EXERCISE 2.2

Q.1 Find the cube roots of $-1, 8, -27, 64$.

(i) Cube roots of -1

Solution:

Let $x = (-1)^{\frac{1}{3}}$

$x^3 = -1$

$x^3 + 1 = 0$

$x^3 + (1)^3 = 0$

$\because (a^3 + b^3) = (a+b)(a^2 - ab + b^2)$

$(x+1)\left[x^2 - (x)(1) + 1^2\right] = 0$

$(x+1)(x^2 - x + 1) = 0$

$x + 1 = 0$ or

$\boxed{x = -1}$ or $\boxed{x^2 - x + 1 = 0}$

Now we solve $x^2 - x + 1 = 0$ by formula

$ax^2 + bx + c = 0$

$a = 1, \ b = -1, \ c = 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2\times1}$$

$$= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2\times1}$$

$$= \frac{1 \pm \sqrt{1-4}}{2}$$

$$= \frac{1 \pm \sqrt{-3}}{2}$$

$$= -1\left(\frac{-1 \pm \sqrt{-3}}{2}\right)$$

$x = -1\left(\frac{-1+\sqrt{-3}}{2}\right)$; $x = -1\left(\frac{-1-\sqrt{-3}}{2}\right)$

$x = -1\omega$ or $x = -1(\omega^2)$

$x = -\omega$ or $x = -\omega^2$

So, cube roots of -1 are -1, $-\omega$ and $-\omega^2$

(ii) Cube roots of 8

Solution:

Let $x = (8)^{\frac{1}{3}}$

$x^3 = 8$

$x^3 - 8 = 0$

$x^3 - 2^3 = 0$

$\because (a^3 - b^3) = (a-b)(a^2 + ab + b^2)$

$(x-2)\left[x^2 + (x)(2) + 2^2\right] = 0$

$(x-2)(x^2 + 2x + 4) = 0$

$x - 2 = 0$ or $x^2 + 2x + 4 = 0$

$x = 2$

Now we solve $x^2 + 2x + 4 = 0$ by formula

$a = 1$, $b = 2$, $c = 4$

$$x = \frac{-2 \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(4)}}{2\times1}$$

$$= \frac{-2 \pm \sqrt{4-16}}{2}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$$= \frac{-2 \pm \sqrt{4\times(-3)}}{2}$$

$$= \frac{-2 \pm \sqrt{4}\sqrt{-3}}{2}$$

$$= \frac{-2 \pm 2\sqrt{-3}}{2}$$

$$x = \frac{2\left(-1 \pm \sqrt{-3}\right)}{2}$$

$x = 2\left(\frac{-1+\sqrt{-3}}{2}\right)$, $x = 2\left(\frac{-1-\sqrt{3}}{2}\right)$

$x = 2\omega$, $x = 2\omega^2$

So cube roots of 8 are 2, 2ω, $2\omega^2$

(iii) **Cube roots of −27**

Solution:

Let $x = (-27)^{\frac{1}{3}}$

$x^3 = -27$

$x^3 + 27 = 0$

$x^3 + 3^3 = 0$

$\because \quad (a^3 + b^3) = (a+b)(a^2 - ab + b^2)$

$(x+3)\left[x^2 - (x)(3) + 3^2\right] = 0$

Either $(x+3)(x^2 - 3x + 9) = 0$

$x + 3 = 0 \quad$ or $\quad x^2 - 3x + 9 = 0$

Now we solve $x^2 - 3x + 9 = 0$ by formula

$a = 1, \; b = -3, \; c = 19$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(19)}}{2 \times 1}$$

$$x = \frac{3 \pm \sqrt{9 - 36}}{2}$$

$$x = \frac{3 \pm \sqrt{-27}}{2}$$

$$x = \frac{3 \pm \sqrt{(9)(-3)}}{2}$$

$$x = \frac{3 \pm 3\sqrt{-3}}{2}$$

$$x = -3\left(\frac{-1 \pm \sqrt{-3}}{2}\right)$$

$$x = -3\left(\frac{-1 + \sqrt{-3}}{2}\right) \quad \text{or} \quad x = -3\left(\frac{-1 - \sqrt{-3}}{2}\right)$$

$x = -3\omega \quad$ or $\quad x = -3\omega^2$

So cube roots of −27 are −3, -3ω and $-3\omega^2$

(iv) **Cube roots of 64**

Solution:

