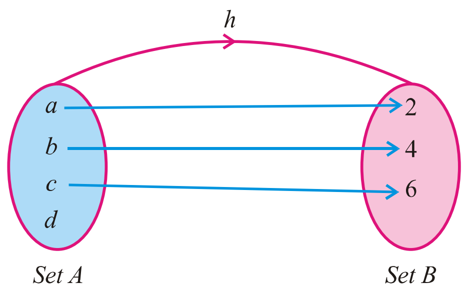
Rang $f = \{1, 2, 3, 4\} \subseteq B$.



The following are the examples of relations but not functions.

g is not a function, because an element $a \in A$ has two images in set B and h is not a function because an element $d \in A$ has no image in set B.





5.3(ii) Demonstrate the following:

(a) Into function:

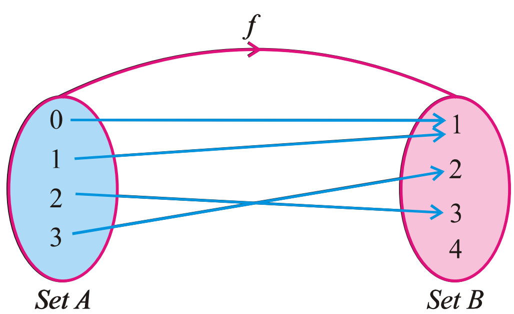
A function $f : A \to B$ is called an into function, if at least one element in B is not an image of some element of set A *i.e.*, Range of $f \subset$ set B.

For example, we define a function $f : A \to B$ such that

$f = \{(0, 1), (1, 1), (2, 3), (3, 2)\}$

where $A = \{0, 1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$

f is an into function.



(b) One-one function:

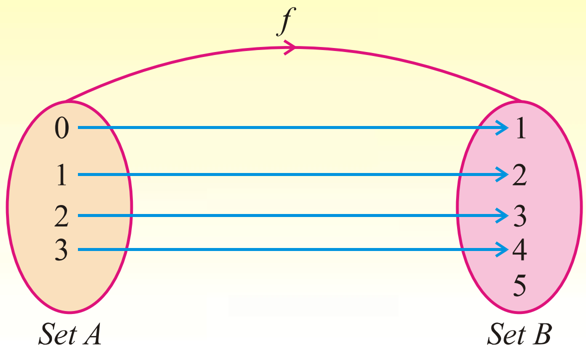
A function $f : A \to B$ is called one-one function, if all distinct elements of A have distinct images in B, *i.e.*, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \in A$ or $\forall\, x_1 \neq x_2 \in A \Rightarrow f(x_1) \neq f(x_2)$

For example, if $A = \{0, 1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$, then we define a function $f : A \rightarrow B$ such that

$f = \{(x, y) \mid y = x + 1, \forall\, x \in A, y \in B\}$.

$= \{(0, 1,), (1, 2), (2, 3), (3, 4)\}$

f is one-one function.



(c) Into and one-one function: (injective function)

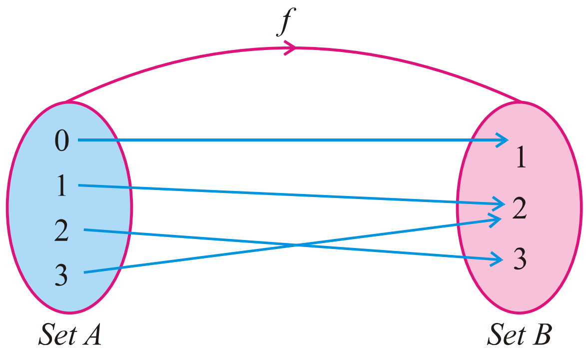
The function f discussed in (b) is also an into function. Thus f is an into and one-one function.

(d) An onto or surjective function:

A function $f : A \rightarrow B$ is called an onto function, if every element of set B is an image of at least one element of set A *i.e.*, Range of $f = B$.

For example, if $A = \{0, 1, 2, 3\}$ and $B = \{1, 2, 3\}$, then $f : A \rightarrow B$ such that $f = \{(0, 1), (1, 2), (2, 3), (3, 2)\}$. Here Rang $f = \{1, 2, 3\} = B$.

Thus f so defined is an onto function.



(e) Bijective function or one to one correspondence:

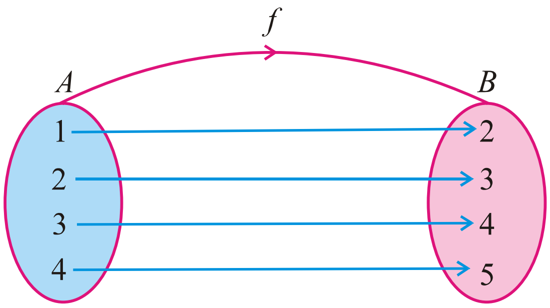
A function $f : A \rightarrow B$ is called bijective function iff function f is one-one and onto.

For example, if $A = \{1, 2, 3, 4\}$ and $B = \{2, 3, 4, 5\}$

We define a function $f : A \rightarrow B$ such that

$f = \{(x, y) \mid y = x + 1, \forall\, x \in A, y \in B\}$

Then $f = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$



Evidently this function is one-one because distinct elements of A have distinct images in B. This is an onto function also because every element of B is the image of atleast one element of A.

Note: (1) Every function is a relation but converse may not be true.
(2) Every function may not be one-one.
(3) Every function may not be onto.

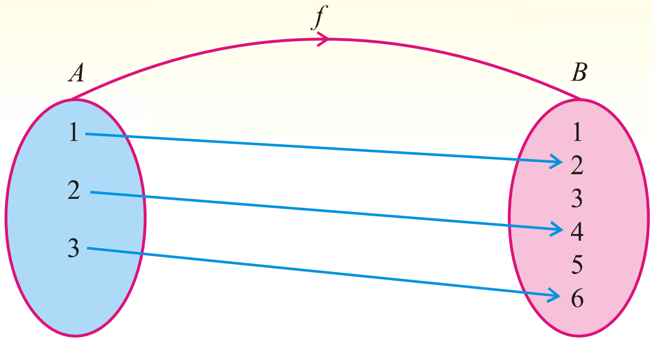
Example: Suppose $A = \{1, 2, 3\}$

$B = \{1, 2, 3, 4, 5, 6\}$

We define a function $f: A \to B = \{(x, y) \mid y = 2x, \forall\, x \in A, y \in B\}$

Then $f = \{(1, 2), (2, 4), (3, 6)\}$

Evidently this function is one-one but not an onto



5.3(iii) Examine whether a given relation is a function:

A relation in which each $x \in$ its domain, has a unique image in its range, is a function.

5.3(iv) Differentiate between one-to-one correspondence and one-one function:

A function f from set A to set B is one-one if distinct elements of A has distinct images in B. The domain of f is A and its range is contained in B.

In one-to-one correspondence between two sets A and B, each element of either set is assigned with exactly one element of the other set. If the sets A and B are finite, then these sets have the same number of elements, that is, $n(A) = n(B)$.

EXERCISE 5.5

1. If $L = \{a, b, c\}$, $M = \{3, 4\}$, then find two binary relations of $L \times M$ and $M \times L$.
2. If $Y = \{-2, 1, 2\}$, then make two binary relations for $Y \times Y$. Also find their domain and range.
3. If $L = \{a, b, c\}$ and $M = \{d, e, f, g\}$, then find two binary relations in each:

   (i) $L \times L$ (ii) $L \times M$ (iii) $M \times M$
4. If set M has 5 elements, then find the number of binary relations in M.
5. If $L = \{x \mid x \in N \wedge x \leq 5\}$, $M = \{y \mid y \in P \wedge y < 10\}$, then make the following relations from L to M

   (i) $R_1 = \{(x, y) \mid y < x\}$ (ii) $R_2 = \{(x, y) \mid y = x\}$

   (iii) $R_3 = \{(x, y) \mid x + y = 6\}$ (iv) $R_4 = \{(x, y) \mid y - x = 2\}$

   Also write the domain and range of each relation.
6. Indicate relations, into function, one-one function, onto function, and bijective function from the following. Also find their domain and the range.

(i) $R_1 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$

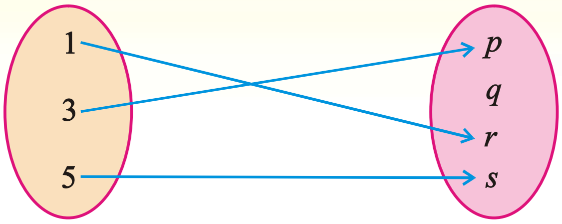
(ii) $R_2 = \{(1, 2), (2, 1), (3, 4), (3, 5)\}$

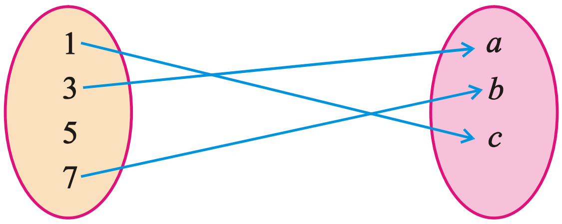
(iii) $R_3 = \{(b, a), (c, a), (d, a)\}$

(iv) $R_4 = \{(1, 1), (2, 3), (3, 4), (4, 3), (5, 4)\}$

(v) $R_5 = \{(a, b), (b, a), (c, d), (d, e)\}$

(vi) $R_6 = \{(1, 2), (2, 3), (1, 3), (3, 4)\}$

(vii) $R_7 =$ 

(viii) $R_8 =$ 

MISCELLANEOUS EXERCISE - 5

1. Multiple Choice Questions

Four possible answers are given for the following questions. Tick mark (✓) the correct answer.

(i) A collection of well-defined objects is called

(a) subset (b) power set
(c) set (d) none of these

(ii) A set $Q = \left\{ \frac{a}{b} \mid a, b \in Z \wedge b \neq 0 \right\}$ is called a set of

(a) Whole numbers (b) Natural numbers
(c) Irrational numbers (d) Rational numbers

(iii) The different number of ways to describe a set are

(a) 1 (b) 2
(c) 3 (d) 4

(iv) A set with no element is called

(a) Subset (b) Empty set
(c) Singleton set (d) Super set

(v) The set $\{ x \mid x \in W \wedge x \leq 101 \}$ is

(a) Infinite set (b) Subset
(c) Null set (d) Finite set

(vi) The set having only one element is called

(a) Null set (b) Power set

(c) Singleton set (d) Subset

(vii) Power set of an empty set is

(a) ϕ (b) $\{a\}$

(c) $\{\phi, \{a\}\}$ (d) $\{\phi\}$

(viii) The number of elements in power set $\{1, 2, 3\}$ is

(a) 4 (b) 6

(c) 8 (d) 9

(ix) If $A \subseteq B$, then $A \cup B$ is equal to

(a) A (b) B

(c) ϕ (d) none of these

(x) If $A \subseteq B$, then $A \cap B$ is equal to

(a) A (b) B

(c) ϕ (d) none of these

(xi) If $A \subseteq B$, then $A - B$ is equal to

(a) A (b) B

(c) ϕ (d) $B - A$

(xii) $(A \cup B) \cup C$ is equal to

(a) $A \cap (B \cup C)$ (b) $(A \cup B) \cap C$

(c) $A \cup (B \cup C)$ (d) $A \cap (B \cap C)$

(xiii) $A \cup (B \cap C)$ is equal to

(a) $(A \cup B) \cap (A \cup C)$ (b) $A \cap (B \cap C)$

(c) $(A \cap B) \cup (A \cap C)$ (d) $A \cup (B \cup C)$

(xiv) If A and B are disjoint sets, then $A \cup B$ is equal to

(a) A (b) B

(c) ϕ (d) $B \cup A$

(xv) If number of elements in set A is 3 and in set B is 4, then number of elements in $A \times B$ is

(a) 3 (b) 4

(c) 12 (d) 7

(xvi) If number of elements in set A is 3 and in set B is 2, then number of binary relations in $A \times B$ is

(a) 2^3 (b) 2^6

(c) 2^8 (d) 2^2

(xvii) The domain of $R = \{(0, 2), (2, 3), (3, 3), (3, 4)\}$ is

(a) $\{0, 3, 4\}$ (b) $\{0, 2, 3\}$

(c) $\{0, 2, 4\}$ (d) $\{2, 3, 4\}$

(xviii) The range of $R = \{(1, 3), (2, 2), (3, 1), (4, 4)\}$ is

(a) $\{1, 2, 4\}$ (b) $\{3, 2, 4\}$

(c) $\{1, 2, 3, 4\}$ (d) $\{1, 3, 4\}$

(xix) Point (–1, 4) lies in the quadrant

(a) I (b) II

(c) III (d) IV

(xx) The relation {(1, 2), (2, 3), (3, 3), (3, 4)} is

(a) onto function (b) into function

(c) not a function (d) one-one function

2. Write short answers of the following questions.

(i) Define a subset and give one example.

(ii) Write all the subsets of the set $\{a, b\}$

(iii) Show $A \cap B$ by Venn diagram. When $A \subseteq B$

(iv) Show by Venn diagram $A \cap (B \cup C)$.

(v) Define intersection of two sets.

(vi) Define a function.

(vii) Define one-one function.

(viii) Define an onto function.

(ix) Define a bijective function.

(x) Write De Morgan's laws.

3. Fill in the blanks

(i) If $A \subseteq B$, then $A \cup B =$ _________.

(ii) If $A \cap B = \phi$ then A and B are _______.

(iii) If $A \subseteq B$ and $B \subseteq A$ then ___________.

(iv) $A \cap (B \cup C) =$ ___________.

(v) $A \cup (B \cap C) =$ _________.

(vi) The complement of U is ____________.

(vii) The complement of ϕ is ____________.

(viii) $A \cap A^c =$ ____________.

(ix) $A \cup A^c =$ ____________.

(x) The set $\{x \mid x \in A \text{ and } x \notin B\} =$ ____________.

(xi) The point (–5, –7) lies in ____________ quadrant.

(xii) The point (4, –6) lies in ____________ quadrant.

(xiii) The ***y*** co-ordinate of every point is ____________ on-x-axis.

(xiv) The ***x*** co-ordinate of every point is ____________ on-y-axis.

(xv) The domain of $\{(a, b), (b, c), (c, d)\}$ is ____________.

(xvi) The range of $\{(a, a), (b, b), (c, c)\}$ is ____________.

(xvii) Venn-diagram was first used by ____________.

(xviii) A subset of $A \times A$ is called the ____________ in A.

(xix) If $f : A \longrightarrow B$ and range of $f = B$, then f is an ____________ function.

(xx) The relation $\{(a, b), (b, c), (a, d)\}$ is ____________ a function.

SUMMARY

- A set is the **well defined collection** of objects with some common properties.
- Union of two sets A and B denoted by $A \cup B$ is the set **containing elements** which either belong to A or to B or to both.
- Intersection of two sets A and B denoted by $A \cap B$ is the set of **common elements** of both A and B. In symbols $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.
- The set difference of B and A denoted by $B - A$ is the set of all those elements of B which **do not belong to A.**
- Complement of a set A w.r.t. **universal** set U denoted by $A^C = A' = U - A$ contains all those elements of U which **do not belong to A.**
- British mathematician John Venn (1834 – 1923) introduced rectangle for a universal set U and its subsets A and B as **closed figures** inside this rectangle.
- An ordered pair of elements is written according to a **specific order** for which the order of elements is strictly maintained.
- Cartesian product of two non-empty sets A and B denoted by $A \times B$ consists of all **ordered pairs** (x, y) such that $x \in A, y \in B$.
- If A and B are any two non-empty sets, then a non empty subset $R \subseteq A \times B$ is called binary relation from set A into set B.
- If A and B are two non empty sets, then **relation** $f : A \to B$ is called a function if (i) Dom f = set A (ii) every $x \in A$ appears in one and only one ordered pair $\in f$.
- Dom f is the set consisting of all **first elements** of each ordered pair $\in f$ and Rang of f is the set consisting of all **second elements** of each ordered pair $\in f$.
- A function $f : A \to B$ is called an into function if at least one element in B is not an image of some element of set A *i.e.*, **range of $f \subseteq B$.**
- A function $f : A \to B$ is called an onto function if every element of set B is an image of at least one element of set A *i.e.*, **range of $f = B$**.
- A function $f : A \to B$ is called one-one function if all **distinct elements** of A have distinct images in B
- A function $f : A \to B$ is called bijective function iff function f is **one-one and onto**.

Unit-6

BASIC STATISTICS

In this unit, students will learn how to

- *construct grouped frequency table.*
- *construct histograms with equal and unequal class intervals.*
- *construct a frequency polygon.*
- *construct a cumulative frequency table.*
- *draw a cumulative frequency polygon.*
- *calculate (for ungrouped and grouped data)*
  - *arithmetic mean by definition and using deviations from assumed mean.*
- *calculate median, mode, geometric mean, harmonic mean.*
- *recognize properties of arithmetic mean.*
- *calculate weighted mean and moving averages.*
- *estimate median, quartiles and mode graphically.*
- *measure range, variance and standard deviation.*

6.1 Frequency Distribution

A frequency distribution is a tabular arrangement for classifying data into different groups and the number of observations falling in each group corresponds to the respective group. The data presented in the form of frequency distribution is called Grouped Data. Hence a frequency distribution is a method to summarize data.

6.1(i) Construction of Frequency Table

On the basis of types of variable or data, there are two types of frequency distribution. These are:

(a) Discrete Frequency Distribution.
(b) Continuous Frequency Distribution.

(a) Discrete Frequency Table

Following steps are involved in making of a discrete frequency distribution:

(i) Find the minimum and maximum observation in the data and write the values of the variable in the variable column from minimum to the maximum.
(ii) Record the observations by using tally marks. (Vertical bar '|')
(iii) Count the tally and write down the frequency in the frequency column.

Example 1: Five coins are tossed 20 times and the number of heads recorded at each toss are given below: 3, 4, 2, 3, 3, 5, 2, 2, 2, 1, 1, 2, 1, 4, 2, 2, 3, 3, 4, 2.

Make frequency distribution of the number of heads observed.

Solution: Let X = Number of Heads. The frequency distribution is given below:

Frequency distribution of number of heads		
X	Tally Marks	Frequency
1	|||	3
2	~~||||~~ |||	8
3	~~||||~~	5
4	|||	3
5	|	1

(b) Continuous Frequency Table

The making of continuous frequency distribution involves the following steps:

(i) Find the Range, where Range = $X_{max} - X_{min}$ (the difference between *maximum* and *minimum* observations).
(ii) Decide about the number of groups (denote it by k) into which the data is to be classified (usually an integer between 5 and 20). Usually it depends upon the range. The larger the range the more the number of groups.
(iii) Determine the size of class (denote by h) by using the formula:

$$h = \frac{\text{Range}}{k} \qquad \text{(Use formula when "}b\text{" is not given)}$$

Note: The rule of approximation is relaxed in determining h. For example, $h = 7.1$ or $h = 7.9$ may be taken as 8.

(iv) Start writing the classes or groups of the frequency distribution usually starting from the minimum observation and keeping in view the size of a class.

(v) Record the observations from the data by using tally marks.

(vi) Count the number of tally marks and record them in the frequency column for each class.

Example 2: The following are the marks obtained by 40 students in mathematics of class X.

Make a frequency distribution with a class interval of size 10.

51, 55, 32, 41, 22, 30, 35, 53, 30, 60, 59, 15, 7, 18, 40, 49, 40, 25, 14, 18,19, 2, 43, 22, 39, 26, 34, 19, 10, 17, 47, 38, 13, 30, 34, 54, 10, 21, 51, 52.

Solution: Let X = marks of a student.

From the above data we have $X_{min} = 2$, $X_{max} = 60$. It is given that $h = 10$. We can either start from 2 or the nearest smallest integer 0 for our convenience. There are two ways to make frequency distribution.

(a) We may write the actual observations falling in the respective groups. This is given as follows:

Classes/Groups	Observations	Frequency
0 —— 9	2, 7	2
10 —— 19	10, 10, 13, 14, 15, 17, 18, 18, 19, 19	10
20 —— 29	21, 22, 22, 25, 26	5
30 —— 39	30, 30, 30, 32, 34, 34, 35, 38, 39	9
40 —— 49	40, 40, 41, 43, 47, 49	6
50 —— 59	51, 51, 52, 53, 54, 55, 59	7
60 —— 69	60	1

(b) Use tally marks for recording each observation in the respective group. This is given in the following table:

Classes/Groups	Tally Marks	Frequency
0 —— 9	II	2
10 —— 19	~~IIII~~ ~~IIII~~	10
20 —— 29	~~IIII~~	5
30 —— 39	~~IIII~~ IIII	9
40 —— 49	~~IIII~~ I	6
50 —— 59	~~IIII~~ II	7
60 —— 69	I	1
Total		40

Note: The solution (b) is usually adopted to construct a frequency distribution.

Concepts involved in a Continuous frequency table:

The following terms are frequently used in a continuous frequency distribution:

(a) Class Limits: The minimum and the maximum values defined for a class or group are called Class limits. The minimum value is called the **lower class limit** and the maximum value is called the **upper class limit** of that class. In example 2, the lower class limits are 0, 10, 20, 30 etc., while the upper class limits are 9, 19, 29, 39 etc.

(b) Class Boundaries: As a continuous frequency distribution is based on measurable characteristic variable which involves the rule of approximation to record any observation. From example 2, some class boundaries are given below:

Class limits	Class Boundaries
0 ---- 9	–0.5 ---- 9.5
10 ---- 19	9.5 ---- 19.5
20 ---- 29	19.5 ---- 29.5

Hence referring to example 2, we may say that the real lower class limit of 10 is 9.5, as all values between 9.5 and 10.49 are recorded as 10. While the upper class limit of 19 is 19.5 as all values between 18.5 and 19.5 are recorded as 19. The real class limits of a class are called **class boundaries**. A **class boundary** is obtained by adding two successive class limits and dividing the sum by 2. The value so obtained is taken as **upper class boundary** for the previous class and **lower class boundary** for the next class.

(c) Midpoint or Class Mark: For a given class the average of that class obtained by dividing the sum of upper and lower class limits by 2, is called the **midpoint** or **class mark** of that class.

(d) Cumulative Frequency: The total of frequency up to an upper class limit or boundary is called the **cumulative frequency**.

The above concepts have been explained with reference to Example 2 below:

Example 3: Compute class boundaries, class marks and cumulative frequency for data of example 2.

Solution: Computation follows,

Class limits	Class Boundaries	Midpoint/ Class mark	Frequency	Cumulative frequency
0 ---- 9	–0.5 ---- 9.5	4.5	2	2
10 ---- 19	9.5 ---- 19.5	14.5	10	2 + 10 = 12
20 ---- 29	19.5 ---- 29.5	24.5	5	12 + 5 = 17
30 ---- 39	29.5 ---- 39.5	34.5	9	17 + 9 = 26
40 ---- 49	39.5 ---- 49.5	44.5	6	26 + 6 = 32
50 ---- 59	49.5 ---- 59.5	54.5	7	32 + 7 = 39
60 ---- 69	59.5 ---- 69.5	64.5	1	39 + 1 = 40
Total			40	

6.1(ii) Construction of Histograms

Histogram

A **Histogram** is a graph of adjacent rectangles constructed on *XY*-plane. It is a graph of frequency distribution. In practice both discrete and continuous frequency distributions are represented by means of histogram. However there is a little difference in the construction procedure. We explain this with the help of examples.

Equal Intervals Histogram:

Example 1: Make a Histogram of the following distribution of the number of heads when 5 coins were tossed.

X (number of heads)	Frequency
0	1
1	3
2	8
3	5
4	3
5	1

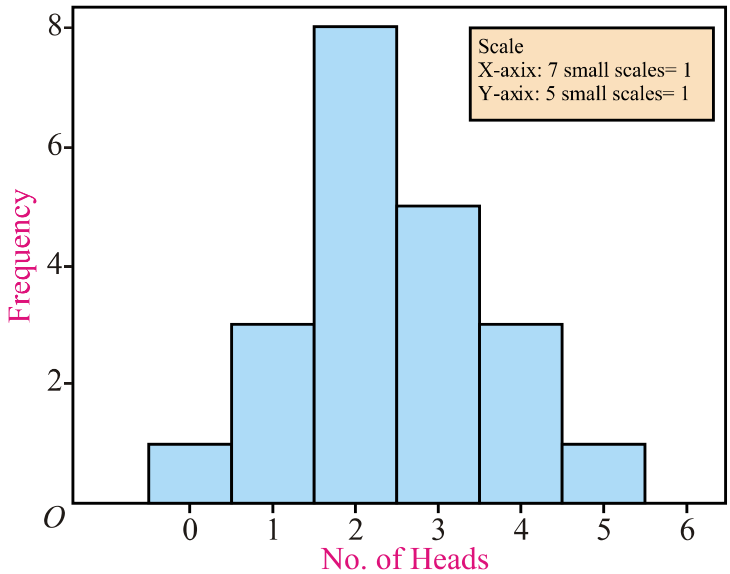
Solution: We proceed as follows,

Step 1: Mark the values of variable *X* along *x*-axis using a suitable interval.

Step 2: Mark the frequency along *y*-axis using a suitable scale.

Step 3: At each interval make a rectangle of height corresponding to the respective frequency of values of the variable *X*.

The resulting Histogram is given below:



Example 2: Make a Histogram for the following distribution of marks.

Class Boundaries	Frequency
–0.5 —— 9.5	2
9.5 —— 19.5	10
19.5 —— 29.5	5
29.5 —— 39.5	6
39.5 —— 49.5	7
49.5 —— 59.5	1

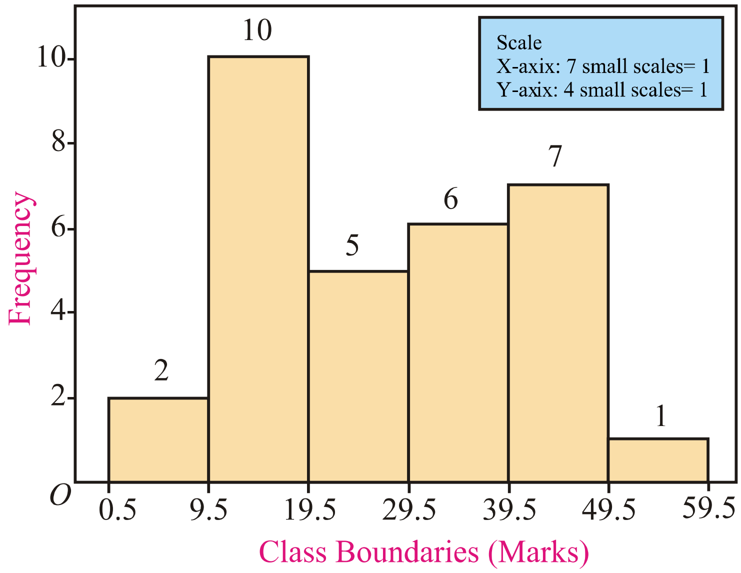
Solution: Since this is a continuous frequency distribution so we proceed as follows:

(a) Mark the class boundaries along x-axis using a suitable scale.

(b) Mark the frequency along y-axis using a suitable scale.

(c) At each class interval construct a rectangle of height corresponding to the frequency of that group.

Histogram



Note: On graphs above 0 and –0.5 are written on positive side of x-axis just to better understand the histogram.

Unequal Intervals Histogram

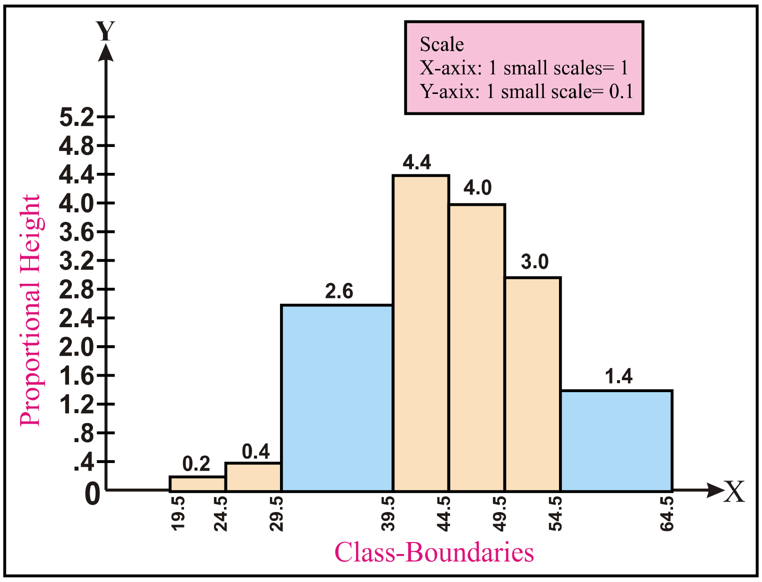
If the class intervals are un-equal, the frequency must be adjusted by dividing each class frequency on its class interval size. If the interval becomes double, then frequency is divided by 2, so that the area of the bar is in proportion to the areas of other bars etc.

Example 3: Draw a histogram illustrating the following data:

Age	Number of Men
20 —— 24	1
25 —— 29	2
30 —— 39	26
40 —— 44	22
45 —— 49	20
50 —— 54	15
55 —— 64	14

Solution: As the class intervals are unequal the height of each rectangle cannot be made equal to the frequency. Therefore, we obtain proportional heights by dividing each frequency with class interval size. This is shown in the following table:

Age	Class Boundaries	Class Interval (h)	Frequency	Proportional Heights
20 —— 24	19.5 —— 24.5	5	1	$1 \div 5 = 0.20$
25 —— 29	24.5 —— 29.5	5	2	$2 \div 5 = 0.4$
30 —— 39	29.5 —— 39.5	10	26	$26 \div 10 = 2.6$
40 —— 44	39.5 —— 44.5	5	22	$22 \div 5 = 4.4$
45 —— 49	44.5 —— 49.5	5	20	$20 \div 5 = 4.0$
50 —— 54	49.5 —— 54.5	5	15	$15 \div 5 = 3.0$
55 —— 64	54.5 —— 64.5	10	14	$14 \div 10 = 1.4$



6.1(iii) Construction of Frequency Polygon

A **Frequency Polygon** is a many sided closed figure. Its construction is explained by the following example:

Example 1: For the following data make a Frequency Polygon.

Class limits	Class Boundaries	Frequency
10—19	9.5—19.5	10
20—29	19.5—29.5	5
30—39	29.5—39.5	9
40—49	39.5—49.5	6
50—59	49.5—59.5	7
60—69	59.5—69.5	1

Solution:

Step 1. Take two additional groups with the same class interval size. One before the very first group and the second after the very last group. Also calculate midpoints for these two groups. These groups will have frequency '0'.

Class limits	Class Boundaries	Frequency
0 — 9	**–0.5 — 9.5**	0
10 — 19	9.5 — 19.5	10
20 — 29	19.5 — 29.5	5
30 — 39	29.5 — 39.5	9
40 — 49	39.5 — 49.5	6
50 — 59	49.5 — 59.5	7
60 — 69	59.5 — 69.5	1
70 — 79	**69.5 — 79.5**	0

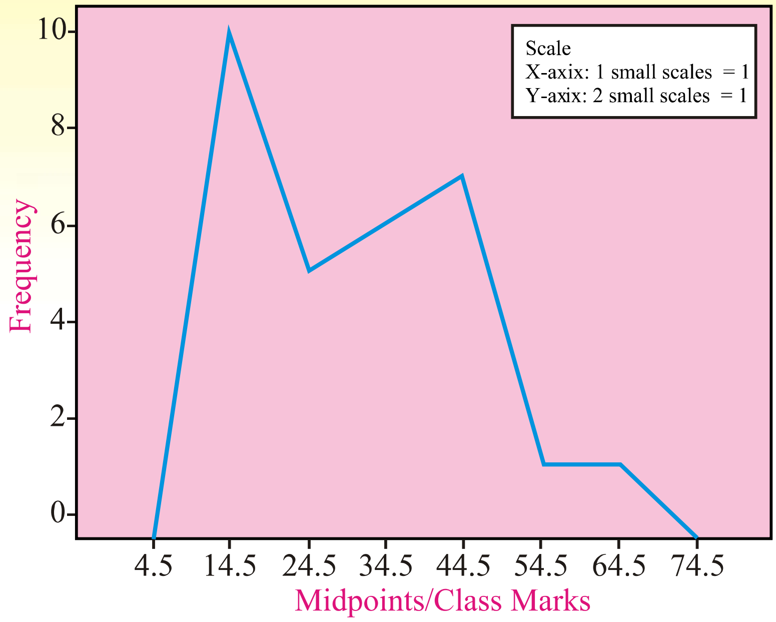
Step 2. Calculate class marks or midpoints for the given distribution.

Midpoints / Class marks	Frequency
4.5	0
14.5	10
24.5	5
34.5	9
44.5	6
54.5	7
64.5	1
74.5	0

Step 3. Mark midpoints at x-axis and frequency along y-axis using appropriate scale.

Step 4. Plot a point against the frequency for each of the corresponding midpoint/class mark.

Step 5. Join all the points by means of line segments.



6.2 Cumulative Frequency Distribution

6.2(i) Construction of Cumulative Frequency Table

A table showing cumulative frequencies against upper class boundaries is called a **cumulative frequency distribution**. It is also called ***a less than cumulative frequency distribution***.

Example 1: Construct a cumulative frequency distribution for the following data.

Classes	20 – 24	25 – 29	30 – 34	35 – 39	40 – 44	45 – 49	50 – 54
Frequency	1	2	26	22	20	15	14

Solution: The **cumulative frequency distribution** is constructed below:

Class Boundaries	Frequency (f)	Cumulative Frequency	Class Boundaries	Cumulative Frequency
10.5 — 19.5	0	0	Less than 19.5	0
19.5 — 24.5	1	0 + 1 = 1	Less than 24.5	1
24.5 — 29.5	2	1 + 2 = 3	Less than 29.5	3
29.5 — 34.5	26	3 + 26 = 29	Less than 34.5	29
34.5 — 39.5	22	29 + 22 = 51	Less than 39.5	51
39.5 — 44.5	20	51 + 20 = 71	Less than 44.5	71
44.5 — 49.5	15	71 + 15 = 86	Less than 49.5	86
49.5 — 54.5	14	86 + 14 = 100	Less than 54.5	100

6.2(ii) Drawing of Cumulative Frequency Polygon or Ogive

A cumulative frequency polygon or ogive is a graph of less than cumulative frequency distribution. It involves the following steps:

Step 1. Mark the class boundaries on x-axis and frequency (cumulative) on y-axis.

Step 2. Plot the points for the given frequencies corresponding to the upper class boundaries.

Step 3. Join the points by means of line segments.

Step 4. Drop perpendicular from the last point to x-axis to make a closed figure.

Example 2: Construct a cumulative frequency polygon for the given data.

Class limits	Frequency
4 — 6	2
7 — 9	4
10 — 12	8
13 — 15	3

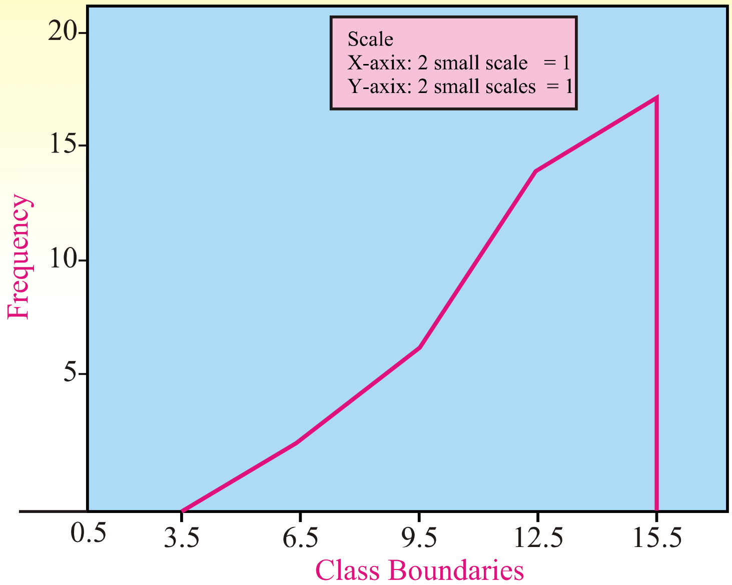
Solution: First we add one group before the first group. Then we make the class boundaries and also calculate the cumulative frequencies.

Class limits	Class Boundaries	Frequency	Cumulative frequency
1 — 3	**0.5 — 3.5**	**0**	**0**
4—6	3.5 — 6.5	2	0 + 2 = 2
7—9	6.5 — 9.5	4	2 + 4 = 6
10—12	9.5 — 12.5	8	6 + 8 = 14
13—15	12.5 — 15.5	3	14 + 3 = 17

Now we write the above frequency distribution in the form of Less than cumulative distribution as given below:

Class Boundaries	Cumulative frequency
Less than 3.5	0
Less than 6.5	2
Less than 9.5	6
Less than 12.5	14
Less than 15.5	17

The Cumulative frequency polygon follows:



EXERCISE 6.1

1. The following data shows the number of members in various families. Construct frequency distribution. Also find cumulative frequencies.

   9, 11, 4, 5, 6, 8, 4, 3, 7, 8, 5, 5, 8, 3, 4, 9, 12, 8, 9, 10, 6, 7, 7, 11, 4, 4, 8, 4, 3, 2, 7, 9, 10, 9, 7, 6, 9, 5, 7.

2. The following data has been obtained after weighing 40 students of class V. Make a frequency distribution taking class interval size as 5. Also find the class boundaries and midpoints.

   34, 26, 33, 32, 24, 21, 37, 40, 41, 28, 28, 31, 33, 34, 37, 23, 27, 31, 31, 36, 29, 35, 36, 37, 38, 22, 27, 28, 29, 31, 35, 35, 40, 21, 32, 33, 27, 29, 30, 23.

   Also make a less than cumulative frequency distribution. (Hint: Make classes 20—24, 25—29…..).

3. From the following data representing the salaries of 30 teachers of a school .Make a frequency distribution taking class interval size of Rs.100**,** 450, 500, 550, 580, 670, 1200,1150, 1120, 950, 1130, 1230, 890, 780, 760, 670, 880, 890, 1050, 980, 970, 1020, 1130, 1220, 760, 690, 710, 750, 1120, 760, 1240.

   (Hint: Make classes 450—549, 550—649,…..).

4. The following data shows the daily load shedding duration in hours in 30 localities of a certain city. Make a frequency distribution of the load shedding duration taking 2 hours as class interval size and answer the following questions.

   6, 12, 5, 7, 8, 3, 6, 7, 10,2, 14, 11, 12, 8, 6, 8, 9, 7, 11, 6, 9, 12, 13, 10, 14, 7, 6, 10, 11, 14, 12.

   (a) Find the most frequent load shedding hours?

(b) Find the least load shedding intervals?

(Hint: Make classes 2—3, 4—5, 6—7….)

5. Construct a Histogram and frequency Polygon for the following data showing weights of students in kg.

Weights	Frequency / No. of students
20—24	5
25—29	8
30—34	13
35—39	22
40—44	15
45—49	10
50—54	8

6.3 Measures of Central Tendency

Introduction

The purpose of frequency distribution and graphical techniques is to view, summarize and understand different aspects of data in a simple manner. But we are also interested to find a '*representative*' of the data under study. In other words to determine a specific value of the variable around which the majority of the observations tend to concentrate. This *representative* shows the tendency or behavior of the distribution of the variable under study. This value is called **average** or **the central value.** The measures or techniques that are used to determine this central value are called **Measures of Central Tendency**. The following measures of central tendency will be discussed in this section:

1. Arithmetic mean
2. Median
3. Mode
4. Geometric mean
5. Harmonic mean
6. Quartiles

All these measures are used under different situations depending upon the nature of the data.

6.3(i-a) Arithmetic Mean

Arithmetic Mean (or simply called Mean) is a measure that determines a value (observation) of the variable under study by dividing the sum of all values (observations) of the variable by their number of observations. We denote Arithmetic mean by $\overline{X}$. In symbols we define:

Arithmetic mean of n observations $\overline{X} = \frac{\sum X}{n} = \frac{\text{Sum of all values of observation}}{\text{No. of obsrvation}}$

Computation of Arithmetic Mean

There are two types of data, ungrouped and grouped. We, therefore have different methods to determine Mean for the two types of data. These are explained with the help of examples.

Ungrouped Data

For ungrouped data we use three approaches to find mean. These are as follows.

(i) Direct Method (By Definition)

The formula under this method is given by:

$$\bar{X} = \frac{\sum X}{n} = \frac{\text{Sum of all observations}}{\text{No. of observations}}$$

Example 1: The marks of seven students in Mathematics are as follows. Calculate the Arithmetic Mean and interpret the result.

Student No	1	2	3	4	5	6	7
Marks	45	60	74	58	65	63	49

Solution: Let X = marks of a student

$$\bar{X} = \frac{\sum X}{n} = \frac{x_1 + x_2 + x_3 + \ldots\ldots + x_7}{7}$$

or $$\bar{X} = \frac{45 + 60 + 74 + 58 + 65 + 63 + 49}{7} = \frac{414}{7} = 59.14 \text{ marks}$$

Explanation: Since the unit of data is marks, so the result is also in marks. Hence we may say that, 'Each of the seven students obtains 59 marks on the average'.

(ii) Indirect, Short Cut or Coding Methods

There are two approaches under Indirect Method. These are used to find mean when data set consists of large values or large number of values. The purpose is to simplify the computation of Mean. These approaches exist in theory but are not used in practice as many Statistical softwares are available now to handle large data. However a student should have knowledge of the two approaches. These are:

(i) using an Assumed or Provisional mean

(ii) using a Provisional mean and changing scale of the variable.

Deviation is defined as difference of any value of the variable from any constant 'A'. For example we say,

Deviation from mean of X = $(x_i - \bar{X}) = (X_i - \bar{X})$ for i = 1, 2, …… n

Deviation from any constant A = $(x_i - A) = (X_i - A)$ for i = 1, 2, …… n

The formulae used under indirect methods are:

(i) $$\bar{X} = A + \frac{\sum_{i=1}^{n} D_i}{n}$$ (ii) $$\bar{X} = A + \frac{\sum_{i=1}^{n} u_i}{n} \times h$$

where

$D_i = (X_i - A)$, A is any assumed value of X called Assumed or Provisional mean.

$u_i = \frac{(X_i - A)}{h}$, "$h$" is the class interval size for unequal intervals.

Example 2: The salaries of five teachers are as follows. Find the mean salary using direct and indirect methods and compare the results. 11500, 12400, 15000, 14500, 14800.

Solution: We proceed as follows:

(a) Using Direct method

$$\overline{X} = \frac{\sum_{i=1}^{5} x_i}{5} = \frac{11500 + 12400 + 15000 + 14500 + 14800}{5}$$

$$= \frac{68200}{5} = 13640 \text{ Rupees.}$$

we assume $A = 13000$, $D_i = (x_i - 13000)$, $h = 100$ and $u_i = \frac{(x_i - A)}{100}$, the computations are shown in the following table:

X	$D_i = (x_i - 13000)$	$u_i = \frac{(x_i - A)}{100}$
11500	−1500	−15
12400	−600	−6
15000	2000	20
14500	1500	15
14800	1800	18
$\Sigma x_i = 68200$	$\Sigma D_i = 3200$	$\Sigma u_i = 32$

(i) Using Indirect methods

$$\overline{X} = 13000 + \frac{3200}{5} = 13000 + 640 = 13640 \text{ Rupees}$$

(ii) Using Indirect method

$$\overline{X} = 13000 + \frac{32}{5} \times 100 = 13640 \text{ Rupees}$$

Grouped Data

A data in the form of frequency distribution is called **grouped data**. For the grouped data we define formulae under Direct and Indirect methods as given below:

(a) Using Direct method,

$$\overline{X} = \frac{\sum fx_i}{\sum f} = \frac{\sum fX}{\sum f}$$

(b) Using Indirect method,

(i) $\overline{X} = A + \frac{\sum fD}{\sum f}$ (ii) $\overline{X} = A + \frac{\sum fu}{\sum f} \times h$

where '$X = x_i$' denotes the midpoint of a class or group if class intervals are given, and 'h' is the class interval size .

Example 3: Find the arithmetic mean using Direct method for the following frequency distribution.

(Number of heads) X	Frequency
1	3
2	8
3	5
4	3
5	1

Solution: We compute mean as follows:

X	f	fX
1	3	3
2	8	16
3	5	15
4	3	12
5	1	5
Total	$\Sigma f = 20$	$\Sigma fX = 49$

$\bar{X} = \dfrac{\sum fX}{\sum f} = \dfrac{49}{20} = 2.45$ or 3 heads (since the variable is discrete).

Example 4: For the following data showing weights of toffee boxes in gms. Determine the mean weight of boxes.

Classes / Groups	Frequency
0 —— 9	2
10 —— 19	10
20 —— 29	5
30 —— 39	9
40 —— 49	6
50 —— 59	7
60 —— 69	1
Total	$\Sigma f = 40$

Solution: First we calculate midpoints for each class and then find arithmetic mean.

Classes / Groups	f	Midpoints X	fX
0 —— 9	2	4.5	9
10 —— 19	10	14.5	145
20 —— 29	5	24.5	122.5
30 —— 39	9	34.5	310.5
40 —— 49	6	44.5	267
50 —— 59	7	54.5	381.5
60 —— 69	1	64.5	64.5
Total	$\Sigma f = 40$		$\Sigma fX = 1300$

$$\bar{X} = \frac{\sum fX}{\sum f} = \frac{1300}{40} = 32.5 \text{ gm}$$

Example 5: Find arithmetic mean using short formulae taking $X = 34.5$ as the provisional mean in example 4.

Solution: We use the following formulae

(i) $\bar{X} = A + \frac{\sum fD}{\sum f}$ (ii) $\bar{X} = A + \frac{\sum fu}{\sum f} \times h$

Given $A = 34.5$, we note that the distribution has equal class interval size of 10. So we may take $h = 10$ and make the following calculations:

Classes/ Groups	f	Midpoints X	$D = X - 34.5$	$u = (X - A)/10$	fD	fu
0 —— 9	2	4.5	–30	–3	–60	–6
10 —— 19	10	14.5	–20	–2	–200	–20
20 —— 29	5	24.5	–10	–1	–50	–5
30 —— 39	9	34.5	0	0	0	0
40 —— 49	6	44.5	10	1	60	6
50 —— 59	7	54.5	20	2	140	14
60 —— 69	1	64.5	30	3	30	3
Total	40				$\Sigma fD = -80$	$\Sigma fu = -8$

Substituting the totals in the above formulae, we get

(i) $\bar{X} = 34.5 + \frac{-80}{40} = 34.5 - 2 = 32.5$ gm

(ii) $\bar{X} = 34.5 + \frac{-8}{40} \times 10 = 34.5 - 2 = 32.5$ gm.

