EXERCISE 2.3

Q.1 Without solving, Find the sum and product of the Roots of following Quadratic Equations.

(i) $x^2-5x+3=0$

Solution: $x^2-5x+3=0$

$ax^2+bx+c=0$

$\Rightarrow$ $a=1$, $b=-5$, $c=3$

Sum of roots $= S = \frac{-b}{a} = -\left(\frac{-5}{1}\right) = 5$

Product of roots $= P = \frac{c}{a} = \frac{3}{1} = 3$

(ii) $3x^2+7x-11=0$

Solution: $3x^2+7x-11=0$

$ax^2+bx+c=0$

$\Rightarrow$ $a=3$, $b=7$, $c=-11$

Sum of roots $= S = \frac{-b}{a} = \frac{-7}{3}$

Product of roots $= P = \frac{c}{a} = \frac{-11}{3}$

(iii) $px^2-qx+r=0$

Solution: $px^2-qx+r=0$

$ax^2+bx+c=0$

$\Rightarrow$ $a=p$, $b=-q$, $c=r$

Sum of roots $= S = \frac{-b}{a} = -\left(\frac{-q}{p}\right) = \frac{q}{p}$

Product of roots $= P = \frac{c}{a} = \frac{r}{p}$

(iv) $(a+b)x^2-ax+b=0$

Solution: $(a+b)x^2-ax+b=0$

$Ax^2+Bx+C=0$

$\Rightarrow$ $A=a+b$, $B=-a$, $C=b$

Sum of roots $= S = \frac{-B}{A} = -\left(\frac{-a}{a+b}\right) = \frac{a}{a+b}$

Product of roots $= P = \frac{C}{A} = \frac{b}{a+b}$

(v) $(\ell+m)x^2+(m+n)x+n-\ell=0$

Solution: $(\ell+m)x^2+(m+n)x+n-\ell=0$

$ax^2+bx+c=0$

$\Rightarrow$ $a=\ell+m$, $b=m+n$, $c=n-\ell$

Sum of roots $= S = \frac{-b}{a} = \frac{-(m+n)}{\ell+m}$

Product of roots $= P = \frac{c}{a} = \frac{n-\ell}{\ell+m}$

(vi) $7x^2-5mx+9n=0$

Solution: $7x^2-5mx+9n=0$

$ax^2+bx+c=0$

$\Rightarrow$ $a=7$, $b=-5m$, $c=9n$

Sum of roots $= \frac{-b}{a} = -\left(\frac{-5m}{7}\right) = \frac{5m}{7}$

Product of roots $= \frac{c}{a} = \frac{9n}{7}$

Q. 2 Find the Value of k if.

(i) Sum of the roots of the equation $2kx^2-3x+4k=0$ is twice the product of the roots.

Solution:

$2kx^2-3x+4k=0$

$ax^2+bx+c=0$

$a=2k$, $b=-3$, $c=4k$

Let α, β be the Roots of Equation

Sum of the roots $= \alpha+\beta = \frac{-b}{a}$

$S = \alpha+\beta = \frac{-(-3)}{2k} = \frac{3}{2k}$

product of the Roots $= \alpha\beta = \frac{c}{a}$

$P = \alpha\beta = \frac{4k}{2k}$

$= 2$

Given condition

$S = 2p$

$$\frac{3}{2k} = 2(2)$$
$$\frac{3}{2k} = 4$$
$3 = 4\,(2k)$
$3 = 8k$
$$\frac{3}{8} = k \quad \Rightarrow \quad \boxed{k = \frac{3}{8}}$$

(ii). **Sum of the roots of the equation $x^2 + (3k-7)x + 5k = 0$ is $\frac{3}{2}$ times the product of roots.**

Solution:

$$1x^2 + (3k-7)x + 5k = 0$$
$$ax^2 + bx + c = 0$$
$a = 1$, $b = 3k-7$, $c = 5k$

Let α, β be the roots of the given equation.

