EXERCISE 2.4

Q.1 **If α, β are the roots of the equations $x^2 + px + q = 0$ then evaluate**

(i) $\alpha^2 + \beta^2$ (ii) $\alpha^3\beta + \alpha\beta^3$ (iii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

Solution:

$a = 1$, $b = p$, $c = q$

Sum of roots,

$$\alpha + \beta = \frac{-b}{a} = \frac{-p}{1} = -p$$

Product of roots

$$\alpha\beta = \frac{c}{a} = \frac{q}{1} = q$$

(i) $\alpha^2 + \beta^2$

As

$$(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$$

or $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$$= (\alpha + \beta)^2 - 2\alpha\beta$$

$$= (-p)^2 - 2(q)$$

$$= p^2 - 2q$$

(ii) $\alpha^3\beta + \alpha\beta^3$

$$= \alpha\beta(\alpha^2 + \beta^2)$$

$$= \alpha\beta\left[(\alpha + \beta)^2 - 2\alpha\beta\right]$$

$$= q\left[(-p)^2 - 2q\right]$$

$$= q(p^2 - 2q)$$

(iii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

$$= \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$= \frac{1}{\alpha\beta}(\alpha^2 + \beta^2)$$

$$= \frac{1}{\alpha\beta}\left[(\alpha + \beta)^2 - 2\alpha\beta\right]$$

$$= \frac{1}{q}\left[(-p)^2 - 2 \times q\right]$$

$$= \frac{1}{q}(p^2 - 2q)$$

Q.2 If α, β are the roots of the equation $4x^2-5x+6=0$, the find the value of

(i) $\frac{1}{\alpha}+\frac{1}{\beta}$ (ii) $\alpha^2\beta^2$ (iii) $\frac{1}{\alpha^2\beta}+\frac{1}{\alpha\beta^2}$ (iv) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$

Solution: $4x^2-5x+6=0$

$ax^2+bx+c=0$

$a=4$, $b=-5$, $c=6$

Sum of roots,

$$\alpha+\beta=\frac{-b}{a}=-\frac{(-5)}{4}=\frac{5}{4}$$

Product of roots,

$$\alpha\beta=\frac{c}{a}=\frac{6}{4}$$

(i) $$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=\frac{5/4}{6/4}=5/6$$

(ii) $$\alpha^2\beta^2=(\alpha\beta)^2=\left(\frac{6}{4}\right)^2=\left(\frac{3}{2}\right)^2=\frac{9}{4}$$

(iii) $$\frac{1}{\alpha^2\beta}+\frac{1}{\alpha\beta^2}=\frac{\beta+\alpha}{\alpha^2\beta^2}=\frac{\alpha+\beta}{(\alpha\beta)^2}$$

$$=\frac{\frac{5}{4}}{\left(\frac{6}{4}\right)^2}=\frac{\frac{5}{4}}{\frac{36}{16}}=\frac{5}{4}\times\frac{16}{36}=\frac{5}{9}$$

(iv) $$\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}=\frac{\alpha^3+\beta^3}{\alpha\beta}$$

Using Formula: $(\alpha+\beta)^3=\alpha^3+\beta^3+3(\alpha\beta)(\alpha+\beta)$

$\Rightarrow \alpha^3+\beta^3=(\alpha+\beta)^3-3(\alpha\beta)(\alpha+\beta)$

$$=\frac{(\alpha+\beta)^3-3(\alpha\beta)(\alpha+\beta)}{\alpha\beta}$$

$$=\frac{\left(\frac{5}{4}\right)^3-3\left(\frac{6}{4}\right)\left(\frac{5}{4}\right)}{\left(\frac{6}{4}\right)}=\left(\frac{125}{64}-\frac{90}{16}\right)\frac{4}{6}$$

$$=\left(\frac{125-360}{64}\right)\frac{4}{6}=\frac{-235}{96}$$

Q.3 If α, β are the roots of the equation $\ell x^2+mx+n=0$ $(\ell\neq 0)$ then find the value of

(i) $\alpha^3\beta^2+\alpha^2\beta^3$ (ii) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$

Solution: $lx^2+mx+n=0$

$ax^2+bx+c=0$

$a=l$, $b=m$, $c=n$

If α,β be the Roots of given equation

Sum of roots $=\frac{-b}{a}$

$$\alpha+\beta=\frac{-m}{l}$$

Product of Roots $=\frac{c}{a}$

$$\alpha\beta=\frac{n}{l}$$

(i) $\alpha^3\beta^2+\alpha^2\beta^3$

$=\alpha^2\beta^2(\alpha+\beta)$

$=(\alpha\beta)^2(\alpha+\beta)$

$$=\left(\frac{n}{l}\right)^2\left(\frac{-m}{l}\right)$$

$$=\left(\frac{n^2}{l^2}\right)\left(\frac{-m}{l}\right)$$

$$=\frac{-mn^2}{l^3}$$

(ii) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$

$$=\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}$$

$$=\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}$$

$$=\frac{\left(\frac{-m}{l}\right)^2-2\left(\frac{n}{l}\right)}{\left(\frac{n}{l}\right)^2}=\frac{\frac{m^2}{l^2}-\frac{2n}{l}}{\frac{n^2}{l^2}}$$

$$=\left(\frac{m^2-2nl}{l^2}\right)\cdot\frac{l^2}{n^2}$$

$$=\left(\frac{m^2-2nl}{n^2}\right)$$

$$=\frac{1}{n^2}(m^2-2nl)$$

Formation of a quadratic equation.