Hence the three methods yield the same answer.

6.3(i) (b) Median

Median is the middle most observation in an arranged data set. It divides the data set into two equal parts. '$\tilde{x}$' is used to represent median. We determine Median by using the following formulae:

Ungrouped data

Case-1: When the number of observations is odd of a set of data arranged in order of magnitude the median (middle most observation) is located by the formula given below:

$$\tilde{X} = \text{size of } \left(\frac{n+1}{2}\right)\text{th} \quad \text{observation}$$

Case-2: When the number of observations is *even* of a set of data arranged in order of magnitude the median is the arithmetic mean of the two middle observations. That is, median is average of $\frac{n}{2}$ and $\left(\frac{n}{2}+1\right)$th values.

$$\tilde{X} = \frac{1}{2}\left[\text{size of } \left(\frac{n}{2}\text{th} + \frac{n+2}{2}\text{ th}\right) \text{observations}\right]$$

Example 1: On 5 term tests in mathematics, a student has made marks of 82, 93, 86, 92 and 79. Find the median for the marks.

Solution: By arranging the grades in ascending order, the arranged data is

79 , 82 , 86 , 92 , 93

Since number of observations is odd *i.e.*, $n = 5$.

$\tilde{X} = \text{size of } \left(\frac{5+1}{2}\right)$ th observation

$\tilde{X} =$ size of 3^{rd} observation

$\tilde{X} = 86$

Example 2: The sugar contents for a random sample of 6 packs of juices of a certain brand are found to be 2.3, 2.7, 2.5, 2.9, 3.1 and 1.9 milligram. Find the median.

Solution: Arrange the values by increasing order of magnitude

1.9 , 2.3 , 2.5 , 2.7 , 2.9 , 3.1

Since number of observations is even *i.e.*, $n = 6$.

$$\tilde{X} = \frac{1}{2}\left[\text{size of } \left(\frac{6}{2}\text{th} + \frac{6+2}{2}\text{ th}\right) \text{observations}\right]$$

$$\tilde{X} = \frac{1}{2}\left[\text{size of } (3^{rd} + 4^{th}) \text{ observations}\right]$$

$$\tilde{X} = \frac{2.5+2.7}{2} = 2.6 \text{ milligram.}$$

Grouped Data (Discrete)

The following steps are involved in determining median for grouped data(discrete):

- Make cumulative frequency column.
- Determine the median observation using cumulative frequency, i.e., the class containing $\left(\frac{n}{2}\right)^{th}$ observation.

Example 3: Find median for the following frequency distribution.

(Number of heads) X	Frequency
1	3
2	8
3	5
4	3
5	1

Solution: We first make cumulative frequency column as given below:

X	Frequency	Cumulative frequency
1	3	3
2	8	11
3	5	16
4	3	19
5	1	20
Total	$\Sigma f = 20$	

Now

Median = the class containing $\left(\frac{n}{2}\right)^{th}$ observation

Median = the class containing $\left(\frac{20}{2}\right)^{th}$ observation

Median = the class containing $(10)^{th}$ observation

Median = 2.

Grouped Data (Continuous)

The following steps are involved in determining median for grouped data (continuous):

- Determine class boundaries
- Make cumulative frequency column
- Determine the median class using cumulative frequency, i.e., the class containing $\left(\frac{n}{2}\right)^{th}$ observation
- Use the formula,

$$\text{Median} = l + \frac{h}{f}\left\{\frac{n}{2} - c\right\}$$

where l : lower class boundary of the median class,
h : class interval size of the median class,
f : frequency of the median class,
c : cumulative frequency of the class preceding the median class.

Example 4: The following data is the time taken by 40 students to solve a problem is recorded. Find the median time taken by the students.

138	164	150	132	144	125	149	157
146	158	140	147	136	148	152	144
168	126	138	176	163	119	154	165
146	173	142	147	135	153	140	135
161	145	135	142	150	156	145	128

Solution:

(*a*)

Class Intervals	Frequency	Class Boundaries	Cumulative Frequency	
118 –– 126	3	117.5 – 126.5	3	
127 –– 135	5	126.5 – 135.5	8	
136 –– 144	9	135.5 – 144.5	17	
145 –– 153	12	144.5 – 153.5	29	Median class
154 –– 162	5	153.5 – 162.5	34	
163 –– 171	4	162.5 – 171.5	38	
172 –– 180	2	171.5 – 180.5	40	
Total	$\Sigma f = 40$	—	—	

Now

median class = class containing $\left(\frac{n}{2}\right)^{th}$ observation

median class = class containing $\left(\frac{40}{2}\right)^{th} = 20^{th}$ observation

median class = 144.5 – 153.5

l = lower class boundary of median class = 144.5

c = cumulative frequency preceding the median class = 17

f = frequency of median class = 12

h = size of median class interval = 9

So, $\tilde{X} = l + \frac{h}{f}\left(\frac{n}{2} - c\right) = 144.5 + \frac{9}{12}(20 - 17) = 146.8$

6.3(i) (c) Mode

Mode is defined as the most frequent occurring observation in the data. It is the observation that occurs maximum number of times in given data. The following formula is used to determine Mode:

(i) Ungrouped data and Discrete Grouped data

Mode = the most frequent observation

(ii) Grouped Data (Continuous)

The following steps are involved in determining mode for grouped data:

- Find the group that has the maximum frequency.
- Use the formula

$$\text{Mode} = l + \frac{f_m - f_1}{2f_m - f_1 - f_2} \times h,$$

Where l : lower class boundary of the modal class or group,

h : class interval size of the modal class,

f_m : frequency of the modal class,

f_1 : frequency of the class preceding the modal class and

f_2 : frequency of the class succeeding the modal class.

Example 1: Find the modal size of shoe for the following data:

4, 4.5, 5, 6, 6, 6, 7, 7.5, 7.5, 8, 8, 8, 6, 5, 6.5, 7.

Solution: We note the most occurring observation in the given data and find that,

mode = 6.

Example 2: Find Mode for the following frequency distribution.

(Number of heads) X	Frequency
1	3
2	8
3	5
4	3
5	1

Solution: Since the given data is discrete grouped data so that,

mode = 2,

(Since for X = 2 the frequency is maximum, means 2 heads appear the maximum number of times i e .,8) .

Example 3: For the following data showing weights of toffee boxes in gm. Determine the modal weight of boxes.

Classes / Groups	Frequency
0 —– 9	2
10 —– 19	10
20 —– 29	5
30 —– 39	9
40 —– 49	6
50 —– 59	7
60 —– 69	1

Solution: Since the data is continuous so we proceed as follows:

(a) Determine the class boundaries first,

(b) Find the class with maximum frequency

Classes/Groups	Class Boundaries	Frequency	
0 ---- 9	−0.5 ---- 9.5	2	
10 ---- 19	9.5 ---- 19.5	10	Modal class
20 ---- 29	19.5 ---- 29.5	5	
30 ---- 39	29.5 ---- 39.5	9	
40 ---- 49	39.5 ---- 49.5	6	
50 ---- 59	49.5 ---- 59.5	7	
60 ---- 69	59.5 ---- 69.5	1	
Total		$\sum f = 40$	

From the above table we get, modal class or group = 9.5 — 19.5.

$$f_m = 10,\ l = 9.5,\ \text{h} = 10,\ \ f_1 = 2 \text{ and } f_2 = 5.$$

$$Mode = 9.5 + \frac{10 - 2}{2(10) - 2 - 5} \times 10$$

$$Mode = 9.5 + \frac{80}{13} = 9.5 + 6.134 = 15.654 \text{ gm}$$

6.3(i) (d) Geometric Mean

Geometric mean of a variable X is the n^th^ positive root of the product of the $x_1, x_2, x_3, \ldots\ldots\ldots, x_n$ observations. In symbols, we write

$$\text{G.M.} = (x_1, x_2, x_3, \ldots x_n)^{1/n}$$

The above formula can also be written by using logarithm.

For Ungrouped data

$$\text{G.M.} = Anti \log\left(\frac{\sum \log X}{n}\right)$$

For Grouped data

$$\text{G.M.} = Anti \log\left(\frac{\sum f \log X}{\sum f}\right)$$

Example 1: Find the geometric mean of the observations 2, 4, 8 using (i) basic formula and (ii) using logarithmic formula.

Solution: (i) Using basic formula

$$G.M = (2\times4\times8)^{1/3} = (64)^{1/3} = 4.$$

X	log *X*
2	0.3010
4	0.6021
8	0.9031
Total	$\Sigma \log X = 1.8062$

$$G.M. = \textit{Anti}\log\left(\frac{1.8062}{3}\right)$$

$$= \textit{Anti}\log(0.6021) = 4.00003 = 4$$

Example 2: Find the geometric mean for the following data:

Marks in percentage	Frequency/ (No of Students)
33 –––– 40	28
41 –––– 50	31
51 –––– 60	12
61 –––– 70	9
71 –––– 75	5

Solution: We proceed as follows:

Classes	*f*	*X*	log *X*	*f* log *X*
33 –––– 40	28	36.5	1.562293	43.7442
41 –––– 50	31	45.5	1.658011	51.39835
51 –––– 60	12	55.5	1.744293	20.93152
61 –––– 70	9	65.5	1.816241	16.34617
71 –––– 75	5	73	1.863323	9.316614
Total	$\Sigma f = 85$			$\Sigma f \log X = 141.7369$

$$G.M = \textit{Anti}\log\left(\frac{141.7369}{85}\right)$$

$$= \textit{Anti}\log(1.66749) = 46.50\ \%\ \text{marks}$$

6.3(i) (e) Harmonic Mean

Harmonic mean refers to the value obtained by reciprocating the mean of the reciprocal of $x_1, x_2, x_3, ..., x_n$ observations. In symbols, for ungrouped data,

$$H.M. = \frac{n}{\sum \frac{1}{X}}$$

and for grouped data,

$$\text{H.M.} = \frac{n}{\sum \frac{f}{X}}$$

Example 1: For the following data find the Harmonic mean.

X	12	5	8	4

Solution:

X	$^1/_X$
12	0.0833
5	0.2
8	0.125
4	0.25
Total	0.6583

$$\text{H.M.} = \frac{4}{0.6583} = 6.076$$

Example 2: Find the Harmonic mean for the following data.

Classes	No. of Students
33 –— 40	28
41 –— 50	31
51 –— 60	12
61 –— 70	9
71 –— 75	5

Solution:

Classes	*f*	*X*	*f* / *X*
33 –— 40	28	36.5	0.767123
41 –— 50	31	45.5	0.681319
51 –— 60	12	55.5	0.216216
61 –— 70	9	65.5	0.137405
71 –— 75	5	73	0.068493
Total	$\Sigma f = 85$		$\Sigma \frac{f}{X} = 1.870556$

$$\text{H.M.} = \frac{\Sigma f}{\Sigma \frac{f}{X}} = \frac{85}{1.8706} = 45.441$$

6.3(ii) Properties of Arithmetic Mean

(i) Mean of a variable with similar observations say constant k is the constant k itself.

(ii) Mean is affected by change in origin.

(iii) Mean is affected by change in scale.

(iv) Sum of the deviations of the variable X from its mean is always zero.

Example 1: Find the mean of observations: 34, 34, 34, 34,34, 34.

Solution: Since the variable say X here is taking same observation so by property (i)

$$\bar{X} = 34 .$$

Example 2: A variable X takes the following values 4, 5, 8, 6, 2. Find the mean of X. Also find the mean when (a) 5 is added to each observation (b) 10 is multiplied with each observation (c) Prove sum of the deviation from mean is zero.

Solution: Given the values of X,

X: 4 5 8 6 2.

We may introduce two new variables Y and Z under (*a*) and (*b*) respectively. So we are given that (*a*) $Y = X + 5$ (*b*) $Z = 10X$. The following table shows the desired result:

	X	$Y = X + 5$	$Z = 10X$	$X - \bar{X}$
	4	9	40	−1
	5	10	50	0
	8	13	80	3
	6	11	60	1
	2	7	20	−3
Total	$\Sigma X = 25$	$\Sigma Y = 50$	$\Sigma Z = 250$	$\sum(X - \bar{X}) = 0$

From the above table we get,

$$\bar{X} = \frac{25}{5} = 5 \quad ; \quad \bar{Y} = \frac{50}{5} = 10 \quad ; \quad \bar{Z} = \frac{250}{5} = 50$$

We note that (a) $\bar{Y} = 10 = 5 + 5 = \bar{X} + 5$

(b) $\bar{Z} = 50 = 10\ (5) = 10\bar{X}$

Which shows that mean is affected by change in origin and scale.

(c) From the last column of the table we note that $\sum(X - \bar{X}) = 0$, the sum of the deviations from mean is zero.

6.3 (iii) Calculation of Weighted Mean and Moving Averages

a. The Weighted Arithmetic Mean

The relative importance of a number is called its weight. When numbers $x_1, x_2, \ldots x_n$ are not equally important, we associate them with certain weights, $w_1, w_2, w_3, \ldots, w_n$ depending on the importance or significance.

$$\bar{x}_w = \frac{w_1 x_1 + w_2 x_2 + \ldots + w_n x_n}{w_1 + w_2 + \ldots + w_n} = \frac{\Sigma wx}{\Sigma w}$$

is called the weighted arithmetic mean.

Example 1: The following table gives the monthly earnings and the number of workers in a factory, compute the weighted average.

No. of employees	Monthly earnings. Rs.
4	800
22	45
20	100
30	30
80	35
300	15

Solution: Number of employees are treated as a weight (w) and monthly earnings as variable (x)

No. of employees (w)	Monthly earning in Rs. (x)	(xw)
4	800	3200
22	45	990
20	100	2000
30	30	900
80	35	2800
300	15	4500
$\Sigma w = 456$	———	$\Sigma xw = 14390$

$$\bar{x}_w = \frac{\Sigma xw}{\Sigma w} = \frac{14390}{456} = 31.5$$

b. Moving Averages

Moving averages are defined as the successive averages (arithmetic means) which are computed for a sequence of days/months/years at a time. If we want to find 3-days moving average, we find the average of first 3-days, then dropping the first day and add the succeeding day to this group. Place the average of each 3-days against the mid of 3-days. This process continues until all the days, beginning from first to the last, are exhausted.

Example 2: Calculate three days moving average for the following record of attendance:

Week	Sun	Mon	Tue	Wed	Thu	Fri	Sat
1	24	55	28	45	51	54	60

Solution:

Week and Days	Attendance	3-days moving	
		Total	Average
Sun.	24	—	—
Mon.	55	107	107/3 = 35.67
Tue.	28	128	128/3 = 42.67
Wed.	45	124	124/3 = 41.33
Thu.	51	150	150/3 = 50.00
Fri.	54	165	165/3 = 55.00
Sat.	60	—	—

By adding the first three values, we get 107, which is placed at the center of these values *i.e.*, Monday and then dropping the first observation *i.e.*, 24 and adding the next 3 values, we get 128 and placed at the middle of these three values and so on. For average values, divide 3 days moving total by "3" which shows in last column of the table.

6.3(iv) Graphical Location of Median, Quartiles and Mode

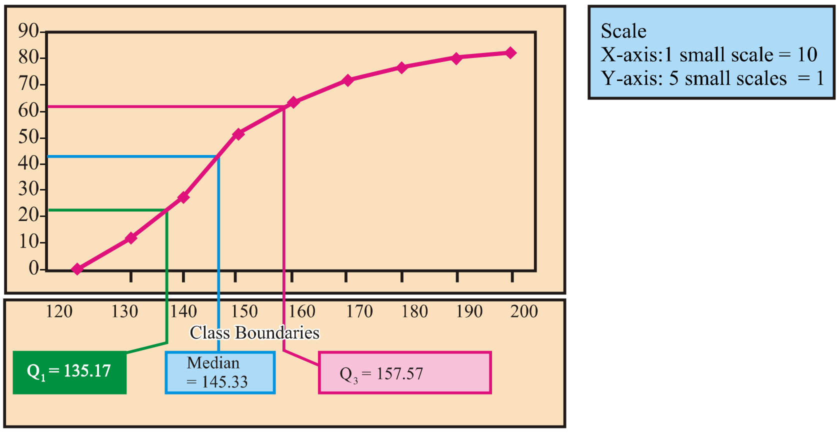
We explain the graphical location of Median, Quartiles and Mode by the help of following examples:

Example 1: For the following distribution, locate Median and Quartiles on graph:

Class Boundaries	Cumulative frequency
Less than 120	0
Less than 130	12
Less than 140	27
Less than 150	51
Less than 160	64
Less than 170	71
Less than 180	76
Less than 190	80
Less than 200	82

Solution:

We locate Median and Quartiles by using the following cumulative frequency polygon.



Finding

Q_1:

a) Find $(n/4)^{th}$ observation which is 82/4=20.5.

b) On the graph locate 20.5 on y-axis and draw a horizontal line segment parallel to x-axis touching the polygon.

c) Draw a vertical line segment from this point touching x-axis.

d) Read the value of first quartile at the point where the line segment meets x-axis which is 135.17.

Finding

Q_2 or Median :

a) Find $2(^{n}/4)^{th}$ observation which is 2(82/4)=41.

b) On the graph locate 41 on y-axis and draw a horizontal line segment parallel to x-axis touching the polygon.

c) Draw a vertical line segment from this point touching x-axis.

d) Read the value of Median at the point where the line segment meets *x*-axis which is 145.33.

Finding

Q_3 :

a) Find $3(n/4)^{th}$ observation which is 3(82/4)=61.5.

b) On the graph locate 61.5 on y-axis and draw a horizontal line segment parallel to x-axis touching the polygon.

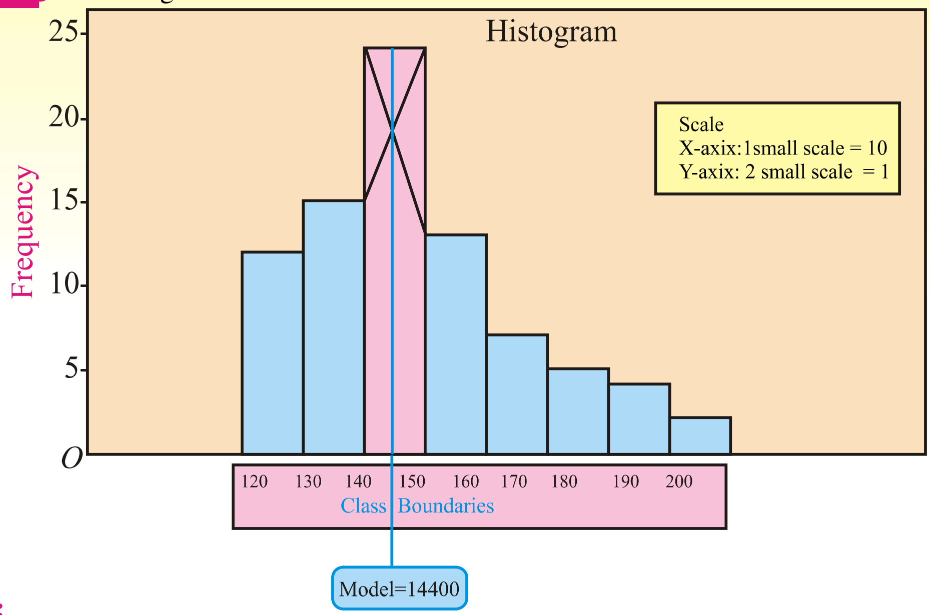
c) Draw a vertical line segment from this point touching x-axis.

d) Read the value of Median at the point where the line segment meets x-axis which is 157.57.

Example 2: For the following distribution locate Mode on graph.

Salaries in Rupees	No. of teachers
120 —— 130	12
130 —— 140	15
140 —— 150	24
150 —— 160	13
160 —— 170	7
170 —— 180	5
180 —— 190	4
190 —— 200	2

Solution: On Histogram the mode is located on X-axis as shown below:



Steps:

- Determine the rectangle having the highest peak indicating the modal class.
- Draw a line segment from the top left corner of the rectangle to the top left corner of the succeeding rectangle.
- Draw another line segment from the top right corner of the rectangle to the top right corner of the preceding rectangle.
- Drop perpendicular from the top of the rectangle to the x-axis passing through the point of intersection of the two line segments.
- Read the value at the point where the perpendicular meets the x-axis. This is the Mode of the data which is 144.

EXERCISE 6.2

1. What do you understand by measures of central tendency.
2. Define Arithmetic mean, Geometric mean, Harmonic mean, mode and median.
3. Find arithmetic mean by direct method for the following set of data:

   (i) 12, 14, 17, 20, 24, 29, 35, 45.

   (ii) 200, 225, 350, 375, 270, 320, 290.
4. For each of the data in Q. no 3., compute arithmetic mean using indirect method.

5. The marks obtained by students of class XI in mathematics are given below. Compute arithmetic mean by direct and indirect methods.

Classes / Groups	Frequency
0—9	2
10—19	10
20—29	5
30—39	9
40—49	6
50—59	7
60—69	1

6. The following data relates to the ages of children in a school. Compute the mean age by direct and short-cut method taking any provisional mean. (Hint. Take A = 8)

Class limits	Frequency
4—6	10
7—9	20
10—12	13
13—15	7
Total	50

Also Compute Geometric mean and Harmonic mean.

7. The following data shows the number of children in various families. Find mode and median.

9, 11, 4, 5, 6, 8, 4, 3, 7, 8, 5, 5, 8, 3, 4, 9, 12, 8, 9, 10, 6, 7, 7, 11, 4, 4, 8, 4, 3, 2, 7, 9, 10, 9, 7, 6, 9, 5.

8. Find Modal number of heads for the following distribution showing the number of heads when 5 coins are tossed. Also determine median.

X (number of heads)	Frequency (number of times)
1	3
2	8
3	5
4	3
5	1

9. The following frequency distribution the weights of boys in kilogram. Compute mean, median, mode.

Class Intervals	Frequency
1—3	2
4—6	3
7—9	5
10—12	4
13—15	6
16—18	2
19—21	1

10. A student obtained the following marks at a certain examination: English 73, Urdu 82, Mathematics 80, History 67 and Science 62.
    (i) If the weights accorded these marks are 4, 3, 3, 2 and 2, respectively, what is an appropriate average mark?
    (ii) What is the average mark if equal weights are used?

11. On a vacation trip a family bought 21.3 liters of petrol at 39.90 rupees per liter, 18.7 liters at 42.90 rupees per liter, and 23.5 liters at 40.90 rupees per liter. Find the mean price paid per liter.

12. Calculate simple moving average of 3 years from the following data:

Years	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
Values	102	108	130	140	158	180	196	210	220	230

13. Determine graphically for the following data and check your answer by using formulae.
    (i) Median and Quartiles using cumulative frequency polygon.
    (ii) Mode using Histogram.

Class Boundaries	Frequency
10—20	2
20—30	5
30—40	9
40—50	6
50—60	4
60—70	1

6.4 Measures of Dispersion

Statistically, Dispersion means the spread or scatterness of observations in a data set. The spread or scatterness in a data set can be seen in two ways:

(i) The spread between two extreme observations in a data set.
(ii) The spread of observations around an average say their arithmetic mean.

The purpose of finding Dispersion is to study the behavior of each unit of population around the average value. This also helps in comparing two sets of data in more detail.

The measures that are used to determine the degree or extent of variation in a data set are called **Measures of Dispersion.**

We shall discuss only some important absolute measures of dispersion now.

(i) Range

Range measures the extent of variation between two extreme observations of a data set. It is given by the formula:

Range = $X_{max} - X_{min} = X_m - X_0$

where $X_{max} = X_m$: the maximum, highest or largest observation.

$X_{min} = X_0$: the minimum ,lowest or smallest observation.

The formula to find range for grouped continuous data is given below:

Range = (Upper class boundary of last group)-(lower class boundary of first group

Find Range for the following weights of students:

110, 109, 84, 89, 77, 104, 74, 97, 49, 59, 103, 62.

Solution: Given that $X_m = 110$, $X_0 = 49$, Range = 110 – 49 = 61

Example 2: Find the Range for the following distribution.

Classes / Groups	f
10 — 19	10
20 — 29	7
30 — 39	9
40 — 49	6
50 — 59	7
60 — 69	1
Total	40

Solution: We find class boundaries and class marks for the given data as follows:

Class limits	Class Boundaries	Frequency
10 — 19	9.5—19.5	10
20 — 29	19.5—29.5	7
30 — 39	29.5—39.5	9
40 — 49	39.5—49.5	6
50 — 59	49.5—59.5	7
60 — 69	59.5—69.5	1
		$\Sigma f = 40$

Range = 69.5 – 9.5 = 60

(ii) Variance

Variance is defined as the mean of the squared deviations of x_i (i = 1, 2, ….., n) observations from their arithmetic mean. In symbols,

$$\text{Variance of } X = \text{Var}(X) = S^2 = \frac{\sum(X-\bar{X})^2}{n}$$

(iii) Standard Deviation

Standard deviation is defined as the positive square root of mean of the squared deviations of X_i ($i = 1, 2, \ldots, n$) observations from their arithmetic mean. In symbols we write,

$$\text{Standard Deviation of X} = \text{S.D(X)} = S = \sqrt{\frac{\sum(X-\bar{X})^2}{n}}$$

Computation of Variance and Standard Deviation

We use the following formulae to compute Variance and Standard Deviation for Ungrouped and Grouped Data.

Ungrouped Data

The formula of Variance is given by:

$$\text{Var}(X) = S^2 = \frac{\sum X^2}{n} - \left(\frac{\sum X}{n}\right)^2$$

and Standard deviation

$$\text{S.D}(X) = S = \sqrt{\left[\frac{\sum X^2}{n} - \left(\frac{\sum X}{n}\right)^2\right]}$$

Example 3: The marks of six students in Mathematics are as follows. Determine Variance and Standard deviation.

Student No	1	2	3	4	5	6
Marks	60	70	30	90	80	42

Solution: Let X = marks of a student. We make the following computations for finding Variance and Standard deviation.

X	X^2	$X-\bar{X}$	$(X-\bar{X})^2$
60	3600	−2	4
70	4900	8	64
30	900	−32	1024
90	8100	28	784
80	6400	18	324
42	1764	−20	400
$\Sigma X = 372$	$\Sigma X^2 = 25664$	$\Sigma(X-\bar{X}) = 0$	$\Sigma(X-\bar{X})^2 = 2600$

So, $\bar{X} = \frac{372}{6} = 62$ marks

and $\text{Var}(X) = S^2 = \frac{2600}{6} \approx 433.3333$ (square marks)

Using computational formula

$$\text{Var}(X) = S^2 = \frac{25664}{6} - \left(\frac{372}{6}\right)^2$$

$$\approx 4277.3333 - 3844 = 433.3333 \text{ (square marks)}$$

$$\text{S.D}(X) = S \approx \sqrt{4277.3333 - 3844} = \sqrt{433.3333}$$

$$\approx 20.81666 \text{ Marks}$$

Grouped Data

The formula of Variance is given by:

$$S^2 = \frac{\Sigma fX^2}{\Sigma f} - \left(\frac{\Sigma fX}{\Sigma f}\right)^2$$

and standard deviation

$$S = \sqrt{\left[\frac{\Sigma fX^2}{\Sigma f} - \left(\frac{\Sigma fX}{\Sigma f}\right)^2\right]}$$

Example 4: For the following data showing weights of toffee boxes in gm. determine the variance and standard deviation by using direct methods.

X (gm)	f
4.5	2
14.5	10
24.5	5
34.5	9
44.5	6
54.5	7
64.5	1

Solution: We make the following computations:

X	f	$X-\bar{X}$	$(X-\bar{X})^2$	$f(X-\bar{X})^2$	fX	fX^2
4.5	2	−28	784	1568	9	40.5
14.5	10	−18	324	3240	145	2102.5
24.5	5	−8	64	320	122.5	3001.25
34.5	9	2	4	36	310.5	10712.25
44.5	6	12	144	864	267	11881.5
54.5	7	22	484	3388	381.5	20791.75
64.5	1	32	1024	1024	64.5	4160.25
Total			$\Sigma(X-\bar{X})^2 = 2600$	$\Sigma f(X-\bar{X})^2 = 10440$	$\Sigma fX = 1300$	$\Sigma fX^2 = 52690$

Using definitional formula

$$S^2 = \frac{10440}{40} = 261 \text{ sq. gm.}$$

Using computational formula we get,

$$S^2 = \frac{52690}{40} - \left(\frac{1300}{40}\right)^2 = 1317.25 - (32.5)2 = 1317.25 - 1056.25 = 261 \text{ sq. gm.,}$$

and standard deviation is given by,

$$S = \sqrt{\frac{10440}{40}} = \sqrt{261} = 16.155 \text{ gm.}$$

$$S = \sqrt{\frac{52690}{40} - \left(\frac{1300}{40}\right)^2} = \sqrt{261} = 16.155 \text{ gm.}$$

Example 5: Compare the variation about mean for the two groups of students who obtained the following marks in Statistics:

***X* = Marks (section *A*)**	***Y* = Marks section *B***
60	62
70	62
30	65
90	68
80	67
40	48

Solution: In order to compare variation about mean we compute standard deviation for the two groups as follows:

X	Y	$X-\bar{X}$	$(X-\bar{X})^2$	$Y-\bar{Y}$	$(Y-\bar{Y})^2$
60	62	–2	4	0	0
70	62	8	64	0	0
30	65	–32	1024	3	9
90	68	28	784	6	36
80	67	18	324	5	25
40	48	–20	400	–14	196
$\Sigma X = 370$	$\Sigma Y = 372$		$\Sigma(X-\bar{X})^2 = 2600$		$\Sigma(Y-\bar{Y})^2 = 266$

Mean for group A = $\bar{X} = \frac{\sum X}{n} = \frac{370}{6} = 61.67 \simeq 62$ Marks

Mean for group B = $\bar{Y} = \frac{\sum Y}{n} = \frac{372}{6} = 62$ Marks

$$S.D\ (X) = \sqrt{\frac{\sum(X-\bar{X})^2}{n}} = \sqrt{\frac{2600}{6}} = \sqrt{433.333} = 20.82 \text{ Marks}$$

$$S.D\ (Y) = \sqrt{\frac{\sum(Y-\bar{Y})^2}{n}} = \sqrt{\frac{266}{6}} = \sqrt{44.333} = 6.66 \text{ Marks}$$

Comment: We note that the variation in Group B is smaller than that of Group A. This implies that the marks of students in Group B are closer to their Mean than that of Group A.

EXERCISE 6.3

1. What do you understand by Dispersion?
2. How do you define measures of dispersion?
3. Define Range, Standard deviation and Variance.
4. The salaries of five teachers in Rupees are as follows.

   11500, 12400, 15000, 14500, 14800.

   Find Range and standard deviation.
5. a- Find the standard deviation "S" of each set of numbers:

   (*i*) 12, 6, 7, 3, 15, 10, 18, 5

   (*ii*) 9, 3, 8, 8, 9, 8, 9, 18.

   b- Calculate variance for the data: 10, 8, 9, 7, 5, 12, 8, 6, 8, 2.
6. The length of 32 items are given below. Find the mean length and standard deviation of the distribution.

Length	20–22	23–25	26–28	29–31	32–34
Frequency	3	6	12	9	2

7. For the following distribution of marks calculate Range.

Marks in percentage	**Frequency/ (No of Students)**
31 — 40	28
41 — 50	31
51 — 60	12
61 — 70	9
71 — 75	5

MISCELLANEOUS EXERCISES

1. Multiple Choice Questions

Three possible answers are given for the following questions. Tick (✓) the correct answer.

(i) A grouped frequency table is also called
(a) data (b) frequency distribution
(c) frequency polygon

(ii) A histogram is a set of adjacent
(a) squares (b) rectangles
(c) circles

(iii) A frequency polygon is a many sided
(a) closed figure (b) rectangle
(c) square

(iv) A cumulative frequency table is also called
(a) frequency distribution (b) data
(c) less than cumulative frequency distribution

(v) In a cumulative frequency polygon frequencies are plotted against
(a) midpoints (b) upper class boundaries
(c) class limits

(vi) Arithmetic mean is a measure that determines a value of the variable under study by dividing the sum of all values of the variable by their
(a) number (b) group
(c) denominator

(vii) A Deviation is defined as a difference of any value of the variable from a
(a) constant (b) histogram
(c) sum

(viii) A data in the form of frequency distribution is called
(a) Grouped data (b) Ungrouped data
(c) Histogram

(ix) Mean of a variable with similar observations say constant k is
(a) negative (b) k itself
(c) zero

(x) Mean is affected by change in
(a) value (b) ratio
(c) origin

(xi) Mean is affected by change in
(a) place (b) scale
(c) rate

(xii) Sum of the deviations of the variable X from its mean is always
(a) zero (b) one
(c) same

(xiii) The n[th] positive root of the product of the $x_1, x_2, x_3, \ldots\ldots\ldots, x_n$ observations is called

(a) Mode (b) Mean

(c) Geometric mean

(xiv) The value obtained by reciprocating the mean of the reciprocal of $x_1, x_2, x_3, \ldots\ldots\ldots, x_n$ observations is called

(a) Geometric mean (b) Median

(c) Harmonic mean

(xv) The most frequent occurring observation in a data set is called

(a) mode (b) median

(c) harmonic mean

(xvi) The measure which determines the middlemost observation in a data set is called

(a) median (b) mode

(c) mean

(xvii) The observations that divide a data set into four equal parts are called

(a) deciles (b) quartiles

(c) percentiles

(xviii) The spread or scatterness of observations in a data set is called

(a) average (b) dispersion

(c) central tendency

(xix) The measures that are used to determine the degree or extent of variation in a data set are called measures of

(a) dispersion (b) central tendency

(c) average

(xx) The extent of variation between two extreme observations of a data set is measured by

(a) average (b) range

(c) quartiles

(xxi) The mean of the squared deviations of x_i $(i = 1, 2, \ldots\ldots, n)$ observations from their arithmetic mean is called

(a) variance (b) standard deviation

(c) range

(xxii) The positive square root of mean of the squared deviations of X_i $(i = 1, 2, \ldots\ldots, n)$ observations from their arithmetic mean is called

(a) harmonic mean (b) range

(c) standard deviation

2. Write short answers of the following questions.

(i) Define class limits.

(ii) Define class mark.

(iii) What is cumulative frequency?

(iv) Define a frequency distribution.

(v) What is a Histogram?

(vi) Name two measures of central tendency.
(vii) Define Arithmetic mean.
(viii) Write three properties of Arithmetic mean.
(ix) Define Median.
(x) Define Mode?
(xi) What do you mean by Harmonic mean?
(xii) Define Geometric mean.
(xiii) What is Range?
(xiv) Define Standard deviation.

SUMMARY

- **Range** is the difference between *maximum* and *minimum* observation.
- The minimum and the maximum values defined for a class or group are called **class limits**.
- The total of frequency up to an upper class limit or boundary is called the **cumulative frequency**.
- A **frequency distribution** is a tabular arrangement classifying data into different groups.
- A **Histogram** is a graph of adjacent rectangles constructed on *XY*-plane. A **cumulative frequency polygon** or **ogive** is a graph of less than cumulative frequency distribution.
- **Arithmetic mean** is a measure that determines a value of the variable under study by dividing the sum of all values of the variable by their number.
- A **Deviation** is defined as 'a difference of any value of the variable from any constant'. $D_i = x_i - A$.
- **Geometric mean** of a variable X is the nth positive root of the product of the $x_1, x_2, x_3, ..., x_n$ observations.
- **Harmonic mean** refers to the value obtained by reciprocating the mean of the reciprocal of $x_1, x_2, x_3, ..., x_n$ observations.
- **Mode** is defined as the most frequent occurring observation of the variable or data.
- **Median** is the measure which determines the middlemost observation in a data set.
- Statistically, **Dispersion** means the spread or scatterness of observations in a data set.
- **Range** measures the extent of variation between two extreme observations of a data set.
- **Variance** is defined as the mean of the squared deviations of x_i (i = 1, 2, ….., n) observations from their arithmetic mean.
- **Standard deviation** is defined as the positive square root of mean of the squared deviations of x_i (i = 1, 2, ….., n) observations from their arithmetic mean.

Unit-7

INTRODUCTION TO TRIGONOMETRY

In this unit, students will learn how to

- *measure an angle in degree, minute and second.*
- *convert an angle given in degrees, minutes and seconds into decimal form and vice versa.*
- *define a radian (measure of an angle in circular system) and prove the relationship between radians and degrees.*
- *establish the rule $l = r\theta$, where r is the radius of the circle, l the length of circular arc and θ the central angle measured in radians.*
- *prove that the area of a sector of a circle is $\frac{1}{2}r^2\theta$.*
- *define and identify:*
  - *general angle (coterminal angles),*
  - *angle in standard position.*
- *recognize quadrants and quadrantal angles.*
- *define trigonometric ratios and their reciprocals with the help of a unit circle.*
- *recall the values of trigonometric ratios for 45°, 30°, 60°.*
- *recognise signs of trigonometric ratios in different quadrants.*
- *find values of remaining trigonometric ratios if one trigonometric ratio is given.*
- *calculate the values of trigonometric ratios for 0°, 90°, 180°, 270°, 360°.*
- *prove the trigonometric identities and apply them to show different trigonometric relations.*
- *find angle of elevation and depression.*
- *solve real life problems involving angle of elevation and depression.*

7.1 Measurement of an Angle

An **angle** is defined as the union of two non-collinear rays with some common end point. The rays are called **arms** of the angle and the common end point is known as **vertex** of the angle.

It is easy if we make an angle by rotating a ray from one position to another. When we form an angle in this way, the original position of the ray is called **initial side** and final position of the ray is called the **terminal side** of the angle. If the rotation of the ray is anti-clockwise or clockwise, the angle has positive or negative measure respectively.

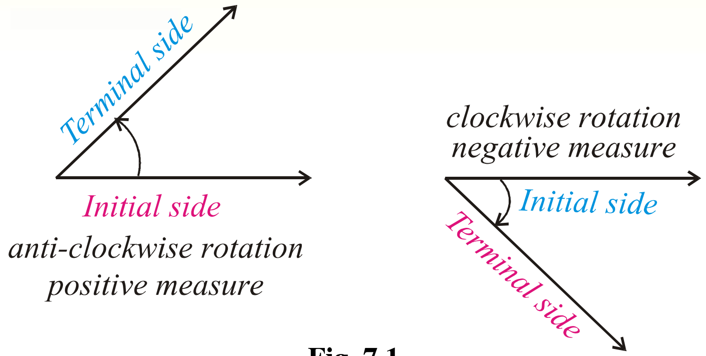


Fig. 7.1

7.1(i) Measurement of an angle in sexagesimal system (degree, minute and second)

Degree: We divide the circumference of a circle into 360 equal arcs. The angle subtended at the centre of the circle by one arc is called one degree and is denoted by 1^{o}.

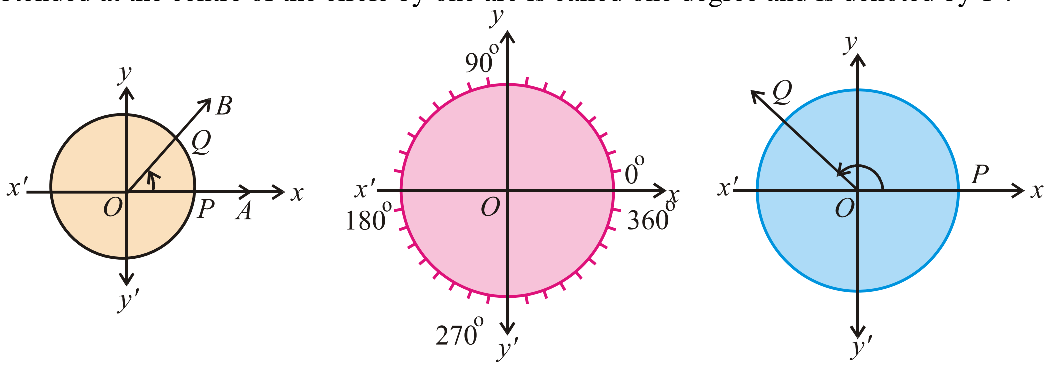


Fig. 7.1.1

The symbols 1^{o}, $1'$ and $1''$ are used to denote a degree, a minute and a second respectively.

Thus 60 seconds $(60'')$ make one minute $(1')$

60 minutes $(60')$ make one degree (1^{o})

90 degrees (90^{o}) make one right angle

360 degrees (360^{o}) make 4 right angles

An angle of 360^{o} denotes a complete circle or one revolution. We use coordinate system to locate any angle to a **standard position,** where its initial side is the positive x-axis and its vertex is the origin.

Example: Locate (*a*) −45° (*b*) 120° (*c*) 45° (*d*) −270°

Solution: The angles are shown in Figure 7 1 2

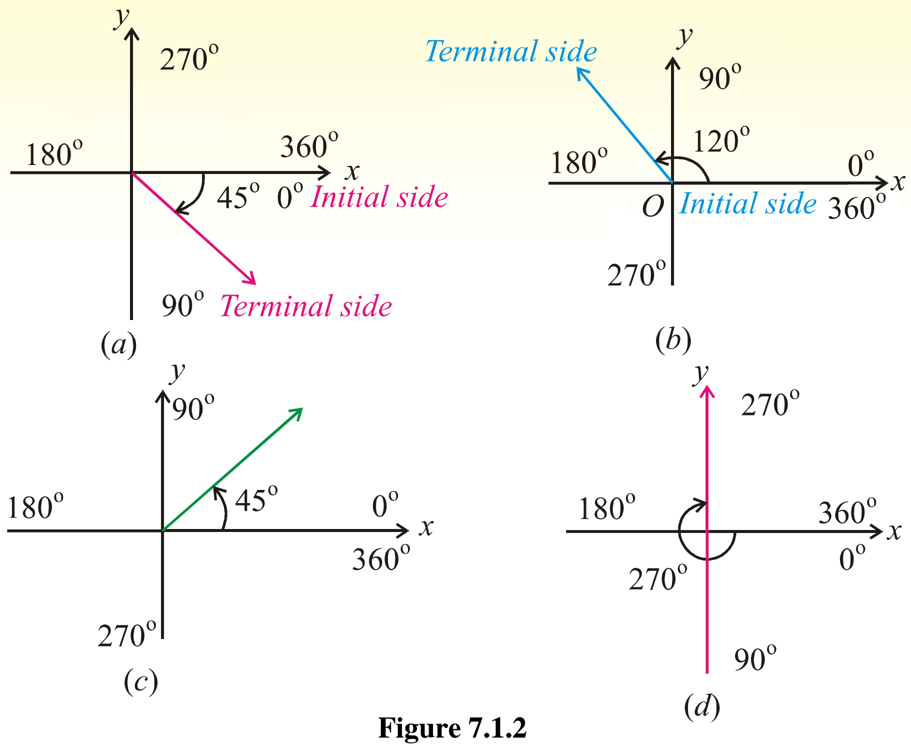


Figure 7.1.2

7.1(ii) Conversion of an angle given in $D^o\ M'S''$ form into decimal form and vice versa

The conversion is explained through examples.

Example 1: (i) Convert 25°30′ to decimal degrees.

(ii) Convert 32.25° to $D^o\ M'S''$ form.

Solution:

(i) $25^o30' = 25^o + \left(\frac{30}{60}\right)^o = 25^o + 0.5^o = 25.5^o$

(ii) $32.25^o = 32^o + 0.25^o = 32^o + \left(\frac{25}{100}\right)^o$

$= 32^o + \frac{1^o}{4} = 32^o + \left(\frac{1}{4}\times 60\right)' = 32^o\ 15'$

Example 2: Convert 12°23′35″ to decimal degrees correct to three decimal places.

Solution: $12^o23'35'' = 12^o + \frac{23^o}{60} + \frac{35^o}{60\times 60} = \left(12^o + \frac{23^o}{60} + \frac{35^o}{3600}\right)$

$\approx\ 12^o + 0.3833^o + 0.00972^o$

$\approx 12.3930^o = 12.393^o$

Example 3: Convert 45.36^o to $D^o\, M'S''$ form.

Solution: $(45.36)^o = 45^o + (0.36)^o$

$$= 45^o + \left(\frac{36}{100}\right)^o = 45^o + \left(\frac{9}{25} \times 60'\right)$$

$$= 45^o + 21.6' = 45^o + 21' + (0.6 \times 60)''$$

$$= 45^o 21' 36''$$

7.1(iii) Radian measure of an angle (circular system).

Another system of measurement of an angle known as circular system is of most importance and is used in all the higher branches of Mathematics.

Radian: The angle subtended at the centre of the circle by an arc, whose length is equal to the radius of the circle is called one **Radian**.

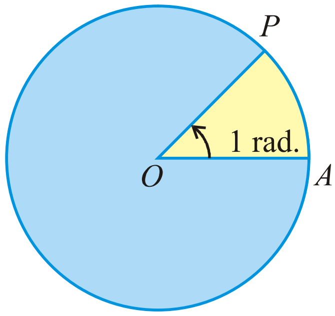


Fig. 7.1.3

Consider a circle of radius r whose centre is O. From any point A on the circle cut off an arc AP whose length is equal to the radius of the circle. Join O with A and O with P. The $\angle AOP$ is one radian. This means that when length of arc $\widehat{AP}$ = length of radius $\overline{OA}$ then $m\angle AOP = 1$ radian

Relationship between radians and degrees

We know that circumference of a circle is $2\pi r$ where r is the radius of the circle. Since a circle is an arc whose length is $2\pi r$. The radian measure of an angle that form a complete circle is $\frac{2\pi r}{r} = 2\pi$

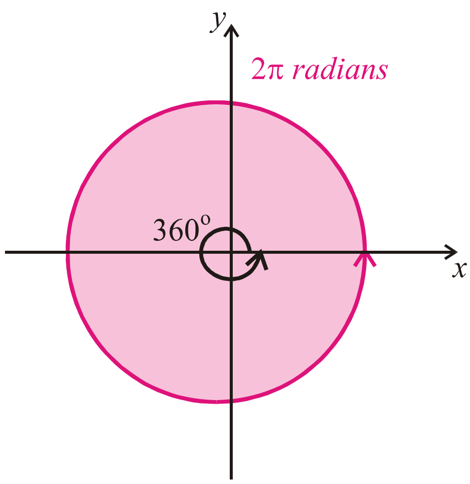


Fig. 7.1.4

From this, we see that $360^o = 2\pi$ radians

or $180^o = \pi$ radians ...... (i)

Using this relation, we can convert degrees into radians and radians into degrees as follows:

$$180^o = \pi \text{ radians} \Rightarrow 1^o = \frac{\pi}{180} \text{ radian},$$

$$x^o = x \,.\, 1^o = x\left(\frac{\pi}{180}\right) \text{ radian} \qquad \text{...... (ii)}$$

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^o, \; y \text{ radian} = y\left(\frac{180}{\pi}\right) \text{ degrees} \qquad \text{...... (iii)}$$

Some special angles in degree and radians.