Sum of the roots

$$S = \alpha + \beta = \frac{-b}{a}$$
$$S = \frac{-(3K-7)}{1}$$
$S = -3K + 7$

Product of the roots $= P = \alpha\ \beta = \frac{c}{a}$

$$P = \frac{5k}{1}$$
$P = 5k$

Given condition

$$S = \frac{3}{2}P$$
$$-3k + 7 = \frac{3}{2}(5k)$$
$2(-3k+7) = 3\,(5k)$
$-6k + 14 = 15k$
$14 = 15k + 6k$
$14 = 21k$
$$k = \frac{\cancel{14}^{2}}{\cancel{21}_{3}} = \frac{2}{3}$$
$$\boxed{k = \frac{2}{3}}$$

Q. 3 Find k if,

(i) Sum of the squares of the roots of the equation. $4kx^2 + 3kx - 8 = 0$ is 2

Solution:

$$4kx^2 + 3kx - 8 = 0 \text{ is } 2$$
$$ax^2 + bx + c = 0$$
$a = 4k$, $b = 3k$, $c = -8$

Sum of roots

$$S = \alpha + \beta = \frac{-b}{a}$$
$$\alpha + \beta = \frac{-3k}{4k}$$
$$\alpha + \beta = \frac{-3}{4}$$

Product of roots $= P = \alpha\beta = \frac{c}{a}$

$$\alpha\beta = \frac{-8}{4k}$$
$$\alpha\beta = \frac{-2}{k}$$

Given that sum of square of roots is 2 i.e.

$$\alpha^2 + \beta^2 = 2$$
$$\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta = 2$$
$$(\alpha+\beta)^2 - 2\alpha\beta = 2$$

Put $\alpha + \beta = \frac{-3}{4}$ and $\alpha\beta = \frac{-2}{k}$

$$\left(\frac{-3}{4}\right)^2 - 2\left(\frac{-2}{k}\right) = 2$$
$$\frac{9}{16} + \frac{4}{k} = 2$$
$$\frac{9k + 64}{16k} = 2$$
$9k + 64 = 2\,(16k)$
$9k + 64 = 32k$
$64 = 32k - 9k$
$\Rightarrow$ $23k = 64$
$$k = \frac{64}{23}$$

Required value of $k = \frac{64}{23}$

(ii) Sum of square of the roots of the equation $x^2 - 2kx + (2k+1) = 0$ is 6.

Solution: $1x^2 - 2kx + (2k+1) = 0$

$$ax^2 + bx + c = 0$$

$a = 1$, $b = -2k$, $c = 2k+1$

α, β be the Roots of given equation.

$$S = \alpha + \beta = -\frac{b}{a}$$

$$\alpha + \beta = \frac{-(-2k)}{1} = 2k$$

$$\alpha\beta = \frac{c}{a}$$

$$P = \alpha\beta = \frac{2k+1}{1} = 2k+1$$

Given condition

$$\alpha^2 + \beta^2 = 6$$

$$\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta = 6$$

$$(\alpha+\beta)^2 - 2(\alpha\beta) = 6$$

$$(-2k)^2 - 2(2k+1) = 6$$

$$4k^2 - 4k - 2 = 6$$

$$4k^2 - 4k - 2 - 6 = 0$$

$$4k^2 - 4k - 8 = 0$$

$$4(k^2 - k - 2) = 0$$

$\because$ $k^2 - k - 2 = 0$ $(\because 4 \neq 0)$

$$k^2 - 2k + k - 2 = 0$$

$$k(k-2) + 1(k-2) = 0$$

$$(k-2)(k+1) = 0$$

$k - 2 = 0$ or $k + 1 = 0$

$k = 2$ or $k = -1$

$k = -1, 2$

Q.4 Find p if

(i) The roots of the equation $x^2 - x + p^2 = 0$ differ by unity. 02(045)

Solution: $x^2 - x + p^2 = 0$

$$ax^2 + bx + c = 0$$

$a = 1$, $b = -1$, $c = p^2$

Let the roots are 'α and $\alpha - 1$'

$$\text{Sum of roots} = S = \alpha + \alpha - 1 = \frac{-b}{a} = -\left(\frac{-1}{1}\right) = 1$$

$$2\alpha - 1 = 1$$

$$\Rightarrow 2\alpha = 1 + 1$$

$$\Rightarrow 2\alpha = 2 \Rightarrow \boxed{\alpha = 1}$$

$$\text{Product of roots} = P = \alpha(\alpha - 1) = \frac{c}{a} = \frac{p^2}{1} = p^2$$

$$\alpha(\alpha - 1) = p^2$$

Putting the value of $\alpha = 1$

$$1(1-1) = p^2$$

$$\Rightarrow \quad p^2 = 0$$

$$\Rightarrow \quad \boxed{p = 0}$$

(ii) Find p if the roots of the equation $x^2 + 3x + p - 2 = 0$ differ by 2.