Let $x = (64)^{\frac{1}{3}}$

$x^3 = 64$

$x^3 - 64 = 0$

$x^3 - 4^3 = 0$

$\because \quad a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$(x-4)\left[x^2 + (x)(4) + 4^2\right] = 0$

$(x-4)(x^2 + 4x + 16) = 0$

Either $x - 4 = 0 \quad$ or $\quad x^2 + 4x + 16 = 0$

$x = 4$

Now we solve $x^2 + 4x + 16 = 0$ by formula

$a = 1, \; b = 4, \; c = 16$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times 16}}{2(1)}$$

$$x = \frac{-4 \pm \sqrt{16 - 64}}{2}$$

$$x = \frac{-4 \pm \sqrt{-48}}{2}$$

$$x = \frac{-4 \pm \sqrt{16(-3)}}{2}$$

$$x = \frac{-4 \pm 4\sqrt{-3}}{2}$$

$$x = \frac{4\left(-1 \pm \sqrt{-3}\right)}{2}$$

Either

$$x = 4\left(\frac{-1 + \sqrt{-3}}{2}\right) \quad \text{or} \quad x = 4\left(\frac{-1 - \sqrt{-3}}{2}\right)$$

Here $\omega = \frac{-1 + \sqrt{-3}}{2}$, $\omega^2 = \frac{-1 - \sqrt{-3}}{2}$

Therefore,

$x = 4\omega, \quad$ or $\quad x = 4\omega^2$

So cube roots of 64 are 4, 4ω *and* $4\omega^2$

Q.2 Evaluate

(i) $(1-\omega-\omega^2)^7$

Solution: $(1-\omega-\omega^2)^7$

$=\left[1-(\omega+\omega^2)\right]^7$

$\because 1+\omega+\omega^2=0$
$\omega+\omega^2=-1$

$=\left[1-(-1)\right]^7$

$=(1+1)^7$

$=2^7$

$=128$

(ii) $(1-3\omega-3\omega^2)^5$

Solution: $(1-3\omega-3\omega^2)^5$

$=\left[1-3(\omega+\omega^2)\right]^5$

$\because 1+\omega+\omega^2=0$
$\omega+\omega^2=-1$

$=\left[1-3(-1)\right]^5$

$=(1+3)^5$

$=4^5=1024$

(iii) $(9+4\omega+4\omega^2)^3$

Solution: $(9+4\omega+4\omega^2)^3$

$=\left[9+4(\omega+\omega^2)\right]^3$

$=\left[9+4(-1)\right]^3 \quad (\because \omega+\omega^2=-1)$

$=\left[9-4\right]^3$

$=5^3$

$=125$

(iv) $(2+2\omega-2\omega^2)(3-3\omega+3\omega^2)$

Solution: $(2+2\omega-2\omega^2)(3-3\omega+3\omega^2)$

$=[2(1+\omega)-2\omega^2][3+3\omega^2-3\omega]$

$=[2(1+\omega)-2\omega^2][3(1+\omega^2)-3\omega]$

$\because 1+\omega+\omega^2=0$

$1+\omega=-\omega^2 \qquad 1+\omega^2=-\omega$

$\therefore \quad =[2(-\omega^2)-2\omega^2][3(-\omega)-3\omega]$

$=(-2\omega^2-2\omega^2)(-3\omega-3\omega)$

$=(-4\omega^2)(-6\omega)$

$=24\omega^3 \qquad$ As $\omega^3=1$

$=24(1)$

$=24$

(v) $\left(-1+\sqrt{-3}\right)^6+\left(-1-\sqrt{-3}\right)^6$

Solution:

$\left(-1+\sqrt{-3}\right)^6+\left(-1-\sqrt{-3}\right)^6 \quad ....(i)$

As $\dfrac{-1+\sqrt{-3}}{2}=\omega$ *and* $\dfrac{-1-\sqrt{-3}}{2}=\omega^2$

$-1+\sqrt{-3}=2\omega$ *and* $-1-\sqrt{-3}=2\omega^2$

Now equation (i) becomes

$=(2\omega)^6+(2\omega^2)^6$

$=2^6\omega^6+2^6\omega^{12}$

$=2^6\left[(\omega^3)^2+(\omega^3)^4\right] \qquad$ As $\omega^3=1$

$=2^6\left[(1)^2+(1)^4\right]$

$=64(1+1)$

$=64(2)$

$=128$

(vi) $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^9+\left(\dfrac{-1-\sqrt{-3}}{2}\right)^9$