$180^\circ = 1\,(180^\circ) = \pi$ radians

$90^\circ = \frac{1}{2}(180^\circ) = \frac{\pi}{2}$ radians

$60^\circ = \frac{1}{3}(180^\circ) = \frac{\pi}{3}$ radians

$45^\circ = \frac{1}{4}(180^\circ) = \frac{\pi}{4}$ radians

$30^\circ = \frac{1}{6}(180^\circ) = \frac{\pi}{6}$ radians

$270^\circ = \frac{3}{2}(180^\circ) = \frac{3\pi}{2}$ radians

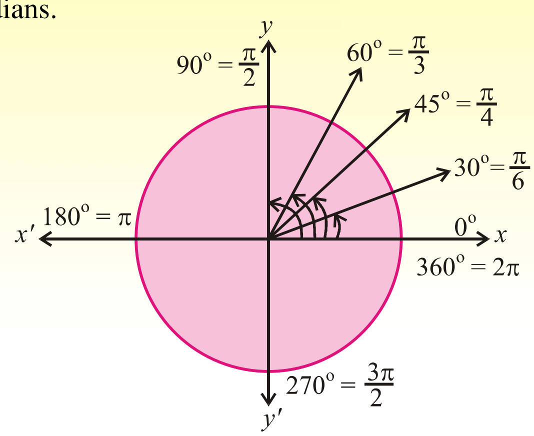


Fig. 7.1.5

Example 4: Convert the following angles into radian measure:

(*a*) 15° (*b*) $124^\circ 22'$

Solution: (*a*) $15^\circ = 15\left(\frac{\pi}{180}\text{ rads}\right)$ by using (*i*)

$= \frac{\pi}{12}$ radians

(*b*) $124^\circ 22' = \left(124 + \frac{22}{60}\right)^\circ = (124.3666)\left(\frac{\pi}{180}\right)$ radians

≈ 2.171 radians

Example 5: Express the following into degree.

(*a*) $\frac{2\pi}{3}$ radians (*b*) 6.1 radians

Solution:

(*a*) $\frac{2\pi}{3}$ radians $= \frac{2\pi}{3}\left(\frac{180}{\pi}\right)$ degrees

$= 120^\circ$

(*b*) 6.1 radians $= (6.1)\left(\frac{180}{\pi}\right)$ degrees

$= 6.1\,(57.295779) = 349.5043$ degrees

Remember that:

1 radian $\approx \left(\frac{180}{3.1416}\right)^\circ \approx 57.295795^\circ \approx 57^\circ 17' 45''$, $1^\circ \approx \frac{3.1416}{180} \approx 0.0175$ radians

EXERCISE 7.1

1. Locate the following angles:

   (i) 30° (ii) $22\frac{1}{2}^{\circ}$ (iii) 135° (iv) 225°

   (v) -60° (vi) -120° (vii) -150° (viii) -225°

2. Express the following sexagesimal measures of angles in decimal form.

   (i) $45^{\circ}30'$ (ii) $60^{\circ}30'30''$ (iii) $125^{\circ}22'50''$

3. Express the following into $D^{\circ}\ M'\ S''$ form.

   (i) 47.36° (ii) 125.45° (iii) 225.75° (iv) -22.5°

   (v) -67.58° (vi) 315.18°

4. Express the following angles into radians.

   (i) 30° (ii) $(60)^{\circ}$ (iii) 135° (iv) 225° (v) -150°

   (vi) -225° (vii) 300° (viii) 315°

5. Convert each of following to degrees.

   (i) $\frac{3\pi}{4}$ (ii) $\frac{5\pi}{6}$ (iii) $\frac{7\pi}{8}$ (iv) $\frac{13\pi}{16}$ (v) 3

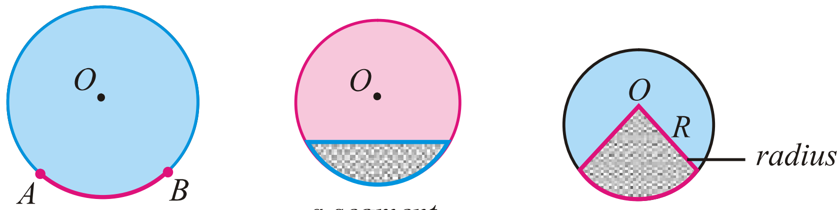
   (vi) 4.5 (vii) $\frac{-7\pi}{8}$ (viii) $-\frac{13}{16}\pi$

7.2 Sector of a Circle

(i) A part of the circumference of a circle is called an **arc**.

(ii) A part of the circle bounded by an arc and a chord is called **segment of a circle**.

(iii) A part of the circle bounded by the two radii and an arc is called **sector of the circle**.



an arc *a segment* *a sector*

7.2(i) To establish the rule $l = r\,\theta$, where r is the radius of the circle, l the length of circular arc and θ the central angle measured in radians.

Let an arc AB denoted by l subtends an angle θ radian at the centre of the circle. It is a fact of plane geometry that measure of central angles of the arcs of a circle are proportional to the lengths of their arcs.

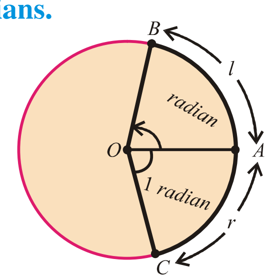
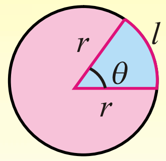


Fig. 7.2.1

$$\frac{m\angle AOB}{m\angle AOC} = \frac{m\widehat{AB}}{m\widehat{AC}}$$



$$\Rightarrow \quad \frac{\theta \text{ radian}}{1 \text{ radian}} = \frac{l}{r} \quad \Rightarrow \quad \frac{l}{r} = \theta \quad \text{or} \quad l = r\theta$$

Example 1: In a circle of radius 10m,

(a) find the length of an arc intercepted by a central angle of 1.6 radian.

(b) find the length of an arc intercepted by a central angle of 60^o.

Solution: (a) Here $\theta = 1.6$ radian , $r = 10$m and $l = ?$

$\because$ Since $l = r\,\theta \quad \Rightarrow \quad l = 10 \times 1.6 = 16$ m

(b) $\theta = 60^o = 60 \times \frac{\pi}{180} = \frac{\pi}{3}$ rad.

$\because \quad l = r\,\theta = 10 \times \frac{\pi}{3} = \frac{10\pi}{3}$ m.

Example 2: Find the distance travelled by a cyclist moving on a circle of radius 15m, if he makes 3.5 revolutions.

Solution: 1 revolution $= 2\pi$ radians

3.5 revolution $= 2\pi \times 3.5$ m

Distance traveled $= l = r\,\theta$

$$l = 15 \times 2\pi \times 3.5 = 105\pi \text{ m}$$

7.2(ii) Area of a circular sector

Consider a circle of radius r units and an arc of length l units, subtends an angle θ at O.

Area of the circle $= \pi r^2$

Angle of the circle $= 2\pi$

Angle of the sector $= \theta$ radian

Then by elementary geometry we can use the proportion,

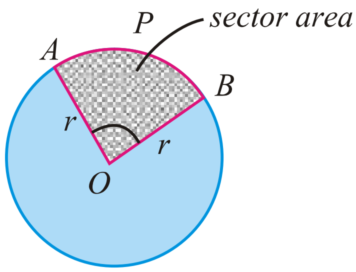


Fig. 7.2.2

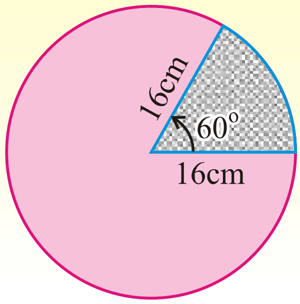
$$\frac{\text{area of sector } AOBP}{\text{area of circle}} = \frac{\text{angle of the sector}}{\text{angle of the circle}}$$

or $$\frac{\text{area of sector } AOBP}{\pi r^2} = \frac{\theta}{2\pi}$$

$$\Rightarrow \quad \text{Area of sector } AOBP = \frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2} r^2\theta$$

$$\boxed{\text{Area of sector } AOBP = \frac{1}{2} r^2\theta}$$

Example 3: Find area of the sector of a circle of radius 16cm if the angle at the centre is 60°.



Solution: Area of sector $= \frac{1}{2} r^2 \theta$

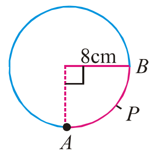
Now $\quad \theta = 60^\circ \times \frac{\pi}{180} = \frac{\pi}{3}$ rad , $r = 16$cm

Area of sector $= \frac{1}{2}(16)^2 \left(\frac{\pi}{3}\right)$

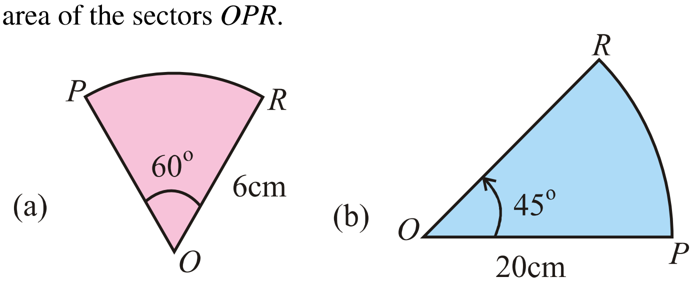
$= \frac{1}{2}(256) \times \left(\frac{22}{7 \times 3}\right) \simeq 134.1 \text{ cm}^2$

EXERCISE 7.2

1. Find θ, when:
   (i) $l = 2$cm , $r = 3.5$cm $\qquad$ (ii) $l = 4.5$m , $r = 2.5$m
2. Find l, when:
   (i) $\theta = 180^\circ$, $r = 4.9$cm $\qquad$ (ii) $\theta = 60^\circ 30'$, $r = 15$mm
3. Find r, when:
   (i) $l = 4$cm , $\theta = \frac{1}{4}$ radian $\qquad$ (ii) $l = 52$cm , $\theta = 45^\circ$
4. In a circle of radius 12m, find the length of an arc which subtends a central angle $\theta = 1.5$ radian.
5. In a circle of radius 10m, find the distance travelled by a point moving on this circle if the point makes 3.5 revolution. (3.5 revolution = 7π).
6. What is the circular measure of the angle between the hands of the watch at 3 o'clock?
7. What is the length of the arc APB?

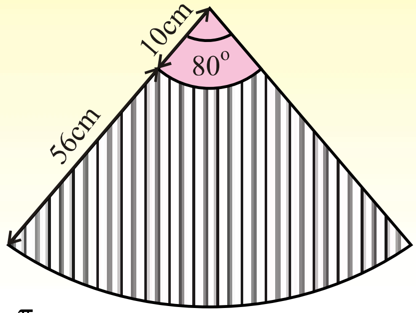


8. In a circle of radius 12cm, how long an arc subtends a central angle of 84°.
9. Find the area of the sectors OPR.



10. Find area of the sector inside a central angle of 20° in a circle of radius 7m.

11. Sehar is making a skirt. Each panel of this skirt is of the shape shown shaded in the diagram. How much material (cloth) is required for each panel?



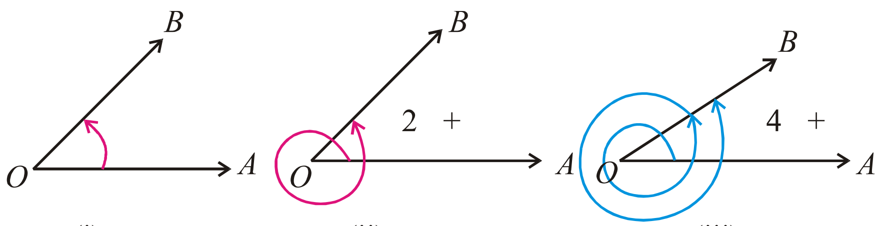
12. Find the area of the sector with central angle of $\frac{\pi}{5}$ radian in a circle of radius 10cm.

13. The area of the sector with a central angle θ in a circle of radius 2m is 10 square meter. Find θ in radians.

7.3 Trigonometric Ratios

7.3(i-a) General Angles (Coterminal angles)

An angle is indicated by a curved arrow that shows the direction of rotation from initial to the terminal side. Two or more than two angles may have the same initial and terminal sides. Consider an angle $\angle AOB$ with $\overline{OA}$ as initial side and $\overline{OB}$ as terminal side with vertex O. Let $m\angle AOB = \theta$ radian, where $0 \le \theta \le 2\pi$.



(i) (ii) (iii)

If the terminal side $\overline{OB}$ comes to its original position after, one, two or more than two complete revolutions in the anti-clockwise direction, then $m\angle AOB$ in above three cases will be

(i)	θ rad	after zero revolution
(ii)	$(2\pi + \theta)$ rad.	after one revolution
(iii)	$(4\pi + \theta)$ rad.	after two revolutions

Coterminal Angles: Two or more than two angles with the same initial and terminal sides are called **coterminal angles.**

It means that terminal side comes to its original position after every revolution of 2π radian in anti=clockwise or clockwise direction. In general, if θ is in degrees, then $360^\circ k + \theta$, where $k \in Z$, is an angle coterminal with θ. If angle θ is in radian measure, then $2k\pi + \theta$ where $k \in Z$, is an angle coterminal with θ.

Thus, the general angle $\theta = 2(k)\,\pi + \theta$, where $k \in Z$.

Example: Which of following angles are coterminal with 120^o?

$$-240^o, 480^o, \frac{14\pi}{3} \text{ and } -\frac{14\pi}{3}$$

Solution: -240^o is conterminal with 120^o as their terminal side is same

$480^o = 360^o + 120^o$, the angle 480^o terminates at 120^o after one complete revolution.

$\frac{14}{3}\pi \equiv 4\pi + \frac{2\pi}{3} = 720^o + 120^o$ the angle $\frac{14\pi}{3}$ is coterminal with 120^o.

$\frac{-14\pi}{3} = -4\pi + \frac{-2\pi}{3} = -720^o - 120^o$ so $\frac{-14\pi}{3}$ is not coterminal with 120^o.

7.3(i-b) Angle in Standard Position

A general angle is said to be in standard position if its vertex is at the origin and its initial side is directed along the positive direction of the x-axis of a rectangular coordinate system.

Because all the angles in standard position have the same initial side, the location of the terminal side is of importance. The position of the terminal side of an angle in standard position remains the same if measure of the angle is increased or decreased by a multiple of 2π.

Some standard angles are shown in the following figures:

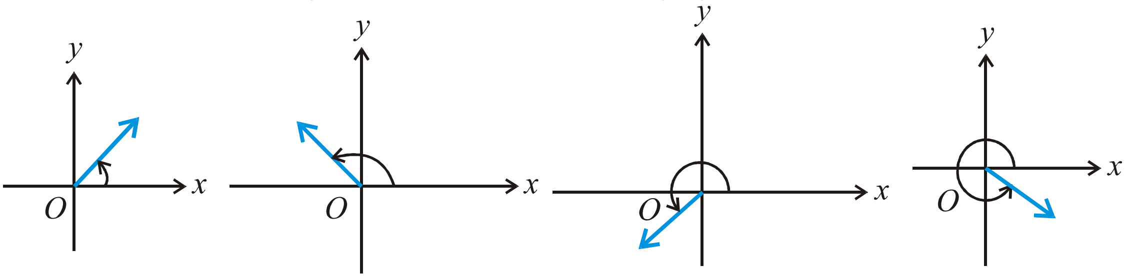


Fig. 7.3.1 (a)

Example: Locate each angle in standard position.

(i) 240^o (ii) 490^o (iii) -270^o

Solution: The angles are shown in Figure 7.3.1 (b)

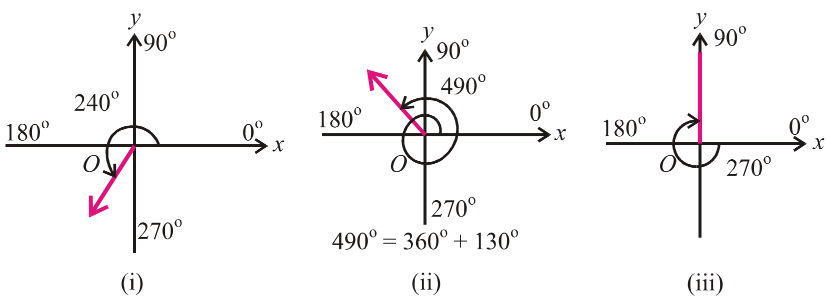
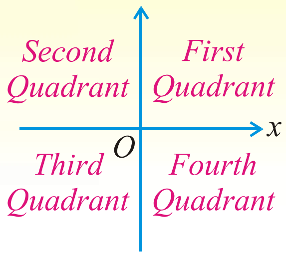


Fig. 7.3.1 (b)

7.3(ii) The Quadrants and Quadrantal Angles

The x-axis and y-axis divides the plane in four regions, called **quadrants**, when they intersect each other at right angle. The point of intersection is called **origin** and is denoted by O.



Angles between 0^o and 90^o are in the first quadrant.
Angles between 90^o and 180^o are in the second quadrant.
Angles between 180^o and 270^o are in the third quadrant.
Angles between 270^o to 360^o are in the fourth quadrant.
An angle in standard position is said to lie in a quadrant if its terminal side lies in that quadrant. Angles α, β, γ and θ lie in I, II, III and IV quadrant respectively in figure 7.3.1.

Quadrantal Angles

If the terminal side of an angle in standard position falls on x-axis or y-axis, then it is called a **quadrantal angle** *i.e.,* 90^o, 180^o, 270^o and 360^o are quadrantal angles. The quadrantal angles are shown as below:

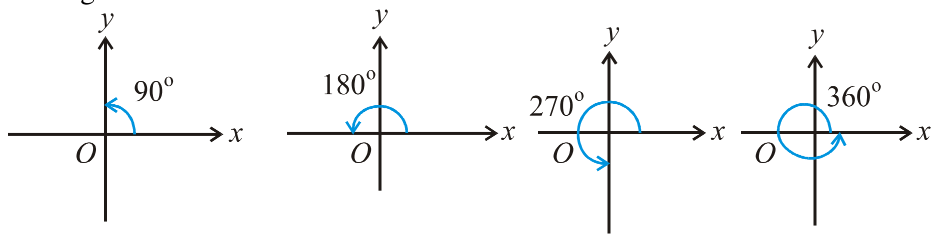


Fig. 7.3.2

7.3(iii) Trigonometric ratios and their reciprocals with the help of a unit circle.

There are six fundamental trigonometric ratios called sine, cosine, tangent, cotangent, secant and cosecant. To define these functions we use circular approach which involves the unit circle.

Let θ be a real number, which represents the radian measure of an angle in standard position. Let P (x, y) be any point on the unit circle lying on terminal side of θ as shown in the figure.

We define sine of θ, written as $\sin\theta$ and cosine of θ written as $\cos\theta$, as:

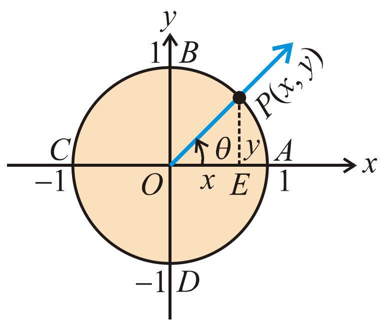


Fig. 7.3.3

$$\sin\theta = \frac{EP}{OP} = \frac{y}{1} \quad \Rightarrow \quad \sin\theta = y$$

and $$\cos\theta = \frac{OE}{OP} = \frac{x}{1} \quad \Rightarrow \quad \cos\theta = x$$

i.e., $\cos\theta$ and $\sin\theta$ are the x-coordinate and y-coordinate of the point P on the unit circle. The equations $x = \cos\theta$ and $y = \sin\theta$ are called **circular** or **trigonometric functions**.

The remaining trigonometric functions tangent, cotangent, secant and cosecant will be denoted by $\tan\theta$, $\cot\theta$, $\sec\theta$ and $\text{cosec}\,\theta$ for any real angle θ.

$$\tan\,\theta = \frac{EP}{OE} = \frac{y}{x} \quad \Rightarrow \quad \tan\theta = \frac{y}{x} \qquad (x \neq 0)$$

since $\quad y = \sin\theta$ and $x = \cos\theta \;\Rightarrow\; \tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$$\cot\theta = \frac{x}{y} \quad (y \neq 0) \quad \Rightarrow \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sec\,\theta = \frac{1}{x}\,(x \neq 0) \quad \text{and} \quad \text{cosec}\,\theta = \frac{1}{y} \quad (y \neq 0)$$

$$= \frac{1}{\cos\,\theta} \qquad\qquad = \frac{1}{\sin\,\theta}$$

Reciprocal Identities

$\sin\theta = \dfrac{1}{\text{cosec}\,\theta}$	or	$\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$
$\cos\theta = \dfrac{1}{\sec\theta}$	or	$\sec\theta = \dfrac{1}{\cos\theta}$
$\tan\theta = \dfrac{1}{\cot\theta}$	or	$\cot\theta = \dfrac{1}{\tan\theta}$

Example 2: Find the value of the trigonometric ratios at θ if point (3, 4) is on the terminal sides of θ.

Solution: We have $x = 3$ and $y = 4$

We shall also need value of r, which is found by using the fact that

$$r = \sqrt{x^2 + y^2} \quad ; \quad r = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5 \text{ where } r = |OP|$$

Thus $\quad \sin\theta = \dfrac{y}{r} = \dfrac{4}{5} \quad ; \quad \text{cosec}\,\theta = \dfrac{5}{4}$

$$\cos\theta = \frac{x}{r} = \frac{3}{5} \quad ; \quad \sec\theta = \frac{5}{3}$$

$$\tan\theta = \frac{y}{x} = \frac{4}{3} \quad ; \quad \cot\theta = \frac{3}{4}$$

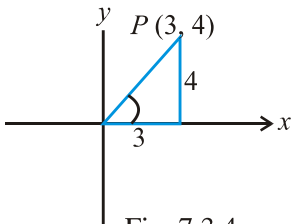


Fig. 7.3.4

7.3(iv) The values of trigonometric ratio for 45°, 30°, 60°.

Consider a right triangle ABC with $m\angle C = 90^\circ$. The sides opposite to the vertices A, B and C are denoted by a, b and c respectively.

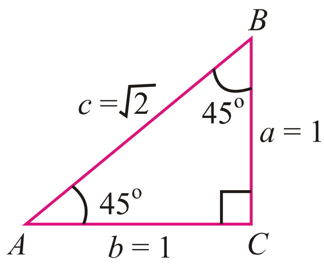


Fig. 7.3.5

Case I When $m\angle A = 45^\circ$, where $45^\circ = \dfrac{\pi}{4}$ radian. Since the sum of angles in a triangle is 180°, so $m\angle B = 45^\circ$.

As values of trigonometric functions depend on the size of the angle only and not on the size of triangle. For convenience, we take $a = b = 1$. In this case the triangle is isosceles right triangle.

By Pythagorean theorem.

$$c^2 = a^2 + b^2 \quad \Rightarrow \quad c^2 = (1)^2 + (1)^2 = 2$$

$$c^2 = 2 \quad \Rightarrow \quad c = \sqrt{2}$$

From this triangle, we have

$$\sin 45^o = \sin\frac{\pi}{4} = \frac{a}{c} = \frac{1}{\sqrt{2}} \quad ; \quad \operatorname{cosec} 45^o = \frac{1}{\sin 45^o} = \sqrt{2}$$

$$\cos 45^o = \cos\frac{\pi}{4} = \frac{b}{c} = \frac{1}{\sqrt{2}} \quad ; \quad \sec 45^o = \frac{1}{\cos 45} = \sqrt{2}$$

$$\tan 45^o = \tan\frac{\pi}{4} = \frac{a}{b} = \frac{1}{1} = 1 \; ; \quad \cot 45^o = \frac{1}{\tan 45^o} = 1$$

Case II When $m\angle A = 30^o$ or $m\angle A = 60^o$

Consider an equilateral triangle with sides $a = b = c = 2$ for convenience. Since the angles in an equilateral triangle are equal and their sum is 180^o, each angle has measure 60^o. Bisecting an angle in the triangle, we obtain two right triangles with 30^o and 60^o angles. The height $|AD|$ of these triangles may be found by Pythagorean theorem, *i.e.*,

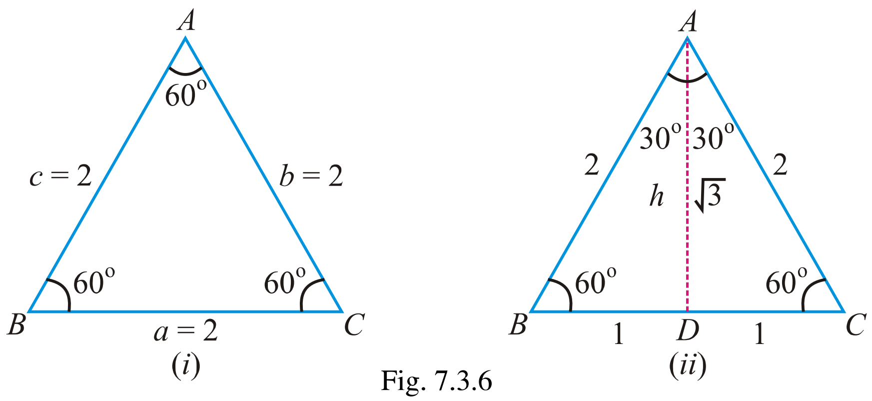


Fig. 7.3.6

$$(AD)^2 + (BD)^2 = (AB)^2 \quad \Rightarrow \quad (AD)^2 = (AB)^2 - (BD)^2$$

$$h^2 = (2)^2 - (1)^2 = 3$$

$$\Rightarrow \quad h = \sqrt{3}$$

$\therefore$ Using triangle ADB with $m\angle A = 30^o$, we have

$$\sin 30^o = \sin\frac{\pi}{6} = \frac{BD}{AB} = \frac{1}{2} \qquad \operatorname{cosec} 30^o = \frac{1}{\sin 30^o} = 2$$

$$\cos 30^o = \cos\frac{\pi}{6} = \frac{AD}{AB} = \frac{\sqrt{3}}{2} \qquad \sec 30^o = \frac{1}{\cos 30^o} = \frac{2}{\sqrt{3}}$$

$$\tan 30^o = \tan\frac{\pi}{6} = \frac{BD}{AD} = \frac{1}{\sqrt{3}} \qquad \cot 30^o = \frac{1}{\tan 30^o} = \sqrt{3}$$

Now using triangle ABD with $m\angle B = 60^o$.

$$\sin 60^o = \frac{AD}{AB} = \frac{\sqrt{3}}{2} \qquad \operatorname{cosec} 60^o = \frac{1}{\sin 60^o} = \frac{2}{\sqrt{3}}$$

$$\cos 60^\circ = \frac{BD}{AB} = \frac{1}{2} \qquad \sec 60^\circ = \frac{1}{\cos 60^\circ} = 2$$

$$\tan 60^\circ = \frac{AD}{BD} = \frac{\sqrt{3}}{1} \qquad \cot 60^\circ = \frac{1}{\tan 60^\circ} = \frac{1}{\sqrt{3}}$$

7.3(v) Signs of trigonometric ratios in different quadrants

In case of trigonometric ratios like $\sin\theta$, $\cos\theta$ and $\tan\theta$ if θ is not a quadrantal angle, then θ will lie in a particular quadrant. Since $r = \sqrt{x^2 + y^2}$ is always $+ve$, the signs of ratios can be found if the quadrant of θ is known.

(*i*) If θ lies in first quadrant, then a point P (x, y) on its terminal side has x and y co-ordinate positive.

Therefore, all trigonometric functions are positive in quadrant I.

(*ii*) If θ lies in 2nd quadrant, then point $P(x, y)$ on its terminal side has negative x-coordinate and positive y-coordinate.

$\therefore$ $\sin\theta = \frac{y}{r}$ is $+ve$ or > 0 $\quad$ $\cos\theta = \frac{x}{r}$ is $-ve$ or < 0 and $\tan\theta = \frac{y}{x}$ is $-ve$ or < 0

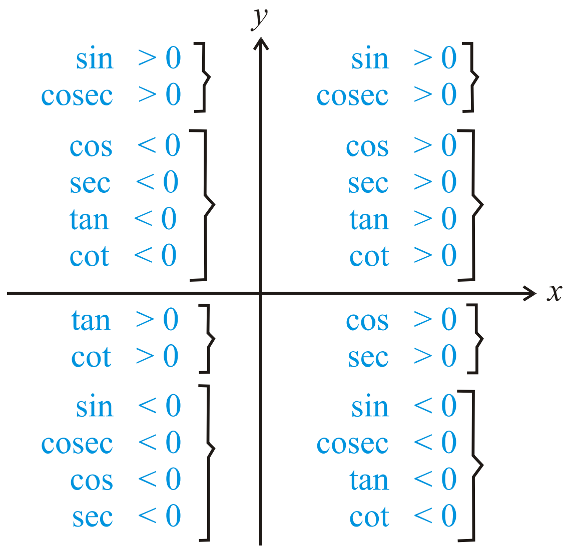
(*iii*) When θ lies in third quadrant, then a point P (x, y) on its terminal side has negative x-coordinate and negative y-coordinate.

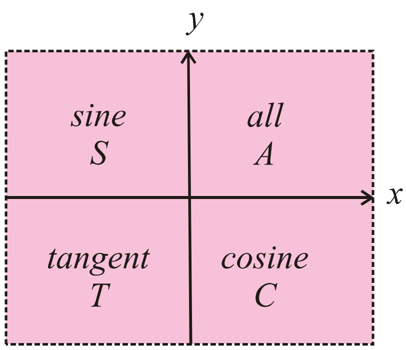
$\therefore$ $\sin\theta = \frac{y}{r}$ is $-ve$ or < 0, $\cos\theta = \frac{x}{r}$ is $-ve$ or < 0 and $\tan\theta = \frac{y}{x}$ is $+ve$ or > 0

(*iv*) When θ lies in fourth quadrant, then the point P (x, y) on the terminal side of θ has positive x-coordinate and negative y-coordinate.

$\therefore$ $\sin\theta = \frac{y}{r}$ is $-ve$ or < 0, $\cos\theta = \frac{x}{r}$ is $+ve$ or > 0 and $\tan\theta = \frac{y}{x}$ is $-ve$ or < 0

The signs of all trigonometric functions are summarized as below.





Read 'ASTC' as
'Add Sugar To Coffee'

7.3(vi) Values of remaining trigonometric ratios if one trigonometric ratio is given

The method is illustrated by the following examples:

Example 1: If $\sin\theta = \frac{-3}{4}$ and $\cos\theta = \frac{\sqrt{7}}{4}$, then find the values of $\tan\theta$, $\cot\theta$, $\sec\theta$ and $\text{cosec}\,\theta$.

Solution: Applying the identities that express the remaining trigonometric functions in terms of sine and cosine, we have

$$\because \quad \sin\theta = \frac{-3}{4} \quad \therefore \quad \text{cosec}\,\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{-3}{4}} = \frac{-4}{3} \quad \Rightarrow \quad \text{cosec}\,\theta = \frac{-4}{3}$$

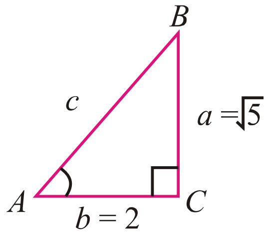
$$\because \quad \cos\theta = \frac{\sqrt{7}}{4} \quad \therefore \quad \sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{\sqrt{7}}{4}} \quad \Rightarrow \quad \sec\theta = \frac{4}{\sqrt{7}}$$

$$\text{Now} \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{-3}{4}}{\frac{\sqrt{7}}{4}} = \frac{-3}{\sqrt{7}} \quad \Rightarrow \quad \tan\theta = \frac{-3}{\sqrt{7}}$$

$$\text{and} \quad \cot\theta = \frac{1}{\tan\theta} = \frac{-\sqrt{7}}{3}$$

Example 2: If $\tan\theta = \frac{\sqrt{5}}{2}$, then find the values of other trigonometric ratios at θ.

Solution: In any right triangle ABC,



$$\tan\theta = \frac{\sqrt{5}}{2} = \frac{a}{b} \quad \Rightarrow \quad a = \sqrt{5},\ b = 2$$

Now by Pythagorean theorem

$$a^2 + b^2 = c^2 \quad \Rightarrow \quad (\sqrt{5})^2 + (2)^2 = c^2$$

$$c^2 = 5 + 4 = 9 \quad \Rightarrow \quad c = \pm 3 \quad \text{or} \quad c = 3$$

$$\because \quad \cot\theta = \frac{1}{\tan\theta}$$

$$\therefore \quad \cot\theta = \frac{1}{\frac{\sqrt{5}}{2}} \quad \Rightarrow \quad \cot\theta = \frac{2}{\sqrt{5}}$$

$$\sin\theta = \frac{a}{c} = \frac{\sqrt{5}}{3}, \quad \text{cosec}\,\theta = \frac{1}{\sin\theta} \quad \Rightarrow \quad \text{cosec}\,\theta = \frac{1}{\frac{\sqrt{5}}{3}} \quad \therefore \quad \text{cosec}\,\theta = \frac{3}{\sqrt{5}}$$

$$\text{Also} \quad \cos\theta = \frac{b}{c} = \frac{2}{3}, \quad \sec\theta = \frac{1}{\cos\theta} \quad \Rightarrow \quad \sec\theta = \frac{1}{\frac{2}{3}} \quad \therefore \quad \sec\theta = \frac{3}{2}$$

7.3(vii) Calculate the values of trigonometric ratios for 0°, 90°, 180°, 270°, 360°

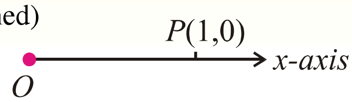
We have discussed quadrantal angles in section 7.3.2. An angle θ is called a quadrantal angle if its terminal side lies on the x-axis or on the y-axis.

Case I When $\theta = 0^o$

The point (1, 0) lies on the terminal side of angle θ^o. We may consider the point on the unit circle on the terminal side of the angle.

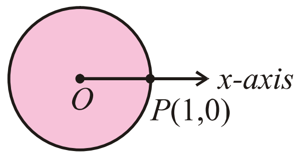
$$P\,(1, 0) \quad \Rightarrow \quad x = 1 \text{ and } y = 0 \quad \text{so} \quad r = \sqrt{x^2 + y^2} = \sqrt{1 + 0} = 1$$

$\therefore \quad \sin 0^o = \dfrac{y}{r} = \dfrac{0}{1} = 0 \quad , \quad \text{cosec}\,0^o = \dfrac{1}{\sin 0^o} = \dfrac{1}{0} = \infty$ (undefined)



$\cos 0^o = \dfrac{x}{r} = \dfrac{1}{1} = 1 \quad , \quad \sec 0^o = \dfrac{1}{\cos 0^o} = 1$

$\tan 0^o = \dfrac{y}{x} = \dfrac{0}{1} = 0 \quad , \quad \cot 0^o = \dfrac{1}{\tan 0^o} = \dfrac{1}{0} = \infty$ (undefined)

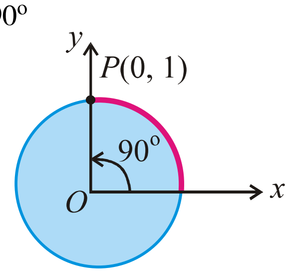


Case II When $\theta = 90^o$

The point P (0, 1) lies on the terminal side of angle 90°

Here $\quad x = 0$ and $y = 1 \quad \Rightarrow \quad r = \sqrt{0^2 + (1)^2} = 1$

$\therefore \quad \sin 90^o = \dfrac{y}{r} = \dfrac{1}{1} = 1$



i.e., $\sin 90^o = 1$ and $\text{cosec}\ 90^o = \dfrac{r}{y} = 1$

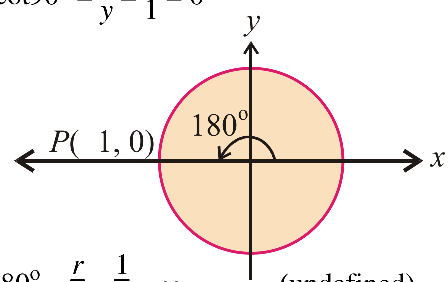
Using reciprocal identities, we have

$\cos 90^o = \dfrac{x}{r} = \dfrac{0}{1} = 0 \quad ; \quad \sec 90^o = \dfrac{r}{x} = \dfrac{1}{0} = \infty$ (undefined)

$\tan 90^o = \dfrac{y}{x} = \dfrac{1}{0} = \infty$ (undefined) , $\cot 90^o = \dfrac{x}{y} = \dfrac{0}{1} = 0$

Case III When $\theta = 180^o$

The point P (−1, 0) lies on x'-axis or on terminal side of angle 180°



Here $\quad x = -1 \quad$ and $\quad y = 0$

$\Rightarrow \quad r = \sqrt{x^2 + y^2} = 1$

$\therefore \quad \sin 180^o = \dfrac{y}{r} = \dfrac{0}{1} = 0 \quad ; \quad \text{cosec}\,180^o = \dfrac{r}{y} = \dfrac{1}{0} = \infty$ (undefined)

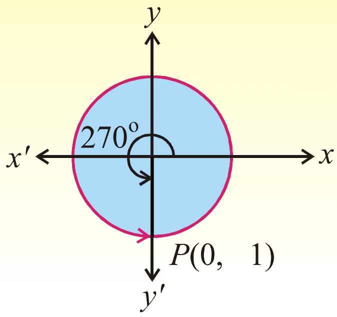
$\cos 180^o = \dfrac{x}{r} = \dfrac{-1}{1} = -1 \quad ; \quad \sec 180^o = \dfrac{r}{x} = \dfrac{1}{-1} = -1$

$\tan 180^o = \dfrac{y}{x} = \dfrac{0}{-1} = 0 \; ; \quad \cot 180^o = \dfrac{x}{y} = \dfrac{-1}{0} = \infty$ (undefined)

Case IV When $\theta = 270^o$ and the point P (0, −1) lies on y'-axis or on the terminal side of angle 270°.

The point P (0, −1) shows that $x = 0$ and $y = -1$

So $\quad r = \sqrt{(0)^2 + (-1)^2} = 1$



$\therefore\ \sin 270^\circ = \frac{y}{r} = \frac{-1}{1} = -1 \quad ; \quad \text{cosec}270^\circ = \frac{r}{y} = \frac{1}{-1} = -1$

$\cos 270^\circ = \frac{x}{r} = \frac{0}{1} = 0 \quad ; \quad \sec 270^\circ = \frac{r}{x} = \frac{1}{0} = \infty$

$\tan 270^\circ = \frac{y}{x} = \frac{-1}{0} = -\infty \quad ; \quad \cot 270^\circ = \frac{x}{y} = \frac{0}{-1} = 0$

Case V When $\theta^\circ = 360^\circ$

Now the point P (1, 0) lies once again on x-axis

We know that $\quad \theta + 2k\pi = \theta \quad$ where $\quad k \in Z.$

Now $\quad \theta = 360^\circ = 0^\circ + (360^\circ)\,1 = 0^\circ \quad$ where $k = 1$

So $\quad \sin 360^\circ = \sin 0^\circ = 0 \quad ; \quad \text{cosec}360^\circ = \frac{1}{\sin 360^\circ} = \frac{1}{\sin 0^\circ} = \frac{1}{0} = \infty$ (undefined)

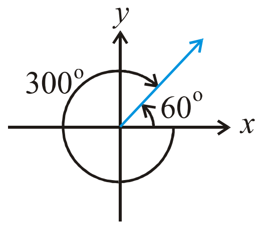
$\cos 360^\circ = \cos 0^\circ = 1 \quad ; \quad \sec 360^\circ = \frac{1}{\cos 0^\circ} = \frac{1}{1} = 1,$

$\tan 360^\circ = \tan 0^\circ = 0 \quad ; \quad \cot 360^\circ = \frac{1}{\tan 0^\circ} = \frac{1}{0} = \infty$ (undefined)

Example: Find each of the following without using table or calculator:

(i) $\cos 540^\circ$ (ii) $\sin 315^\circ$ (iii) $\sec(-300^\circ)$

Solution: We know that $2k\pi + \theta = \theta$, where $k \in Z.$



(i) $\quad 540^\circ = (360^\circ + 180^\circ) = 2(1)\pi + 180^\circ$

$\cos 540^\circ = \cos(2\pi + \pi) = \cos\pi = -1$

(ii) $\quad \sec 315^\circ = \sin(360^\circ - 45^\circ) = \sin\left(2\pi - \frac{\pi}{4}\right)$

$= \sin\left(\frac{-\pi}{4}\right) = -\sin\frac{\pi}{4} = \frac{-1}{\sqrt{2}}$

(iii) $\quad \sec(-300^\circ) = \sec(-360^\circ + 60^\circ)$

$= \sec(2(-1)\pi + 60^\circ)$

$= \sec 60^\circ = \frac{1}{\cos 60^\circ} = \frac{1}{\frac{1}{2}} = 2$

EXERCISE 7.3

1. Locate each of the following angles in standard position using a protractor or fair free hand guess. Also find a positive and a negative angle coterminal with each given angle.

   (i) 170^{o} (ii) 780^{o} (iii) -100^{o} (iv) -500^{o}

2. Identify the closest quadrantal angles between which the following angles lies.

   (i) 156^{o} (ii) 318^{o} (iii) 572^{o} (iv) -330^{o}

3. Write the closest quadrantal angles between which the angle lies. Write your answer in radian measure.

   (i) $\frac{\pi}{3}$ (ii) $\frac{3\pi}{4}$ (iii) $\frac{-\pi}{4}$ (iv) $\frac{-3\pi}{4}$

4. In which quadrant θ lie when

   (i) $\sin\theta > 0$, $\tan\theta < 0$ (ii) $\cos\theta < 0$, $\sin\theta < 0$

   (iii) $\sec\theta > 0$, $\sin\theta < 0$ (iv) $\cos\theta < 0$, $\tan\theta < 0$

   (v) $\operatorname{cosec}\theta > 0$, $\cos\theta > 0$ (vi) $\sin\theta < 0$, $\sec\theta < 0$

5. Fill in the blanks.

   (i) $\cos(-150^{o})$ = ... $\cos 150^{o}$ (ii) $\sin(-310^{o})$ = ... $\sin 310^{o}$

   (iii) $\tan(-210^{o})$ = ... $\tan 210^{o}$ (iv) $\cot(-45^{o})$ = ... $\cot 45^{o}$

   (v) $\sec(-60^{o})$ = ... $\sec 60^{o}$ (vi) $\operatorname{cosec}(-137^{o})$ = ... $\operatorname{cosec} 137^{o}$

6. The given point P lies on the terminal side of θ. Find quadrant of θ and all six trigonometric ratios.

   (i) $(-2, 3)$ (ii) $(-3, -4)$ (iii) $(\sqrt{2}, 1)$

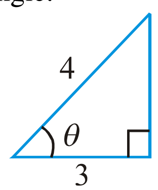
7. If $\cos\theta = \frac{-2}{3}$ and terminal arm of the angle θ is in quadrant II, find the values of remaining trigonometric functions.

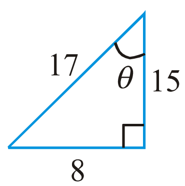
8. If $\tan\theta = \frac{4}{3}$ and $\sin\theta < 0$, find the values of other trigonometric functions at θ.

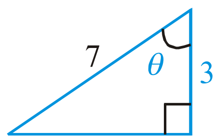
9. If $\sin\theta = \frac{-1}{\sqrt{2}}$ and terminal side of the angle is not in quadrant III, find the values of $\tan\theta$, $\sec\theta$, and $\operatorname{cosec}\theta$.

10. If $\operatorname{cosec}\theta = \frac{13}{12}$ and $\sec\theta > 0$, find the remaining trigonometric functions.

11. Find the values of trigonometric functions at the indicated angle θ in the right triangle.

    (i) 

    (ii) 

    (iii) 

12. Find the values of the trigonometric functions. Do not use trigonometric tables or calculator.

(i) $\tan 30^\circ$ (ii) $\tan 330^\circ$ (iii) $\sec 330^\circ$

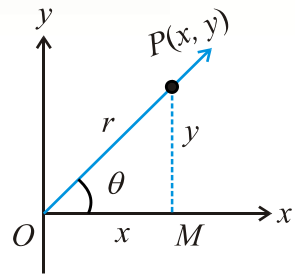
(iv) $\cot \frac{\pi}{4}$ (v) $\cos \frac{2\pi}{3}$ (vi) $\operatorname{cosec} \frac{2\pi}{3}$

(vii) $\cos(-450^\circ)$ (viii) $\tan(-9\pi)$ (ix) $\cos\left(\frac{-5\pi}{6}\right)$

(x) $\sin \frac{7\pi}{6}$ (xi) $\cot \frac{7\pi}{6}$ (xii) $\cos 225^\circ$

7.4 Trigonometric Identities

We have discussed trigonometric functions (ratios) and their reciprocals in section 7.3. Consider an angle $\angle MOP = \theta$ radian in standard position. Let point $P(x, y)$ be on the terminal side of the angle. By Pythagorean theorem, we have from right triangle OMP.



$$OM^2 + MP^2 = OP^2$$

$$x^2 + y^2 = r^2 \quad \ldots\ldots \text{(i)}$$

$\because \sin\theta = \frac{y}{r}, \cos\theta = \frac{x}{r}, \tan\theta = \frac{y}{x}$

Dividing both sides by r^2, we get

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

$$\Rightarrow \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

$$\Rightarrow (\cos\theta)^2 + (\sin\theta)^2 = 1$$

$$\therefore \cos^2\theta + \sin^2\theta = 1 \quad (1)$$

Dividing (i) by x^2, we have

$$\frac{x^2}{x^2} + \frac{y^2}{x^2} = \frac{r^2}{x^2}$$

$$\Rightarrow 1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2 \quad \because \tan\theta = \frac{y}{x} \text{ and } \sec\theta = \frac{r}{x}$$

$$\Rightarrow 1 + (\tan\theta)^2 = (\sec\theta)^2$$

$$\therefore 1 + \tan^2\theta = \sec^2\theta \quad \text{or} \quad \sec^2\theta - \tan^2\theta = 1 \quad (2)$$

Again dividing both sides of (i) by y^2, we get

$$\frac{x^2}{y^2} + \frac{y^2}{y^2} = \frac{r^2}{y^2}$$

$$\left(\frac{x}{y}\right)^2 + 1 = \left(\frac{r}{y}\right)^2 \quad \because \cot\theta = \frac{x}{y} \text{ and } \operatorname{cosec}\theta = \frac{r}{y}$$

$$\Rightarrow (\cot\theta)^2 + 1 = (\operatorname{cosec}\theta)^2$$

$\therefore \quad 1 + \cot^2\theta = \csc^2\theta$ or $\csc^2\theta - \cot^2\theta = 1 \qquad (3)$

The identities (1), (2) and (3) are also known as **Pythagorean Identities.**

The fundamental identities are used to simplify expressions involving trigonometric functions.