Solution: $x^2 + 3x + p - 2 = 0$

$a = 1$, $b = 3$, $c = p - 2$

Let 'α' and $\alpha - 2$ are the roots

Sum of roots

$$S = \alpha + \alpha - 2 = \frac{-b}{a} = \frac{-3}{1} = -3$$

$$2\alpha - 2 = -3$$

$$2\alpha = -3 + 2$$

$$\Rightarrow 2\alpha = -1 \quad \Rightarrow \boxed{\alpha = \frac{-1}{2}} \quad .....(i)$$

Product of roots

$$P = \alpha(\alpha - 2) = \frac{c}{a} = \frac{p-2}{1} = p - 2$$

$$\alpha(\alpha - 2) = p - 2$$

Putting the value of α from equation (i)

$$-\frac{1}{2}\left(\frac{-1}{2} - 2\right) = p - 2$$

$$\frac{-1}{2}\left(\frac{-5}{2}\right) = p - 2$$

$$\frac{5}{4} = p - 2$$

$$\Rightarrow \quad p = \frac{5}{4} + 2$$

$$p = \frac{5+8}{4} \quad \Rightarrow \quad \boxed{p = \frac{13}{4}}$$

Q.5 Find m if

(i) The roots of the equation $x^2 - 7x + 3m - 5 = 0$ Satisfy the relation $3\alpha + 2\beta = 4$

Solution:

Let α, β be the roots of given equation.

$1x^2 - 7x + 3m - 5 = 0$

$a = 1$, $b = -7$, $c = 3m - 5$

Sum of roots

$$\alpha + \beta = \frac{-b}{a} = \frac{-(-7)}{1} = 7 \quad \text{.........} \; (i)$$

Product of roots

$$\alpha\beta = \frac{c}{a} = \frac{3m-5}{1} = 3m - 5 \text{......}(ii)$$

Since $3\alpha + 2\beta = 4 \,(given) \text{......}(iii)$

From equation (i)

$\alpha + \beta = 7$

$\beta = 7 - \alpha$

Put $\beta = 7 - \alpha$ in equation (iii)

$3\alpha + 2(7 - \alpha) = 4$

$3\alpha + 14 - 2\alpha = 4$

$\alpha + 14 = 4$

$\alpha = 4 - 14$

$\boxed{\alpha = -10}$

Put $\alpha = -10$ *in* $\beta = 7 - \alpha$

$\beta = 7 - (-10)$

$\beta = 7 + 10$

$\boxed{\beta = 17}$

Put $\alpha = -10$ *and* $\beta = 17$ in equation (ii)

$\alpha\beta = 3m - 5$

$(-10)(17) = 3m - 5$

$-170 = 3m - 5$

$-170 + 5 = 3m$

$-165 = 3m$

$\frac{-165}{3} = m$

$-55 = m \quad \Rightarrow \boxed{m = -55}$

(ii) Find m if the roots of the equation $x^2 + 7x + 3m - 5 = 0$ satisfy the relation $3\alpha - 2\beta = 4$

Solution:

Let α, β be the roots of the equation.

$1x^2 + 7x + 3m - 5 = 0$

$a = 1$, $b = 7$, $c = 3m - 5$

Sum of roots

$$\alpha + \beta = \frac{-b}{a}$$

$$\alpha + \beta = \frac{-7}{1}$$

$\alpha + \beta = -7 \text{.........}(i)$

Product of roots

$$\alpha\beta = \frac{c}{a}$$

$$\alpha\beta = \frac{3m-5}{1}$$

$\alpha\beta = 3m - 5 \text{..........}(ii)$

$3\alpha - 2\beta = 4 \text{...........}(iii)$ (Given)

From equation (i)

$\alpha + \beta = -7$

$\beta = -7 - \alpha$

Putting the value of β in equation (iii)

$3\alpha - 2\beta = 4$

$3\alpha - 2(-7 - \alpha) = 4$

$3\alpha + 14 + 2\alpha = 4$

$5\alpha = 4 - 14$

$5\alpha = -10$

$$\alpha = \frac{-10}{5}$$

$\boxed{\alpha = -2}$

Put $\alpha = -2$ in equation (i)