Solution:

$\left(\dfrac{-1+\sqrt{-3}}{2}\right)^9+\left(\dfrac{-1-\sqrt{-3}}{2}\right)^9 \quad ......(i)$

As $\dfrac{-1+\sqrt{-3}}{2}=\omega$ *and* $\dfrac{-1-\sqrt{-3}}{2}=\omega^2$

Now equation (i) becomes

$=(\omega)^9+(\omega^2)^9$

$=\omega^9+\omega^{18}$

$=(\omega^3)^3+(\omega^3)^6$

$=(1)^3+(1)^6 \qquad$ As $\omega^3=1$

$=1+1$

$=2$

(vii) $\omega^{37}+\omega^{38}-5$

Solution:

$\omega^{37}+\omega^{38}-5$

$=\omega^{36}\omega+\omega^{36}\omega^2-5$

$=(\omega^3)^{12}\omega+(\omega^3)^{12}\omega^2-5$

$=(1)^{12}\omega+(1)^{12}\omega^2-5 \qquad$ As $\omega^3=1$

$=1\omega+1\omega^2-5$

$=(\omega+\omega^2)-5$

$=(-1)-5 \qquad$ As $1+\omega+\omega^2=0$

$=-1-5=-6 \qquad \omega+\omega^2=-1$

(viii) $\omega^{-13} + \omega^{-17}$

Solution:

$$\omega^{-13} + \omega^{-17}$$

$$= \frac{1}{\omega^{13}} + \frac{1}{\omega^{17}}$$

$$= \frac{1}{\omega^{12}\omega} + \frac{1}{\omega^{15}\omega^2}$$

$$= \frac{1}{(\omega^3)^4\omega} + \frac{1}{(\omega^3)^5\omega^2}$$

$$= \frac{1}{(1)^4\omega} + \frac{1}{(1)^5\omega^2} \qquad \because \omega^3 = 1$$

$$= \frac{1}{\omega} + \frac{1}{\omega^2}$$

$$= \frac{\omega^2 + \omega}{(\omega)(\omega^2)}$$

$$= \frac{-1}{\omega^3} \qquad \because 1 + \omega + \omega^2 = 0 \quad \omega + \omega^2 = -1$$

$$= \frac{-1}{1}$$

$$= -1$$

Q.3. Prove that,

$$x^3 + y^3 = (x + y)(x + \omega y)(x + \omega^2 y)$$ 02(032)

Solution: Let,

$$\text{R.H.S.} = (x + y)(x + \omega y)(x + \omega^2 y)$$

$$= (x + y)(x^2 + \omega^2 xy + \omega xy + \omega^3 y^2)$$

$$= (x + y)[x^2 + (\omega^2 + \omega)xy + \omega^3 y^2]$$

$$\because 1+\omega+\omega^2 = 0 \Rightarrow \omega+\omega^2 = -1 \quad \text{and} \quad \omega^3 = 1$$

$$= (x + y)(x^2 + (-1)xy + 1y^2)$$

$$= (x + y)(x^2 - xy + y^2)$$

Using Formula: $a^3 + b^3 = (a+b)(a^2-ab+b^2)$

$$= x^3 + y^3 = \text{L.H.S.}$$

Q.4. Prove that

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$$

Solution:

Let: R.H.S

$$= (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$$

$$= (x + y + z)(x^2 + \omega^2 xy + \omega xz + \omega yx + \omega^3 y^2 + \omega^2 yz + \omega^2 xz + \omega^4 zy + \omega^3 z^2)$$

$$= (x + y + z)[(x^2 + \omega^3 y^2 + \omega^3 z^2 + (\omega^2 + \omega)xy + (\omega^2 + \omega^4)yz + (\omega + \omega^2)zx]$$

$$\because 1+\omega+\omega^2 = 0 \Rightarrow \omega+\omega^2 = -1 \quad \text{and} \quad \omega^3 = 1$$

$$= (x+y+z)\,[x^2 + 1y^2 + 1z^2 + (-1)xy + (\omega^2 + \omega^3\omega)yz + (-1)zx]$$

$$= (x+y+z)\,[x^2 + 1y^2 + 1z^2 + (-1)xy + (\omega^2 + 1\omega)yz + (-1)zx]$$

$$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

$$= x^3 + y^3 + z^3 - 3xyz = \text{L.H.S}$$

Using Formula

$(a+b+c)(a^2+b^2+c^2-ab-bc-ca) = a^3 + b^3 + c^3 - 3abc$

Q.5. Prove that 02(034)