Example 1: Verify that $\cot\theta\sec\theta = \csc\theta$

Solution: Expressing left hand side in terms of sine and cosine, we have

$$\text{L.H.S} = \cot\theta\sec\theta = \frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\cos\theta}$$

$$= \frac{1}{\sin\theta} = \csc\theta$$

$$= \text{R.H.S}$$

Example 2: Verify that $\tan^4\theta + \tan^2\theta = \tan^2\theta\sec^2\theta$

Solution: $\text{L.H.S} = \tan^4\theta + \tan^2\theta = \tan^2\theta(\tan^2\theta + 1) \qquad \because \tan^2\theta + 1 = \sec^2\theta$

$$= \tan^2\theta\sec^2\theta$$

$$= \text{R.H.S}$$

Example 3: Show that $\dfrac{\cot^2\alpha}{\csc\alpha - 1} = \csc\alpha + 1$

Solution: $\text{L.H.S} = \dfrac{\cot^2\alpha}{\csc\alpha - 1} \qquad \left(\begin{array}{l}\because \csc^2\theta - \cot^2\theta = 1\\ \cot^2\theta = \csc^2\theta - 1\end{array}\right)$

$$= \frac{(\csc^2\alpha - 1)}{\csc\alpha - 1} = \frac{(\csc\alpha - 1)(\csc\alpha + 1)}{(\csc\alpha - 1)} = \csc\alpha + 1 = \text{R.H.S}$$

Example 4: Express the trigonometric functions in terms of $\tan\theta$.

Solution: By using reciprocal identity, we can express $\cot\theta$ in terms of $\tan\theta$.

i.e., $\quad \cot\theta = \dfrac{1}{\tan\theta}$

By solving the identity $\quad 1 + \tan^2\theta = \sec^2\theta$

We have expressed $\sec\theta$ in terms of $\tan\theta$.

$$\sec\theta = \pm\sqrt{\tan^2\theta + 1}$$

Note: We can express all the trigonometric functions in terms of one trigonometric function.

$$\because \quad \cos\theta = \frac{1}{\sec\theta} \quad \Rightarrow \quad \cos\theta = \frac{1}{\pm\sqrt{\tan^2\theta + 1}}$$

Because $\sin\theta = \tan\theta\cos\theta$, we have

$$\sin\theta = \tan\theta\left(\frac{1}{\pm\sqrt{\tan^2\theta + 1}}\right) = \frac{\tan\theta}{\pm\sqrt{\tan^2\theta + 1}}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{\pm\sqrt{\tan^2\theta + 1}}{\tan\theta}$$

EXERCISE 7.4

In Problems 1–6, simplify each expression to a single trigonometric function.

1. $\dfrac{\sin^2 x}{\cos^2 x}$

2. $\tan x \sin x \sec x$

3. $\dfrac{\tan x}{\sec x}$

4. $1 - \cos^2 x$

5. $\sec^2 x - 1$

6. $\sin^2 x \, . \, \cot^2 x.$

In problems 7-24, verify the identities.

7. $(1 - \sin\theta)(1 + \sin\theta) = \cos^2\theta$

8. $\dfrac{\sin\theta + \cos\theta}{\cos\theta} = 1 + \tan\theta$

9. $(\tan\theta + \cot\theta)\tan\theta = \sec^2\theta$

10. $(\cot\theta + \mathrm{cosec}\,\theta)(\tan\theta - \sin\theta) = \sec\theta - \cos\theta$

11. $\dfrac{\sin\theta + \cos\theta}{\tan^2\theta - 1} = \dfrac{\cos^2\theta}{\sin\theta - \cos\theta}$

12. $\dfrac{\cos^2\theta}{\sin\theta} + \sin\theta = \mathrm{cosec}\,\theta$

13. $\sec\theta - \cos\theta = \tan\theta\sin\theta$

14. $\dfrac{\sin^2\theta}{\cos\theta} + \cos\theta = \sec\theta$

15. $\tan\theta + \cot\theta = \sec\theta\,\mathrm{cosec}\,\theta$

16. $(\tan\theta + \cot\theta)(\cos\theta + \sin\theta) = \sec\theta + \mathrm{cosec}\,\theta$

17. $\sin\theta(\tan\theta + \cot\theta) = \sec\theta$

18. $\dfrac{1 + \cos\theta}{\sin\theta} + \dfrac{\sin\theta}{1 + \cos\theta} = 2\mathrm{cosec}\,\theta$

19. $\dfrac{1}{1 - \cos\theta} + \dfrac{1}{1 + \cos\theta} = 2\mathrm{cosec}^2\theta$

20. $\dfrac{1 + \sin\theta}{1 - \sin\theta} - \dfrac{1 - \sin\theta}{1 + \sin\theta} = 4\tan\theta\sec\theta$

21. $\sin^3\theta = \sin\theta - \sin\theta\cos^2\theta$

22. $\cos^4\theta - \sin^4\theta = (\cos^2\theta - \sin^2\theta)$

23. $\sqrt{\dfrac{1 + \cos\theta}{1 - \cos\theta}} = \dfrac{\sin\theta}{1 - \cos\theta}$

24. $\sqrt{\dfrac{\sec\theta + 1}{\sec\theta - 1}} = \dfrac{\sec\theta + 1}{\tan\theta}$

7.5 Angle of Elevation and Angle of Depression

One of the objects of trigonometry is to find the distances between points or the heights of objects, without actually measuring these distances or heights.

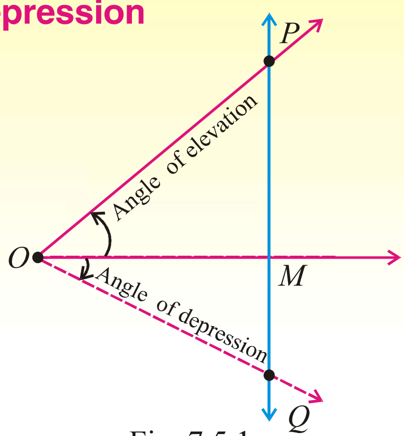


Fig. 7.5.1

Angle of elevation: Suppose O, P and Q are three points, P being at a higher level of O and Q being at lower level than O. Let a horizontal line drawn through O meet in M, the vertical line drawn through P and Q.

The angle MOP is called the **angle of elevation** of point P as seen from O. For looking at Q below the horizontal line we have to lower our eyes and $\angle MOQ$ is called the **angle of depression.**

We measure an angle of elevation from a horizontal line up to an object or an angle of depression from a horizontal line down to an object, see figure 7.5.2.

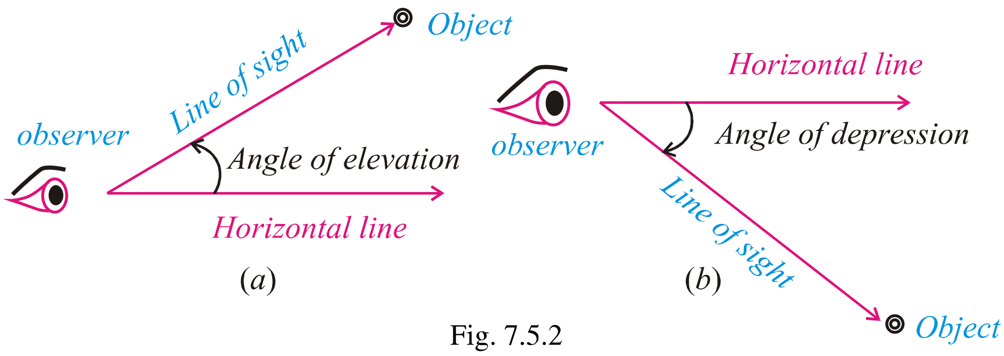
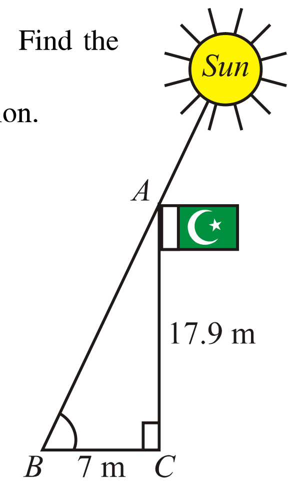


Fig. 7.5.2

7.5(i) Find angle of elevation and angle of depression:

For finding distances, heights and angles by the use of trigonometric functions, consider the following examples:

Example 1: A flagpole 17.9 meter high casts a 7 meter shadow Find the angle of elevation of the sun.



Solution: From the figure, we observe that α is the angle of elevation.

Using the fact that

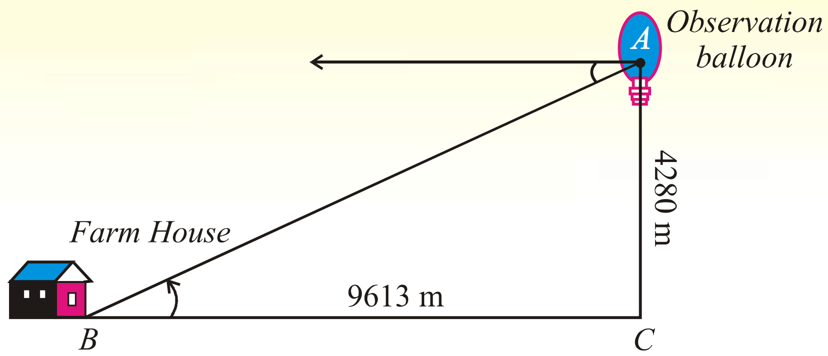
$$\tan\alpha = \frac{AC}{BC} = \frac{17.9}{7} \approx 2.55714$$

Solving for α gives us

$$\alpha \approx \tan^{-1}(2.55714)$$
$$\approx (68.6666)^{\circ} \approx 68^{\circ}\,40'$$
$$\Rightarrow \quad \alpha \approx 68^{\circ}\,40'$$

Example 2: An observation balloon is 4280 meter above the ground and 9613 meter away from a farmhouse. Find the angle of depression of the farmhouse as observed from the observation balloon.

Solution:



For problems of this type the angle of elevation of *A* from *B* is considered equal to the angle of depression of *B* from *A*, as shown in the diagram.

$$\tan\alpha = \frac{AC}{BC} = \frac{4280}{9613} = 0.44523$$

$$\alpha = \tan^{-1}(0.44523) = 24^{\circ}$$

So, angle of depression is 24°.

EXERCISE 7.5

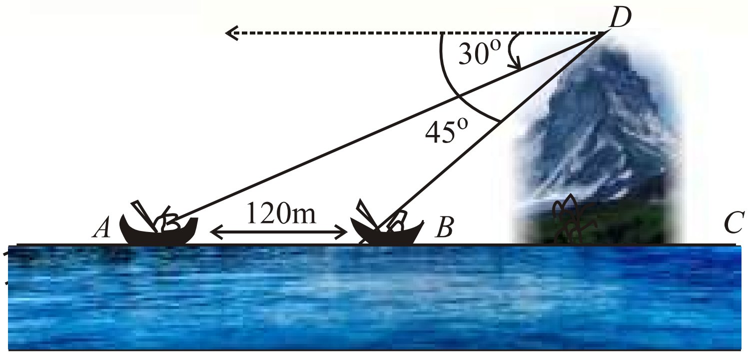
1. Find the angle of elevation of the sun if a 6 feet man casts a 3.5 feet shadow.
2. A tree casts a 40 meter shadow when the angle of elevation of the sun is 25°. Find the height of the tree.
3. A 20 feet long ladder is leaning against a wall. The bottom of the ladder is 5 feet from the base of the wall. Find the acute angle (angle of elevation) the ladder makes with the ground.
4. The base of a rectangle is 25 feet and the height of the rectangle is 13 feet. Find the angle that the diagonal of the rectangle makes with the base.
5. A rocket is launched and climbs at a constant angle of 80°. Find the altitude of the rocket after it travels 5000 meter.
6. An aeroplane pilot flying at an altitude of 4000m wishes to make an approach to an airport at an angle of 50° with the horizontal. How far from the airport will the plane be when the pilot begins to descend?
7. A guy wire (supporting wire) runs from the middle of a utility pole to the ground. The wire makes an angle of 78.2° with the ground and touches the ground 3 meters from the base of the pole. Find the height of the pole.
8. A road is inclined at an angle 5.7°. Suppose that we drive 2 miles up this road starting from sea level. How high above sea level are we?
9. A television antenna of 8 feet height is located on the top of a house. From a point on the ground the angle of elevation to the top of the house is 17° and the angle of elevation to the top of the antenna is 21.8°. Find the height of the house.

10. From an observation point, the angles of depression of two boats in line with this point are found to be 30° and 45°. Find the distance between the two boats if the point of observation is 4000 feet high.

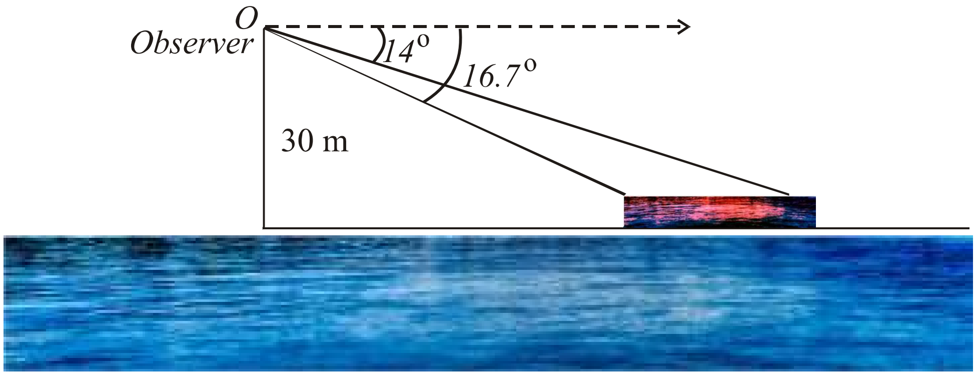
11. Two ships, which are in line with the base of a vertical cliff, are 120 meters apart. The angles of depression from the top of the cliff to the ships are 30° and 45°, as shown in the diagram.

(a) Calculate the distance *BC*

(b) Calculate the height *CD*, of the cliff.



12. Suppose that we are standing on a bridge 30 feet above a river watching a log (piece of wood) floating towards us. If the angle with the horizontal to the front of the log is 16.7° and angle with the horizontal to the back of the log is 14°, how long is the log?



MISCELLANEOUS EXERCISE - 7

1. Multiple Choice Questions

Four possible answers are given for the following questions. Tick (✓) the correct answer.

(i) The union of two non-collinear rays, which have common end point is called

(*a*) an angle (*b*) a degree (*c*) a minute (*d*) a radian

(ii) The system of measurement in which the angle is measured in radians is called

(*a*) CGS system (*b*) sexagesimal system

(*c*) MKS system (*d*) circular system

(iii) $20^\circ =$

(*a*) $360'$ (*b*) $630'$ (*c*) $1200'$ (*d*) $3600'$

(iv) $\frac{3\pi}{4}$ radians =

(*a*) 115° (*b*) 135° (*c*) 150° (*d*) 30°

(v) If $\tan\theta = \sqrt{3}$, then θ is equal to

(*a*) 90° (*b*) 45° (*c*) 60° (*d*) 30°

(vi) $\sec^2\theta =$

(*a*) $1 - \sin^2\theta$ (*b*) $1 + \tan^2\theta$ (*c*) $1 + \cos^2\theta$ (*d*) $1 - \tan^2\theta$

(vii) $\frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta} =$

(*a*) $2\sec^2\theta$ (*b*) $2\cos^2\theta$ (*c*) $\sec^2\theta$ (*d*) $\cos\theta$

(viii) $\frac{1}{2}\,\mathrm{cosec}45^\circ =$

(*a*) $\frac{1}{2\sqrt{2}}$ (*b*) $\frac{1}{\sqrt{2}}$ (*c*) $\sqrt{2}$ (*d*) $\frac{\sqrt{3}}{2}$

(ix) $\sec\theta\cot\theta =$

(*a*) $\sin\theta$ (*b*) $\frac{1}{\cos\theta}$ (*c*) $\frac{1}{\sin\theta}$ (*d*) $\frac{\sin\theta}{\cos\theta}$

(x) $\mathrm{cosec}^2\theta - \cot^2\theta =$

(*a*) -1 (*b*) 1 (*c*) 0 (*d*) $\tan\theta$

2. Write short answers of the following questions.

(i) Define an angle.

(ii) What is the sexagesimal system of measurement of angles?

(iii) How many minutes are there in two right angles?

(iv) Define radian measure of an angle.

(v) Convert $\frac{\pi}{4}$ radians to degree measure.

(vi) Convert 15° to radians.

(vii) What is radian measure of the central angle of an arc 50m long on the circle of radius 25m?

(viii) Find r when $l = 56\ cm$ and $\theta = 45^\circ$

(ix) Find $\tan\theta$ when $\cos\theta = \frac{9}{41}$ and terminal side of the angle θ is in fourth quadrant.

(x) Prove that $(1 - \sin^2\theta)(1 + \tan^2\theta) = 1$

3. Fill in the blanks

(i) π radians = __________degree.

(ii) The terminal side of angle 235° lies in __________quadrant.

(iii) Terminal side of the angle −30° lies in __________quadrant.

(iv) Area of a circular sector is __________.

(v) If $r = 2$ cm and $\theta = 3$ radian, then area of the circular sector is __________.

(vi) The general form of the angle 480° is __________.

(vii) If $\sin\theta = \frac{1}{2}$, then $\theta =$ __________.

(viii) If $\theta = 300^{\circ}$, then $\sec(-300)^{\circ} =$ __________.

(ix) $1 + \cot^2\theta =$ __________.

(x) $\text{Sec}\ \theta - \tan\ \theta =$ __________.

SUMMARY

- If we divide the circumference of a circle into 360 equal arcs, then the angle subtended at the centre of the circle by one arc is called one **degree** and is denoted by 1°.
- The angle subtended at the centre of the circle by an arc, whose length is equal to the radius of the circle, is called one **radian.**
- **Relationship between radian and degree measure**

  $1^{\circ} = \frac{\pi}{180}$ radians , ≈ 0.0175 radians and 1 radian $= \left(\frac{180}{\pi}\right)^{\circ}$, ≈ 57.295 degrees
- Relation between central angle and arc length of a circle: $l = r\theta$
- Area of a circular sector, $A = \frac{1}{2}r^2\theta$
- Two or more than two angles with the same initial and terminal sides are called **coterminal** angles.
- An angle is called a **quadrantal angle,** if its terminal side lies on the x-axis or y-axis.
- A general angle is said to be in **standard position** if its vertex is at the origin and its initial side is directed along the positive direction of the x-axis of a rectangular coordinate system.
- There are six fundamental **trigonometric ratios** (functions) known as sine, cosine, tangent, cotangent, secant and cosecant.
- **Trigonometric Identities:**
  (a) $\cos^2\theta + \sin^2\theta = 1$
  (b) $1 + \tan^2\theta = \sec^2\theta$
  (c) $1 + \cot^2\theta = \text{cosec}^2\theta$

Unit-8

PROJECTION OF A SIDE OF A TRIANGLE

In this unit, students will learn how to

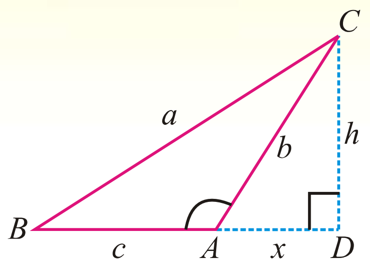
Prove the following theorems along with corollaries and apply them to solve appropriate problems.

- *In an obtuse-angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.*
- *In any triangle, the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing that acute angle diminished by twice the rectangle contained by one of those sides and the projection on it of the other.*
- *In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side (Apollonius' Theorem).*

THEOREM 1

8.1(*i*) In an obtuse angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.

Given: ABC is a triangle having an obtuse angle BAC at A. Draw $\overline{CD}$ perpendicular on $\overline{BA}$ produced. So that $\overline{AD}$ is the projection of $\overline{AC}$ on $\overline{BA}$ produced.



Take $m\overline{BC} = a$, $m\overline{CA} = b$, $m\overline{AB} = c$, $m\overline{AD} = x$ and $m\overline{CD} = h$.

To prove: $(BC)^2 = (AC)^2 + (AB)^2 + 2(m\overline{AB})\,(m\overline{AD})$

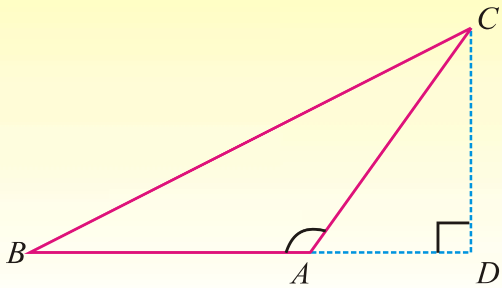
i.e., $a^2 = b^2 + c^2 + 2cx$

Proof:

Statements	Reasons
In $\angle rt\Delta\ CDA$,	
$m\angle CDA = 90^{\circ}$	Given
$\therefore$ $(AC)^2 = (AD)^2 + (CD)^2$	Pythagoras Theorem
or $b^2 = x^2 + h^2$ (i)	
In $\angle rt\Delta\ CDB$,	
$m\angle CDB = 90^{\circ}$	Given
$\therefore$ $(BC)^2 = (BD)^2 + (CD)^2$	Pythagoras Theorem
or $a^2 = (c + x)^2 + h^2$	$\overline{BD} = \overline{BA} + \overline{AD}$
$= c^2 + 2cx + x^2 + h^2$ (ii)	
Hence $a^2 = c^2 + 2cx + b^2$	Using (i) and (ii)
i.e., $a^2 = b^2 + c^2 + 2cx$	
or $(BC)^2 = (AC)^2 + (AB)^2 + 2(m\overline{AB})\,(m\overline{AD})$	

Example: In a ΔABC with obtuse angle at A, if $\overline{CD}$ is an altitude on $\overline{BA}$ produced and $m\overline{AC} = m\overline{AB}$

Then prove that $(BC)^2 = 2(AB)(BD)$



Given: In a ΔABC, $m\angle A$ is obtuse $m\overline{AC} = m\overline{AB}$ and $\overline{CD}$ being altitude on $\overline{BA}$ produced.

To prove: $(BC)^2 = 2(m\overline{AB})(m\overline{BD})$

Proof: In a ΔABC, having obtuse angle BAC at A.

Statements	Reasons
$(BC)^2 = (BA)^2 + (AC)^2 + 2(m\overline{BA})(m\overline{AD})$	By Theorem 1
$= (AB)^2 + (AB)^2 + 2(m\overline{AB})(m\overline{AD})$	Given
$= 2(AB)^2 + 2(m\overline{AB})(m\overline{AD})$	
$(BC)^2 = 2m\overline{AB}(m\overline{AB} + m\overline{AD})$	On the line segment BD,
$= 2m\overline{AB}\ (m\overline{BD})$	Point A is between B and D

EXERCISE 8.1

1. Given $m\overline{AC} = 1\text{cm}$, $m\overline{BC} = 2\text{cm}$, $m\angle C = 120^\circ$.

   Compute the length AB and the area of ΔABC.

   Hint: $(AB)^2 = (AC)^2 + (BC)^2 + 2mAC.\ mCD$

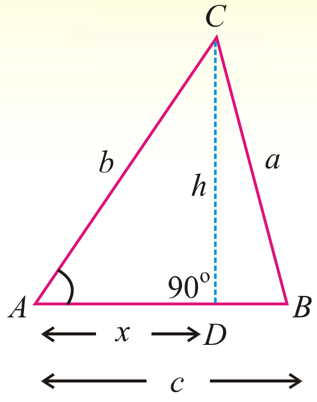
   where $(m\overline{CD}) = (m\overline{BC}) \cos(180^\circ - m\angle C)$ (Use theorem 1).

2. Find $m\ \overline{AC}$ if in ΔABC $m\ \overline{BC} = 6$ cm, $m\ \overline{AB} = 4\sqrt{2}$ cm and $m\angle ABC = 135^\circ$.

THEOREM 2

8.1(*ii*) In any triangle, the square on the side opposite to acute angle is equal to sum of the squares on the sides containing that acute angle diminished by twice the rectangle contained by one of those sides and the projection on it of the other.

Given: ΔABC with an acute angle CAB at A.



Take $m\overline{BC} = a,\ \ m\overline{CA} = b$ and $m\overline{AB} = c$

Draw $\overline{CD} \perp \overline{AB}$ so that $\overline{AD}$ is projection of $\overline{AC}$ on $\overline{AB}$

Also, $m\overline{AD} = x$ and $m\overline{CD} = h$

To prove: $(BC)^2 = (AC)^2 + (AB)^2 - 2(AB)(AD)$

i.e., $a^2 = b^2 + c^2 - 2cx$

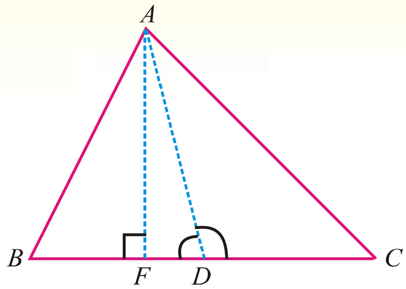
Proof:

Statements	Reasons
In $\angle rt\ \Delta\ CDA$	
$m\angle CDA = 90^o$	Given
$(AC)^2 = (AD)^2 + (CD)^2$	Pythagoras Theorem
i.e., $b^2 = x^2 + h^2$ (i)	
In $\angle rt\ \Delta\ CDB$,	
$m\angle CDB = 90^o$	Given
$(BC)^2 = (BD)^2 + (CD)^2$	Pythagoras Theorem
$a^2 = (c-x)^2 + h^2$	From the figure
or $a^2 = c^2 - 2cx + x^2 + h^2$ (ii)	
$a^2 = c^2 - 2cx + b^2$	Using (i) and (ii)
Hence, $a^2 = b^2 + c^2 - 2cx$	
i.e., $(BC)^2 = (AC)^2 + (AB)^2 - 2(AB)(AD)$	

THEOREM 3

(APOLLONIUS' THEOREM)

8.1.(*iii*) In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side.



Given: In a ΔABC, the median $\overline{AD}$ bisects $\overline{BC}$. i.e., $m\overline{BD} = m\overline{CD}$

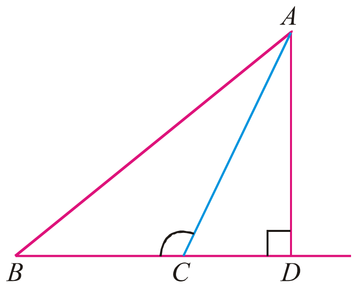
To prove: $(AB)^2 + (AC)^2 = 2(BD)^2 + 2(AD)^2$

Construction: Draw $\overline{AF} \perp \overline{BC}$

Proof:

Statements	Reasons
In ΔADB Since $\angle ADB$ is acute at D	
$\therefore$ $(AB)^2 = (BD)^2 + (AD)^2 - 2\, m\overline{BD}.\, m\overline{FD}$ (i)	Using Theorem 2
Now in ΔADC since $\angle ADC$ is obtuse at D	
$\therefore$ $(AC)^2 = (CD)^2 + (AD)^2 + 2\, m\overline{CD}.\, m\overline{FD}$	Using Theorem 1
$= (BD)^2 + (AD)^2 + 2\, m\overline{BD}\, .\, m\overline{FD}$ (ii)	
Thus $(AB)^2 + (AC)^2 = 2(BD)^2 + 2\,(AD)^2$	Adding (i) and (ii)

Example 1: In ΔABC, $\angle C$ is obtuse, $\overline{AD} \perp \overline{BC}$ produced, whereas $\overline{BD}$ is projection of $\overline{AB}$ on $\overline{BC}$. Prove that $(AC)^2 = (AB)^2 + (BC)^2 - 2m\overline{BC}.m\overline{BD}$



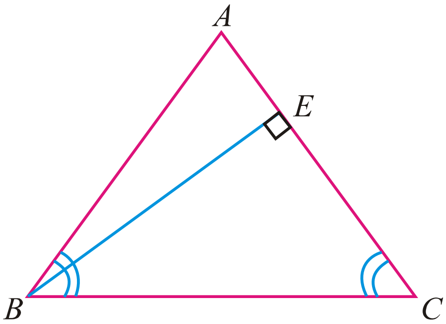
Given: In a ΔABC, $\angle BCA$ is obtuse so that $\angle B$ is acute, $\overline{AD} \perp \overline{BC}$ produced, whereas $\overline{BD}$ is projection of $\overline{AB}$ on $\overline{BC}$ produced.

To prove: $(AC)^2 = (AB)^2 + (BC)^2 - 2m\overline{BC}\, .\, m\overline{BD}$

Proof:

Statements		Reasons
In $\angle rt\ \Delta ABD$		
$(AB)^2 = (AD)^2 + (BD)^2$	(i)	Pythagoras Theorem
In $\angle rt\ \Delta ACD$		
$(AC)^2 = (AD)^2 + (CD)^2$	(ii)	Pythagoras Theorem
or $(AC)^2 = (AD)^2 + (BD - BC)^2$		$m\overline{BC} + m\overline{CD} = m\overline{BD}$
$(AC)^2 = (AD)^2 + (BD)^2 + (BC)^2 - 2BC.BD$	(iii)	
$(AC)^2 = (AB)^2 + (BC)^2 - 2BC.BD$		Using (i) and (iii)

Example 2: In an Isosceles ΔABC, if $m\overline{AB} = m\overline{AC}$ and $\overline{BE} \perp \overline{AC}$, then prove that $(BC)^2 = 2m\overline{AC}.m\overline{CE}$



Given: In an Isosceles ΔABC

$m\overline{AB} = m\overline{AC}$ and $\overline{BE} \perp \overline{AC}$

whereas $\overline{CE}$ is the projection of $\overline{BC}$ upon on $\overline{AC}$.

To prove: $(BC)^2 = 2m\overline{AC}\ .\ m\overline{CE}$

Proof:

Statements	Reasons
In an isosceles ΔABC with $m\overline{AB} = m\overline{AC}$. If $\angle C$ is acute,	
then $(AB)^2 = (AC)^2 + (BC)^2 - 2m\overline{AC}.m\overline{CE}$	By Theorem 2
$(AC)^2 = (AC)^2 + (BC)^2 - 2m\overline{AC}.m\overline{CE}$	Given $m\overline{AB} = m\overline{AC}$
$\Rightarrow$ $(BC)^2 - 2m\overline{AC}.m\overline{CE} = 0$	Cancel $(\overline{AC})^2$ on both sides
or $(BC)^2 = 2m\overline{AC}\ .\ m\overline{CE}$	

EXERCISE 8.2

1. In a ΔABC calculate $m\overline{BC}$ when $m\overline{AB} = 6$cm, $m\overline{AC} = 4$cm and $m\angle A = 60^o$.

2. In a ΔABC, $m\overline{AB} = 6$ cm, $m\overline{BC} = 8$ cm, $m\overline{AC} = 9$ cm and D is the mid point of side $\overline{AC}$. Find length of the median $\overline{BD}$.

3. In a parallelogram $ABCD$ prove that $(AC)^2 + (BD)^2 = 2\,[(AB)^2 + (BC)^2]$

MISCELLANEOUS EXERCISE 8

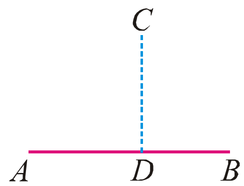
1. In a ΔABC, $m\angle A = 60^{\circ}$, prove that $(BC)^2 = (AB)^2 + (AC)^2 - m\overline{AB}\,.\,m\overline{AC}$.
2. In a ΔABC, $m\angle A = 45^{\circ}$, prove that $(BC)^2 = (AB)^2 + (AC)^2 - \sqrt{2}\, m\overline{AB}\,.\,m\overline{AC}$.
3. In a ΔABC, calculate $m\overline{BC}$ when $m\overline{AB} = 5$ cm, $m\overline{AC} = 4$ cm, $m\angle A = 60^{\circ}$.
4. In a ΔABC, calculate $m\overline{AC}$ when $m\overline{AB} = 5$ cm, $m\overline{BC} = 4\sqrt{2}$ cm , $m\angle B = 45^{\circ}$.
5. In a triangle ABC, $m\overline{BC} = 21$ cm, $m\overline{AC} = 17$ cm, $m\overline{AB} = 10$ cm.

   Measure the length of projection of $\overline{AC}$ upon $\overline{BC}$.
6. In a triangle ABC, $m\overline{BC} = 21$ cm. $m\overline{AC} = 17$ cm, $m\overline{AB} = 10$ cm.

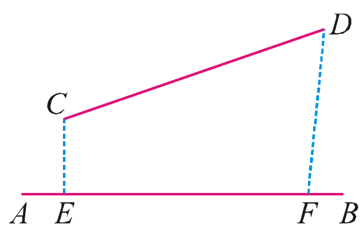
   Calculate the projection of $\overline{AB}$ upon $\overline{BC}$.
7. In a ΔABC, $a = 17$ cm, $b = 15$ cm and $c = 8$ cm. Find $m\angle A$.
8. In a ΔABC, $a = 17$ cm, $b = 15$ cm and $c = 8$ cm find $m\angle B$.
9. Whether the triangle with sides 5 cm, 7 cm, 8 cm is acute, obtuce or right angled.
10. Whether the triangle with sides 8 cm, 15 cm, 17 cm is acute, obtuce or right angled.

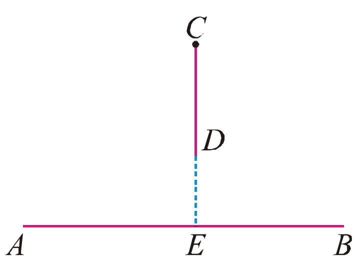
SUMMARY

- The **projection** of a given point on a line segment is the foot of $\perp$ drawn from the point on that line segment. If $\overline{CD} \perp \overline{AB}$, then evidently D is the foot of perpendicular CD from the point C on the line segment AB.



- The projection of a line segment $\overline{CD}$ on a line segment $\overline{AB}$ is the portion $\overline{EF}$ of the latter intercepted between foots of the perpendiculars drawn from C and D. However projection of a vertical line segment $\overline{CD}$ on a line segment $\overline{AB}$ is a point on $\overline{AB}$ which is of **zero dimension**.





- In an obtuse-angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.
- In any triangle, the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing that acute angle diminished by twice the rectangle contained by one of those sides and the projection on it of the other.
- In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side (Apollonius' Theorem).

Unit-9

CHORDS OF A CIRCLE

In this unit, students will learn how to

Prove the following theorems alongwith corollaries and apply them to solve appropriate problems.

- ***One and only one circle can pass through three non collinear points.***
- ***A straight line, drawn from the centre of a circle to bisect a chord (which is not a diameter) is perpendicular to the chord.***
- ***Perpendicular from the centre of a circle on a chord bisects it.***
- ***If two chords of a circle are congruent then they will be equidistant from the centre.***
- ***Two chords of a circle which are equidistant from the centre are congruent.***

Basic concepts of the circle

A **circle** is the locus of a moving point P in a plane which is always equidistant from some fixed point O. The fixed point O not lying on the circle is called the centre, the constant distance OP is its radius whereas the boundary traced by moving point P is called circumference of the circle.

Note that the **radial segment** of a circle is a line segment, determined by the centre and a point on the circle. There is only one centre point whereas all the radii of a circle are equal in length.

In the adjoining figure (i) of the circle, the length of radial segment = $m\overline{OP} = m\overline{OQ} = m\overline{OT}$

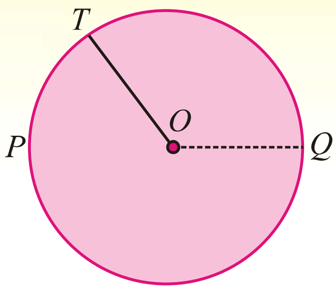


Fig. (i)

$2\pi r$ is the **circumference** of a circle with radius r whereas an irrational number π being the ratio of the circumference and the diameter of a given circle.

An **arc** ACB of a circle is any portion of its circumference.

A **chord** AKB of a circle is a line segment joining any two points A and B on the circumference of a circle. Whereas diameter POQ is the chord passing through the centre of a circle. Evidently diameter bisects a circle.

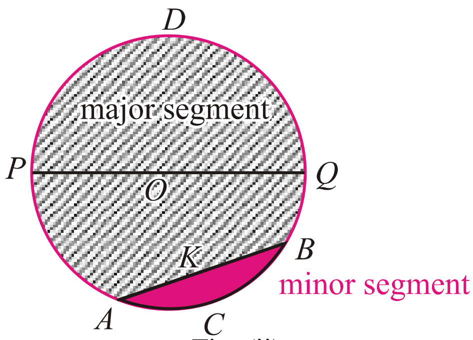


Fig. (ii)

A **segment** is the portion of a circle bounded by an arc and a corresponding chord. Evidently any chord divides a circle into two segments.

In figure (ii) the bigger area shown by slanting line segments is the major segment whereas the smaller area shown by shading is the minor segment.

A **sector** of a circle is the plane figure bounded by two radii and the arc intercepted between them. Any pair of radii divides a circle into two sectors.

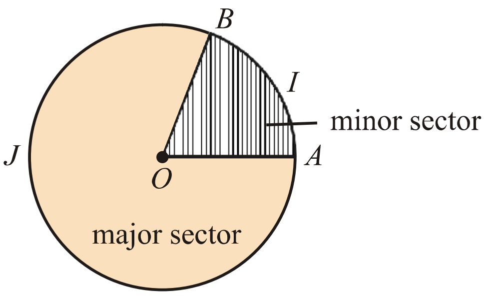


Fig. (iii)

In the figure (iii) $OAIB$ is the minor sector, whereas $OAJB$ is the major sector of the circle.

$\angle AOB$ is the central angle of a circle whose vertex is at the centre O and its arms meet at the end points of the arc AB.

THEOREM 1

9.1(*i*) One and only one circle can pass through three non-collinear points.

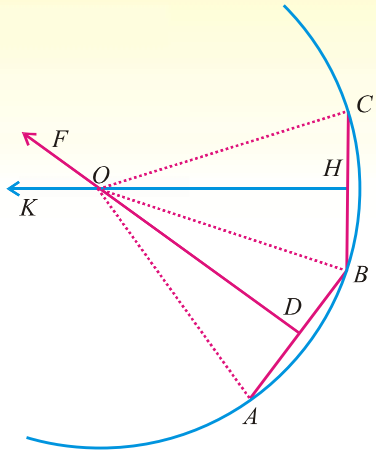
Given: A, B and C are three non collinear points in a plane.

To prove: One and only one circle can pass through three non-collinear points A, B and C.

Construction: Join A with B and B with C.

Draw $\overline{DF} \perp$ bisector to $\overline{AB}$ and $\overline{HK} \perp$ bisector to $\overline{BC}$.

So, $\overline{DF}$ and $\overline{HK}$ are not parallel and they intersect each other at point O. Also join A, B and C with point O.



Proof:

Statements	Reasons
Every point on $\overline{DF}$ is equidistant from A and B.	$\overline{DF}$ $\perp$ bisector to $\overline{AB}$ (construction)
In particular $m\overline{OA} = m\overline{OB}$ (i)	
Similarly every point on $\overline{HK}$ is equidistant from B and C.	$\overline{HK}$ is $\perp$ bisector to $\overline{BC}$ (construction)
In particular $m\overline{OB} = m\overline{OC}$ (ii)	
Now O is the only point common to $\overline{DF}$ and $\overline{HK}$ which is equidistant from A, B and C.	
i.e., $m\overline{OA} = m\overline{OB} = m\overline{OC}$	Using (i) and (ii)
However there is no such other point expect O.	

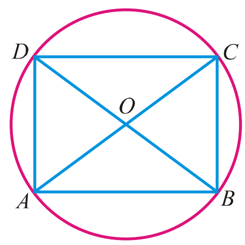
Hence a circle with centre O and radius OA will pass through A, B and C. Ultimately there is only one circle which passes through three given points A, B and C.

Example: Show that only one circle can be drawn to pass through the vertices of any rectangle.

Given: $ABCD$ is a rectangle.

To Prove: Only one circle can be drawn through the vertices of the rectangle $ABCD$.

Construction: Diagonals $\overline{AC}$ and $\overline{BD}$ of the rectangle meet each other at point O.



Proof:

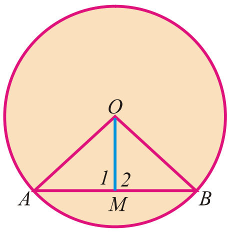
Statements		Reasons
$ABCD$ is a rectangle.		Given
$\therefore \quad m\overline{AC} = m\overline{BD}$	(i)	Diagonals of a rectangle are equal.
$\because \ \overline{AC}$ and $\overline{BD}$ meet each other at O		Construction
$\therefore \ m\overline{OA} = m\overline{OC}$ and $m\overline{OB} = m\overline{OD}$	(ii)	Diagonals of rectangle bisect each other
$\Rightarrow \ m\overline{OA} = m\overline{OB} = m\overline{OC} = m\overline{OD}$	(iii)	Using (i) and (ii)
i.e., point O is equidistant from all vertices of the rectangle $ABCD$. Hence $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ and $\overline{OD}$ are the radii of the circle which is passing through the vertices of the rectangle having centre O.		

THEOREM 2

9.1(*ii*) A straight line, drawn from the centre of a circle to bisect a chord (which is not a diameter) is perpendicular to the chord.

Given: M is the mid point of any chord $\overline{AB}$ of a circle with centre at O.

Where chord $\overline{AB}$ is not the diameter of the circle.



To prove: $\overline{OM} \perp$ the chord $\overline{AB}$.

Construction: Join A and B with centre O.

Write $\angle 1$ and $\angle 2$ as shown in the figure.

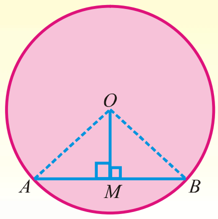
Proof:

Statements		Reasons
In $\Delta\ OAM \leftrightarrow \Delta OBM$		
$m\overline{OA} = m\overline{OB}$		Radii of the same circle
$m\overline{AM} = m\overline{BM}$		Given
$m\overline{OM} = m\overline{OM}$		Common
$\therefore \quad \Delta OAM \cong \Delta OBM$		S.S.S $\cong$ S.S.S
$\Rightarrow \quad m\angle 1 = m\angle 2$	(i)	Corresponding angles of congruent triangles
i.e., $m\angle 1 + m\angle 2 = m\angle AMB = 180^{\circ}$	(ii)	Adjacent supplementary angles
$\therefore \ m\angle 1 = m\angle 2 = 90^{\circ}$		From (i) and (ii)
i.e., $\overline{OM} \perp \overline{AB}$		

THEOREM 3

9.1(*iii*) Perpendicular from the centre of a circle on a chord bisects it.

Given: $\overline{AB}$ is the chord of a circle with centre at O so that $\overline{OM} \perp$ chord $\overline{AB}$.



To prove: M is the mid point of chord $\overline{AB}$

i.e., $m\overline{AM} = m\overline{BM}$

Construction: Join A and B with centre O.

Proof:

Statements	Reasons
In $\angle rt\ \Delta^s\ OAM \leftrightarrow OBM$	
$m\angle OMA = m\angle OMB = 90^o$	Given
hyp. $m\overline{OA}$ = hyp. $m\overline{OB}$.	Radii of the same circle
$m\overline{OM} = m\overline{OM}$	Common
$\therefore$ $\Delta OAM \cong \Delta OBM$	In $\angle rt\ \Delta^s$ H.S $\cong$ H.S
Hence, $m\overline{AM} = m\overline{BM}$	Corresponding sides of congruent triangles
$\Rightarrow$ $\overline{OM}$ bisects the chord $\overline{AB}$.	

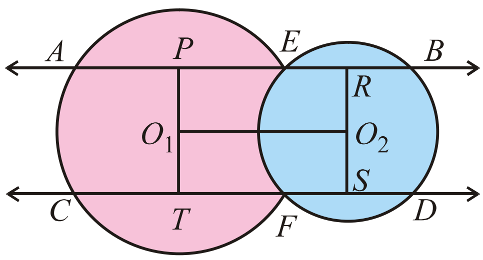
Corollary 1: $\perp$ bisector of the chord of a circle passes through the centre of a circle.

Corollary 2: The diameter of a circle passes through the mid points of two parallel chords of a circle.

Example: Parallel lines passing through the points of intersection of two circles and intercepted by them are equal.

Given: Two circles have centres O_1 and O_2. They intersect each other at points E and F.

Line segment $\overline{AB}$ || Line segment $\overline{CD}$



To Prove: $m\overline{AB} = m\overline{CD}$

Construction: Draw $\overline{PT}$ and $\overline{RS} \perp$ both $\overline{AB}$ and $\overline{CD}$ and join the centres O_1 and O_2.

Proof:

Statements	Reasons
$PRST$ is a rectangle	Construction
$\therefore$ $m\ \overline{PR} = m\ \overline{TS}$ (i)	
Now $m\ \overline{PR} = m\ \overline{PE} + m\ \overline{ER}$	
$= \frac{1}{2} m\ \overline{AE} + \frac{1}{2} m\ \overline{EB}$	By Theorem 3

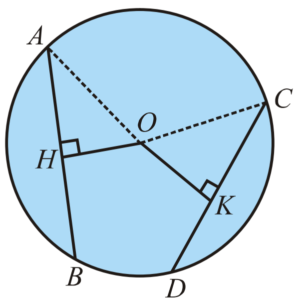
$= \frac{1}{2}(m\,\overline{AE} + m\,\overline{EB})$	
$m\,\overline{PR} = \frac{1}{2}(m\,\overline{AB})$ (ii)	$m\overline{AE} + m\overline{EB} = m\overline{AB}$
Similarly $m\,\overline{TS} = \frac{1}{2}m\overline{CD}$ (iii)	
$\Rightarrow \frac{1}{2}m\overline{AB} = \frac{1}{2}m\overline{CD}$	Using (i), (ii) and (iii)
i.e., $m\overline{AB} = m\overline{CD}$	

EXERCISE 9.1

1. Prove that, the diameters of a circle bisect each other.
2. Two chords of a circle do not pass through the centre. Prove that they cannot bisect each other.
3. If length of the chord $\overline{AB}$ = 8cm. Its distance from the centre is 3 cm, then find the diameter of such circle.
4. Calculate the length of a chord which stands at a distance 5cm from the centre of a circle whose radius is 9cm.