$\alpha + \beta = -7$

$-2 + \beta = -7$

$\beta = -7 + 2$

$\boxed{\beta = -5}$

Now Put $\alpha = -2$ and $\beta = -5$ in equation (ii)

$\alpha\beta = 3m - 5$

$(-2)(-5) = 3m - 5$

$10 = 3m - 5$

$10 + 5 = 3m$

$15 = 3m$

$m = \frac{15}{3} \quad \Rightarrow \boxed{m = 5}$

(iii) Find m if the roots of the equation $3x^2 - 2x + 7m + 2 = 0$ satisfy the relation $7\alpha - 3\beta = 18$

Solution: Let α, β be the roots of the equation

$3x^2 - 2x + 7m + 2 = 0$

$ax^2 + bx + c = 0$

$a = 3, b = -2, c = 7m + 2$

Sum of roots

$$\alpha + \beta = -\frac{b}{a} = \frac{-(-2)}{3} = \frac{2}{3} \quad \text{.........(i)}$$

Product of roots

$$\alpha\beta = \frac{c}{a} = \frac{7m+2}{3} \quad \text{................ (ii)}$$

$$7\alpha - 3\beta = 18 \quad \text{.................. (iii)}$$

From equation (i)

$$\alpha + \beta = \frac{2}{3}$$

$$\beta = \frac{2}{3} - \alpha$$

Putting this value in eq. (iii)

$$7\alpha - 3\beta = 18$$

$$7\alpha - 3\left(\frac{2}{3} - \alpha\right) = 18$$

$$7\alpha - \frac{6^2}{3} + 3\alpha = 18$$

$$7\alpha - 2 + 3\alpha = 18$$

$$10\alpha = 18 + 2$$

$$10\alpha = 20$$

$$\alpha = \frac{20}{10}$$

$$\boxed{\alpha = 2}$$

Putting $\alpha = 2$ in equation (i)

$$\alpha + \beta = \frac{2}{3}$$

$$2 + \beta = \frac{2}{3}$$

$$\beta = \frac{2}{3} - 2$$

$$\beta = \frac{2-6}{3} = \frac{-4}{3}$$

$$\boxed{\beta = \frac{-4}{3}}$$

Now putting the value of α and β in equation (ii)

$$\alpha\beta = \frac{7m+2}{3}$$

$$2\left(-\frac{4}{3}\right) = \frac{7m+2}{3}$$

$$-\frac{8}{3} \times 3 = 7m + 2$$

$$\Rightarrow 7m + 2 = -8$$

$$7m = -8 - 2$$

$$\Rightarrow m = \frac{-10}{7}$$

Q.6 Find m, if sum and product of the roots of the following equations is equal to given number λ.

(i) $(2m+3)x^2 + (7m-5)x + (3m-10) = 0$

Solution:

$(2m+3)x^2 + (7m-5)x + (3m-10) = 0$

$ax^2 + bx + c = 0$

$a = 2m + 3$, $b = 7m - 5$, $c = 3m - 10$

Let α, β are the roots of the given equation, then

Sum of Roots

$$\alpha + \beta = -\frac{b}{a} = -\frac{(7m-5)}{2m+3} = \frac{5-7m}{2m+3}$$

Product of Roots

$$\alpha\beta = \frac{c}{a} = \frac{3m-10}{2m+3}$$

As given that

$\alpha + \beta = \lambda$............(i) , $\alpha\beta = \lambda$............(ii)

From (i) and (ii)

$$\alpha + \beta = \alpha\beta$$

$$\frac{5-7m}{2m+3} = \frac{3m-10}{2m+3}$$

$$5 - 7m = 3m - 10$$

$$5 + 10 = 3m + 7m$$

$$15 = 10m$$

$$\frac{15^3}{10_2} = m$$

$$\boxed{m = \frac{3}{2}}$$

(ii) $4x^2 - (3+5m)x - (9m-17) = 0$

Solution:

Let α, β be the roots of the equation.

$$4x^2 - (3+5m)x - (9m-17) = 0$$

$\because \quad ax^2 + bx + c = 0$

$\therefore \quad a = 4,\ b = -(3+5m),\ c = -(9m-17)$

Sum of roots

$$\alpha+\beta = \frac{-b}{a} = -\left(-\frac{(3+5m)}{4}\right) = \frac{3+5m}{4}$$