$(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)$..........2n factors = 1

Solution:

Let L.H.S.

$$=(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8) \ldots\ldots\ldots \text{ 2n factors}$$

$$=(1+\omega)(1+\omega^2)(1+\omega\omega^3)(1+\omega^6\omega^2) \ldots \text{ 2n factors}$$

$$\because \omega^3 = 1 \Rightarrow \omega^6 = (\omega^3)^2 = (1)^2 = 1$$

$$=(1+\omega)(1+\omega^2)(1+\omega)(1+\omega^2) \ldots\ldots\ldots \text{ 2n factors}$$

$$=\left[(1+\omega)(1+\omega^2)\right]\left[(1+\omega)(1+\omega^2)\right] \ldots\ldots\ldots \text{n factors}$$

$$=\left[(1+\omega)(1+\omega^2)\right]^n$$

$$=\left[1+\omega^2+\omega+\omega^3\right]^n$$

$$=\left[1+\omega+\omega^2+\omega^3\right]^n \qquad \because 1+\omega+\omega^2 = 0, \; \omega^3 = 1$$

$$=[0+1]^n$$

$$=[1]^n$$

$$= 1 = \text{R.H.S} \qquad \Rightarrow \quad \text{L.H.S} = \text{R.H.S}$$

Roots and coefficients of a quadratic equation.

We know that $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$

Are roots of the equation $ax^2 + bx + c = 0$ where a, b are coefficients of x^2 and x respectively. While c is the constant term.

Relation between roots and co–efficients of a quadratic equation.

If $\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$

then we can find the sum and the product of the roots as follows.

Sum of the roots

$$\alpha+\beta = \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\alpha+\beta = \frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}$$

$$\alpha+\beta = \frac{-2b}{2a} = -\frac{b}{a}$$

Product of the roots

$$\alpha\beta = \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$\alpha\beta = \frac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2}$$

$$\alpha\beta = \frac{b^2-(b^2-4ac)}{4a^2}$$

$$\alpha\beta = \frac{b^2-b^2+4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

If we denote the sum of roots and product of roots by S and P respectively, then

$$S = \frac{-b}{a} = -\frac{Co-efficient\ of\ x}{Co-efficient\ of\ x^2}$$

$$P = \frac{c}{a} = \frac{\text{constant term}}{\text{Co-efficient of } x^2}$$

The sum and the product of the roots of a given quadratic equation without solving it.

Example 1: Without solving, find the sum and product of the roots of the equations.

(a) $3x^2 - 5x + 7 = 0$ (b) $x^2 + 4x - 9 = 0$

Solution: (a) Let α and β be the roots of the equation $3x^2 - 5x + 7 = 0$

Then sum of roots $= \alpha+\beta = -\frac{b}{a} = -\left(\frac{-5}{3}\right) = \frac{5}{3}$

and product of roots $= \alpha\beta = \frac{c}{a} = \frac{7}{3}$

(b) Let α and β be the roots of the equation $x^2 + 4x - 9 = 0$

Then $\alpha+\beta = -\frac{b}{a} = -\frac{4}{1} = -4$

and $\alpha\beta = \frac{c}{a} = \frac{-9}{1} = -9$

To find unknown values involved in a given quadratic equation.

The procedure is illustrated through the following examples.

(a) Sum of the roots is equal to a multiple of the product of the roots.

Example 1:

Find the value of h, if the sum of the roots is equal to 3–times the product of the roots of the equation $3x^2 + (9 - 6h)x + 5h = 0$.

Solution: Let α, β be the roots of the equation

$$3x^2 + (9-6h)x + 5h = 0$$

Then $\alpha+\beta = -\frac{b}{a} = -\left(\frac{9-6h}{3}\right) = \frac{6h-9}{3}$

$$\alpha\beta = \frac{c}{a} = \frac{5h}{3}$$

Since $\alpha+\beta = 3(\alpha\beta)$

$$\frac{6h-9}{3} = 3\left(\frac{5h}{3}\right)$$

or $\frac{3(2h-3)}{3} = 5h$

$$2h-3 = 5h \Rightarrow 2h-5h = 3$$

$$-3h = 5 \Rightarrow h = \frac{3}{-3} = -1$$

(b) Sum of the squares of the roots is equal to a given number.