THEOREM 4

9.1(*iv*) If two chords of a circle are congruent then they will be equidistant from the centre.



Given: $\overline{AB}$ and $\overline{CD}$ are two equal chords of a circle with centre at O.

So that $\overline{OH} \perp \overline{AB}$ and $\overline{OK} \perp \overline{CD}$.

To prove: $m\overline{OH} = m\overline{OK}$

Construction: Join O with A and O with C.

So that we have $\angle rt\Delta^s$ OAH and OCK.

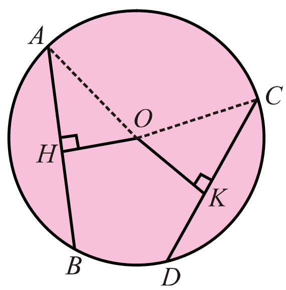
Proof:

Statements	Reasons
$\overline{OH}$ bisects chord $\overline{AB}$	$\overline{OH} \perp \overline{AB}$ By Theorem 3
i.e., $m\overline{AH} = \frac{1}{2}m\overline{AB}$ (i)	
Similarly $\overline{OK}$ bisects chord $\overline{CD}$	$\overline{OK} \perp \overline{CD}$ By Theorem 3
i.e., $m\overline{CK} = \frac{1}{2}m\overline{CD}$ (ii)	

But $m\overline{AB} = m\overline{CD}$	(iii)	Given
Hence $m\overline{AH} = m\overline{CK}$	(iv)	Using (i), (ii) & (iii)
Now in $\angle rt\ \Delta^s\ OAH \leftrightarrow OCK$		Given $\overline{OH} \perp \overline{AB}$ and $\overline{OK} \perp \overline{CD}$
hyp $\overline{OA}$ = hyp $\overline{OC}$		Radii of the same circle
$m\overline{AH} = m\overline{CK}$		Already proved in (iv)
$\therefore$ $\Delta OAH \cong \Delta OCK$		H. S postulate
$\Rightarrow$ $m\overline{OH} = m\overline{OK}$		

THEOREM 5

9.1(v) Two chords of a circle which are equidistant from the centre, are congruent.



Given: $\overline{AB}$ and $\overline{CD}$ are two chords of a circle with centre at O.

$\overline{OH} \perp \overline{AB}$ and $\overline{OK} \perp \overline{CD}$, so that $m\overline{OH} = m\overline{OK}$

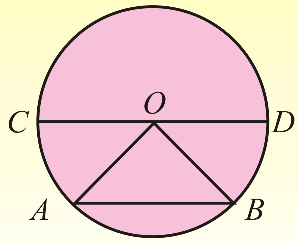
To prove: $m\overline{AB} = m\overline{CD}$

Construction: Join A and C with O. So that we can form $\angle rt\Delta^s\ OAH$ and OCK.

Proof:

Statements		Reasons
In $\angle rt\ \Delta^s\ OAH \leftrightarrow OCK$.		
$\because$ hyp $\overline{OA}$ = hyp $\overline{OC}$		Radii of the same circle.
$m\overline{OH} = m\overline{OK}$		Given
$\therefore$ $\Delta\ OAH \cong \Delta\ OCK$		H.S Postulate
So $m\overline{AH} = m\overline{CK}$	(i)	Corresponding sides of congruent triangles
But $m\overline{AH} = \frac{1}{2} m\overline{AB}$	(ii)	$\overline{OH} \perp$ chord $\overline{AB}$ (Given)
Similarly $m\overline{CK} = \frac{1}{2} m\overline{CD}$	(iii)	$\overline{OK} \perp$ chord $\overline{CD}$ (Given)
Since $m\overline{AH} = m\overline{CK}$		Already proved in (i)
$\therefore$ $\frac{1}{2} m\ \overline{AB} = \frac{1}{2} m\ \overline{CD}$		Using (ii) & (iii)
or $m\ \overline{AB} = m\ \overline{CD}$		

Example: Prove that the largest chord in a circle is the diameter.



Given: $\overline{AB}$ is a chord and $\overline{CD}$ is the diameter of a circle with centre point O.

To prove: If $\overline{AB}$ and $\overline{CD}$ are distinct, then $m\overline{CD} > m\overline{AB}$.

Construction: Join O with A and O with B then form a $\Delta\, OAB$.

Proof: Sum of two sides of a triangle is greater than its third side.

$\therefore$ In $\Delta OAB \Rightarrow m\overline{OA} + m\overline{OB} > m\overline{AB}$ (i)

But $\overline{OA}$ and $\overline{OB}$ are the radii of the same circle with centre O.

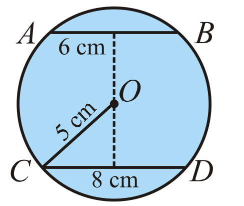
So that $m\overline{OA} + m\overline{OB} = m\overline{CD}$ (ii)

$\Rightarrow$ Diameter $\overline{CD}$ > chord $\overline{AB}$ using (i) & (ii).

Hence, diameter CD is greater than any other chord drawn in the circle.

EXERCISE 9.2

1. Two equal chords of a circle intersect, show that the segments of the one are equal corresponding to the segments of the other.
2. AB is the chord of a circle and the diameter CD is perpendicular bisector of AB. Prove that $m\widehat{AC} = m\widehat{BC}$.
3. As shown in the figure, find the distance between two parallel chords AB and CD.



MISCELLANEOUS EXERCISE 9

Multiple Choice Questions

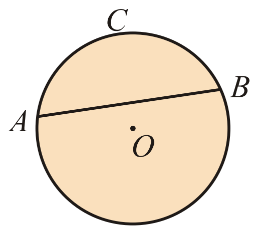
1. Four possible answers are given for the following questions.

Tick (✓) the correct answer.

(*i*) In the circular figure, ADB is called

(*a*) an arc (*b*) a secant

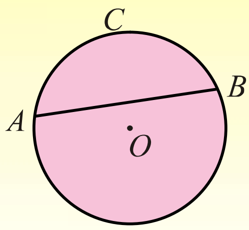
(*c*) a chord (*d*) a diameter



(*ii*) In the circular figure, $\overset{\frown}{ACB}$ is called

(*a*) an arc (*b*) a secant

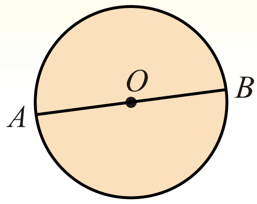
(*c*) a chord (*d*) a diameter



(*iii*) In the circular figure, *AOB* is called

(*a*) an arc (*b*) a secant

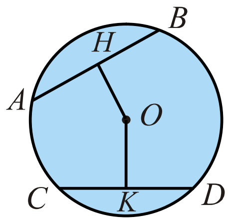
(*c*) a chord (*d*) a diameter



(*iv*) In a circular figure, two chords $\overline{AB}$ and $\overline{CD}$ are equidistant from the centre. They will be

(*a*) parallel (*b*) non congruent

(*c*) congruent (*d*) perpendicular



(*v*) Radii of a circle are

(*a*) all equal (*b*) double of the diameter

(*c*) all unequal (*d*) half of any chord

(*vi*) A chord passing through the centre of a circle is called

(*a*) radius (*b*) diameter

(*c*) circumference (*d*) secant

(*vii*) Right bisector of the chord of a circle always passes through the

(*a*) radius (*b*) circumference

(*c*) centre (*d*) diameter

(*viii*) The circular region bounded by two radii and the corresponding arc is called

(*a*) circumference of a circle (*b*) sector of a circle

(*c*) diameter of a circle (*d*) segment of a circle

(*ix*) The distance of any point of the circle to its centre is called

(*a*) radius (*b*) diameter (*c*) a chord (*d*) an arc

(*x*) Line segment joining any point of the circle to the centre is called

(*a*) circumference (*b*) diameter

(*c*) radial segment (*d*) perimeter

(*xi*) Locus of a point in a plane equidistant from a fixed point is called

(*a*) radius (*b*) circle (*c*) circumference (*d*) diameter

(*xii*) The symbol for a triangle is denoted by

(*a*) $\angle$ (*b*) Δ (*c*) $\perp$ (*d*) $\odot$

(*xiii*) A complete circle is divided into

(*a*) 90 degrees (*b*) 180 degrees (*c*) 270 degrees (*d*) 360 degrees

(*xiv*) Through how many non collinear points, can a circle pass?

(*a*) one (*b*) two (*c*) three (*d*) none

Q.2. Differentiate between the following terms and illustrate them by diagrams.

(*i*) A circle and a circumference.

(*ii*) A chord and the diameter of a circle.

(*iii*) A chord and an arc of a circle.

(*iv*) Minor arc and major arc of a circle.

(*v*) Interior and exterior of a circle.

(*vi*) A sector and a segment of a circle.

SUMMARY

- **$2\pi r$** is the **circumference** of a circle with radius r.
- **πr^2** is the **area** of a circle with radius r.
- Three or more points lying on the same line are called **collinear points** otherwise they are **non-collinear points**.
- The circle passing through the vertices of a triangle is called its **circumcircle** whereas ⊥ bisectors of sides of the triangle provide the centre.
- One and only one circle can pass through three non-collinear points.
- A straight line, drawn from the centre of a circle to bisect a chord (which is not a diameter) is perpendicular to the chord.
- Perpendicular from the centre of a circle on a chord bisects it.
- If two chords of a circle are congruent, then they will be equidistant from the centre.
- Two chords of a circle which are equidistant from the centre are congruent.

Unit-10

TANGENT TO A CIRCLE

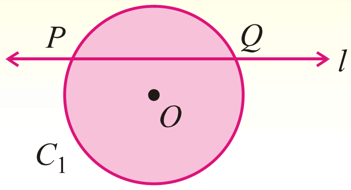
In this unit, students will learn how to

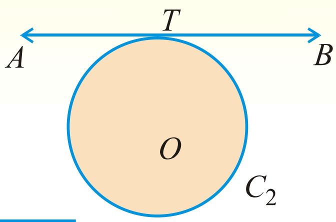
Prove the following theorems alongwith corollaries and apply them to solve appropriate problems.

- *If a line is drawn perpendicular to a radial segment of a circle at its outer end point, it is tangent to the circle at that point.*
- *The tangent to a circle and the radial segment joining the point of contact and the centre are perpendicular to each other.*
- *The two tangents drawn to a circle from a point outside it, are equal in length.*
- *If two circles touch externally or internally, the distance between their centres is respectively equal to the sum or difference of their radii.*

Definition: A secant is a straight line which cuts the circumference of a circle in two distinct points. In the figure l indicates the secant line to the circle C_1.

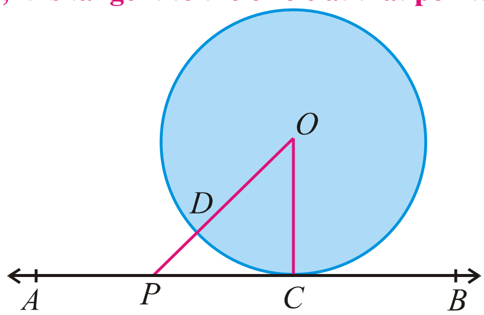
Definition: A tangent to a circle is the straight line which touches the circumference at a single point only. The point of tangency is also known as the point of contact. In the figure $\overleftrightarrow{AB}$ indicates the tangent line to the circle C_2.





THEOREM 1

10.1(i) If a line is drawn perpendicular to a radial segment of a circle at its outer end point, it is tangent to the circle at that point.



Given: A circle with centre O and $\overline{OC}$ is the radial segment. $\overleftrightarrow{AB}$ is perpendicular to $\overline{OC}$ at its outer end C.

To prove: $\overleftrightarrow{AB}$ is a tangent to the circle at C.

Construction: Take any point P other than C on $\overleftrightarrow{AB}$. join O with P.

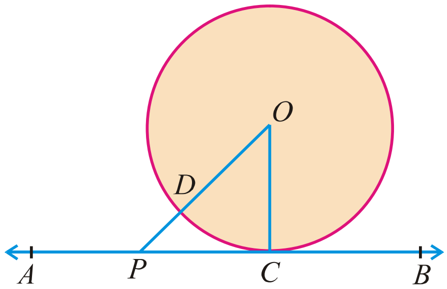
Proof:

Statements	Reasons
In ΔOCP,	
$m\angle OCP = 90^{o}$	$\overleftrightarrow{AB} \perp \overline{OC}$ (given)
and $m\angle OPC < 90^{o}$	Acute angle of right angled triangle.
$m\overline{OP} > m\overline{OC}$	Greater angle has greater side opposite to it.

∴ P is a point outside the circle. Similarly, every point on $\overleftrightarrow{AB}$ except C lies outside the circle. Hence $\overleftrightarrow{AB}$ intersects the circle at one point C only. *i.e.,* $\overleftrightarrow{AB}$ is a tangent to the circle at one point only.	$\overline{OC}$ is the radial segment.

THEOREM 2

10.1(*ii*) The tangent to a circle and the radial segment joining the point of contact and the centre are perpendicular to each other.



Given: In a circle with centre O has radius $\overline{OC}$,
$\overleftrightarrow{AB}$ is the tangent to the circle at point C.

To prove: $\overleftrightarrow{AB}$ and radial segment $\overline{OC}$ are perpendicular to each other.

Construction: Take any point P other than C on the tangent line $\overleftrightarrow{AB}$.
Join O with P so that $\overline{OP}$ meets the circle at D.

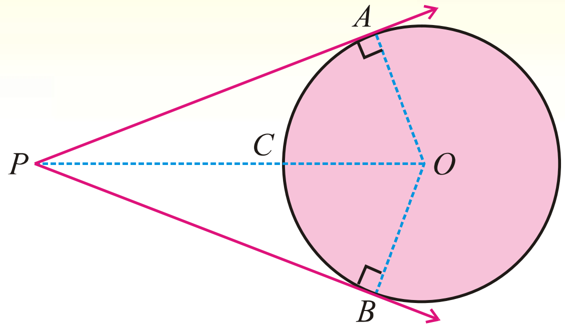
Proof:

Statements	Reasons
$\overleftrightarrow{AB}$ is the tangent to the circle at point C. Whereas	Given
$\overline{OP}$ cuts the circle at D.	Construction
∴ $m\overline{OC} = m\overline{OD}$ (i)	Radii of the same circle
But $m\overline{OD} < m\,\overline{OP}$ (ii)	Point P is outside the circle.
∴ $m\overline{OC} < m\overline{OP}$	Using (i) and (ii)
So radius $\overline{OC}$ is shortest of all lines that can be drawn from O to the tangent line $\overleftrightarrow{AB}$	
Also $\overline{OC} \perp \overleftrightarrow{AB}$	
Hence, radial segment $\overline{OC}$ is perpendicular to the tangent $\overleftrightarrow{AB}$.	

Corollary: There can only be one perpendicular drawn to the radial segment $\overline{OC}$ at the point C of the circle. It follows that one and only one tangent can be drawn to the circle at the given point C on its circumference.

THEOREM 3

10.1(*iii*) Two tangents drawn to a circle from a point outside it, are equal in length.



Given: Two tangents $\overrightarrow{PA}$ and $\overrightarrow{PB}$ are drawn from an external point P to the circle with centre O.

To prove: $m\overline{PA} = m\overline{PB}$

Construction: Join O with A , B and P, so that we form $\angle rt\Delta^s$ OAP and OBP.

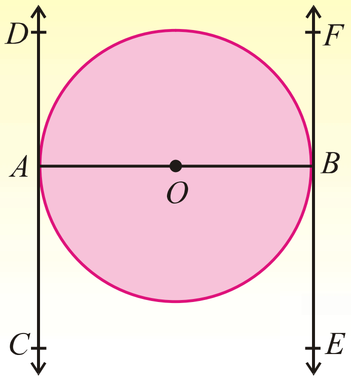
Proof:

Statements	Reasons
In $\angle rt\ \Delta^s\ OAP \leftrightarrow OBP$	
$m\angle OAP = m\angle OBP = 90^o$	Radii $\perp$ to the tangents $\overrightarrow{PA}$ and $\overrightarrow{PB}$
hyp. $\overline{OP}$ = hyp. $\overline{OP}$	Common
$m\ \overline{OA} = m\ \overline{OB}$	Radii of the same circle.
$\therefore\ \Delta\ OAP \cong \Delta\ OBP$	In $\angle rt\ \Delta^s$ H.S $\cong$ H.S
Hence, $m\overline{PA} = m\overline{PB}$	

Note: The length of a tangent to a circle is measured from the given point to the point of contact.

Corollary: If O is the centre of a circle and two tangents $\overrightarrow{PA}$ and $\overrightarrow{PB}$ are drawn from an external point P then $\overline{OP}$ is the right bisector of the chord of contact $\overline{AB}$.

Example 1: $\overline{AB}$ is a diameter of a given circle with centre O. Tangents are drawn at the end points A and B. Show that the two tangents are parallel.



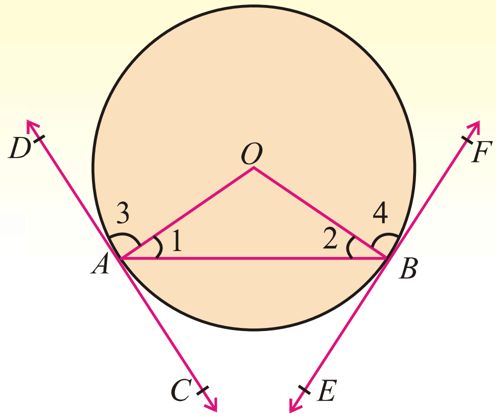
Given: $\overline{AB}$ is a diameter of a given circle with centre O.
CD is the tangent to the circle at point A
and EF is an other tangent at point B.

To prove: $\overline{CD} \parallel \overline{EF}$

Proof:

Statements	Reasons
$\overline{AB}$ is the diameter of a circle with centre O.	Given
$\therefore$ $\overline{OA}$ and $\overline{OB}$ are radii of the same circle.	
Moreover $\overleftrightarrow{CD}$ is a tangent to the circle at A.	Given
$\therefore$ $\overline{OA} \perp \overleftrightarrow{CD}$	By Theorem 1
$\overline{AB} \perp \overleftrightarrow{CD}$ (i)	
Similarly $\overleftrightarrow{EF}$ is tangent at point B.	Given
So $\overline{OB} \perp \overleftrightarrow{EF}$	By Theorem 1
$\Rightarrow$ $\overrightarrow{AB} \perp \overrightarrow{EF}$ (ii)	
Hence $\overleftrightarrow{CD} \parallel \overleftrightarrow{EF}$	From (i) and (ii) ($\overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ are perpendicular to $\overline{AB}$)

Example 2: In a circle, the tangents drawn at the ends of a chord, make equal angles with that chord.



Given: $\overline{AB}$ is the chord of a circle with centre O.

$\overleftrightarrow{CAD}$ is the tangent at point A and $\overleftrightarrow{EBF}$ is an other tangent at point B.

To prove: $m\angle BAD = m\angle ABF$

Construction: Join O with A and B so that we form a ΔOAB then write $\angle 1$, $\angle 2$, $\angle 3$ and $\angle 4$ as shown in the figure.

Proof:

Statements		Reasons
In $\Delta\, OAB$		Construction
$\because\ m\,\overline{OA} = m\overline{OB}$		Radii of the same circle.
$\therefore\ m\angle 1 = m\,\angle 2$	(i)	Angles opp. to equal sides of ΔOAB
Also $\because\ \overline{OA} \perp \overleftrightarrow{CD}$		Radius is $\perp$ to the tangent line
$\therefore\ m\angle 3 = m\angle OAD = 90^{\circ}$	(ii)	
Similarly $\overline{OB} \perp \overleftrightarrow{EF}$		Radius is $\perp$ to the tangent
$\therefore\ m\angle 4 = m\angle OBF = 90^{\circ}$	(iii)	
Hence $m\,\angle 3 = m\,\angle 4$	(iv)	Using (ii) and (iii)
$\Rightarrow\ m\angle 1 + m\angle 3 = m\angle 2 + m\angle 4$		Adding (i) and (iv)
i.e., $m\angle BAD = m\angle ABF$		

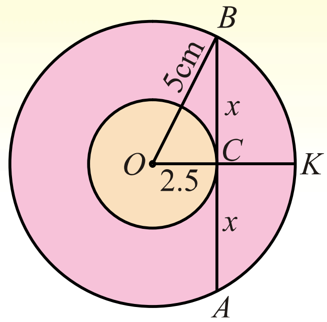
EXERCISE 10.1

1. Prove that the tangents drawn at the ends of a diameter in a given circle must be parallel.

2. The diameters of two concentric circles are 10cm and 5cm respectively. Look for the length of any chord of the outer circle which touches the inner one.

   (Hint) From the figure

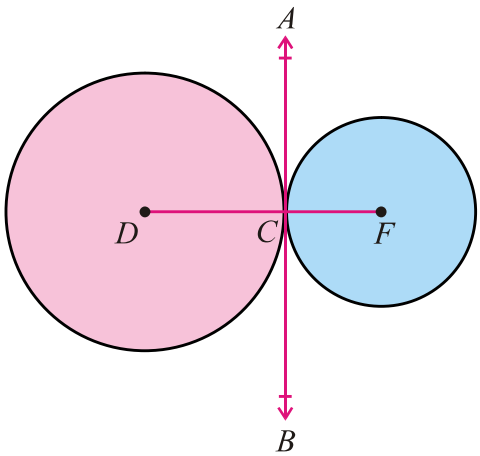
   $m\overline{AB} = 2x = 2\sqrt{25 - 6.25} = 2\sqrt{18.75} \simeq 8.7\text{cm}$



3. $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are the common tangents drawn to the pair of circles. If A and C are the points of tangency of 1st circle where B and D are the points of tangency of 2nd circle, then prove that $\overline{AC} \parallel \overline{BD}$.

THEOREM 4(A)

10.1(iv) If two circles touch externally then the distance between their centres is equal to the sum of their radii.



Given: Two circles with centres D and F respectively touch each other externally at point C. So that $\overline{CD}$ and $\overline{CF}$ are respectively the radii of the two circles.

To prove: Point C lies on the join of centres D and F and $m\overline{DF} = m\overline{DC} + m\overline{CF}$

Construction: Draw $\overleftrightarrow{ACB}$ as a common tangent to the pair of circles at C.

Proof:

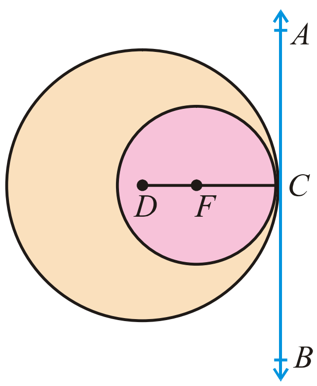
Statements	Reasons
Both circles touch externally at C whereas $\overline{CD}$ is radial segment and $\overleftrightarrow{ACB}$ is the common tangent.	
$\therefore \quad m\angle ACD = 90^o \quad$ (i)	Radial segment $\overline{CD} \perp$ the tangent line $\overline{AB}$
Similarly $\overline{CF}$ is radial segment and $\overleftrightarrow{ACB}$ is the common tangent	
$\therefore \quad m\angle ACF = 90^o \quad$ (ii)	Radial segment $\overline{CF} \perp$ the tangent line $\overline{AB}$
$m\angle ACD + m\angle ACF = 90^o + 90^o$	Adding (i) and (ii)
$m\angle DCF = 180^o \quad$ (iii)	Sum of supplementary adjacent angles
Hence DCF is a line segment with point C between D and F	
and $\quad m\overline{DF} = m\overline{DC} + m\overline{CF}$	

EXERCISE 10.2

1. $\overline{AB}$ and $\overline{CD}$ are two equal chords in a circle with centre O. H and K are respectively the mid points of the chords. Prove that $\overline{HK}$ makes equal angles with $\overline{AB}$ and $\overline{CD}$.
2. The radius of a circle is 2.5 cm. $\overline{AB}$ and $\overline{CD}$ are two chords 3.9cm apart. If $m\overline{AB}$ = 1.4 cm, then measure the other chord.
3. The radii of two intersecting circles are 10cm and 8cm. If the length of their common chord is 6cm then find the distance between the centres.
4. Show that greatest chord in a circle is its diameter.

THEOREM 4(B)

10.1(v) If two circles touch each other internally, then the point of contact lies on the line segment through their centres and distance between their centres is equal to the difference of their radii.



Given: Two circles with centres D and F touch each other internally at point C.

So that $\overline{CD}$ and $\overline{CF}$ are the radii of two circles.

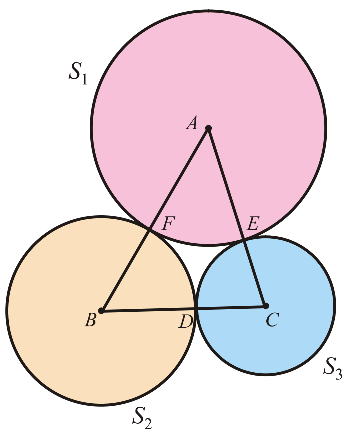
To prove: Point C lies on the join of centres D and F extended and $m\overline{DF} = m\overline{DC} - m\overline{CF}$

Construction: Draw $\overleftrightarrow{ACB}$ as the common tangent to the pair of circles at C.

Proof:

Statements	Reasons
Both circles touch internally at C whereas $\overleftrightarrow{ACB}$ is the common tangent and $\overline{CD}$ is the radial segment of the first circle. $\therefore \quad m\angle ACD = 90^{o} \quad$ (i)	Radial segment $\overline{CD} \perp$ the tangent line $\overline{AB}$
Similarly $\overleftrightarrow{ACB}$ is the common tangent and $\overline{CF}$ is the radial segment of the second circle. $\therefore \quad m\angle ACF = 90^{o} \quad$ (ii)	Radial segment $\overline{CF} \perp$ the tangent line $\overline{AB}$.
$\Rightarrow \quad m\angle ACD = m\angle ACF = 90^{o}$ Where $\angle ACD$ and $\angle ACF$ coincide each other with point F between D and C. Hence $\quad m\overline{DC} = m\overline{DF} + m\overline{FC}$ *i.e.,* $\quad m\overline{DC} - m\overline{FC} = m\overline{DF}$ or $\quad m\overline{DF} = m\overline{DC} - m\overline{FC}$	Using (i) and (ii)

Example 1: Three circles touch in pairs externally. Prove that the perimeter of a triangle formed by joining centres is equal to the sum of their diameters.



Given: Three circles have centres A, B and C their radii are r_1, r_2 and r_3 respectively. They touch in pairs externally at D, E and F. So that ΔABC is formed by joining the centres of these circles.

To prove:

Perimeter of $\Delta ABC = 2r_1 + 2r_2 + 2r_3 = d_1 + d_2 + d_3$

= Sum of the diameters of these circles.

Proof:

Statements	Reasons
Three circles with centres A, B and C touch in pairs externally at the points, D, E and F.	Given
$\therefore \quad m\overline{AB} = m\overline{AF} + m\overline{FB}$ (i)	
$m\overline{BC} = m\overline{BD} + m\overline{DC}$ (ii)	
and $\quad m\overline{CA} = m\overline{CE} + m\overline{EA}$ (iii)	
$m\overline{AB} + m\overline{BC} + m\overline{CA} = m\overline{AF} + m\overline{FB} + m\overline{BD}$ $+ m\overline{DC} + m\overline{CE} + m\overline{EA}$ $= (m\overline{AF} + m\overline{EA}) + (m\overline{FB} + m\overline{BD})$ $+ (m\overline{CD} + m\overline{CE})$	Adding (i), (ii) and (iii)
Perimeter of $\Delta ABC = 2r_1 + 2r_2 + 2r_3$ $= d_1 + d_2 + d_3$ = Sum of diameters of the circles.	$d_1 = 2r_1$, $d_2 = 2r_2$ and $d_3 = 2r_3$ are diameters of the circles.

EXERCISE 10.3

1. Two circles with radii 5cm and 4cm touch each other externally. Draw another circle with radius 2.5cm touching the first pair, externally.
2. If the distance between the centres of two circles is the sum or the difference of their radii they will touch each other.

MISCELLANEOUS EXERCISE 10

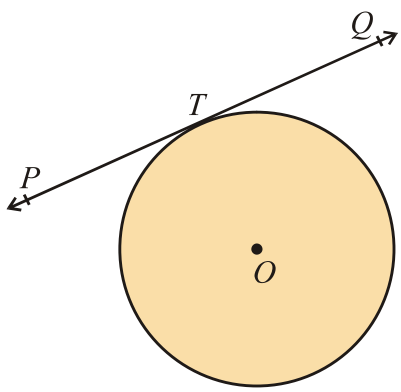
Multiple Choice Questions

1. Four possible answers are given for the following questions. Tick (✓) the correct answer.

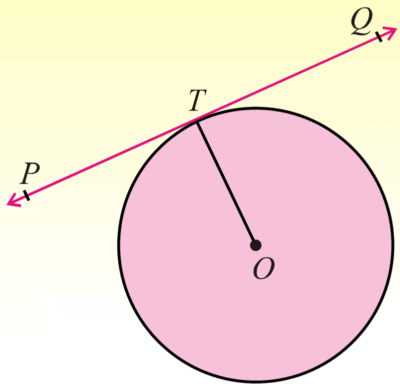
(i) In the adjacent figure of the circle, the line $\overleftrightarrow{PTQ}$ is named as

(a) an arc (b) a chord

(c) a tangent (d) a secant



(ii) In a circle with centre O, if $\overline{OT}$ is the radial segment and $\overleftrightarrow{PTQ}$ is the tangent line, then

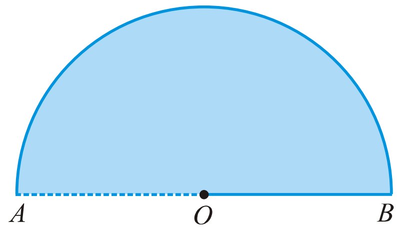


(*a*) $\overline{OT} \perp \overleftrightarrow{PQ}$ (*b*) $\overline{OT} \not\perp \overleftrightarrow{PQ}$

(*c*) $\overline{OT} \parallel \overleftrightarrow{PQ}$

(*d*) $\overline{OT}$ is right bisector of $\overleftrightarrow{PQ}$

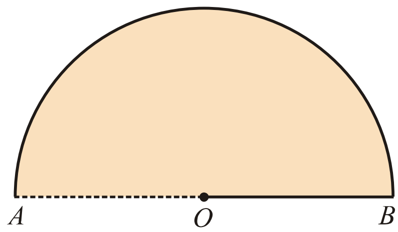
(iii) In the adjacent figure, find semicircular area if $\pi \simeq 3.1416$ and $m\overline{OA} = 20$cm.



(*a*) 62.83sq cm (*b*) 314.16sq cm

(*c*) 436.20sq cm (*d*) 628.32sq cm

(iv) In the adjacent figure, find half the perimeter of circle with centre O if $\pi \simeq 3.1416$ and $m\overline{OA} = 20$cm.



(*a*) 31.42 cm (*b*) 62.832 cm

(*c*) 125.65 cm (*d*) 188.50 cm

(v) A line which has two points in common with a circle is called:

(*a*) sine of a circle (*b*) cosine of a circle

(*c*) tangent of a circle (*d*) secant of a circle

(vi) A line which has only one point in common with a circle is called:

(*a*) sine of a circle (*b*) cosine of a circle

(*c*) tangent of a circle (*d*) secant of a circle

(vii) Two tangents drawn to a circle from a point outside it are of .... in length.

(*a*) half (*b*) equal (*c*) double (*d*) triple

(viii) A circle has only one:

(*a*) secant (*b*) chord (*c*) diameter (*d*) centre

(ix) A tangent line intersects the circle at:

(*a*) three points (*b*) two points (*c*) single point (*d*) no point at all

(x) Tangents drawn at the ends of diameter of a circle are .... to each other.

(*a*) parallel (*b*) non-parallel (*c*) collinear (*d*) perpendicular

(xi) The distance between the centres of two congruent touching circles externally is:

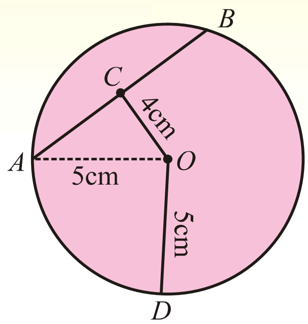
(*a*) of zero length (*b*) the radius of each circle

(*c*) the diameter of each circle (*d*) twice the diameter of each circle

(xii) In the adjacent circular figure with centre O and radius 5cm, the length of the chord intercepted at 4cm away from the centre of this circle is:

(*a*) 4cm (*b*) 6cm

(*c*) 7cm (*d*) 9cm

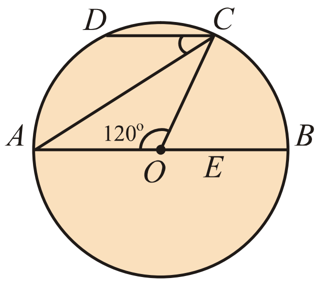


(xiii) In the adjoining figure, there is a circle with centre O.

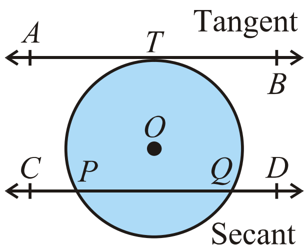
If $\overline{DC}$ // diameter $\overline{AB}$ and $m\angle AOC = 120^{o}$, then $m\angle ACD$ is:

(*a*) 40^{o} (*b*) 30^{o}

(*c*) 50^{o} (*d*) 60^{o}



SUMMARY



- A **secant** is a straight line which cuts the circumference of a circle in two distinct points. In the figure, the secant $\overleftrightarrow{CD}$ cuts the circle at two distinct points P and Q.
- A **tangent** to a circle is the straight line which touches the circumference at one point only. The point of tangency is also known as the point of contact in the figure. $\overleftrightarrow{AB}$ is the tangent line to the circle at the point T.

- The **length of a tangent** to a circle is measured from the given point to the point of contact.
- If a line is drawn perpendicular to a radial segment of a circle at its outer end point, it is tangent to the circle at that point.
- The tangent to a circle and the radial segment joining the point of contact and the centre are perpendicular to each other.
- The two tangents drawn to a circle from a point outside it, are equal in length.
- If two circles touch externally or internally, the distance between their centres is respectively equal to the sum or difference of their radii.

Unit-11

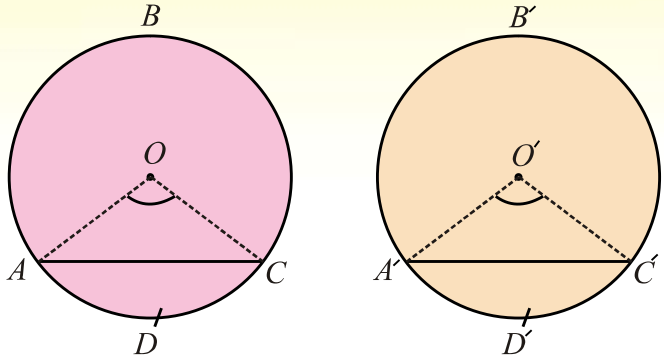
CHORDS AND ARCS

In this unit, students will learn

- ***If two arcs of a circle (or of congruent circles) are congruent, then the corresponding chords are equal.***
- ***If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major or semi-circular) are congruent.***
- ***Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).***
- ***If the angles subtended by two chords of a circle (or congruent circles) at the centre (corresponding centres) are equal, the chords are equal.***

THEOREM 1

11.1(*i*) If two arcs of a circle (or of congruent circles) are congruent then the corresponding chords are equal.



Given: $ABCD$ and $A'B'C'D'$ are two congruent circles with centres O and O' respectively. So that $m\overset{\frown}{ADC} = m\overset{\frown}{A'D'C'}$

To prove: $m\overline{AC} = m\overline{A'C'}$

Construction: Join O with A, O with C, O' with A' and O' with C'. So that we can form Δ^s OAC and $O'A'C'$.

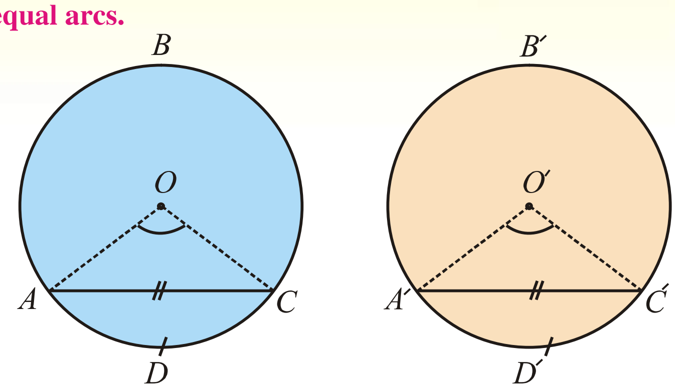
Proof:

Statements	Reasons
In two equal circles $ABCD$ and $A'B'C'D'$ with centres O and O' respectively.	Given
$m\overset{\frown}{ADC} = m\overset{\frown}{A'D'C'}$	Given
$\therefore$ $m\angle AOC = m\angle A'O'C'$	Central angles subtended by equal arcs of the equal circles.
Now in $\Delta AOC \leftrightarrow \Delta A'O'C'$	
$m\overline{OA} = m\overline{O'A'}$	Radii of equal circles
$m\angle AOC = m\angle A'O'C'$	Already Proved
$m\overline{OC} = m\overline{O'C'}$	Radii of equal circles
$\therefore$ $\Delta AOC \cong \Delta A'O'C'$	S.A.S $\cong$ S.A.S
and in particular $m\overline{AC} = m\overline{A'C'}$	
Similarly we can prove the theorem in the same circle.	

THEOREM 2

Converse of Theorem 1

11.1(ii) If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major or semi-circular) are congruent. In equal circles or in the same circle, if two chords are equal, they cut off equal arcs.



Given: $ABCD$ and $A'B'C'D'$ are two congruent circles with centres O and O' respectively.

So that chord $m\overline{AC} = m\overline{A'C'}$.

To prove: $m\widehat{ADC} = m\widehat{A'D'C'}$

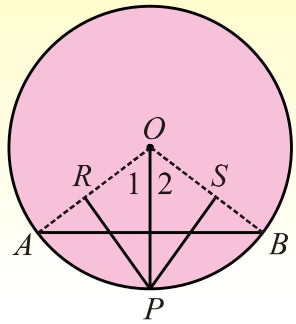
Construction: Join O with A, O with C, O' with A' and O' with C'.

Proof:

Statements	Reasons
In $\Delta AOC \leftrightarrow \Delta A'O'C'$	
$m\overline{OA} = m\overline{O'A'}$	Radii of equal circles
$m\overline{OC} = m\overline{O'C'}$	Radii of equal circles
$m\,\overline{AC} = m\,\overline{A'C'}$	Given
$\therefore$ $\Delta AOC \cong \Delta A'O'C'$	S.S.S $\cong$ S.S.S
$\Rightarrow$ $m\angle AOC = m\angle A'O'C'$	
Hence $m\widehat{ADC} = m\widehat{A'D'C'}$	Arcs corresponding to equal central angles.

Example 1: A point P on the circumference is equidistant from the radii $\overline{OA}$ and OB.

Prove that $m\,\widehat{AP} = m\,\widehat{BP}$



Given: AB is the chord of a circle with centre O. Point P on the circumference of the circle is equidistant from the radii $\overline{OA}$ and $\overline{OB}$ so that $m\overline{PR} = m\overline{PS}$.

To prove: $m\,\widehat{AP} = m\,\widehat{BP}$

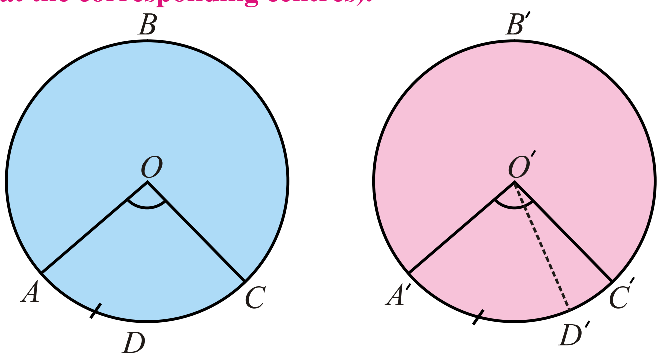
Construction: Join O with P. Write $\angle 1$ and $\angle 2$ as shown in the figure.

Proof:

Statements	Reasons
In $\angle$rt ΔOPR and $\angle rt\ \Delta OPS$	
$m\,\overline{OP} = m\,\overline{OP}$	Common
$m\,\overline{PR} = m\,\overline{PS}$	Point P is equidistance from radii (Given)
$\therefore$ $\Delta OPR \cong \Delta\ OPS$	(In $\angle rt\Delta^s$ H.S $\cong$ H.S)
So $m\,\angle 1 = m\,\angle 2$	Central angles of a circle
$\Rightarrow$ Chord $AP \cong$ Chord BP	
Hence $\text{m}\,\widehat{AP} = \text{m}\,\widehat{BP}$	Arcs corresponding to equal chords in a circle.

THEOREM 3

11.1(*iii*) Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).



Given: ABC and $A'B'C'$ are two congruent circles with centres O and O' respectively.

So that $\overline{AC} = \overline{A'C'}$

To prove: $\angle AOC \cong \angle A'O'C'$

Construction: Let if possible $m\angle AOC \neq m\angle A'O'C'$ then consider $\angle AOC \cong \angle A'O'D'$

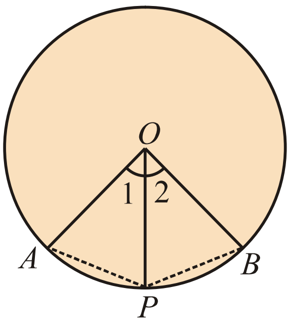
Proof:

Statements			Reasons
	$\angle AOC \cong \angle A'O'D'$		Construction
$\therefore$	$\overset{\frown}{AC} \cong \overset{\frown}{A'D'}$	(i)	Arcs subtended by equal Central angles in congruent circles
	$m\overline{AC} = m\overline{A'D'}$	(ii)	Using Theorem 1
But	$m\overline{AC} = m\overline{A'C'}$	(iii)	Given
$\therefore$	$m\overline{A'C'} = m\overline{A'D'}$		Using (ii) and (iii)
Which is only possible, if C' concides with D'.			
Hence	$m\angle A'O'C' = m\angle A'O'D'$	(iv)	
But	$m\angle AOC = m\angle A'O'D'$	(v)	Construction
$\Rightarrow$	$m\angle AOC = m\angle A'O'C'$		Using (iv) and (v)

Corollary 1. In congruent circles or in the same circle, if central angles are equal then corresponding sectors are equal.

Corollary 2. In congruent circles or in the same circle, unequal arcs will subtend unequal central angles.

Example 1: The internal bisector of a central angle in a circle bisects an arc on which it stands.



Solution: In a circle with centre O. $\overline{OP}$ is an internal bisector of central angle AOB.

To prove: $\overset{\frown}{AP} \cong \overset{\frown}{BP}$

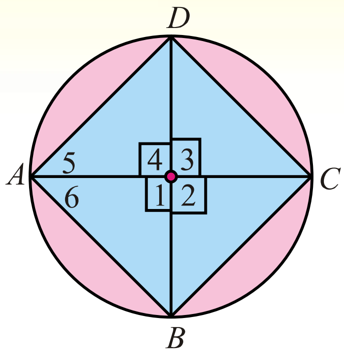
Construction: Draw $\overline{AP}$ and $\overline{BP}$, then write $\angle 1$ and $\angle 2$ as shown in the figure.

Proof:

Statements		Reasons
	In $\Delta OAP \leftrightarrow \Delta OBP$	
	$m\,\overline{OA} = m\,\overline{OB}$	Radii of the same circle
	$m\angle 1 = m\angle 2$	Given $\overline{OP}$ as an angle bisector of $\angle AOB$
and	$m\,\overline{OP} = m\,\overline{OP}$	Common
		(S.A.S $\cong$ S.A.S)

$\Delta OAP \cong \Delta OBP$	
Hence $\overline{AP} \cong \overline{BP}$	
$\Rightarrow \quad \overset{\frown}{AP} \cong \overset{\frown}{BP}$	Arcs corresponding to equal chords in a circle.

Example 2: In a circle if any pair of diameters are $\perp$ to each other then the lines joining its ends in order, form a square.



Given: $\overline{AC}$ and $\overline{BD}$ are two perpendicular diameters of a circle with centre O. So $ABCD$ is a quadrilateral.

To prove: $ABCD$ is a square

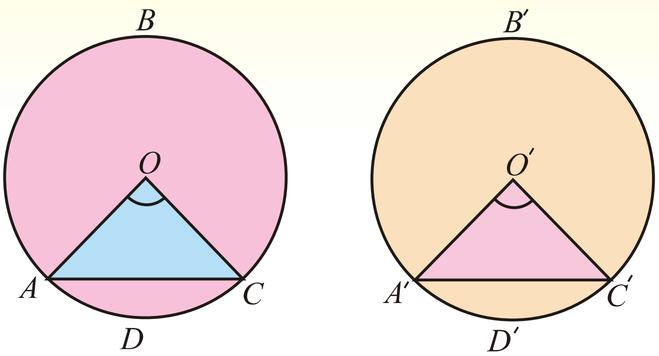
Construction: Write $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$, $\angle 5$ and $\angle 6$ as shown in the figure.

Proof:

Statements	Reasons
$\overline{AC}$ and $\overline{BD}$ are two $\perp$ diameters of a circle with centre O	Given
$\therefore \quad m\angle 1 = m\angle 2 = m\angle 3 = m\angle 4 = 90^o$	Pair of diameters are $\perp$ to each other.
$\therefore \quad m\overset{\frown}{AB} = m\overset{\frown}{BC} = m\overset{\frown}{CD} = m\overset{\frown}{DA}$	Arcs opposite to the equal central angles in a circle.
$\Rightarrow \quad m\,\overline{AB} = m\,\overline{BC} = m\overline{CD} = m\overline{DA}$ (i)	Chords corresponding to equal arcs.
Moreover $m\angle A = m\angle 5 + m\angle 6$	
$= 45^o + 45^o = 90^o$ (ii)	
Similarly $m\angle B = m\angle C = m\angle D = 90^o$ (iii)	
Hence $ABCD$ is a square	Using (i), (ii) and (iii).