Product of roots

$$\alpha\beta = \frac{c}{a} = -\left(\frac{9m-17}{4}\right)$$

Let $\alpha+\beta = \lambda$.....(i) and $\alpha\beta = \lambda$.....(ii)

from (i) and (ii)

$$\alpha+\beta = \alpha\beta$$

$$\frac{3+5m}{4} = \frac{-(9m-17)}{4}$$

$$3+5m = -9m+17$$

$$9m+5m = 17-3$$

$$14m = 14$$

$$\Rightarrow m = \frac{14}{14} \Rightarrow \boxed{m=1}$$

Symmetric functions of the roots of a quadratic equation.

Definition:

Symmetric Functions are those functions in which the roots involved are such that the value of the expressions involving them remain unaltered, when roots are interchanged.

For example, if

$f(\alpha,\beta) = \alpha^2+\beta^2$, then

$f(\beta,\alpha) = \beta^2+\alpha^2 = \alpha^2+\beta^2$

$(\because \beta^2+\alpha^2 = \alpha^2+\beta^2)$

$= f(\alpha,\beta)$

Example: **Find the value of $\alpha^3+\beta^3+3\alpha\beta$, if $\alpha = 2, \beta = 1$. Also find the value of $\alpha^3+\beta^3+3\alpha\beta$ if $\alpha = 1, \beta = 2$.**

Solution:

(i) When $\alpha = 2$ and $\beta = 1$,

$$\alpha^3+\beta^3+3\alpha\beta = (2)^3+(1)^3+3(2)(1) = 8+1+6 = 15$$

(ii) When $\alpha = 1$ and $\beta = 2$,

$$\alpha^3+\beta^3+3\alpha\beta = (1)^3+(2)^3+3(1)(2) = 1+8+6 = 15$$

The expression $\alpha^3+\beta^3+3\alpha\beta$ represents a symmetric function of α and β.

Evaluate a symmetric function of roots of a quadratic equation in terms of its co–efficients.

If α, β are the roots of the quadratic equation

$$ax^2+bx+c = 0,\ (a \neq 0)........(i)$$

Then $\alpha+\beta = \dfrac{-b}{a}$ ....(ii)

and $\alpha\beta = \dfrac{c}{a}$ ....(iii)

The functions given in equations (ii) and (iii) are the symmetric functions for the quadratic equation (i).

Some more symmetric functions of two variables α, β are given below:

(i) $\alpha^2+\beta^2$ (ii) $\alpha^3+\beta^3$

(iii) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$ (iv) $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}$

Example 1: **If α, β are the roots of the quadratic equation $px^2+qx+r = 0$, $(p \neq 0)$ then evaluate $\alpha^2\beta+\alpha\beta^2$**

Solution:

Since α, β are the roots of $px^2+qx+r = 0$, therefore,

$$\alpha+\beta = \frac{-b}{a} = -\frac{q}{p} \quad \text{and} \quad \alpha\beta = \frac{r}{p}$$

$$\alpha^2\beta+\alpha\beta^2 = \alpha\beta(\alpha+\beta)$$

$$= \frac{r}{p}\left(-\frac{q}{p}\right) = \frac{-qr}{p^2}$$

Example 2: If α, β are the roots of the equation $2x^2 + 3x + 4 = 0$, then find the value of

(i) $\alpha^2 + \beta^2$ (ii) $\frac{1}{\alpha} + \frac{1}{\beta}$

Solution:

Since, α, β are the roots of the equation $2x^2 + 3x + 4 = 0$, therefore,

$$\alpha + \beta = \frac{-b}{a}$$

$$= -\frac{3}{2}$$

$$\alpha\beta = \frac{c}{a}$$

$$= \frac{4}{2} = 2$$

(i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$$= \left(\frac{-3}{2}\right)^2 - 2(2)$$

$$= \frac{9}{4} - 4$$

$$= \frac{9-16}{4}$$

$$= \frac{-7}{4}$$

(ii) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{\alpha + \beta}{\alpha\beta}$

$$= (\alpha + \beta)\frac{1}{\alpha\beta}$$

$$= \left(-\frac{3}{2}\right)\left(\frac{1}{2}\right) = \frac{-3}{4}$$