Example 2:

Find p, if the sum of the squares of the roots of the equation $4x^2 + 3px + p^2 = 0$ is unity.

Solution:

If α, β are the roots of $4x^2 + 3px + p^2 = 0$

Then $\alpha + \beta = -\frac{b}{a} = -\frac{3p}{4}$

and $\alpha\beta = \frac{c}{a} = \frac{p^2}{4}$

Since $\alpha^2 + \beta^2 = 1$ (Given)

$(\alpha + \beta)^2 - 2\alpha\beta = 1$

$\Rightarrow \left(\frac{-3p}{4}\right)^2 - 2\left(\frac{p^2}{4}\right) = 1$

or $\frac{9p^2}{16} - \frac{p^2}{2} = 1$

$\Rightarrow 9p^2 - 8p^2 = 16$

$\Rightarrow p^2 = 16 \Rightarrow p = \pm 4$

(c) Two roots differ by a given number.

Example 3: **Find h, if the roots of the equation $x^2 - hx + 10 = 0$ differ by 3.**

Solution: Let α and $\alpha - 3$ be the roots of $x^2 - hx + 10 = 0$.

Then $\alpha + \alpha - 3 = -\frac{b}{\alpha} = -\left(\frac{-h}{1}\right) = h$

$2\alpha - 3 = h$

$\Rightarrow 2\alpha = h + 3$

$\Rightarrow \alpha = \frac{h+3}{2}$ ....(i)

$\alpha(\alpha - 3) = \frac{c}{a} = \frac{10}{1} = 10$

and or $\alpha(\alpha - 3) = 10$ ....(ii)

Putting value of α from equation (i) in equation (ii), we get

$\left(\frac{h+3}{2}\right)\left(\frac{h+3}{2} - 3\right) = 10$

$\Rightarrow \left(\frac{h+3}{2}\right)\left(\frac{h+3-6}{2}\right) = 10$

$\left(\frac{h+3}{2}\right)\left(\frac{h-3}{2}\right) = 10$

$\frac{(h)^2 - (3)^2}{4} = 10$

$\Rightarrow h^2 - 9 = 40$, that is,

$h^2 = 49 \Rightarrow \boxed{h = \pm 7}$

(d) Roots satisfy a given relation

Example 4:

Find p, if the roots α, β of the equation $x^2 - 5x + p = 0$, satisfy the relation $2\alpha + 5\beta = 7$.

Solution: If α, β are the roots of the equation $x^2 - 5x + p = 0$.

Then $\alpha + \beta = -\frac{b}{a} = -\left(\frac{-5}{1}\right) = 5$

$\alpha + \beta = 5 \Rightarrow \beta = 5 - \alpha$ ....(i)

and $\alpha\beta = \frac{c}{a} = \frac{p}{1} = p \Rightarrow \alpha\beta = p$ ....(ii)

Since $2\alpha + 5\beta = 7$ (Given) ....(iii)

Put the value of β from equation (i) in equation (iii)

$2\alpha + 5(5 - \alpha) = 7$

$2\alpha + 25 - 5\alpha = 7$

$-3\alpha = 7 - 25$, that is

$-3\alpha = -18 \Rightarrow \boxed{\alpha = 6}$ .... (iv)

Put $\alpha = 6$ in equation (i)

$\beta = 5 - \alpha$

$\beta = 5 - 6 = -1 \Rightarrow \boxed{\beta = -1}$

Put the value of α and β in equation..... (ii)

$\alpha\beta = p$

$6(-1) = p \Rightarrow \boxed{p = -6}$

(e) Both sum and product of the roots are equal to a given number.

Example 5: **Find m, if sum and product of the roots of the equation $5x^2 + (7 - 2m)x + 3 = 0$ is equal to a given number, say λ.**

Solution: Let α, β be the roots of the equation $5x^2 + (7 - 2m)x + 3 = 0$

Then $\alpha + \beta = -\frac{b}{a} = -\frac{7-2m}{5} = \frac{2m-7}{5}$

and $\alpha\beta = \frac{c}{a} = \frac{3}{5}$

Let $\alpha + \beta = \lambda$...... (i) and $\alpha\beta = \lambda$ ....(ii)

Then from (i) and (ii)

$\alpha + \beta = \alpha\beta$, that is,

$\frac{2m-7}{5} = \frac{3}{5}$

$\Rightarrow 2m - 7 = 3$

$\Rightarrow 2m = 10$

$\Rightarrow m = 5$