THEOREM 4

11.1(*iv*) If the angles subtended by two chords of a circle (or congruent circles) at the centre (corresponding centres) are equal, the chords are equal.



Given: $ABCD$ and $A'B'C'D'$ are two congruent circles with centres O and O' respectively. $\overline{AC}$ and $\overline{A'C'}$ are chords of circles $ABCD$ and $A'B'C'D'$ respectively and $m\angle AOC = m\angle A'O'C'$

To prove: $m\overline{AC} = m\overline{A'C'}$

Proof:

Statements	Reasons
In $\Delta OAC \longleftrightarrow \Delta O'A'C'$	
$m\overline{OA} = m\overline{O'A'}$	Radii of congruent circles
$m\angle AOC = m\angle A'O'C'$	Given
$m\overline{OC} = m\overline{O'C'}$	Radii of congruent circles
$\therefore$ $\Delta OAC \cong \Delta O'A'C'$	$SAS \cong SAS$
Hence $m\overline{AC} = m\overline{A'C'}$	

EXERCISE 11.1

1. In a circle two equal diameters $\overline{AB}$ and $\overline{CD}$ intersect each other.
   Prove that $m\overline{AD} = m\overline{BC}$.
2. In a circle prove that the arcs between two parallel and equal chords are equal.
3. Give a geometric proof that a pair of bisecting chords are the diameters of a circle.
4. If C is the mid point of an arc ACB in a circle with centre O. Show that line segment OC bisects the chord AB.

MISCELLANEOUS EXERCISE 11

1. Multiple Choice Questions

Four possible answers are given for the following questions.
Tick (✓) the correct answer.

(*i*) A 4 cm long chord subtends a central angle of 60°. The radial segment of this circle is:
(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 4

(*ii*) The length of a chord and the radial segment of a circle are congruent, the central angle made by the chord will be:
(*a*) 30° (*b*) 45° (*c*) 60° (*d*) 75°

(*iii*) Out of two congruent arcs of a circle, if one arc makes a central angle of 30° then the other arc will subtend the central angle of:
(*a*) 15° (*b*) 30° (*c*) 45° (*d*) 60°

(*iv*) An arc subtends a central angle of 40° then the corresponding chord will subtend a central angle of:
(*a*) 20° (*b*) 40° (*c*) 60° (*d*) 80°

(*v*) A pair of chords of a circle subtending two congruent central angles is:
(*a*) congruent (*b*) incongruent (*c*) over lapping (*d*) parallel

(*vi*) If an arc of a circle subtends a central angle of 60°, then the corresponding chord of the arc will make the central angle of:
(*a*) 20° (*b*) 40° (*c*) 60° (*d*) 80°

(*vii*) The semi circumference and the diameter of a circle both subtend a central angle of:
(*a*) 90° (*b*) 180° (*c*) 270° (*d*) 360°

(*viii*) The chord length of a circle subtending a central angle of 180° is always:
(*a*) less than radial segment (*b*) equal to the radial segment
(*c*) double of the radial segment (*d*) none of these

(*ix*) If a chord of a circle subtends a central angle of 60°, then the length of the chord and the radial segment are:
(*a*) congruent (*b*) incongruent (*c*) parallel (*d*) perpendicular

(*x*) The arcs opposite to incongruent central angles of a circle arc always:
(*a*) congruent (*b*) incongruent (*c*) parallel (*d*) perpendicular

SUMMARY

- The boundary traced by a moving point in a circle is called its circumference whereas any portion of the circumference will be known as an arc of the circle.
- The straight line joining any two points of the circumference is called a chord of the circle.
- The portion of a circle bounded by an arc and a chord is known as the segment of a circle.
- The circular region bounded by an arc of a circle and its two corresponding radial segments is called a sector of the circle.
- A straight line, drawn from the centre of a circle bisecting a chord is perpendicular to the chord and conversely perpendicular drawn from the centre of a circle on a chord, bisects it.
- If two arcs of a circle (or of congruent circles) are congruent, then the corresponding chords are equal.
- If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major or semi-circular) are congruent.
- Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).
- If the angles subtended by two chords of a circle (or congruent circles) at the centre (corresponding centres) are equal, the chords are equal.

Unit-12

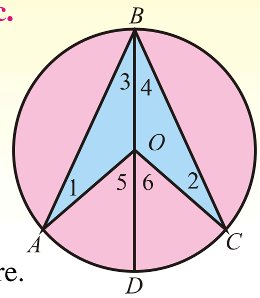
ANGLE IN A SEGMENT OF A CIRCLE

In this unit, students will learn

- *The measure of a central angle of a minor arc of a circle, is double that of the angle subtended by the corresponding major arc.*
- *Any two angles in the same segment of a circle are equal.*
- *The angle*
  - *in a semi-circle is a right angle,*
  - *in a segment greater than a semi circle is less than a right angle,*
  - *in a segment less than a semi-circle is greater than a right angle.*
- *The opposite angles of any quadrilateral inscribed in a circle are supplementary.*

THEOREM 1

12.1(*i*) The measure of a central angle of a minor arc of a circle, is double that of the angle subtended by the corresponding major arc.



Given: $\overset{\frown}{AC}$ is an arc of a circle with centre O.

Whereas $\angle AOC$ is the central angle

and $\angle ABC$ is circum angle.

To prove: $m\angle AOC = 2m\angle ABC$

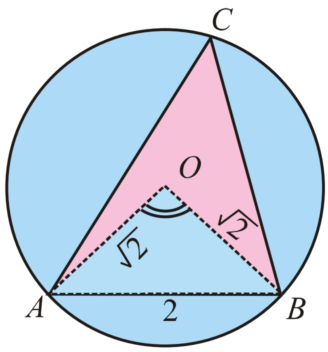
Construction: Join B with O and produce it to meet the circle at D.

Write angles $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$ $\angle 5$ and $\angle 6$ as shown in the figure.

Proof:

Statements	Reasons
As $m\angle 1 = m\angle 3$ (i)	Angles opposite to equal sides in ΔOAB
and $m\angle 2 = m\angle 4$ (ii)	Angles opposite to equal sides in ΔOBC
Now $m\angle 5 = m\angle 1 + m\angle 3$ (iii)	External angle is the sum of internal opposite angles.
Similarly $m\angle 6 = m\angle 2 + m\angle 4$ (iv)	
Again $m\angle 5 = m\angle 3 + m\angle 3 = 2m\angle 3$ (v)	Using (i) and (iii)
and $m\angle 6 = m\angle 4 + m\angle 4 = 2m\angle 4$ (vi)	Using (ii) and (iv)
Then from figure	
$\Rightarrow$ $m\angle 5 + m\angle 6 = 2m\angle 3 + 2m\angle 4$	Adding (v) and (vi)
$\Rightarrow$ $m\angle AOC = 2(m\angle 3 + m\angle 4) = 2m\angle ABC$	

Example: The radius of a circle is $\sqrt{2}$ cm. A chord 2 cm in length divides the circle into two segments. Prove that the angle of larger segment is 45^{o}.



Given: In a circle with centre O and radius $m\overline{OA} = m\overline{OB} = \sqrt{2}$ cm,

The length of chord $\overline{AB}$ = 2 cm divides the circle into two segments with ACB as larger one.

To prove: $m\angle ACB = 45^o$

Construction: Join O with A and O with B.

Proof:

Statements	Reasons
In $\Delta\ OAB$	$m\overline{OA} = m\overline{OB} = \sqrt{2}$ cm
$(OA)^2 + (OB)^2 = (\sqrt{2})^2 + (\sqrt{2})^2$	
$= 2 + 2 = 4$	
$= (2)^2 = (AB)^2$	Given $m\ \overline{AB} = 2$ cm
$\therefore$ ΔAOB is right angled triangle	
With $m\ \angle AOB = 90^o$	Which being a central angle standing on an arc AB
Then $m\angle ACB = \frac{1}{2} m\angle AOB$	By theorem 1
$= \frac{1}{2}(90^o) = 45^o$	Circum angle is half of the central angle.

THEOREM 2

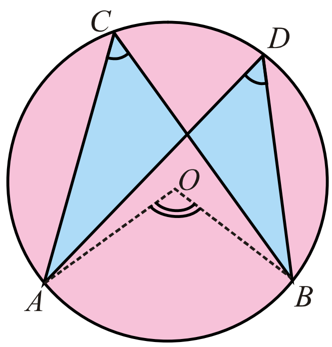
12.1(*ii*) Any two angles in the same segment of a circle are equal.

Given: $\angle\ ACB$ and $\angle\ ADB$ are the circum angles in the same segment of a circle with centre O.

To prove: $m\ \angle ACB = m\ \angle ADB$

Construction: Join O with A and O with B.

So that $\angle AOB$ is the central angle.



Proof:

Statements	Reasons
Standing on the same arc AB of a circle.	
$\angle AOB$ is the central angle whereas	Construction
$\angle ACB$ and $\angle ADB$ are circum angles	Given
$\therefore$ $m\angle AOB = 2m\angle ACB$ (i)	By theorem 1
and $m\angle AOB = 2m\angle ADB$ (ii)	By theorem 1
$\Rightarrow$ $2m\angle ACB = 2m\angle ADB$	Using (i) and (ii)
Hence, $m\angle ACB = m\angle ADB$	

THEOREM 3

12.1(iii) The angle

- **in a semi-circle is a right angle,**
- **in a segment greater than a semi circle is less than a right angle,**
- **in a segment less than a semi-circle is greater than a right angle.**

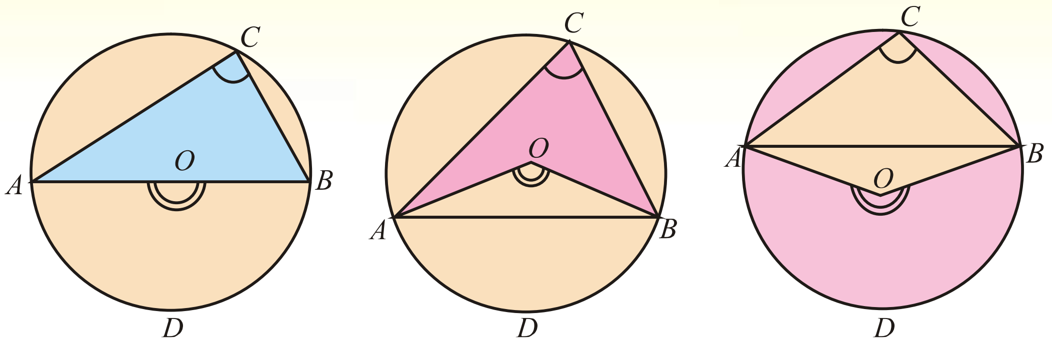


Fig. I **Fig. II** **Fig. III**

Given: $\overline{AB}$ is the chord corresponding to an arc ADB
Whereas $\angle AOB$ is a central angle and $\angle ACB$ is
a circum angle of a circle with centre O.

To prove: In fig (I) If sector ACB is a semi circle
then $m\angle ACB = 1\angle rt$

In fig (II) If sector ACB is greater than a semi circle
then $m\angle ACB < 1\angle rt$

In fig (III) If sector ACB is less than a semi circle
then $m\angle ACB > 1\angle rt$

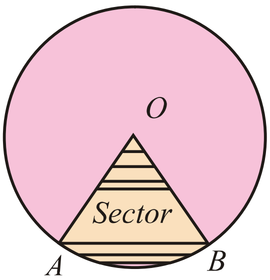
Proof:

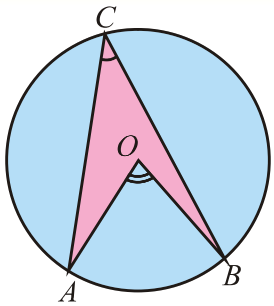
Statements	Reasons
In each figure, $\overline{AB}$ is the chord of a circle with centre O.	Given
$\angle AOB$ is the central angle standing on an arc ADB. Whereas $\angle ACB$ is the circum angle	Given
Such that $m\angle AOB = 2m\angle ACB$ (i)	By theorem 1
Now in fig (I) $m\angle AOB = 180^o$	A straight angle
$\therefore$ $m\angle AOB = 2\angle rt$ (ii)	
$\Rightarrow$ $m\angle ACB = 1\angle rt$	Using (i) and (ii)
In fig (II) $m\angle AOB < 180^o$	
$\therefore$ $m\angle AOB < 2\angle rt$ (iii)	

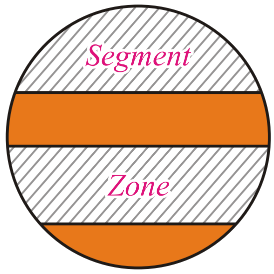
$\Rightarrow$	$m\angle ACB < 1\angle rt$		Using (i) and (iii)
In fig (III)	$m\angle AOB > 180^{\circ}$		
$\therefore$	$m\angle AOB > 2\angle rt$	(iv)	
$\Rightarrow$	$2m\angle ACB > 2\angle rt$		Using (i) and (iv)
$\Rightarrow$	$m\angle ACB > 1\angle rt$		

Corollary 1. The angles subtended by an arc at the circumference of a circle are equal.

Corollary 2. The angles in the same segment of a circle are congruent.

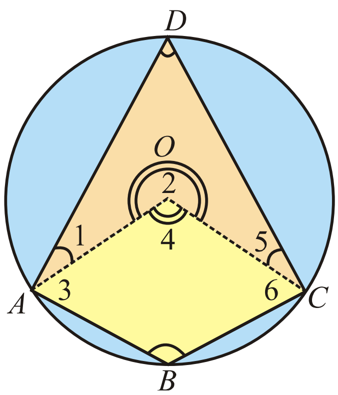






THEOREM 4

12.1(*iv*) The opposite angles of any quadrilateral inscribed in a circle are supplementary.



Given: $ABCD$ is a quadrilateral inscribed in a circle with centre O.

To prove: $\begin{cases} m\angle A + m\angle C = 2\angle rts \\ m\angle B + m\angle D = 2\angle rts \end{cases}$

Construction: Draw $\overline{OA}$ and $\overline{OC}$.

Write $\angle 1, \angle 2, \angle 3, \angle 4, \angle 5$ and $\angle 6$ as shown in the figure.

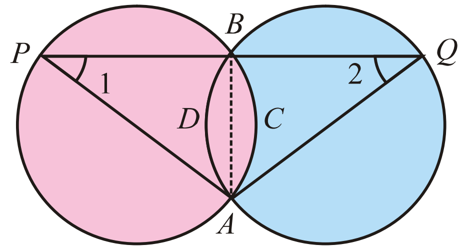
Proof:

Statements	Reasons
Standing on the same arc ADC, $\angle 2$ is a central angle Whereas $\angle B$ is the circum angle	Arc ADC of the circle with centre O.
$\therefore \quad m\angle B = \frac{1}{2}(m\angle 2)$ (i)	By theorem 1
Standing on the same arc ABC, $\angle 4$ is a central angle whereas $\angle D$ is the circum angle	Arc ABC of the circle with centre O.
$\therefore \quad m\angle D = \frac{1}{2}(m\angle 4)$ (ii)	By theorem 1
$\Rightarrow \quad m\angle B + m\angle D = \frac{1}{2}m\angle 2 + \frac{1}{2}m\angle 4$ $= \frac{1}{2}(m\angle 2 + m\angle 4) = \frac{1}{2}(\text{Total central angle})$	Adding (i) and (ii)
i.e., $\quad m\angle B + m\angle D = \frac{1}{2}(4\angle rt) = 2\angle rt$ Similarly $m\angle A + m\angle C = 2\angle rt$	

Corollary 1. In equal circles or in the same circle if two minor arcs are equal then angles inscribed by their corresponding major arcs are also equal.

Corollary 2. In equal circles or in the same circle, two equal arcs subtend equal angles at the circumference and vice versa.

Example 1: Two equal circles intersect in A and B. Through B, a straight line is drawn to meet the circumferences at P and Q respectively. Prove that $m\overline{AP} = m\overline{AQ}$.



Given: Two equal circles cut each other at points A and B. A straight line PBQ drawn through B meets the circles at P and Q respectively.

To prove: $m\overline{AP} = m\overline{AQ}$

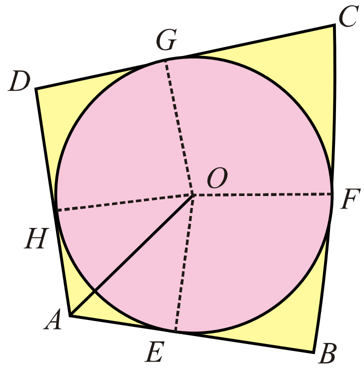
Construction: Join the points A and B. Also draw $\overline{AP}$ and $\overline{AQ}$.
Write $\angle 1$ and $\angle 2$ as shown in the figure.

Proof:

Statements	Reasons
$\because \quad m\overset{\frown}{ACB} = m\overset{\frown}{ADB}$	Arcs about the common chord AB.
$\therefore \quad m\angle 1 = m\angle 2$	Corresponding angles made by opposite arcs.
So $\quad m\overline{AQ} = m\overline{AP}$	Sides opposite to equal angles in $\Delta\, APQ$.
or $\quad m\overline{AP} = m\overline{AQ}$	

Example 2: $ABCD$ is a quadrilateral circumscribed about a circle.

Show that $m\overline{AB} + m\overline{CD} = m\overline{BC} + m\overline{DA}$



Given: $ABCD$ is a quadrilateral circumscribed about a circle with centre O.

So that each side becomes tangent to the circle.

To prove: $m\overline{AB} + m\overline{CD} = m\overline{BC} + m\overline{DA}$

Construction: Drawn $\overline{OE} \perp \overline{AB}$, $\overline{OF} \perp \overline{BC}$

$\overline{OG} \perp \overline{CD}$ and $\overline{OH} \perp \overline{DA}$

Proof:

Statements	Reasons
$\therefore \quad m\overline{AE} = m\overline{HA} \;;\; m\overline{EB} = m\overline{BF} \quad \ldots$ (i) $m\overline{CG} = m\overline{FC}$ and $m\overline{GD} = m\overline{DH} \quad \ldots$ (ii)	Since tangents drawn from a point to the circle are equal in length
$(m\overline{AE} + m\overline{EB}) + (m\overline{CG} + m\overline{GD}) = (m\overline{BF} + m\overline{FC}) + (m\overline{DH} + m\overline{HA})$	Adding (i) & (ii)
or $\quad m\overline{AB} + m\overline{CD} = m\overline{BC} + m\overline{DA}$	

EXERCISE 12.1

1. Prove that in a given cyclic quadrilateral, sum of opposite angles is two right angles and conversely.
2. Show that parallelogram inscribed in a circle will be a rectangle.
3. AOB and COD are two intersecting chords of a circle. Show that Δ^s AOD and BOC are equiangular.
4. $\overline{AD}$, and $\overline{BC}$ are two parallel chords of a circle. Prove that arc $AB \cong$ arc CD and arc $AC \cong$ arc BD.

MISCELLANEOUS EXERCISE 12

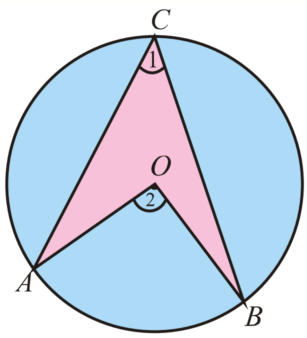
1. Multiple Choice Questions

Four possible answers are given for the following questions. Tick (✓) the correct answer.

(*i*) A circle passes through the vertices of a right angled ΔABC with $m\overline{AC}$ = 3cm and $m\overline{BC}$ = 4 cm, $m\angle C = 90^o$. Radius of the circle is:

(*a*) 1.5 cm (*b*) 2.0 cm (*c*) 2.5 cm (*d*) 3.5 cm

(*ii*) In the adjacent circular figure, central and inscribed angles stand on the same arc *AB*. Then



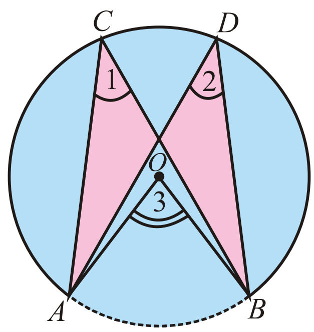
(*a*) $m\angle 1 = m\angle 2$ (*b*) $m\angle 1 = 2m\angle 2$

(*c*) $m\angle 2 = 3m\angle 1$ (*d*) $m\angle 2 = 2m\angle 1$

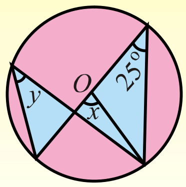
(*iii*) In the adjacent figure if $m\angle 3 = 75^o$, then find $m\angle 1$ and $m\angle 2$.

(*a*) $37\frac{1}{2}^o$, $37\frac{1}{2}^o$ (*b*) $37\frac{1}{2}^o$, 75^o

(*c*) 75^o, $37\frac{1}{2}^o$ (*d*) 75^o, 75^o

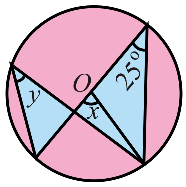


(*iv*) Given that O is the centre of the circle. The angle marked x will be:



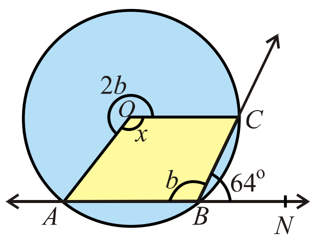
(*a*) $12\frac{1}{2}^{o}$ (*b*) 25° (*c*) 50° (*d*) 75°

(*v*) Given that O is the centre of the circle the angle marked y will be:



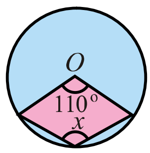
(*a*) $12\frac{1}{2}^{o}$ (*b*) 25° (*c*) 50° (*d*) 75°

(*vi*) In the figure, O is the centre of the circle and $\overleftrightarrow{ABN}$ is a straight line. The obtuse angle $AOC = x$ is:



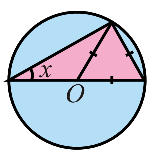
(*a*) 32° (*b*) 64° (*c*) 96° (*d*) 128°

(*vii*) In the figure, O is the centre of the circle, then the angle x is:



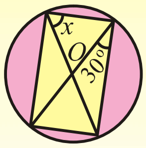
(*a*) 55° (*b*) 110° (*c*) 220° (*d*) 125°

(*viii*) In the figure, O is the centre of the circle then angle x is:



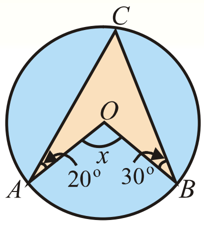
(*a*) 15° (*b*) 30° (*c*) 45° (*d*) 60°

(*ix*) In the figure, O is the centre of the circle then the angle x is:



(*a*) 15° (*b*) 30° (*c*) 45° (*d*) 60°

(*x*) In the figure, O is the centre of the circle then the angle x is:



(*a*) 50° (*b*) 75° (*c*) 100° (*d*) 125°

SUMMARY

- The angle subtended by an arc at the centre of a circle is called is **central angle**.
- A **central angle** is subtended by two radii with the vertex at the centre of the circle.
- The angle subtended by an arc of a circle at its circumference is called a **circumangle**.
- A **circumangle** is subtended between any two chords of a circle, having common point on its circumference.
- A quadrilateral is called **cyclic** when a circle can be drawn through its four vertices.
- The measure of a central angle of a minor arc of a circle, is double that of the angle subtended by the corresponding major arc.
- Any two angles in the same segment of a circle are equal.
- The angle
  - in a semi-circle is a right angle,
  - in a segment greater than a semi-circle is less than a right angle,
  - in a segment less than a semi-circle is greater than a right angle.
- The opposite angles of any quadrilateral inscribed in a circle are supplementary.

Unit-13

PRACTICAL GEOMETRY-CIRCLES

In this unit, students will learn how to

- *locate the centre of a given circle.*
- *draw a circle passing through three given non-collinear points.*
- *complete the circle when a part of its circumference is given,*
  (i) by finding the centre,
  (ii) without finding the centre.
- *circumscribe a circle about a given triangle.*
- *inscribe a circle in a given triangle.*
- *escribe a circle in a given triangle.*
- *circumscribe an equilateral triangle about a given circle.*
- *inscribe an equilateral triangle in a given circle.*
- *circumscribe a square about a given circle.*
- *inscribe a square in a given circle.*
- *circumscribe a regular hexagon about a given circle.*
- *inscribe a regular hexagon in a given circle.*
- *draw a tangent to a given arc, without using the centre, through a given point p when p is the middle point of the arc, p is at the end of the arc and p is outside the arc.*
- *draw a tangent to a given circle from a point P when P is on the circumference and when p is outside the circle.*
- *draw two tangents to a circle meeting each other at a given angle.*
- *draw direct common tangent or external tangents to two equal circles and draw transverse common tangents or internal tangents to two equal circles.*
- *draw direct common tangents or external tangents to two unequal circles and draw transverse common tangents or internal tangents to two unequal circles.*
- *draw a tangent to two unequal touching circles and two unequal intersecting circles.*
- *draw a circle which touches*
  (i) both the arms of a given angle.
  (ii) two converging lines and passes through a given point between them.
  (iii) three converging lines.

INTRODUCTION:

The word geometry is derived from two Greek words namely Geo (earth) and Metron (measurement). Infact, geometry means measurement of the earth or land. Geometry is an important branch of mathematics, which deals with the shape, size and position of geometric figures. We will concentrate upon simple figures namely point, straight line, triangle, polygon and circle in this unit.

The Greek mathematicians (600-300 BC) contributed a lot. In particular "Euclid's Elements" have been taught as text book all over the world for centuries.

13.1 Construction of a Circle

A circle of any radius can be constructed by rotating a compass about a fixed point O.

13.1(i) To locate the centre of a given circle

Given: A circle

Steps of Construction:

1. Draw two chords $\overline{AB}$ and $\overline{CD}$.
2. Draw $\overleftrightarrow{EFG}$ as perpendicular bisector of chord $\overline{AB}$.
3. Draw $\overleftrightarrow{PQR}$ as perpendicular bisector of chord $\overline{CD}$.
4. Perpendicular bisectors $\overleftrightarrow{EFG}$ and $\overleftrightarrow{PQR}$ intersect each other at O. O is the centre of circle.

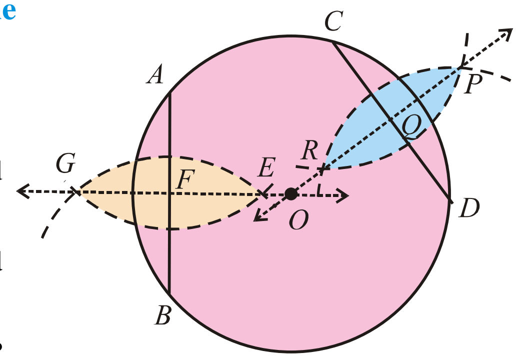


Fig. 13.1.1

13.1(ii)Draw a circle passing through three given non-collinear points:

Given: Three non-collinear points A, B and C.

Steps of Construction:

1. Join A with B and B with C.
2. Draw $\overleftrightarrow{LM}$ and $\overleftrightarrow{PQ}$ right bisectors of $\overline{AB}$ and $\overline{BC}$ respectively. $\overleftrightarrow{LM}'$and $\overleftrightarrow{PQ}'$intersect at point O.
3. Draw a circle with radius $\overline{OA}$ = $\overline{OB}$ = $\overline{OC}$ having centre at O, which is the required circle.

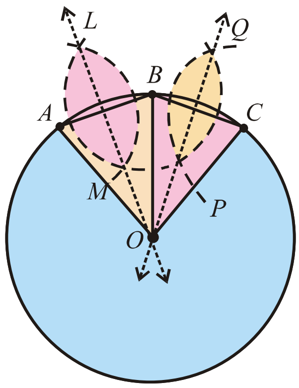


Fig. 13.1.2

13.1(iii-a) To complete the circle by finding the centre when a part of a circumference is given

Given: $\overset{\frown}{AB}$ is Part of circumference of a circle

Steps of Construction:

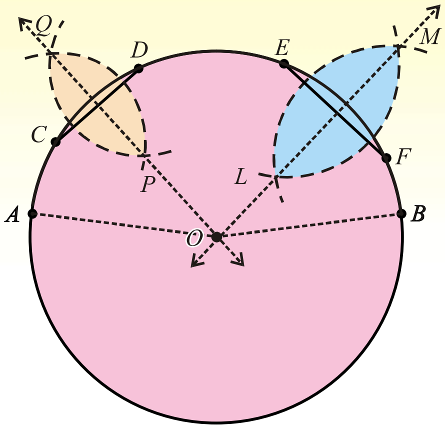


Fig. 13.1.3

1. Let C, D, E and F be the four points on the given arc AB.
2. Draw chord $\overline{CD}$ and $\overline{EF}$.
3. Draw $\overleftrightarrow{PQ}$ as perpendicular bisector of $\overline{CD}$ and $\overleftrightarrow{LM}$ as perpendicular bisector of $\overline{EF}$.
4. $\overleftrightarrow{LM}$ and $\overleftrightarrow{PQ}$ intersect at O.
   $\therefore$ O is equidistant from points A, B, C, D, E and F.
5. Complete the circle with centre O and radius ($\overline{OA} = \overline{OB} = \overline{OC} = \overline{OD} = \overline{OE} = \overline{OF}$). This will pass through all the points A, B, C, D, E and F on the given part of the circumference.

13.1(iii-b) To complete the circle without finding the centre when a part of its circumference is given

Given: $\overset{\frown}{AB}$ is the part of circumference of a circle
Steps of Construction:

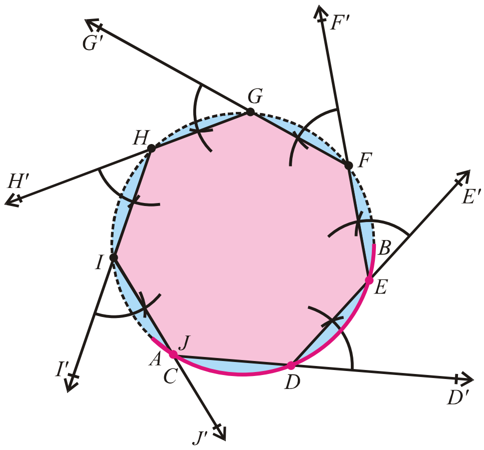


Fig. 13.1.4

1. Take two chords $\overline{CD}$ and $\overline{DE}$ of the suitable same length such that these are chords of $\overset{\frown}{AB}$.
2. Produce $\overline{CD}$ to D' and $\overline{DE}$ to E' such that to get the external angle $D'DE'$.
3. Construct $\angle E'EF \cong \angle D'DE'$ and take $m\overline{EF} = m\overline{CD} = m\overline{DE}$. Produce $\overline{EF}$ to F'.
4. Construct $\angle F'FG \cong \angle E'EF'$ and take $m\overline{FG} = m\overline{CD}$. Produce $\overline{FG}$ to G'.
5. Points F and G are on the circumference of the required circle. The dotted arcs $\overset{\frown}{EF}$ and $\overset{\frown}{FG}$ are shown in the figure.
6. Continue this process of external angles of equal measure to complete the circumference of the circle as shown in the figure.

Note: Constructing internal angles of equal measure, the circumference of the circle can also be completed.

EXERCISE 13.1

1. Divide an arc of any length
   (i) into two equal parts.
   (ii) into four equal parts.
2. Practically find the centre of an arc ABC.
3. (i) If $|\overline{AB}| = 3$ cm and $|\overline{BC}| = 4$ cm are the lengths of two chords of an arc, then locate the centre of the arc.

   (ii) If $|\overline{AB}| = 3.5$ cm and $|\overline{BC}| = 5$ cm are the lengths of two chords of an arc, then locate the centre of the arc.
4. For an arc draw two perpendicular bisectors of the chords $\overline{PQ}$ and $\overline{QR}$ of this arc, construct a circle through P, Q and R.
5. Describe a circle of radius 5 cm passing through points A and B, 6 cm apart. Also find distance from the centre to the line segment AB.
6. If $|\overline{AB}| = 4$ cm and $|\overline{BC}| = 6$ cm, such that $\overline{AB}$ is perpendicular to $\overline{BC}$, construct a circle through points A, B and C. Also measure its radius.

13.2 CIRCLES ATTACHED TO POLYGONS

13.2(i) Circumscribe a circle about a given triangle.

Given: Triangle ABC.

Steps of Construction:

1. Draw $\overleftrightarrow{LMN}$ as perpendicular bisector of side $\overline{AB}$.
2. Draw $\overleftrightarrow{PQR}$ as perpendicular bisector of side $\overline{AC}$.
3. $\overleftrightarrow{LN}$ and $\overleftrightarrow{PR}$ intersect at point O.
4. With centre O and radius

   $m\overline{OA} = m\overline{OB} = m\overline{OC}$, draw a circle.

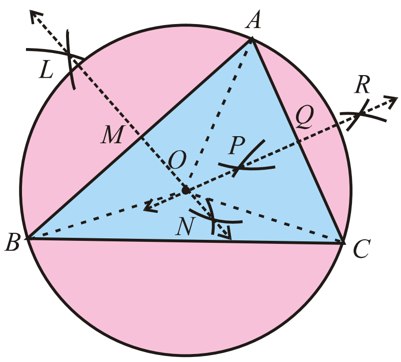


Fig. 13.2.1

This circle will pass through A, B and C whereas O is the circumcentre of the circumscribed circle.

Remember: The circle passing through the vertices of triangle ABC is known as circumcircle, its radius as circumradius and centre as circumcentre.

13.2(ii) Inscribe a circle in a given triangle:

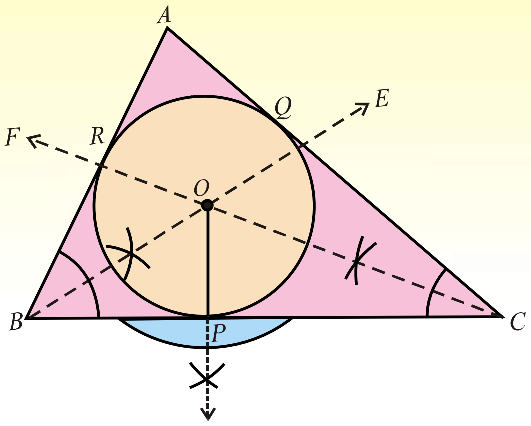


Fig. 13.2.2

Given: A triangle ABC.

Steps of Construction:

1. Draw $\overrightarrow{BE}$ and $\overrightarrow{CF}$ to bisect the angles ABC and ACB respectively. Rays $\overrightarrow{BE}$ and $\overrightarrow{CF}$ intersect each other at point O.
2. O is the centre of the inscribed circle.
3. From O draw $\overrightarrow{OP}$ perpendicular to $\overline{BC}$.

   With centre O and radius $\overline{OP}$ draw a circle. This circle is the inscribed circle of triangle ABC.

Remember:

A circle which touches the three sides of a triangle internally is known as incircle, its radius as in-radius and centre as in-centre.

13.2(iii) Escribe a circle to a given triangle:

Given: A triangle ABC

Steps of Construction:

1. Produce the sides $\overline{AB}$ and $\overline{AC}$ of ΔABC.
2. Draw bisectors of exterior angles ABC and ACB. These bisectors of exterior angles meet at I_1.
3. From I_1 draw perpendicular on side $\overline{BC}$ of ΔABC. Which $\overline{I_1D}$ intersect $\overline{BC}$ at D. I_1D is the radius of the escribed circle with centre at I_1.
4. Draw the circle with radius $\overline{I_1D}$ and centre at I_1 that will touch the side BC of the ΔABC externally and the produced sides AB and AC.

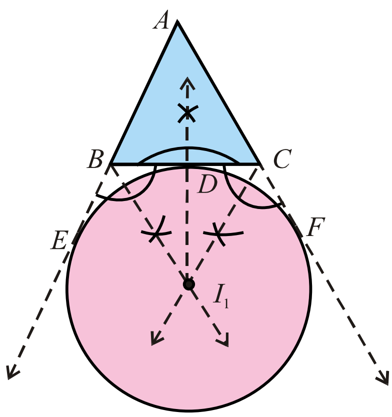


Fig. 13.2.3

Escribed circle: The circle touching one side of the triangle externally and two produced sides internally is called escribed circle (e-circle). The centre of e-circle is called e-centre and radius is called e-radius.

13.2(iv) Circumscribe an equilateral triangle about a given circle

Given: A circle with centre O of reasonable radius.

Steps of Construction:

1. Draw $\overline{AB}$, the diameter of the circle for locating.
2. Draw an arc of radius m $\overline{OA}$ with centre at A for locating points C and D on the circle.
3. Join O to the points C and D.
4. Draw tangents to the circle at points B, C and D.
5. These tangents intersect at points E, F and G.

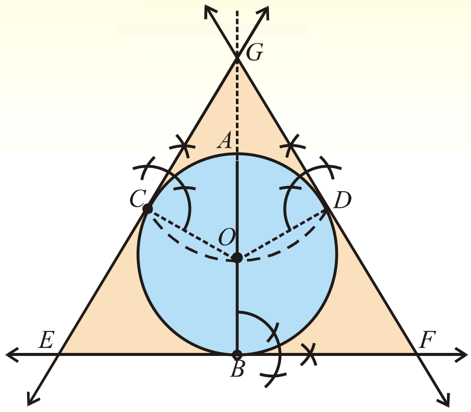


Fig. 13.2.4

13.2(v) Inscribe an equilateral triangle in a given circle.

Given: A circle with centre at O.

Steps of Construction:

1. Draw any diameter $\overline{AB}$ of the circle.
2. Draw an arc of radius OA from point A. The arc cuts the circle at points C and D.
3. Join the points B, C and D to form straight line segments $\overline{BC}$, $\overline{CD}$ and $\overline{BD}$.

Triangle BCD is the required inscribed equilateral triangle.

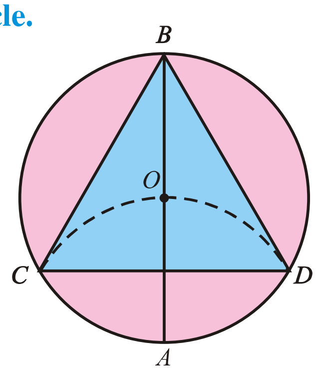


Fig. 13.2.5

13.2(vi) Circumscribe a square about a given circle.

Given: A circle with centre at O.

Steps of Construction:

1. Draw two diameters $\overline{PR}$ and $\overline{QS}$ which bisect each other at right angle.
2. At points P, Q, R and S draw tangents to the circle.
3. Produce the tangents to meet each other at A, B, C and D. $ABCD$ is the required circumscribed square.

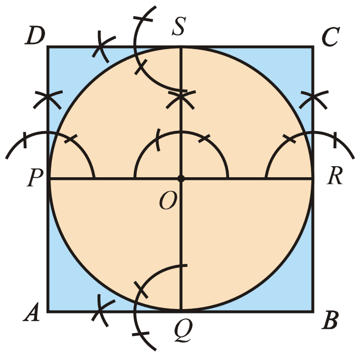


Fig. 13.2.6

13.2(vii) Inscribe a square in a given circle

Given: A circle, with centre at O.

Steps of Construction:

1. Through O draw two diameters $\overline{AC}$ and $\overline{BD}$ which bisect each other at right angle.
2. Join A with B, B with C, C with D, and D with A.

$ABCD$ is the required square inscribed in the circle.

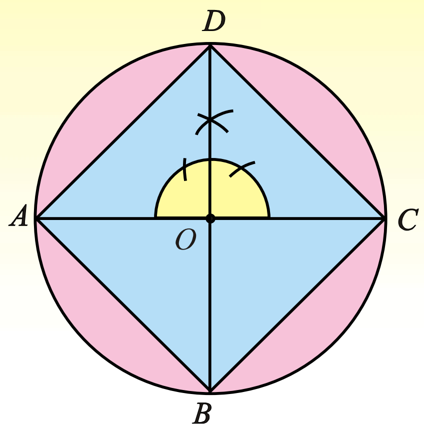


Fig. 13.2.7

13.2(viii) Circumscribe a regular hexagon about a given circle.

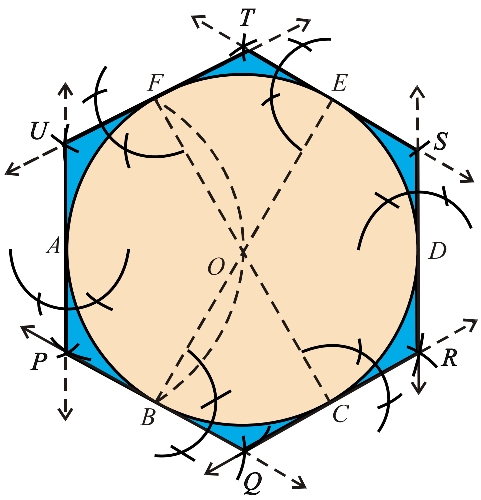


Fig. 13.2.8

Given: A circle with centre at O.

Steps of Construction:

1. Draw any diameter $\overline{AD}$.
2. From point A draw an arc of radius $\overline{AO}$ (the radius of the circle), which cuts the circle at points B and F.
3. Join B with O and extend it to meet the circle at E.
4. Join F with O and extend it to meet the circle at C.
5. Draw tangents to the circle at points A, B, C, D, E and F intersecting one another at points P, Q, R, S, T and U respectively.

Thus $PQRSTU$ is the circumscribed regular hexagon.

13.2(ix) Inscribe a regular hexagon in a given circle:

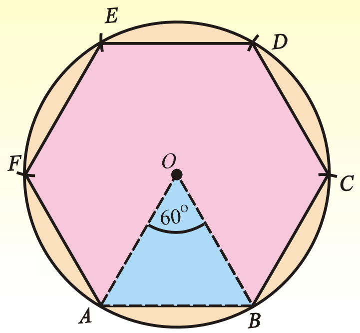


Fig. 13.2.9(a)

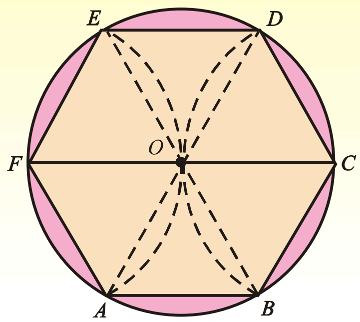


Fig. 13.2.9(b)

Given: A circle, with centre at O.

Steps of Construction:

1. Take any point A on the circle and point with O.
2. From point A, draw an arc of radius $\overline{OA}$ which intersects the circle at point B and F.
3. Join O and A with points B and F.
4. ΔOAB and ΔOAF are equilateral triangles therefore $\angle AOB$ and $\angle AOF$ are of measure 60^o *i.e.,* $m\overline{OA} = m\overline{AB} = m\overline{AF}$.
5. Produce $\overline{FO}$ to meet the circle at C. Join B to C. Since in $\angle BOC = 60$ therefore $m\overline{BC} = m\overline{OA}$.
6. From C and F, draw arcs of radius $\overline{OA}$, which intersect the circle at points D and E.
7. Join C to D, D to E and E to F ultimately. We have

$$m\overline{OA} = m\overline{OB} = m\overline{OC} = m\overline{OD} = m\overline{OE} = m\overline{OF}$$

Thus the figure $ABCDEF$ is a regular hexagon inscribed in the circle.

EXERCISE 13.2

1. Circumscribe a circle about a triangle ABC with sides
   $|AB| = 6$ cm , $|BC| = 3$cm , $|CA| = 4$cm
   Also measure its circum radius.
2. Inscribe a circle in a triangle ABC with sides
   $|AB| = 5$ cm, $|BC| = 3$ cm, $|CA| = 3$ cm. Also measure its in-radius.
3. Escribe a circle opposite to vertex A to a triangle ABC with sides
   $|AB| = 6$ cm, $|BC| = 4$ cm, $|CA| = 3$ cm. Find its radius also.
4. Circumscribe a circle about an equilateral triangle ABC with each side of length 4cm.
5. Inscribe a circle in an equilateral triangle ABC with each side of length 5cm.
6. Circumscribe and inscribe circles with regard to a right angle triangle with sides, 3cm, 4cm and 5cm.
7. In and around the circle of radius 4cm draw a square.
8. In and around the circle of radius 3.5cm draw a regular hexagon.
9. Circumscribe a regular hexagon about a circle of radius 3cm.

13.3 TANGENT TO THE CIRCLE

13.3(i) To draw a tangent to a given arc without using the centre through a given point *P*:

Case (i) When *P* is the middle point of the arc

Given: *P* is the mid-point of an arc *AB*.

Steps of Construction:

1. Join *A* and *B*, to form the chord $\overline{AB}$.
2. Draw the perpendicular bisector of chord $\overline{AB}$ which passes through mid point *P* of $\overset{\frown}{AB}$ and mid point *R* of $\overline{AB}$.
3. At points *P* construct a right angle *TPR*.
4. Produce $\overrightarrow{PT}$ in the direction of *P* beyond point *S*. Thus $\overleftrightarrow{TP}$ is the required tangent to the arc *AB* at point *P*.

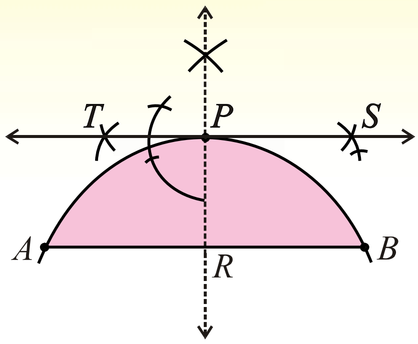


Fig. 13.3.1(*a*)

Case (ii) When *P* is at end point of the arc

Given: *P* is the end point of arc *PQR*.

Steps of Construction:

1. Take a point *A* on the arc *PQR*.
2. Join the points *A* and *P*.
3. Draw perpendicular $\overrightarrow{AS}$ at *A* which intersects the arc *PQR* at *B*.
4. Join the points *B* and *P*.
5. Draw $\angle APD$ of measure equal to that of $\angle ABP$.
6. Now $m\angle BPD = m\angle BPA + m\angle APD$
   $= m\angle BPA + m\angle ABP$ $[\because\ m\angle APD = m\angle ABP]$
   $= 90^o$

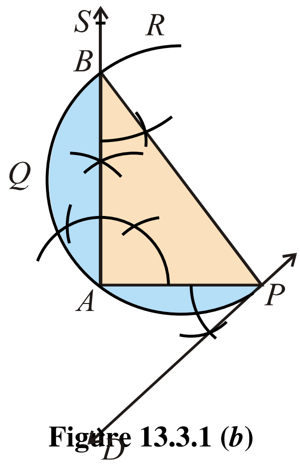


Figure 13.3.1 (*b*)

Thus $\overleftrightarrow{PD}$ is the required tangent.

Case (iii): When point P is outside the arc.

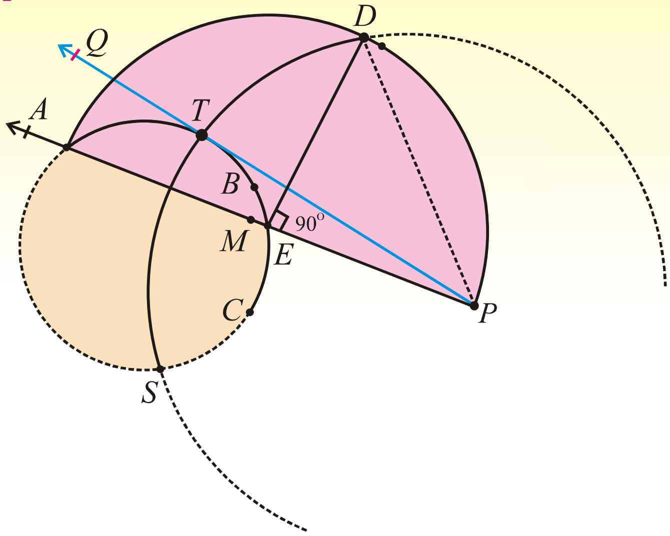


Fig. 13.3.1 (c)

Given: Point P is a line segmetn out side the arc ABC without knowing its centre.

Steps of Construction:

1. Join A to P. $\overline{AP}$ cuts the arc at E.
2. Find mid-point M of AP.
3. Draw a semi circle of radius $|AM| = |MP|$ with center at M.
4. Draw perpendicular at point E which meets the semi circle at D.
5. Draw an arc of radius $|\overline{PD}|$ with P as its center.
6. This arc cuts the given arc ABC at points T.
7. Join P with T.

$\overrightarrow{PTQ}$ is the required tangent.

13.3(ii-a) To draw a tangent to a circle from a given point P at a given point on the circumference:

Given: A circle with the centre O and some point P lies on the circumference.

Steps of Construction:

1. Join point P to the centre O, so that $\overline{OP}$ is the radius of the circle.
2. Draw a line TPS which is perpendicular to the radius $\overline{OP}$.

$\therefore$ $\overleftrightarrow{TPS}$ is the required tangent to the circle at given point P.

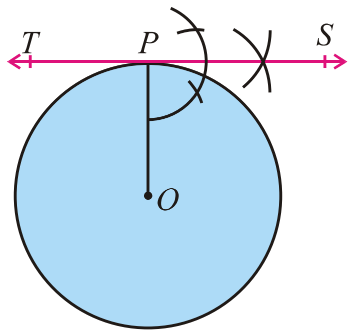


Fig. 13.3.2(a)

13.3(ii-b) To draw a tangent to a circle from a given point P which lies outside the circle:

Given: A circle with centre O and some point P out side the circle.

Steps of Construction:

1. Join point P to the centre O.
2. Find M, the mid point of $\overline{OP}$.
3. Construct a semi circle on diameter $\overline{OP}$, with M as its centre. This semi circle cuts the given circle at T.
4. Join P with T and produce $\overline{PT}$ on both sides, then $\overleftrightarrow{PT}$ is the required tangent.

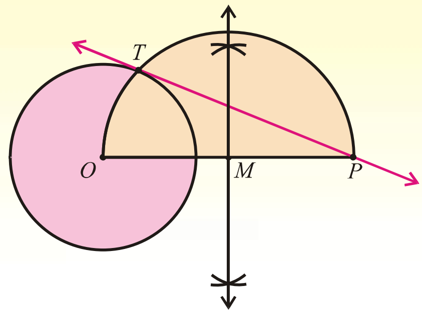


Fig. 13.3.2 (*b*)

13.3(iii) To draw two tangents to a circle meeting each other at a given angle:

Given: A circle with centre O, $\angle MNS$ is a given angle.

Steps of Construction:

1. Take a point A on the circumference of circle having centre O.
2. Join the points O and A.
3. Draw $\angle COA$ of measure equal to that of $\angle MNS$.
4. Produce $\overline{CO}$ to meet the circle at B.
5. $m\angle AOB = 180^{\circ} - m\angle COA$
6. Draw $\overleftrightarrow{AD}$ perpendicular to $\overline{OA}$.
7. Draw $\overleftrightarrow{BE}$ perpendicular to $\overline{OB}$.
8. $\overleftrightarrow{AD}$ and $\overleftrightarrow{BE}$ intersect at P.
9. $m\angle AOB + m\angle APB = 180^{\circ}$, that is, $m\angle AOB = 180^{\circ} - m\angle APB$
10. From step 5 and step 9, we have

$$180^{\circ} - m\angle COA = 180^{\circ} - m\angle APB \Rightarrow m\angle COA = m\angle APB$$

$$\Rightarrow \quad m\angle APB = m\angle MNS \qquad (\because m\angle COA = m\angle MNS)$$

11. $\overleftrightarrow{AP}$ and $\overleftrightarrow{BP}$ are the required tangents meeting at the given $\angle MNS$.

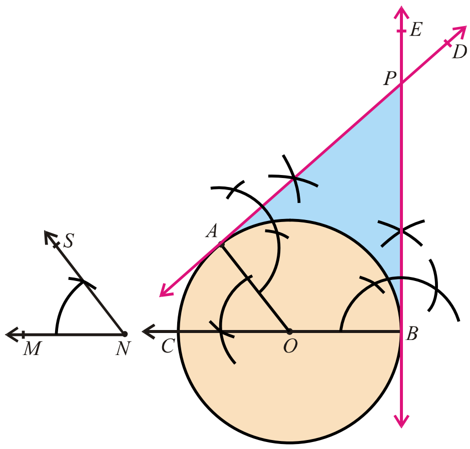


Fig. 13.3.3

13.3(iv-a) To draw direct or (external) common tangents to equal circles:

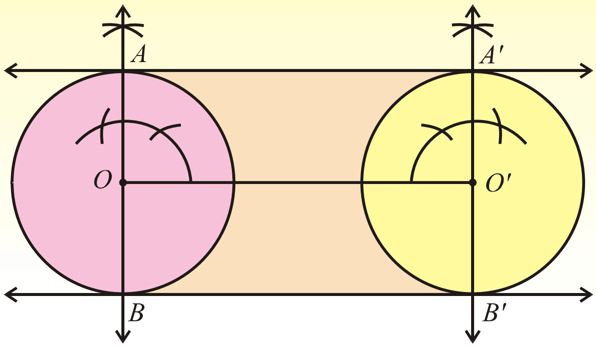


Fig. 13.3.4 (*a*)

Given: Two circles of equal radii with centres O and O' respectively.

Steps of Construction:

1. Join the centres O and O'.
2. Draw diameter AOB of the first circle so that $\overline{AOB} \perp \overline{OO'}$.
3. Draw diameter $A'O'B'$ of the second circle so that $\overline{A'O'B'} \perp \overline{OO'}$.
4. Draw $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$ which are the required common tangents.

13.3(iv-b) To draw transverse or (internal) common tangents to two equal circles:

Given: Two equal circles with centres O and O' respectively.

Steps of Construction:

1. Join the centres O and O'.
2. Find mid-point M of $\overline{OO'}$.
3. Find mid-point N of $\overline{MO'}$.
4. Taking point N as centre and radius equal to $m\overline{MN}$, draw a circle intersecting the circle with centre O' at points P and P'.
5. Draw a line through the points M and P touching the second circle at the point Q.
6. Draw a line through the points M and P' touching the second circle at the point Q'.

   Thus $\overleftrightarrow{PQ}$ and $\overleftrightarrow{P'Q'}$ are the required transverse common tangents to the given circles.

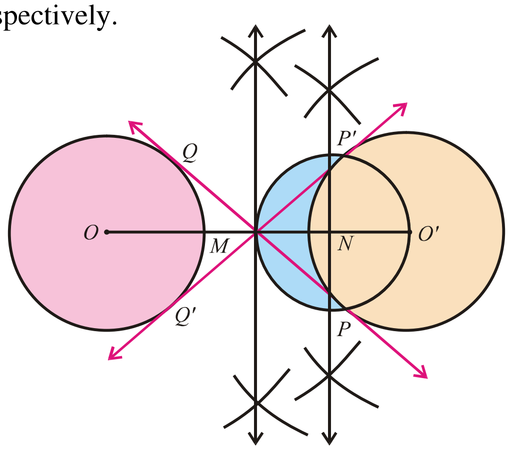


Fig. 13.3.4 (*b*)

13.3(v-a) To Draw direct or (external) common tangents to (two) unequal circles:

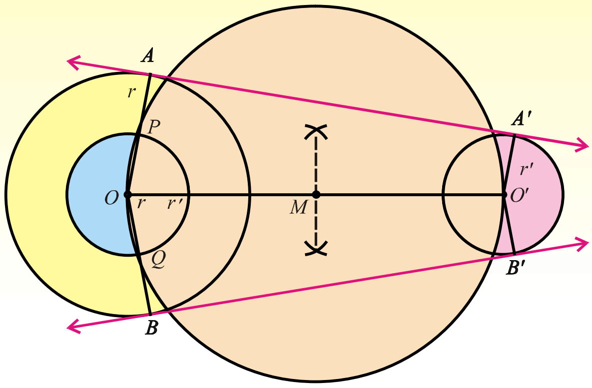


Fig. 13.3.5 (*a*)

Given: Two unequal circles with centres O, O' and radii r, r' $(r > r')$ respectively.

Steps of Construction:

1. Join the points O and O'.
2. On diameter $\overline{OO'}$, construct a new circle with centre M, the mid-point of $\overline{OO'}$.
3. Draw another circle with centre at O and radius $= r - r'$, cutting the circle with diameter $\overline{OO'}$ at P and Q.
4. Produce $\overline{OP}$ and $\overline{OQ}$ to meet the first circle at A and B respectively.
5. Draw $\overrightarrow{O'A'} \parallel \overline{OA}$ and $\overrightarrow{O'B'} \parallel \overline{OB}$.
6. Join AA' and BB' which are the required direct common tangents.

   Thus $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$ are the required common tangents.

13.3(v-b) To draw to transverse or internal common tangents to two unequal circles:

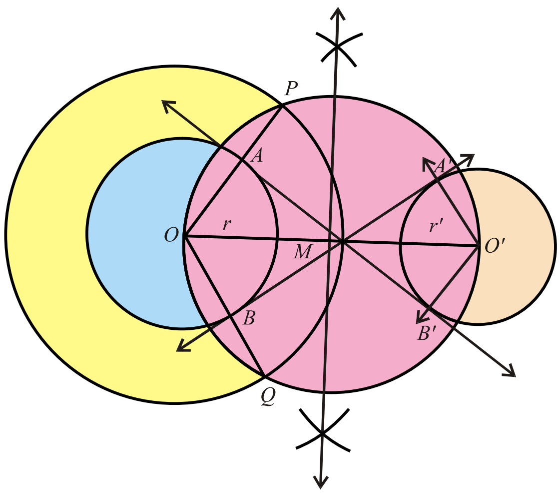


Fig. 13.3.5 (*b*)

Given: Two unequal circles with centres O, O' and radii r, r' respectively.

Steps of Construction:

1. Join the centres O and O' of the given circles.
2. Find the mid point M of $\overline{OO'}$.
3. On diameter OO', construct a new circle with centre M.
4. Draw an other circle with centre at O and radius $= r + r'$ intersecting the circle of diameter OO' at P and Q.
5. Join O with P and Q. $\overline{OP}$ and $\overline{OQ}$ meet the circle with radius r at A and B respectively.
6. Draw $\overrightarrow{OB'} \parallel \overline{OA}$ and $\overrightarrow{OA'} \parallel \overline{OB}$.
7. Join A with B' and A' with B. Thus $\overleftrightarrow{AB'}$ and $\overleftrightarrow{A'B}$ are the required transverse common tangents.

13.3(vi-a) To draw a tangent to two unequal touching circles:

Case I:

Given: Two unequal touching circles with centres O and O'.

Steps of Construction:

1. Join O with O' and produce $\overline{OO'}$ to meet the circles at the point A where these circles touch each other. Fig. 1
2. Tangent is perpendicular to the line segment $\overline{OA}$.
3. Draw perpendicular to $\overline{OA}$ at the point A which is the required tangent.

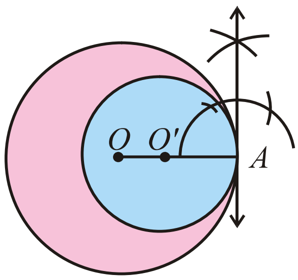


Fig. case-I

Case II:

Given: Two unequal touching circles with centres O and O'.

Steps of Construction:

1. Join O with O'. $\overline{OO'}$ intersects the circles at the point B where these circles touch each other. See Fig. 2.
2. Tangent is perpendicular to line segment containing the centres of the circles.
3. Draw perpendicular to $\overline{OO'}$ at the point B which is the required tangent.

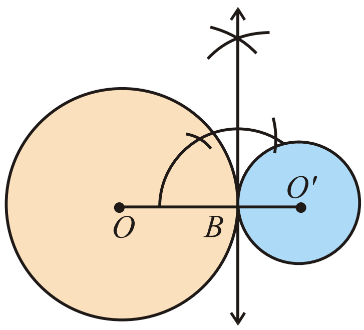


Fig. case-II

Fig. 13.3.6 (*a*)

13.3(vi-b) To draw a tangent to two unequal intersecting circles:

Given: Two intersecting circles with centres A and B.

Steps of Construction:

1. Take a line segment $\overline{AB}$.
2. Draw two circles of radii r and r_1 (where $r > r_1$) with centres at A and B respectively.
3. Taking centre at A, draw a circle of radius $r - r_1$.
4. Bisect the line segment AB at point M.
5. Taking centre at M and radius $= m\overline{AM} = m\overline{BM}$, draw a circle intersecting the circle of radius $r - r_1$ at P and Q.
6. Join the point A with P and produce it to meet the circle with centre A at D. Also join A with Q and produce it to meet the circle with centre A at C.
7. Draw $\overrightarrow{BN}$ parallel to $\overline{AD}$, intersecting the circle with centre B at T.

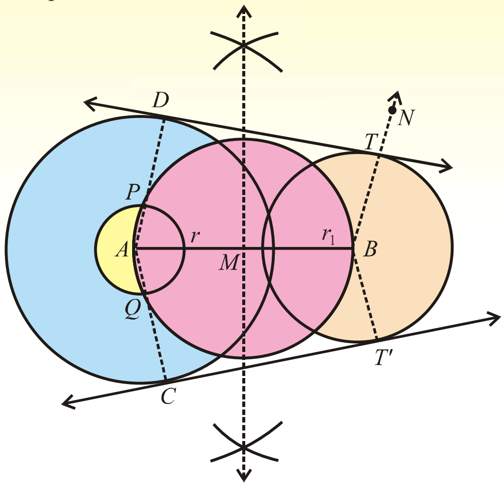


Fig. 13.3.6 (*b*)

8. Draw a line joining the points D and T. $\overleftrightarrow{DT}$ is a common tangent to the given two circles.
9. Repeat the same process on the other side of $\overline{AB}$. $\overleftrightarrow{CT'}$ is also a common tangent to the given two circles.

13.3(vii-a) To draw a circle which touches both the arms of a given angle:

Given: An angle $\angle BAC$.

Steps of Construction:

1. Draw $\overrightarrow{AD}$ bisecting $\angle BAC$.
2. Take any point E on $\overrightarrow{AD}$.
3. Draw $\overrightarrow{ET}$ perpendicular to $\overrightarrow{AC}$ intersecting $\overrightarrow{AC}$ at the point F.
4. Draw a circle with centre E and radius m$\overline{EF}$.

   This circle touches both the arms of $\angle BAC$.

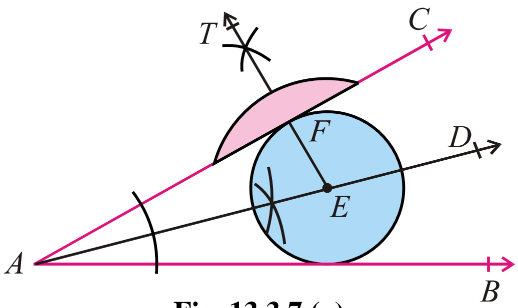


Fig. 13.3.7 (*a*)

13.3(vii-b) To draw a circle touching two convergent lines and passing through a given point between them:

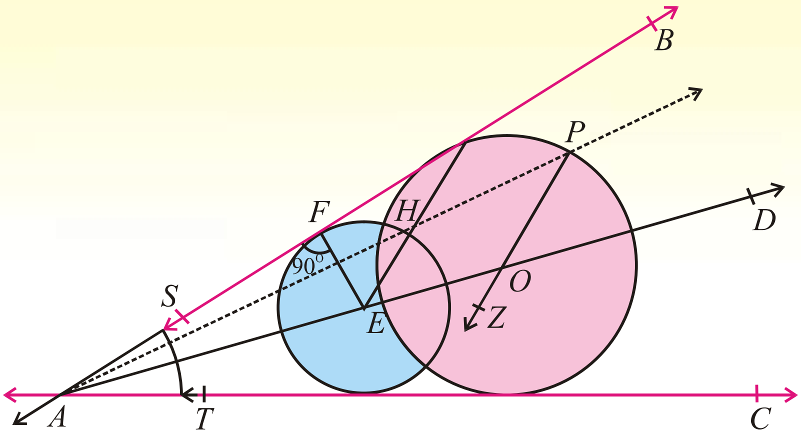


Fig. 13.3.7 (*b*)

Given: $\overleftrightarrow{BS}$ and $\overleftrightarrow{CT}$ are two **converging lines**.

Steps of Construction:

1. Produced $\overleftrightarrow{BS}$ and produced $\overleftrightarrow{CT}$ intersect at A.
2. Draw $\overrightarrow{AD}$ bisecting $\angle BAC$.
3. Take any point E on $\overrightarrow{AD}$.
4. Draw $\overline{EF}$ perpendicular to $\overleftrightarrow{AB}$.
5. Draw a circle with centre E and radius $m\overline{EF}$.
6. This circle touches $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$.
7. Draw $\overrightarrow{AP}$ which cuts this circle at the point H. Join E and H.
8. Through P, draw $\overrightarrow{PZ} \parallel \overline{HE}$ intersecting $\overrightarrow{AD}$ at the point O.
9. Draw a new circle with centre O and radius $m\overline{OP}$. This circle touches both the lines.

13.3(vii-c) To draw a circle which touches three converging lines

Note: It is not possible to draw a circle touching three converging lines.

EXERCISE 13.3

1. In an arc *ABC* the length of the chord $|BC| = 2$ cm. Draw a secant $|PBC| = 8$ cm, where *P* is the point outside the arc. Draw a tangent through point *P* to the arc.
2. Construct a circle with diameter 8 cm. Indicate a point *C*, 5 cm away from its circumference. Draw a tangent from point *C* to the circle without using its centre.
3. Construct a circle of radius 2 cm. Draw two tangents making an angle of 60° with each other.
4. Draw two perpendicular tangents to a circle of radius 3 cm.
5. Two equal circles are at 8 cm apart. Draw two direct common tangents of this pair of circles.
6. Draw two equal circles of each radius 2.4 cm. If the distance between their centres is 6 cm, then draw their transverse tangents.
7. Draw two circles with radii 2.5 cm and 3 cm. If their centres are 6.5 cm apart, then draw two direct common tangents.
8. Draw two circles with radii 3.5 cm and 2 cm. If their centres are 6 cm apart, then draw two transverse common tangents.
9. Draw two common tangents to two touching circles of radii 2.5 cm and 3.5 cm.
10. Draw two common tangents to two intersecting circle of radii 3 cm and 4 cm.
11. Draw circles which touches both the arms of angles (i) 45° (ii) 60°.

MISCELLANEOUS EXERCISE - 13

1. Multiple Choice Questions

Three possible answers are given for the following questions. Tick (✓) the correct answer.

(i) The circumference of a circle is called

(*a*) chord (*b*) segment (*c*) boundary

(ii) A line intersecting a circle is called

(*a*) tangent (*b*) secant (*c*) chord

(iii) The portion of a circle between two radii and an arc is called

(*a*) sector (*b*) segment (*c*) chord

(iv) Angle inscribed in a semi-circle is

(*a*) $\frac{\pi}{2}$ (*b*) $\frac{\pi}{3}$ (*c*) $\frac{\pi}{4}$

(v) The length of the diameter of a circle is how many times the radius of the circle

(*a*) 1 (*b*) 2 (*c*) 3

(vi) The tangent and radius of a circle at the point of contact are

(*a*) parallel (*b*) not perpendicular (*c*) perpendicular

(vii) Circles having three points in common

(*a*) over lapping (*b*) collinear (*c*) not coincide

(viii) If two circles touch each other, their centres and point of contact are

(*a*) coincident (*b*) non-collinear (*c*) collinear

(ix) The measure of the external angle of a regular hexagon is

(*a*) $\frac{\pi}{3}$ (*b*) $\frac{\pi}{4}$ (*c*) $\frac{\pi}{6}$

(x) If the incentre and circumcentre of a triangle coincide, the triangle is

(*a*) an isoscenes (*b*) a right triangle (*c*) an equilateral

(xi) The measure of the external angle of a regular octagon is

(*a*) $\frac{\pi}{4}$ (*b*) $\frac{\pi}{6}$ (*c*) $\frac{\pi}{8}$

(xii) Tangents drawn at the end points of the diameter of a circle are

(*a*) parallel (*b*) perpendicular (*c*) Intersecting

(xiii) The lengths of two transverse tangents to a pair of circles are

(*a*) unequal (*b*) equal (*c*) overlapping

(xiv) How many tangents can be drawn from a point outside the circle?

(*a*) 1 (*b*) 2 (*c*) 3

(xv) If the distance between the centers of two circles is equal to the sum of their radii, then the circles will

(*a*) intersect (*b*) do not intersect

(*c*) touch each other externaly

(xvi) If the two circles touches externally, then the distance between their centers is equal to the

(*a*) difference of their radii (*b*) sum of their radii

(*c*) product of their radii

(xvii) How many common tangents can be drawn for two touching circles?

(*a*) 2 (*b*) 3 (*c*) 4

(xviii) How many common tangents can be drawn for two disjoint circles?

(*a*) 2 (*b*) 3 (*c*) 4

2. Write short answers of the following questions

(i) Define and draw the following geometric figures:

(*a*) The segment of a circle. (*b*) The tangent to a circle.

(*c*) The sector of a circle. (*d*) The inscribed circle.

(*e*) The circumscribed circle. (*f*) The escribed circle.

(ii) The length of each side of a regular octagon is 3 cm. Measure its perimeter.

(iii) Write down the formula for finding the angle subtended by the side of a n-sided polygon at the centre of the circle.

(iv) The length of the side of a regular pentagon is 5 cm what is its perimeter?

3. Fill in the blanks

(i) The boundary of a circle is called __________.

(ii) The circumference of a circle is called ___________ of the circle.

(iii) The line joining the two points of circle is called ___________.

(iv) The point of intersection of perpendicular bisectors of two non-parallel chords of a circle is called the ____________.

(v) Circles having three points in common will ____________.

(vi) The distance of a point inside the circle from its centre is ________ than the radius.

(vii) The distance of a point outside the circle from its centre is _______ than the radius.

(viii) A circle has only ___________ centre.

(ix) One and only one circle can be drawn through three __________ points.

(x) Angle inscribed in a semi-circle is a __________ angle.

(xi) If two circles touch each other, the point of ________ and their ______ are collinear.

(xii) If two circles touch each other, their point of contact and centres are _______.

(xiii) From a point outside the circle ________ tangents can be drawn.

(xiv) A tangent is _______ to the radius of a circle at its point of contact.

(xv) The straight line drawn ⊥ to the radius of a circle is called the ______ to the circle.

(xvi) Two circles can not cut each other at more than _____ points.

(xvii) The ⊥-bisector of a chord of a circle passes through the _______.

(xviii) The length of two direct common tangents to two circles are ______ to each other.

(xix) The length of two transverse common tangents to two circles are ______ to each other.

(xx) If the in-centre and circum-centre of a triangle coincide the triangle is ________.

(xxi) Two intersecting circles are not ________.

(xxii) The centre of an inscribed circle is called __________.

(xxiii) The centre of a circumscribed circle is called __________.

(xxiv) The radius of an inscribed circle is called _____________.

(xxv) The radius of a circumscribed circle is called _____________.

SUMMARY

- A circle of any radius can be traced by rotating a compass about fixed point.
- The perpendicular bisectors of two non-parallel chords of a circle intersect at a point which is known as centre of circle.
- A circle can be drawn through given three non-collinear points.
- When a part of circumference of a circle is given, the circle can be completed.
- If a triangle, the circumscribed circle, inscribed circle and escribed circle opposite to each vertex can be constructed.
- If a circle is given, then the circumscribed and inscribed equilateral triangles can be constructed.
- For a given circle, the circumscribed and inscribed squares can be drawn.

- For a given circle, the circumscribed and inscribed regular hexagon can be constructed.
- We can draw tangents to a given arc as its mid point, its any end point, and a point not on the arc.
- Tangents can be drawn to a given circle, when a point is an its circumference and from a point outside the circle.
- Tangents to two unequal touching circles can be traced.
- Direct or transverse common tangents of two equal circles or two unequal circles can be drawn.
- We can construct a circle touching the arms of a given angle.
- A circle passing through a given point between two converging lines and touching each of them, can be traced.

ANSWERS

Unit 1: Quadratic Equations

EXERCISE 1.1

1. (i) quadratic, $x^2 + 4x - 14 = 0$ (ii) quadratic, $7x^2 - 3x + 7 = 0$
(iii) quadratic, $4x^2 + 4x - 1 = 0$ (iv) pure, $x^2 - 1 = 0$
(v) pure, $x^2 - 20 = 0$ (vi) quadratic, $x^2 + 29x + 66 = 0$

2. (i) $\{-4, 5\}$ (ii) $\left\{0, \frac{-5}{2}\right\}$ (iii) $\left\{-2, \frac{2}{17}\right\}$
(iv) $\{-8, 19\}$ (v) $\{3, -4\}$ (vi) $\left\{\frac{3}{2}, 5\right\}$

3. (i) $\left\{\frac{-1 \pm 2\sqrt{2}}{7}\right\}$ (ii) $\left\{\frac{-2 \pm \sqrt{a^2+4}}{a}\right\}$ (iii) $\left\{3, \frac{1}{11}\right\}$
(iv) $\left\{\frac{-m \pm \sqrt{m^2 - 4ln}}{2l}\right\}$ (v) $\left\{0, \frac{-7}{3}\right\}$ (vi) $\{-13, 15\}$
(vii) $\left\{-5, \frac{3}{2}\right\}$ (viii) $\left\{-\frac{1}{2}, -\frac{33}{2}\right\}$ (ix) $\{1,3\}$
(x) $\{-3a, 4a\}$

EXERCISE 1.2

1. (i) $\left\{\frac{-7 \pm \sqrt{57}}{2}\right\}$ (ii) $\left\{\frac{-4 \pm \sqrt{11}}{5}\right\}$ (iii) $\left\{\sqrt{3}, -\frac{4}{\sqrt{3}}\right\}$
(iv) $\left\{\frac{3 \pm \sqrt{233}}{8}\right\}$ (v) $\left\{-\frac{1}{3}, \frac{3}{2}\right\}$ (vi) $\left\{\frac{-4 \pm \sqrt{10}}{3}\right\}$
(vii) $\{3,7\}$ (viii) $\left\{3, \frac{-4}{5}\right\}$
(ix) $\left\{(a+b), \frac{1}{2}(a+b)\right\}$ (x) $\left\{1, \frac{l+m}{l}\right\}$

EXERCISE 1.3

1. $\left\{\pm\frac{1}{\sqrt{2}}, \pm\sqrt{5}\right\}$ 2. $\left\{\pm\frac{1}{\sqrt{2}}, \pm 2\right\}$ 3. $\left\{\frac{16}{625}, 1\right\}$
4. $\{216, 729\}$ 5. $\left\{\frac{3}{5}, 1\right\}$ 6. $\{-1, 0, 1\}$
7. $\{6\}$ 8. $\left\{\pm\frac{5}{4}\right\}$ 9. $\left\{-7a, \frac{a}{7}\right\}$
10. $\{\pm 1, 1 \pm \sqrt{2}\}$ 11. $\left\{1, -2, -\frac{1}{2}\right\}$ 12. $\{-3,0\}$

13.	{0, −1}	14.	{2,4}	15.	$\{1, 3, 2 \pm \sqrt{33}\}$
16.	{−4,−2,5,7}				

EXERCISE 1.4

1. $\left\{-1, -\frac{9}{4}\right\}$ 2. $\{1\}, \left(\frac{-2}{9} \text{ Extraneous}\right)$ 3. $\left\{\frac{5}{16}\right\}$, (−1 Extraneous)

4. {7}, (−12 Extraneous) 5. {4} 6. {3} 7. ϕ or { }

8. {0}, ($-3a$ Extraneous) 9. $\left\{\frac{-1 \pm \sqrt{6}}{2}\right\}$ 10. $\left\{\frac{-3 \pm \sqrt{2}}{2}\right\}$ 11. {−3, 0}

MISCELLANEOUS EXERCISE 1

1. Multiple choice questions:

(i) (*b*) (ii) (*c*) (iii) (*c*) (iv) (*a*)
(v) (*c*) (vi) (*b*) (vii) (*a*) (viii) (*c*)
(ix) (*a*)

2. Short answers:

(i) $-1 \pm \sqrt{3}$ (ii) 0, 3 (iii) $3x^2 - 2x - 48 = 0$

(iv) (a) Factorization (b) Completing square (c) Quadratic formula

(v) $\frac{-1}{2}, 1$ (vi) −3, 6

3. Fill in the blanks:

(a) $ax^2 + bx + c = 0$ (ii) 3

(iii) Completing square (iv) $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(v) $\left\{\pm \frac{1}{5}\right\}$ (vi) Exponential (vii) {± 3}

(viii) Reciprocal (ix) Extraneous (x) Radical sign

Unit 2: Theory Of Quadratic Equations

EXERCISE 2.1

1. (i) 17 (ii) −8 (iii) 0 (iv) 81

2. (i) real, rational and unequal, $x = 8, 15$ (ii) imaginary, $x = \frac{-3 \pm \sqrt{-47}}{4}$

(iii) real and equal, $x = \frac{3}{4}$

(iv) real, irrational and unequal, $x = \frac{-7 \pm \sqrt{205}}{6}$

3.	$k = -\frac{1}{3}, 1$	4. (i)	$k = 2, \frac{2}{3}$	(ii)	$k = -1, 0$	(iii)	$k = 1$
6.	$a = mc$						

EXERCISE 2.2

1.	(i)	$-1, -\omega, -\omega^2$			(ii)	$2, 2\omega, 2\omega^2$		
	(iii)	$-3, -3\omega, -3\omega^2$			(iv)	$4, 4\omega, 4\omega^2$		
2.	(i)	128	(ii)	1024	(iii)	125	(iv)	24
	(v)	128	(vi)	2	(vii)	−6	(viii)	−1

EXERCISE 2.3

1.	(i)	$S = 5, P = 3$	(ii)	$S = -\frac{7}{3}, P = \frac{-11}{3}$		
	(iii)	$S = \frac{q}{p}, P = \frac{r}{p}$	(iv)	$S = \frac{a}{a+b}, P = \frac{b}{a+b}$		
	(v)	$S = -\frac{m+n}{l+m}, P = \frac{n-l}{l+m}$	(vi)	$S = \frac{5m}{7}, P = \frac{9n}{7}$		
2.	(i)	$k = \frac{3}{8}$	(ii)	$k = \frac{2}{3}$		
3.	(i)	$k = \frac{64}{23}$	(ii)	$k = -1, 2$		
4.	(i)	$p = 0$	(ii)	$p = \frac{13}{4}$		
5.	(i)	$m = -55$	(ii)	$m = 5$	(iii)	$m = -\frac{10}{7}$
6.	(i)	$m = \frac{3}{2}$	(ii)	$m = 1$		

EXERCISE 2.4

1.	(i)	$p^2 - 2q$	(ii)	$q(p^2 - 2q)$			(iii)	$\frac{1}{q}(p^2 - 2q)$
2.	(i)	$\frac{5}{6}$	(ii)	$\frac{9}{4}$	(iii)	$\frac{5}{9}$	(iv)	$-\frac{235}{96}$
3.	(i)	$\frac{-mn^2}{l^3}$	(ii)	$\frac{1}{n^2}[m^2 - 2ln]$				

EXERCISE 2.5

1. (a) $x^2-6x+5=0$ (b) $x^2-13x+36=0$
(c) $x^2-x-6=0$ (d) $x^2+3x=0$
(e) $x^2+4x-12=0$ (f) $x^2+8x+7=0$
(g) $x^2-2x+2=0$ (h) $x^2-6x+7=0$

2. (a) $x^2-8x+31=0$ (b) $x^2+3x+36=0$
(c) $6x^2-3x+1=0$ (d) $2x^2+x+2=0$
(e) $2x^2-7x+3=0$

3. (a) $x^2-(p^2-2q)\,x+q^2=0$ (b) $qx^2-(p^2-2q)\,x+q=0$

EXERCISE 2.6

1. (i) $Q(x)=x+6\ ;\ R=-7$ (ii) $Q(x)=4x^2-12x+31\ ;\ R=-78$
(iii) $Q(x)=x^2+3x+3\ ;\ R=8$

2. (i) $h=\frac{7}{3}$ (ii) $h=6$ (iii) $h=-5$

3. (i) $l=-\frac{3}{2},\ m=-18$ (ii) $l=2\ ,\ m=-\frac{1}{2}$

4. (i) $-6,2,4$ (ii) $-2,\frac{1}{2},3$ (iii) $\frac{-3}{4},-1,2$

5. (i) $-3,-1,1,3$ (ii) $-4,-2,1,3$

EXERCISE 2.7

1. $\{(4,1),(-6,11)\}$ 2. $\{(1,1),(-5,-8)\}$

3. $\left\{(2,-5),\left(\frac{7}{2},\frac{-7}{2}\right)\right\}$ 4. $\left\{(a,-b),\left(\frac{a-b}{2},\frac{a-b}{2}\right)\right\}$

5. $\{(-3,2),(-1,-2)\}$ 6. $\{(0,1),(-3,-2)\}$

7. $\{(\pm 2,\pm 3)\}$ 8. $\{(\pm 2,\pm\sqrt{2})\}$

9. $\{(\pm 1,\pm 1)\}$ 10. $\left\{\left(\frac{5}{3},\frac{-1}{3}\right),\left(\frac{-5}{3},\frac{1}{3}\right),(1,1),(-1,-1)\right\}$

11. $\left\{(3,1),(-3,-1),\left(\frac{-4\sqrt{6}}{3},\sqrt{6}\right),\left(\frac{4\sqrt{6}}{3},-\sqrt{6}\right)\right\}$

12. $\left\{\left(\frac{5}{2\sqrt{2}},\frac{3}{2\sqrt{2}}\right),\left(\frac{-5}{2\sqrt{2}},\frac{-3}{2\sqrt{2}}\right)\right\}$

13. $\left\{\left(\frac{7}{\sqrt{5}},\frac{1}{\sqrt{5}}\right),\left(\frac{-7}{\sqrt{5}},\frac{-1}{\sqrt{5}}\right),\left(-\sqrt{3},\frac{2}{\sqrt{3}}\right),\left(\sqrt{3},\frac{-2}{\sqrt{3}}\right)\right\}$

EXERCISE 2.8

1.	13, 14	2.	4, 5, 6.	3.	12
4.	$\frac{-1}{12}, 2$	5.	$4, -\frac{1}{4}$	6.	81
7.	(3, 6), (6, 3)	8.	$x = 5, y = 4$	9.	11, 7

10. 25cm by 15 cm or 15 cm by 25 cm

MISCELLANEOUS EXERCISE 2

1. Multiple choice questions:

(i)	(c)	(ii)	(b)	(iii)	(b)	(iv)	(a)
(v)	(a)	(vi)	(b)	(vii)	(c)	(viii)	(c)
(ix)	(d)	(x)	(c)	(xi)	(a)	(xii)	(a)
(xiii)	(c)	(xiv)	(d)	(xv)	(d)	(xvi)	(a)

2. Short questions:

(i) (*a*) imaginary (*b*) (real) rational, unequal

(c) (real) irrational, unequal (d) (real) rational equal

(ii) $w^2 = \frac{-1-\sqrt{-3}}{2}$ (iv) 1

(vi) 0 (vii) 64 (viii) $x^2 + 3x + 9 = 0$

(ix) $Q(x) = x^2 + 5x + 10,\ R = 22$ (xi) Sum $= -\frac{3q}{2p}$, Product $= -\frac{2r}{p}$

(xii) $\frac{10}{9}$ (xiii) (a) $\frac{-39}{16}$ (b) $-\frac{13}{8}$ (c) $\frac{\sqrt{-87}}{4}$

(xiv) (a) $x^2 + 5x + 7 = 0$ (b) $x^2 - 10x + 28 = 0$

3. Fill in the blanks:

(i)	$b^2 - 4ac$	(ii)	equal	(iii)	real	(iv)	imaginary
(v)	rational	(vi)	irrational	(vii)	$-\frac{b}{a}$	(viii)	$\frac{c}{a}$
(ix)	$\frac{5}{7}$	(x)	$\frac{-9}{5}$	(xi)	$\frac{1}{\alpha\beta}$	(xii)	$1, w, w^2$
(xiii)	zero	(xiv)	w^2	(xv)	$x^2 - (\alpha + \beta)x + \alpha\beta = 0$		
(xvi)	$x^2 + 2x + 4 = 0$						

Unit 3: Variations

EXERCISE 3.1

1. (i) $3:5;\frac{3}{5}$ (ii) $3:2;\frac{3}{2}$ (iii) $16:11;\frac{16}{11}$
   (iv) $11:24;\frac{11}{24}$ (v) $1:3;\frac{1}{3}$
2. (i) $7:12$ (ii) $7:5$
3. $4:5$
4. $p = 8$
5. $x = 1$
6. $x = 3$; 15 and 24
7. $x = 2$; 8 and 26
8. Rs. 400
9. $51:7$
10. (i) 7 (ii) $9bx$ (iii) $4l$
11. (i) $x = 2$ (ii) $x = 1$ (iii) $x = 38$
    (iv) $x = p^2 - q^2$ (v) $x = 4$

EXERCISE 3.2

1. (i) $y = 4x$ (ii) $y = 20$ (iii) $x = 7$
2. (i) $y = \frac{7}{3}x$ (ii) $x = 15, y = 42$
3. $R = \frac{5}{8}T$, $R = 40$, $T = 32$
4. $R = 32$
5. $V = \frac{5}{27}R^3$, $R = 15$
6. $w = 3u^3$, $w = 375$
7. $y = \frac{14}{x}$, $y = \frac{1}{9}$
8. $y = \frac{12}{x}$, $x = \frac{1}{2}$
9. $w = \frac{35}{z}$, $w = \frac{4}{5}$
10. $A = \frac{18}{r^2}$, $r = \pm\frac{1}{2}$
11. $a = \frac{48}{b^2}$, $a = \frac{3}{4}$
12. $V = \frac{135}{r^3}$, $V = \frac{5}{8}$, $r = \frac{3}{4}$
13. $m = \frac{128}{n^3}$, $m = \frac{16}{27}$, $n = \frac{2}{3}$

EXERCISE 3.3

1. (i) 24 (ii) $9a$ (iii) $\frac{a-b}{a+b}$
   (iv) $(x^2+xy+y^2)^2$ (v) $(x-2y)^2$ (vi) $\frac{p-q}{p^2-pq+q^2}$
2. (i) 24 (ii) $9x^4$ (iii) $14b^2$
   (iv) $5x^3$ (v) $p-q$ (vi) p^2-pq+q^2
3. (i) ± 30 (ii) $\pm 10x^5y^3$ (iii) $\pm 45p^2q^3r^5$
   (iv) $\pm(x-y)$
4. (i) $p = \pm 15$ (ii) $x = \pm 12$ (iii) $p = 8, -4$
   (iv) $m = 17, -11$

EXERCISE 3.4

2. (i) 2 (ii) 2 (iii) $\frac{4(b-a)}{a+b}$ (iv) $\frac{2(z^2-y^2)}{yz}$

(v) 2 (vi) $\left\{\frac{9}{2}, \frac{11}{3}\right\}$ (vii) $\pm\sqrt{\frac{5}{2}}$ (extraneous root), ϕ or $\{\ \}$

(viii) $\{2p, -2p\}$ (ix) $\{7\}$

EXERCISE 3.5

1. $s = \frac{14u^2}{9v}, \frac{28}{5}$ 2. $w = \frac{1}{36}xy^2z, \frac{49}{3}$ 3. $y = \frac{3x^3}{z^2t}, \frac{2}{3}$

4. $u = \frac{7x^2}{4yz^3}, \frac{21}{8}$ 5. $v = \frac{7xy^3}{8z^2}, \frac{14}{3}$ 6. $w = \frac{135}{u^3}, \frac{5}{8}$

EXERCISE 3.7

1. (*i*) $A = 48$ Sq. Units (*ii*) $l = 2$
2. $S = 4\pi r^2$, $r = 3$
3. (*i*) $S = 2.5$ in (*ii*) $F = 16lb$
4. $I = 45$cp 5. $d = 20$ ft 6. Rs. 297000
7. $l = 20$ ft 8. $p = 12$ hp 9. 968000

MISCELLANEOUS EXERCISE 3

1. Multiple choice questions:

(i) (b)	(ii) (c)	(iii) (b)	(iv) (a)
(v) (c)	(vi) (a)	(vii) (d)	(viii) (b)
(ix) (a)	(x) (a)	(xi) (c)	(xii) (b)
(xiii) (a)	(xiv) (d)	(xv) (a)	

2. Short Questions:

(vi) $x = 10$	(vii) $y = \pm\frac{4}{3}$	(viii) $v = 2$
(ix) $\frac{21}{4}$	(x) ± 28	(xi) $\frac{4}{7}$
(xii) $y = \frac{8x^2}{7z}$	(xiii) $z = 6xy$	(xiv) $\frac{18}{v^2}$

3. Fill in the blanks:

(i) $\frac{x+y}{x-y}$	(ii) Antecedent	(iii) Consequent
(iv) Extremes	(v) Means	(vi) $p = 14$
(vii) $m = 8$	(viii) ky	(ix) $\frac{v}{k}$

(x) p^2w (xi) $\frac{4}{3}$ (xii) 2

(xiii) $\pm 2mn^2p^3$ (xiv) $m = \pm 6$

Unit 4: Partial Fractions

EXERCISE 4.1

1. $\frac{4}{x+1}+\frac{3}{x-3}$
2. $\frac{-1}{x-4}+\frac{2}{x+3}$
3. $\frac{1}{x-1}+\frac{2}{x+1}$
4. $\frac{-1}{x-1}+\frac{2}{x+3}$
5. $\frac{2}{x-1}+\frac{1}{x+2}$
6. $\frac{3}{x-4}+\frac{4}{x-3}$
7. $1+\frac{9}{5(x-2)}-\frac{4}{5(x+3)}$
8. $2x+3+\frac{5}{3x+1}+\frac{1}{x-1}$

EXERCISE 4.2

1. $\frac{2}{x-1}+\frac{1}{(x-1)^2}-\frac{1}{x-2}$
2. $\frac{2}{x+2}+\frac{1}{(x+2)^2}-\frac{1}{x+3}$
3. $\frac{1}{x-1}-\frac{1}{x+2}-\frac{3}{(x+2)^2}$
4. $x+1-\frac{1}{x}-\frac{1}{x^2}+\frac{2}{x-1}$
5. $\frac{-6}{3x+2}+\frac{2}{x+1}+\frac{3}{(x+1)^2}$
6. $\frac{1}{4(x+1)}-\frac{1}{4(x-1)}+\frac{1}{2(x-1)^2}$
7. $3+\frac{3}{x+2}-\frac{2}{(x+2)^2}$
8. $\frac{1}{4(x-1)}-\frac{1}{4(x+1)}-\frac{1}{2(x+1)^2}$

EXERCISE 4.3

1. $\frac{-2}{x+3}+\frac{2x-3}{x^2+1}$
2. $\frac{x+12}{5(x^2+1)}-\frac{1}{5(x+3)}$
3. $\frac{1}{2(x+1)}-\frac{x-1}{2(1+x^2)}$
4. $\frac{17x-6}{5(x^2+1)}-\frac{17}{5(x+3)}$
5. $\frac{-2}{13(x+3)}+\frac{2x+33}{13(x^2+4)}$
6. $\frac{1}{2(x+2)}+\frac{x-2}{2(x^2+4)}$
7. $\frac{1}{3(x+1)}-\frac{x-2}{3(x^2-x+1)}$
8. $\frac{2}{3(x+1)}+\frac{x+1}{3(x^2-x+1)}$

EXERCISE 4.4

1. $\frac{x}{x^2+4}-\frac{4x}{(x^2+4)^2}$
2. $\frac{1}{(x+1)}+\frac{x}{(x^2+1)^2}$
3. $\frac{1}{4(1+x)}-\frac{x-1}{4(x^2+1)}+\frac{x-1}{2(x^2+1)^2}$
4. $\frac{1}{4(x-1)}-\frac{x+1}{4(x^2+1)}+\frac{x+1}{2(1+x^2)^2}$

5. $1 - \frac{4}{x^2+2} + \frac{4}{(x^2+2)^2}$

6. $x - \frac{2x}{x^2+1} + \frac{x}{(x^2+1)^2}$

Miscellaneous Exercise 4

1. (i) (*c*) (ii) (*c*) (iii) (*b*) (iv) (*d*) (v) (*c*)
(vi) (*c*) (vii) (*b*) (viii) (*a*) (ix) (*b*) (x) (*c*)

2. (v) $\frac{-4}{x+2} + \frac{5}{x+3}$ (vi) $\frac{1}{2(x-1)} - \frac{1}{2(x+1)}$

(vii) $\frac{3}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$ (viii) $\frac{1}{x-3} + \frac{3}{(x-3)^2}$

(ix) $\frac{1}{2}\left[\frac{1}{x+a} + \frac{1}{x-a}\right]$ (x) Yes it is an identity.

Unit 5: Sets and Functions

EXERCISE 5.1

1. (i) {1, 2, 4, 5, 7, 9} (ii) {4, 9} (iii) {1, 2, 4, 5, 7, 9}
(iv) {4, 9}

2. (i) $Y \cup \{13, 17\}$ (ii) $Y \cup \{13, 17\}$ (iii) {2, 3, 5, 7, 11}
(iv) {2, 3, 5, 7, 11}

3. (i) $Y \cup \{13, 17\}$ (ii) T (iii) Y
(iv) Φ (v) Φ (vi) T

4. (i) {18, 20, 21, 22, 24, 25} (ii) {18, 20, 21, 22, 24, 25}
(iii) {4, 5, ⋯, 10, 12, 14, 15, 16, 18, ⋯, 25}
(iv) {4, 5, ⋯, 10, 12, 14, 15, 16, 18, ⋯, 25}

5. (i) {2, 6, 10, 14, 18} (ii) {24}

6. (i) Φ (ii) {0}

EXERCISE 5.2

1. (i) {0, 1, 2, 3, ⋯, 20, 23} (ii) {0, 1, 2, 3, ⋯, 20, 23} (iii) Φ
(iv) Φ (v) {1, 2, 3, 5, 7, ⋯, 19}
(vi) {1, 2, 3, 5, 7, ⋯, 19} (vii) {3, 5, 7, 11, 13, 17, 19}
(viii) {3, 5, 7, 11, 13, 17, 19}

EXERCISE 5.4

1. $A \times B = \{(a, c), (a, d), (b, c), (b, d)\}$
$B \times A = \{(c, a), (c, b), (d, a), (d, b)\}$

2. $A \times B = \{(0, -1), (0, 3), (2, -1), (2, 3), (4, -1), (4, 3)\}$
$B \times A = \{(-1, 0), (-1, 2), (-1, 4), (3, 0), (3, 2), (3, 4)\}$

$A \times A = \{(0, 0), (0, 2), (0, 4), (2, 0), (2, 2), (2, 4), (4, 0), (4, 2), (4, 4)\}$

$B \times B = \{(-1, -1), (-1, 3), (3, -1), (3, 3)\}$

3. (i) $a = 6, \quad b = 3$ (ii) $a = 1, \quad b = 7$ (iii) $a = \frac{10}{3}, \quad b = -6$

4. $X = \{a, b, c, d\}; \quad Y = \{a\}$

5. (i) 6 (ii) 6 (iii) 9

EXERCISE 5.5

1. $R_1 = \{(a, 3), (b, 4), (c, 3)\}$

   $R_2 = \{(a, 4), (b, 3), (c, 4)\}$

   $R_3 = \{(3, a), (4, a)\}$

   $R_4 = \{(3, b), (4, b), (3, c), (4, c)\}$

2. $R_1 = \{(-2, -2), (-2, 1), (1, 2), (2, 2)\}$,

   Dom $R_1 = \{-2, 1, 2\} = L$, Range $R_1 = \{-2, 1, 2\}$

   $R_2 = \{(-2, 1), (1, 1), (-2, 2)\}$;

   Dom $R_2 = \{-2, 1\}$, Range $R_2 = \{1, 2\}$

3. $R_1 = \{(a, a), (a, b)$; $R_2 = \{(b, c), (c, c)\}$

   $R_1 = \{(a, d), (b, g)\}$; $R_2 = \{(a, f), (b, e), (c, f)\}$

   $R_1 = \{(d, e), (d, f)\}$; $R_2 = \{(e, e), (f, f), (g, g)\}$

4. $2^{5 \times 5} = 2^{25}$

5. (i) $R_1 = \{(3, 2), (4, 2), (5, 2), (4, 3), (5, 3)\}$

   (ii) $R_2 = \{(2, 2), (3, 3), (5, 5)\}$

   (iii) $R_3 = \{(1, 5), (3, 3), (4, 2)\}$

   (iv) $R_4 = \{(1, 3), (3, 5), (5, 7)\}$

6. (i) Bijective

   Dom $R_1 = \{1, 2, 3, 4\}$, Range $R_1 = \{1, 2, 3, 4\}$

   (ii) Relation

   Dom $R_2 = \{1, 2, 3\}$, Range $R_2 = \{1, 2, 4, 5\}$

   (iii) Function

   Dom $R_3 = \{\text{b, c, d}\}$, Range $R_3 = \{\text{a}\}$

   (iv) Onto function

   Dom $R_4 = \{1, 2, 3, 4, 5\}$, Range $R_4 = \{1, 3, 4\}$

   (v) One-one function

   Dom $R_5 = \{a, b, c, d\}$, Range $R_5 = \{a, b, d, e\}$

(vi) Relation

Dom $R_6 = \{1, 2, 3\}$, Range $R_6 = \{2, 3, 4\}$

(vii) One-one function

Dom $R_7 = \{1, 3, 5\}$, Range $R_7 = \{p, r, s\}$

(viii) Relation

Dom $R_8 = \{1, 3, 7\}$, Range $R_8 = \{a, b, c\}$

MISCELLANEOUS EXERCISE 5

1. MCQ's

(i)	(c)	(ii)	(d)	(iii)	(c)	(iv)	(b)	(v)	(d)
(vi)	(c)	(vii)	(d)	(viii)	(c)	(ix)	(b)	(x)	(a)
(xi)	(c)	(xii)	(c)	(xiii)	(a)	(xiv)	(d)	(xv)	(c)
(xvi)	(b)	(xvii)	(b)	(xviii)	(c)	(xix)	(b)	(xx)	(c)

2. Short Questions:

(i) Def. Ex. $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4, 5\}$. A is a subset of B.

(ii) ϕ, $\{a\}$, $\{b\}$, $\{a, b\}$

(x) (i) $(A \cap B)' = A' \cup B'$ (ii) $(A \cup B)' = A' \cap B'$

3. Fill in the Blanks:

(i) B (ii) Disjoint sets (iii) $A = B$

(iv) $(A \cap B) \cup (A \cap C)$ (v) $(A \cup B) \cap (A \cup C)$

(vi) ϕ (vii) U (viii) ϕ

(ix) U (x) $A \setminus B$ (xi) IIIrd quadrant

(xii) IVth quadrant (xiii) Zero (xiv) Zero

(xv) $\{a, b, c\}$ (xvi) $\{a, b, c\}$ (xvii) John Venn

(xviii) Binary relation (xix) onto (xx) not

Unit 6: Basic Statistics

EXERCISE 6.1

4.

Classes	2—3	4—5	6—7	8—9	10—11	12—13	14—15
Frequency	2	1	9	5	6	5	3

a) 6 — 7 b) 4 — 5

EXERCISE 6.2

3. (i) 24.5 (ii) 290

4. (i) 24.5 (ii) 290

5. 32.5

6. A.M = 9.620 G.M=8.553 H.M = 8.089

6. A.M = 9.620 G.M=8.553 H.M = 8.089

7. Mode = 9, 4 Median=7

8. Mode = 2 Median = 2

9. Mean = 10.478 Median = 10.625 Mode = 13.5

10. (i) Weighted Mean = 74 marks (ii) Mean =72.8 marks

11. Weighted Mean = 41.15 rupees per litre

12.

2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
-------	113.33	126	142.66	159.33	178	195.33	208.67	220	-------

EXERCISE 6.3

4. Range = 3500 S.D. ≈ 1417.886

5. a- (i) S.D. = 4.87 (ii) S.D. = 3.87 b- Variance = 6.85

6. Mean = 27.0935 S.D. =3.136

7. Range = 43

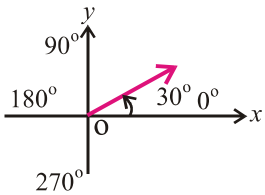
MISCELLANEOUS EXERCISE 6

1. (i) (b) (ii) (b) (iii) (a) (iv) (c) (v) (b)
(vi) (a) (vii) (a) (viii) (a) (ix) (b) (x) (c)
(xi) (b) (xii) (a) (xiii) (c) (xiv) (c) (xv) (a)
(xvi) (a) (xvii) (b) (xviii) (b) (xix) (a) (xx) (b)
(xxi) (a) (xxii) (c)

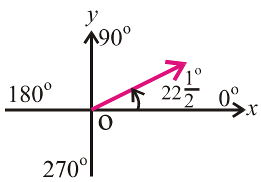
Unit 7: Introduction to Trigonometry

EXERCISE 7.1

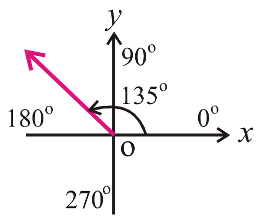
1. (i)



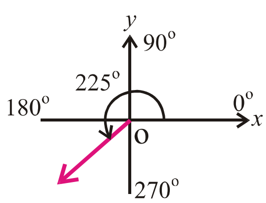
(ii)



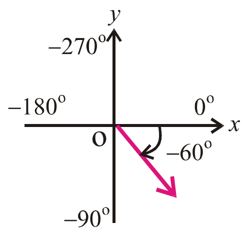
(iii)



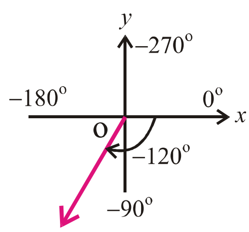
(iv)

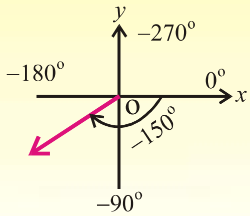


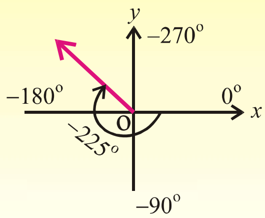
(v)



(vi)



(vii) 

(viii) 

2. (i) 45.5° (ii) 60.5083° (iii) 125.3805°

3. (i) 47°21′36″ (ii) 125°27′ (iii) 225°45′ (iv) −22°30′ (v) −67°34′48″ (vi) 315°10′48″

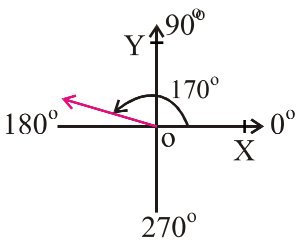
4. (i) $\frac{\pi}{6}$ (ii) $\frac{\pi}{3}$ (iii) $\frac{3\pi}{4}$ (iv) $\frac{5\pi}{4}$ (v) $\frac{-5\pi}{6}$ (vi) $\frac{-5\pi}{4}$ (vii) $\frac{5\pi}{3}$ (viii) $\frac{7\pi}{4}$

5. (i) 135° (ii) 150° (iii) 157.5° (iv) 146.25° (v) 171.8869° (vi) 257.83° (vii) −157.5° (viii) −146.25°

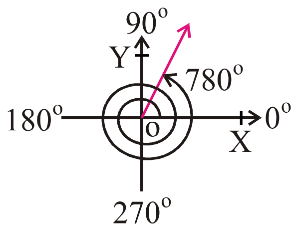
EXERCISE 7.2

1. (i) 0.57rad (ii) 1.8rad
2. (i) 15.4cm (ii) 15.84 mm
3. (i) 16cm (ii) 66.21 cm
4. 18m
5. 220m
6. $\frac{\pi}{2}$ rad
7. 12.57cm
8. 105.56 cm^2
9.(a) 18.85 cm^2 (b) 157.08 cm^2
10. $\frac{49\pi}{18} m^2$ or $8.55m^2$
11. 2972.39 cm^2
12. 31.42 cm^2
13. 5 rad.

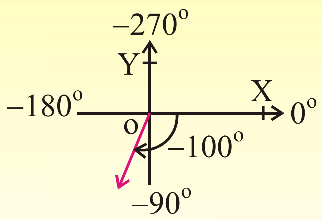
EXERCISE 7.3

1. (i) 

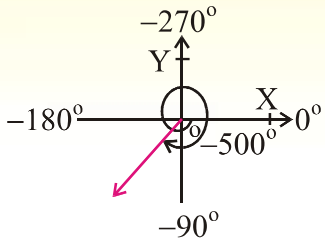
Positive coterminal angle 360° + 170° = 530°
negative coterminal angle −190°

(ii) 

Positive coterminal angle 60°
negative coterminal angle is −300°

(iii) 

Positive coterminal angle is 260°
negative coterminal angle –360° – 100° = –460°

(iv) 

Positive coterminal angle 220°
negative coterminal angle –140°

2. (i) 90°, 180° (ii) 270°, 360° (iii) 540°, 630° (iv) 0°, 90°

3. (i) $0, \frac{\pi}{2}$ (ii) $\frac{\pi}{2}, \pi$ (iii) $0, \frac{-\pi}{2}$ (iv) $\frac{-\pi}{2}, -\pi$

4. (i) II (ii) III (iii) IV (iv) II (v) I (vi) III

5. (i) +*ve* (ii) –*ve* (iii) –*ve* (iv) –*ve* (v) +*ve* (vi) –*ve*

6. (i) II, $\sin\theta = \frac{3}{\sqrt{13}}$; $\operatorname{cosec}\theta = \frac{\sqrt{13}}{3}$; $\cos\theta = \frac{-2}{\sqrt{13}}$; $\sec\theta = -\frac{\sqrt{13}}{2}$; $\tan\theta = \frac{-3}{2}$; $\cot\theta = \frac{-2}{3}$

(ii) III, $\sin\theta = \frac{-4}{5}$; $\operatorname{cosec}\theta = \frac{-5}{4}$; $\cos\theta = \frac{-3}{5}$; $\sec\theta = \frac{-5}{3}$; $\tan\theta = \frac{4}{3}$; $\cot\theta = \frac{3}{4}$

(iii) I, $\sin\theta = \frac{1}{\sqrt{3}}$; $\operatorname{cosec}\theta = \sqrt{3}$; $\cos\theta = \sqrt{\frac{2}{3}}$; $\sec\theta = \sqrt{\frac{3}{2}}$; $\tan\theta = \frac{1}{\sqrt{2}}$; $\cot\theta = \sqrt{2}$

7. $\sec\theta = \frac{-3}{2}$; $\sin\theta = \frac{\sqrt{5}}{3}$; $\operatorname{cosec}\theta = \frac{3}{\sqrt{5}}$ or $\frac{3\sqrt{5}}{5}$; $\tan\theta = \frac{-\sqrt{5}}{2}$; $\cot\theta = \frac{-2}{\sqrt{5}}$

8. $\sin\theta = \frac{-4}{5}$; $\operatorname{cosec}\theta = \frac{-5}{4}$; $\cos\theta = \frac{-3}{5}$; $\sec\theta = \frac{-5}{3}$; $\cot\theta = \frac{3}{4}$

9. $\tan\theta = -1$; $\sec\theta = \sqrt{2}$; $\operatorname{cosec}\theta = -\sqrt{2}$

10. $\sin\theta = \frac{12}{13}$; $\cos\theta = \frac{5}{13}$; $\sec\theta = \frac{13}{5}$; $\tan\theta = \frac{12}{5}$; $\cot\theta = \frac{5}{12}$

11. (*i*) $\sin\theta = \frac{\sqrt{7}}{4}$; $\operatorname{cosec}\theta = \frac{4}{\sqrt{7}}$; $\cos\theta = \frac{3}{4}$; $\sec\theta = \frac{4}{3}$; $\tan\theta = \frac{\sqrt{7}}{3}$; $\cot\theta = \frac{3}{\sqrt{7}}$

(ii) $\sin\theta = \frac{8}{17}$; $\operatorname{cosec}\theta = \frac{17}{8}$; $\cos\theta = \frac{15}{17}$; $\sec\theta = \frac{17}{15}$; $\tan\theta = \frac{8}{15}$; $\cot\theta = \frac{15}{8}$

(iii) $\sin\theta = \frac{2\sqrt{10}}{7}$; $\operatorname{cosec}\theta = \frac{7}{2\sqrt{10}}$; $\cos\theta = \frac{3}{7}$; $\sec\theta = \frac{7}{3}$; $\tan\theta = \frac{2\sqrt{10}}{3}$; $\cot\theta = \frac{3}{2\sqrt{10}}$

12. (i) $\frac{1}{\sqrt{3}}$ (ii) $\frac{-1}{\sqrt{3}}$ (iii) $\frac{2}{\sqrt{3}}$ (iv) 1 (v) $\frac{-1}{2}$ (vi) $\frac{2}{\sqrt{3}}$ (vii) 0 (viii) 0

(ix) $\frac{-\sqrt{3}}{2}$ (x) $\frac{-1}{2}$ (xi) $\frac{1}{\sqrt{3}}$ (xii) $\frac{-1}{\sqrt{2}}$

EXERCISE 7.4

1. $\tan^2 x$ 2. $\tan^2 x$ 3. $\sin x$ 4. $\sin^2 x$
5. $\tan^2 x$ 6. $\cos^2 x$

EXERCISE 7.5

1. 59.74° 2. 18.652m 3. 75.5° or $75^{\circ}30'$
4. 27.47° 5. 4924.04m 6. 3356.4 m 7. 28.72m
8. 0.199 miles 9. 25.94 feet 10. 2928.2 feet 11. 164m ; 164m (or 163.93)
12. 20.33 meter

MISCELLANEOUS EXERCISE 7

Q.1. (i) (*a*) (ii) (*d*) (iii) (*c*) (iv) (*b*) (v) (*c*)
(vi) (*b*) (vii) (*a*) (viii) (*b*) (ix) (*c*) (x) (*b*)

Q.2. (iii) 10800′ (v) 45^{o} (vi) $\frac{\pi}{12}$ rad. (vii) 2 rad. (viii) 71.27cm (x) $\frac{40}{9}$

Q.3. (i) 180^{o} (ii) III (iii) IV (iv) $\frac{1}{2}r^2\theta$ (v) 6 cm^2

(vi) $2k\pi + 120^{o}$ where $k = 1$ (vii) $\theta = 30^{o}$ or $\frac{\pi}{6}$ rad (viii) 2

(ix) $\operatorname{cosec}^2\theta$ (x) $\frac{1-\sin\theta}{\cos\theta}$

Unit 8: Projection of a Side of a Triangle

EXERCISE 8.1

1. 2.646 cm , $\frac{\sqrt{3}}{2}$ sq cm 2. $m\overline{AC} = 2\sqrt{29}$ cm

EXERCISE 8.2

1. $m\overline{BC} \approx 5.29$ cm 2. 5.45 cm

MISCELLANEOUS EXERCISE 8

3. $\simeq$ 4.58 cm 4. $\simeq$ 4.12 cm 5. 15 cm
6. 6 cm 7. 90^{o} 8. $\simeq (61.9)^0$
9. Acute angled 10. Right angled

Unit 9: Chords of a Circle

EXERCISE 9.1

3. 10 cm   4. ≃ 14.97 cm

EXERCISE 9.2

3. 7 cm

MISCELLANEOUS EXERCISE 9

1.	(i) (*c*)	(ii) (*a*)	(iii) (*d*)	(iv) (*c*)
	(v) (*a*)	(vi) (*b*)	(vii) (*c*)	(viii) (*b*)
	(ix) (*a*)	(x) (*c*)	(xi) (*b*)	(xii) (*b*)
	(xiii) (*d*)	(xiv) (*c*)		

Unit 10: Tangent of a Circle

EXERCISE 10.2

2. 4 cm   3. ≃ 16.96 cm

MISCELLANEOUS EXERCISE 10

1.	(i) (*c*)	(ii) (*a*)	(iii) (*d*)	(iv) (*b*)
	(v) (*d*)	(vi) (c)	(vii) (*b*)	(viii) (*d*)
	(ix) (*c*)	(x) (*a*)	(xi) (*c*)	(xii) (*b*)
	(xiii) (*b*)			

Unit 11: Chords and Arcs

MISCELLANEOUS EXERCISE 11

1.	(i) (*d*)	(ii) (*c*)	(iii) (*b*)	(iv) (*b*)
	(v) (*a*)	(vi) (*c*)	(vii) (*b*)	(viii) (*c*)
	(ix) (*a*)	(x) (*b*)		

Unit 12: Angle in a Segment of a Circle

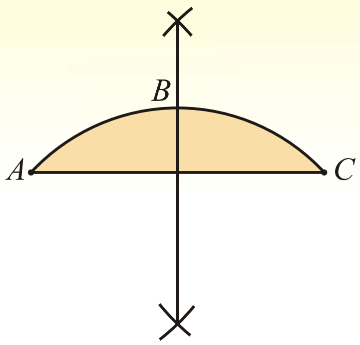
MISCELLANEOUS EXERCISE 12

1.	(i) (*c*)	(ii) (*d*)	(iii) (*a*)	(iv) (*c*)
	(v) (*b*)	(vi) (*d*)	(vii) (*d*)	(viii) (*b*)
	(ix) (*d*)	(x) (*c*)		

EXERCISE 13.1

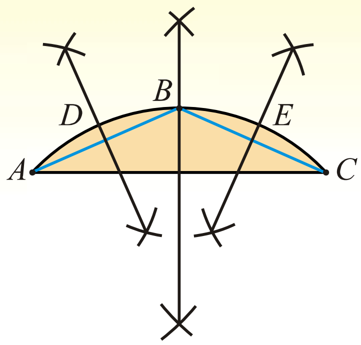
1

(*i*)



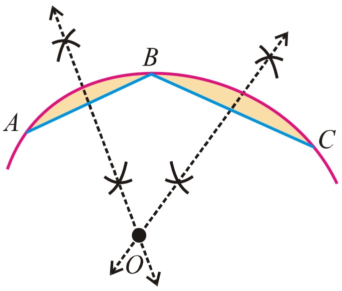
Two equal parts of the arc AC are $\overset{\frown}{AB}$ and $\overset{\frown}{BC}$

(*ii*)

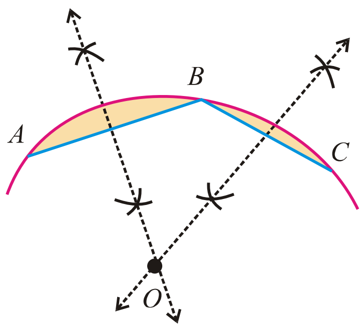


Four equal parts of the arc AC are $\overset{\frown}{AD}$, $\overset{\frown}{DB}$, $\overset{\frown}{BE}$, $\overset{\frown}{EC}$

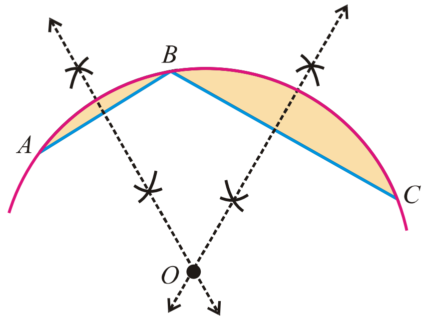
2



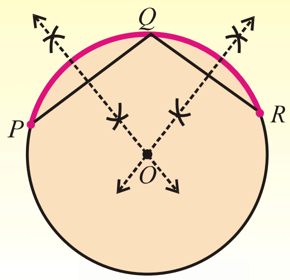
3 (*i*)



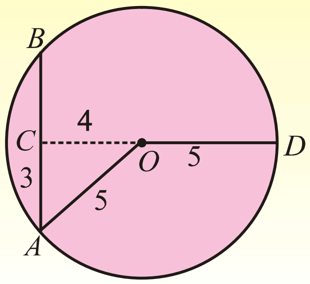
(*ii*)



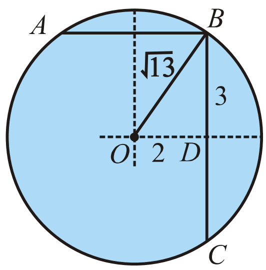
4



5.



6.



EXERCISE 13.2

1. radius = 3.3 cm. 2. 1 cm (approximately) 3. 2.3 cm

MISCELLANEOUS EXERCISE 13

1. MCQ's

(i)	(*c*)	(ii)	(*b*)	(iii)	(*a*)	(iv)	(*a*)	(v)	(*b*)
(vi)	(*c*)	(vii)	(*a*)	(viii)	(*c*)	(ix)	(*a*)	(x)	(*c*)
(xi)	(*a*)	(xii)	(*a*)	(xiii)	(*b*)	(xiv)	(*b*)	(xv)	(*c*)
(xvi)	(*b*)	(xvii)	(*b*)	(xviii)	(*c*)				

2. (ii) 24 cm (iii) $\frac{360^{o}}{n}$ (iv) 25 cm

3. Fill in the Blanks:

i.	circumference	ii.	boundary	iii.	chord
iv.	centre	v.	coincide	vi.	less
vii.	greater	viii.	one	ix.	non-collinear
x.	right	xi.	contact, centres	xii.	collinear
xiii.	two	xiv.	perpendicular	xv.	tangent
xvi.	two	xvii.	centre	xviii.	equal
xix.	equal	xx.	equilateral	xxi.	concentric
xxii.	incentre	xxiii.	circumcentre	xxiv.	in-radius
xxv.	circum-radius				

SYMBOLS AND ABBREVIATIONS

Adj. A	Adjoint of A	$\because$	Since or because
A^t	Transpose of A	det A or $\|A\|$	determinant of A
A^{-1}	Inverse of A	π	pi
Add	Addition, adding	$a \times 10^n$	form for scientific notation
$\log_a x$	Logarithm of x to the base a	pt	Point
i	ieota, no. whose square is -1	w.r.t.	With respect to
+ve	positive	−ve	Negative
$\in$	Belongs to	$\notin$	does not belong to
$\forall$	For all	$=$	Equal to
$\exists$	There exist	$\neq$	Not equal to
Alt	Alternate	$\therefore$	Therefore
Constr	Construction	i.e.	that is
Cor	Corollary	$\Rightarrow$	implies that
Corresp	Corresponding	$^\circ$	degree
Def	Definition	$'$	minute or foot
Ext	Exterior	$''$	second or inch
Fig	Figure	cm	centimeter
Iff	If and only if	$\approx$	nearly equal to
Iso	Isosceles	$\cong$	is congruent to
Mid pt.	Middle point	$\leftrightarrow$	correspondence
perp	Perpendicular	Δ^s	Triangles
prob.	Problem	$\geq$	greater than or equal to
Quad.	Quadrilateral	$\leq$	less than or equal to
Rect	Rectangle	$\lfloor$rt	right angle
Rhmb	Rhombus	Δ	Triangle
Sq	Square	$\perp$	is perpendicular to
st line	Straight line	$\parallel$	is parallel to
Th	Theorem	$\parallel$ gm	parallelogram
Trap	Trapezium	$\odot$	circle
vert opp.	Vertically opposite	O^{ce}	circumference
Q.E.D	Quod Erat Demonstramdum	$\overset{\frown}{AB}$	arc AB
θ	Theta (angle measure)	$\overline{AB}$	line segment AB.
ω	Omega	Φ	phi

TABLE OF LOGARITHM

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4 4	9 8	13 12	17 16	21 20	26 24	30 28	34 32	38 36
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4 4	8 7	12 11	15 15	19 19	23 22	27 26	31 30	35 33
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3 3	7 7	11 10	14 14	18 17	21 20	25 24	28 27	32 31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3 3	7 7	10 10	13 13	16 16	20 19	23 22	26 25	30 29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3 3	6 6	9 9	12 12	15 15	19 17	22 20	25 23	28 26
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3 3	6 6	9 8	11 11	14 14	16 17	20 19	23 22	26 25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3 3	5 5	8 8	11 10	14 13	17 16	19 18	22 21	24 23
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	3 2	5 5	8 7	10 10	13 12	15 15	18 17	20 20	23 22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2 2	5 5	7 7	9 9	12 11	14 14	16 16	19 18	21 21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2 2	4 4	7 6	9 8	11 11	13 13	16 15	18 17	20 19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3785	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
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41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8

TABLE OF LOGARITHM

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51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8738	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9603	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

TABLE OF ANTILOGARITHM

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.01	1023	1026	1027	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	0	1	1	1	2	2	2	2
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
.06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
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.20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
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.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
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.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
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.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

TABLE OF ANTILOGARITHM

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.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	6	6	7	8
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
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.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
.80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
.82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
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.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
.90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
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.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

GLOSSARY

Unit 1

Quadratic equation: An equation which contains the square of the unknown (variable) quantity, but no higher power, is called a *quadratic equation* or an equation of the *second degree.*

Second degree equation

General or standard form

A *second degree equation* in one variable x, is $ax^2 + bx + c = 0$, $a \neq 0$ and a, b, c are constants is called the *general or standard form* of a quadratic equation. Where a is the co-efficient of x^2, b is the co-efficient of x and constant term is c.

Reciprocal equation: An equation is said to be a *reciprocal equation,* if it remains unchanged, when x is replaced by $\frac{1}{x}$.

Exponential equations: In *exponential equations* variable occurs in exponent.

Radical equation: An equation involving expression under the *radical sign* is called a *radical equation*.

Unit 2

Discriminant: The expression "$b^2 - 4ac$" of the quadratic expression $ax^2 + bx + c$ is called Discriminant.

Cube roots: The *cube roots* of unity are 1, ω and ω^2.

Complex cube roots: *Complex cube roots* of unity are ω and ω^2.

Properties of cube roots of unity

(a) The *product* of three cube roots of unity is one. *i.e.*, $(1)(\omega)(\omega^2) = \omega^3 = 1$

(b) Each of the complex cube roots of unity is *reciprocal* of the other.

(c) Each of the complex cube roots of unity is the *square* of the other.

(d) The *sum* of all the cube roots of unity is zero, *i.e.*, $1 + \omega + \omega^2 = 0$

Roots of the quadratic equation: The *roots* of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ are

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Sum and the product: The *sum* and the *product* of the roots of a quadratic equation

$$\alpha+\beta=\frac{-b}{a} \quad \text{and} \quad \alpha\beta=\frac{c}{a}$$

Symmetric functions: *Symmetric functions* of the roots of a quadratic equation are those functions in which all the roots involved are alike, so that the value of the expression remains unaltered, when roots are interchanged.

Formation of a quadratic equation as:

x^2 – (sum of the roots) x + product of the roots = 0

$\Rightarrow \quad x^2-(\alpha+\beta)x+\alpha\beta=0$

Synthetic division: *Synthetic division* is the process of finding the quotient and remainder, when a polynomial is divided by a linear polynomial.

Simultaneous equations: A system of equations $f(x, y) = 0$ and $g(x, y) = 0$ having a common solution is called a system of *simultaneous equations*.

Unit 3

Ratio: A relation between two quantities of the same kind is called *ratio*.

Proportion: A *proportion* is a statement, which is expressed as equivalence of two ratios.

If two ratios $a : b$ and $c : d$ are equal, then we can write $a : b = c : d$

Direct variation: If two quantities are related in such a way that when one changes in any ratio so does the other is called *direct variation*.

Inverse variation If two quantities are related in such a way that when one quantity increases, the other decreases is called *inverse variation*.

Theorem on proportions:

(1) *Theorem of Invertendo*

If $a : b = c : d$ then $b : a = d : c$

(2) *Theorem of Alternando*

If $a : b = c : d$, then $a : c = b : d$

(3) *Theorem of Componendo*

If $a : b = c : d$, then

(*i*) $a + b : b = c + d : d$

(*ii*) $a : a + b = c : c + d$

(4) *Theorem of Dividendo*

If $a : b = c : d$, then

(i) $a-b:b=c-d:d$

(ii) $a:a-b=c:c-d$

(5) *Theorem of Componendo-dividendo*

If $a:b=c:d$, then

$a+b:a-b=c+d:c-d$

Joint variation: A combination of direct and inverse variations of one or more than one variables forms *joint variation*.

K-Method:

If $\frac{a}{b}=\frac{c}{d}$

Then $k=\frac{a}{b}=\frac{c}{d}$ or $a=kb$ and $c=kd$

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$ then $a=bk$, $c=dk$ and $e=fk$

Unit 4

Fraction: A *fraction* is an indicated quotient of two numbers or algebraic expressions.

Equation: An *equation* is equality between two expressions.

Identity: An *identity* is an equation which is satisfied by all the values of the variables involved.

Rational fractional: An expression of the form $\frac{N(x)}{D(x)}$, where $N(x)$ and $D(x)$ are polynomials in x with real coefficient, is called a *rational fractional.* Every fractional expression can be expressed as a quotient of two polynomials.

Proper rational fraction: A rational fraction $\frac{N(x)}{D(x)}$, with $D(x)\neq 0$ is called a *proper rational fraction* if degree of the polynomial $N(x)$, in the numerator is less than the degree of the polynomial $D(x)$, in the denominator.

Improper fraction: A rational fraction $\frac{N(x)}{D(x)}$, with $D(x)\neq 0$ is called an *improper fraction* if degree of the polynomial $N(x)$ in the numerator is greater or equal to the degree of the polynomial $D(x)$ in the denominator.

Partial fractions: Decomposition of resultant fraction $\frac{N(x)}{D(x)}$, when

(a) denominator $D(x)$ consists of non-repeated linear factors.

(b) denominator $D(x)$ consists of repeated linear factors.

(c) denominator $D(x)$ contains non-repeated irreducible quadratic factor.

(d) denominator $D(x)$ has repeated quadratic factor.

Unit 5

Set	A set is the *well defined collection* of *distinct* objects with some common properties.
Union of sets	Union of two sets A and B denoted by $A \cup B$ is the set *containing elements* which either belong to A or to B or to both.
Intersection of sets	Intersection of two sets A and B denoted by $A \cap B$ is the set of *common elements* of both A and B. In symbols $A \cap B = \{x : \forall\ x \in A \text{ and } x \in B\}$.
Difference of sets:	The set difference of B and A denoted by $B - A$ is the set of all those elements of B but *do not belonging to* A.
Compliment:	*Complement* of a set A w.r.t. universal set U is denoted by $A^C = A'$ $= U - A$ contains all those elements of U which *do not belong to* A.
Closed figures:	British mathematician John Venn (1834 – 1923) introduced rectangle for a universal set U and its subsets A and B as *closed figures* inside this rectangle.
Specific order:	An ordered pair of elements is written according to a *specific order* for which the order of elements is strictly maintained.
Ordered pairs:	Cartesian product of two non empty sets A and B denoted by $A \times B$ consists of all *ordered pairs* (x, y) such that $\forall\ x \in A$ and $\forall\ y \in B$.
Binary Relation:	Suppose A and B are two non empty sets then *relation* $f : A \to B$ is called a function if (i) Dom f = set A (ii) every $x \in A$ appears in one and only ordered pair $\in f$.
Function:	Suppose A and B are two non empty sets then *relation* $f : A \to B$ is called a function if (i) Dom f = set A (ii) $\forall\ x \in A$ we can associate some unique image element $y = f(x) \in B$.
First elements & second elements:	Dom f is the set consisting of all *first elements* of each ordered pair $\in f$ and range f is the set consisting of all *second elements* of each ordered pair $\in f$.
Into function	A function $f : A \to B$ is called an into function if at least one element in B is not an image of some element of set A *i.e., Range of* $f \subseteq$ *set* B.

Onto function

A function $f : A \to B$ is called an onto function if every element of set B is an image of at least one element of set A *i.e.*, *Range of* f = *set* B.

One-one function:

A function $f : A \to B$ is called one-one function if all *distinct elements* of A have distinct images in B

Bijective function:

A rule $f : A \to B$ is called bijective function iff function f is *one-one and onto*.

Constant function:

A function $f : A \to B$ is called a constant function if $\forall\ x \in A$. There is an element $C \in B$ such that $f(x) = C$.

Identity function:

A function $f : A \to A$ is called Identity function if $\forall\ x \in A$ we can associate some *unique image element* x itself such that

$$f(x) = x \quad \forall\, x \in A.$$

Unit 6

Frequency distribution:

A *frequency distribution* is a tabular arrangement classifying data into different groups.

Class limits

(a) The minimum and the maximum values defined for a class or group are called *class limits*.

(b) The real class limits of a class is called *class boundary*. It is obtained by adding two successive class limits and dividing the sum by 2.

(c) For a given class the average of that class obtained by dividing the sum of upper and lower class limit by 2, is called the *midpoint or class mark* of that class.

(d) The total of frequency up to an upper class limit or boundary is called the *cumulative frequency*.

Histogram

A *Histogram* is a graph of adjacent rectangles constructed on XY-plane.

Arithmetic mean

Arithmetic mean is a measure that determines a value of the variable under study by dividing the sum of all values of the variable by their number.

Deviation

A *Deviation* is defined as 'a difference of any value of the variable from any constant'. $D_i = x_i - A$.

Geometric mean

Geometric mean of a variable X is the nth positive root of the product of the $x_1, x_2, x_3, ..., x_n$ observations. In symbols we write,

$$G.M = (x_1.x_2.x_3.......x_n)^{1/n}$$

Harmonic mean *Harmonic mean* refers to the value obtained by reciprocating the mean of the reciprocal of $x_1, x_2, x_3, ..., x_n$ observations.

Mode: *Mode* is defined as the most frequent occurring observation of the variable or data.

$$Mode = L + \frac{f_m - f_1}{2f_m - f_1 - f_2} \times h$$

Median: *Median* is the measure which determines the middlemost observation in a data set. $Median = L + \frac{h}{f}\left\{\frac{n}{2} - c\right\}$

Dispersion: Statistically, *Dispersion* means the spread or scatterness of observations in a data set.

Range: *Range* measures the extent of variation between two extreme observations of a data set. It is given by the formula:

$$\text{Range} = X_{max} - X_{min} = X_m - X_0$$

Variance: *Variance* is defined as the mean of the squared deviations of x_i (i = 1, 2, ….., n) observations from their arithmetic mean. In symbols,

$$\text{Variance of } X = \text{Var}(X) = S^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{X})^2}{n} = \frac{\sum(X - \bar{X})^2}{n}$$

Standard deviation: *Standard deviation* is defined as the positive square root of mean of the squared deviations of x_i (i = 1, 2, ….., n) observations from their arithmetic mean. In symbols we write,

$$\text{Standard Deviation of } X = \text{S.D}(X) = \sqrt{\frac{\sum(X - \bar{X})^2}{n}}$$

Unit 7

Degree: If we divide the circumference of a circle into 360 equal arcs. Then the angle subtended at the centre of the circle by one arc is called one *degree* and is denoted by 1^o.

Radian: The angle subtended at the centre of the circle by an arc, whose length is equal to the radius of the circle, is called one *radian.*

Relationship between radian and degree measure: $1^{o} = \frac{\pi}{180}$ radian , ≈ 0.0175 radian and 1 radian = $\left(\frac{180}{\pi}\right)^{o}$, ≈ 57.295 degrees.

Relation between angle, arc length and radius: Relation between central angle and arc length of a circle: $\boxed{l = r\theta}$

Area of a circular sector: Area of a circular sector, $A = \frac{1}{2}r^2\theta$

Coterminal angle: Two or more than two angles with the same initial and terminal sides are called *coterminal angles.*

Quadrantal angle: An angle is called a *quadrantal angle*, if its terminal side lies on the x-axis or y-axis.

Standard position: A general angle is said to be in *standard position* if its vertex is at the origin and its initial side is directed along the positive direction of the x-axis of a rectangular coordinate system.

Trigonometric ratios: There are six fundamental *trigonometric ratios* (functions) called sine, cosine, tangent, cotangent, secant and cosecant.

Trigonometric Identities: *Trigonometric Identities*(a) $\cos^2\theta + \sin^2\theta = 1$

(b) $\sec^2\theta - \tan^2\theta = 1$

(c) $\text{cosec}^2\theta - \cot^2\theta = 1$

Unit 8

Projection: The *projection* of a given point on a line is the foot of ⊥ drawn from the point on that line. However the projection of given point P on a line AB is the point P itself.

Zero dimension: The projection of a finite line on an other line is the portion of the latter intercepted between the projection of ends of the given finite line. However projection of a vertical line on an other line is the join of these two intersecting lines which is of *zero dimension*.

Obtuse angle: An angle which is greater than 90° is called *obtuse angle.*

Right angle An angle which is equal to 90° is called *right angle.*

Acute: An angle which is less than 90° is called *acute angle.*

Unit 9

Circle: A *circle* is the locus of a moving point P in a plane which is equidistant from some fixed point N. The fixed paint N not lying on the circle is called the *centre* and the constant distance PN is called its *radius*.

Circumference: $2\pi r$ is the *circumference* of a circle with radius r.

Circular area: πr^2 is the *circular area* of a circle of radius r.

Collinear points: The points lying on the same line are *collinear points* otherwise they are *non-collinear points*.

Circumcirlce: The circle passing through the vertices of a triangle is called its *circumcirlce* where $\perp$ bisectors of sides of the triangle provides the centre.

Unit 10

Secant: A *secant* is a st line which cuts the circumference of a circle in two distinct points.

Tangent: A *tangent* to a circle is the St line which meets the circumference at one point only and being produced does not cut it at all. The point of tangency is also known as the point of contact. AB is the tangent line to the circle C.

Length of a tangent: The *length of a tangent* to a circle is measured from the given point to the point of contact.

Unit 12

Sector: The *sector* of a circle is an area bounded by any two radii and the arc intercepted between them.

Central angle: A *central angle* is subtended by two radii at the centre of the circle.

Circumangle: A *circumangle* is subtended between any two chords of a circle, having common point on its circumference.

Chord: The join of any two points on the circumference of the circle is called its chord.

Cyclic quadrilateral: A quadrilateral is called *cyclic* when a circle can be drawn through its four vertices.

In-centre: *In-centre* of a triangle is the centre of a circle inscribed in a triangle.

Unit 13

Circle: A *"circle"* is locus of a moving point in a plane which is equidistant from a fixed point. The fixed point is called *"centre"* of the circle.

Radius: The distance from the centre of the circle to any point on the circle is called *radius* of the circle.

Perimeter: The *perimeter* of a closed geometric figure is the sum of its sides.

circumference The perimeter or length of the boundary of the circle is called the *circumference*.

Diameter: A chord which passes through the centre of the circle is called *diameter* of the circle.

Arc: A part of circumference of a circle is called **an** *arc*.

Triangle: A plane figure formed by three straight edges as its sides is called a *triangle*.

Polygon: A plane figure with three or more straight edges as its sides is called a *polygon*.

Regular polygon: A figure bounded by equal straight lines which has all its angles equal is called a *regular polygon*.

Vertices: The corners of a polygon are called its *vertices*.

Locus: The path of an object moving according to some rule, is the *locus* of the object.

Circumscribed circle: If a circle passes through all the vertices of a polygon the circle is said to be *circumscribed* about the polygon and the polygon is said to be *inscribed* in the circle.

Escribed circle: If a circle touches one side of a triangle externally and the other two produced sides internally, is called *escribed* circle.

Circum circle: The circle passing through the vertices of triangle ABC is known as *circum circle*, its radius as *circum radius* and centre as *circum centre*.

In circle: A circle which touches the three sides of a triangle internally is known as *in-circle* its radius as *in-radius* and centre as *in-centre*.

INDEX

A

Alternando 58
Angle of depression 166
Angle of elevation 166
Angle in standard position 154
Antecedent 50
Appolonius theorem 175
Appropriate problems 24
Arc 180
Area of a circular sector 151
Arithmetic mean 118
Associative property 89

B

Basic Statistics 107
Bijective Function 101
Binary relation 99
Biquadratic 38

C

Cartesian product 98
Central angle 206
Central tendency 118
Central value 118
Chord 180
Circle 180
Circum-centre 224
Circumcirlce 180
Circumference 180
Circumscribed circle 224
Class boundaries 110
Class limits 110
Co-domain 99
Co-efficients 2,30,79
Common tangent 232
Commutative property 88
Complement of set 87
Completing square 3
Complex cube roots 23
Componendo 59
Componendo-dividendo 60
Concentrric circles 180
Convergent lines 236
Congruent 185
Congruent circles 203
Consequent 50
Continued proportion 58
Correspondence 101
Corresponding arcs 204
Coterminal angles 153
Cube roots of unity 22
Cubic equation 37
Cumulative frequency 110,115

D

Data 108
De Morgan's laws 90
Degree 146
Denominator 75
Depressed equation 37
Diameter 183
Difference of sets 86
Direct method 118
Direct variation 53
Discriminant 19
Dispersion 136
Distributive property 89
Dividend 36
Dividendo 60
Domain 99

E

E-centre 226
E-circle 226

Equidistant 184
Equilateral triangle 226
Equal circles 232
E-radii 226
Escribed circle 225
Exponential equations 10
Externally 195
Extraneous roots 12,13
Extremes 52

F

Factorization 2
Formation of quadratic equation 32
Fourth proportional 57
Fraction 75
Frequency distribution 108
Frequency polygon 114
Function 99

G

General angle 153
Geometric mean 118
Grouped data 120

H

Harmonic mean 118
Hexagon 227
Histograms 111

I

Identity 77
Imaginary 19
Improper fraction 75
Indirect methods 119
In-centre 225
Initial side 146
Injective function 101
Inscribed circle 215
Internally 196
Intersecting circles 234
Intersection of sets 86
Inverse variation 55
Invertendo 59
Irrational 19

J, K, L

Joint variation 64
K-method 66
Linear equation 2
Linear factors 2, 78

M

Minor sector 180
Major segment 180
Mapping 99
Mean proportion 57
Mean 118
Median 118
Midpoint 110
Minor arc 204
Major arc 204
Minor segment 180
Mode 118

N

Nature of the roots 19
Non collinear points 181
Non-repeated 80

O

Obtuse 172
Ordered pairs 39,98
Order 50

P

Partial fractions 74,76
Perfect square 19,20
Perpendicular bisectors 183
Polygon 116,222
Polynomial 35
Product of the roots 26, 28
Projection 176
Proper fraction 75

Properties of cube roots of unity 23
Proportion 51
Pure quadratic 2, 3
Pythagorean theorem 157

Q

Quadrantal angle 155
Quadrants 155
Quadratic expression 19
Quadratic equations 2
Quadratic formula 5
Quadrilateral 215
Quartiles 118
Quotient 36

R

Radial segment 180
Radian 148
Radical equation 12
Radius 180
Range 99,147
Ratio 50
Rational fraction 75
Reciprocal equations 9
Reciprocal identities 156
Reducible 8
Remainder 36
Resultant fraction 76
Right bisectors 222
Roots 19

S

Secant 190
Second degree 2
Sector of a circle 150,180
Semi-circle 214
Sets 86,89
Sexagesimal 146
Simultaneous equations 39
Solution 2
Square roots 4,5
Standard deviation 137
Standard form 2
Standard position 154
Sum of the roots 26, 27
Supplementary 215
Surjective function 101
Symmetric functions 30
Synthetic division 35
System of equations 39

T

Tally marks 108
Tangent 229
Tangent to a circle 190,229
Terminal side 146
Third proportional 57
Touching circles 234
Transverse 233
Trigonometric identities 163
Trigonometric ratios 153

U

Unequal circles 233
Ungrouped data 118
Union of sets 89
Units 50

V, Z

Variable 2
Variance 137
Variations 53
Venn diagrams 94
Zeros of polynomial 37

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