If α and β are the roots of the required quadratic equation.

Let $x = \alpha$ and $x = \beta$

i.e., $x - \alpha = 0$, $x - \beta = 0$

and $(x - \alpha)(x - \beta) = 0$

$x^2 - \beta x - \alpha x + \alpha\beta = 0$

$x^2 - \alpha x - + \alpha\beta = 0$

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

which is the required quadratic equation in the standard form.

Find a quadratic equation from given roots and establish the formula x^2 – (sum of the roots)x + product of the roots= 0.

Let α, β be the roots of the quadratic equation

$$ax^2 + bx + c = 0 \quad , \quad (a \neq 0) \ldots (i)$$

Then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

Rewrite equation (i) as $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

or $x^2 - \left(-\frac{b}{a}\right)x + \frac{c}{a} = 0$

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

x^2 – (sum of roots) x + product of roots = 0, that is, $x^2 - Sx + P = 0$,

where $S = \alpha + \beta$ and $P = \alpha\beta$

Example 1: Form a quadratic equation with roots 3 and 4.

Solution: Since 3 and 4 are the roots of the required quadratic equation, therefore,

S = Sum of the roots = 3 + 4 = 7

P = Product of the roots = (3) (4) = 12

As $x^2 - Sx + P = 0$, so the required quadratic equation is $x^2 - 7x + 12 = 0$

Form quadratic equations whose roots are of the type.

(i) $2a + 1, 2\beta + 1$ (ii) α^2, β^2

(iii) $\frac{1}{\alpha}, \frac{1}{\beta}$ (iv) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

(v) $\alpha + \beta, \frac{1}{\alpha} + \frac{1}{\beta}$ where α, β are the roots of a given quadratic equation.

Example 2: If α, β are the roots of the equation $2x^2 - 3x - 5 = 0$, form quadratic equations having roots.

(i) $2\alpha + 1, 2\beta + 1$ (ii) α^2, β^2

(iii) $\frac{1}{\alpha}, \frac{1}{\beta}$ (iv) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

(v) $\alpha+\beta, \frac{1}{\alpha} + \frac{1}{\beta}$

Solution: As α, β are the roots of the equation

$2x^2 - 3x - 5 = 0$,

$ax^2 + bx + c = 0$

$a = 2, b = -3, c = -5$

Therefore, $\alpha + \beta = \frac{-b}{a} = -\left(\frac{-3}{2}\right) = \frac{3}{2}$

and $\alpha\beta = \frac{c}{a} = \frac{-5}{2} = -\frac{5}{2}$

(i) $2\alpha+1, 2\beta+1$

Sum of the roots

$S = 2\alpha + 1 + 2\beta + 1$

$S = 2(\alpha+\beta) + 2$

$S = 2\left(\frac{3}{2}\right) + 2 = 5 \Rightarrow \boxed{S = 5}$

Product of the roots

$P = (2\alpha+1)(2\beta+1)$

$P = 4\alpha\beta + 2\alpha + 2\beta + 1$

$P = 4\alpha\beta + 2(\alpha+\beta) + 1$

$P = 4\left(-\frac{5}{2}\right) + 2\left(\frac{3}{2}\right) + 1$

$P = -10 + 3 + 1 = -6 \Rightarrow \boxed{P = -6}$

Using $x^2 - Sx + p = 0$, we have

$x^2 - 5x - 6 = 0$

(ii) α^2, β^2

Sum of the roots

$S = \alpha^2 + \beta^2$

$S = (\alpha + \beta)^2 - 2\alpha\beta$

$S = \left(\frac{3}{2}\right)^2 - 2\left(-\frac{5}{2}\right)$

$S = \frac{9}{4} + 5 = \frac{9+20}{4}$

$\boxed{S = \frac{29}{4}}$

Product $P = \alpha^2 . \beta^2 = (\alpha\beta)^2$

$$P = \left(-\frac{5}{2}\right)^2 = \frac{25}{4} \Rightarrow \boxed{P = \frac{25}{4}}$$

Using $x^2 - Sx + p = 0$, we have

$$x^2 - \frac{29}{4}x + \frac{25}{4} = 0 \qquad \Rightarrow 4x^2 - 29x + 25 = 0$$

(iii) $\frac{1}{\alpha}, \frac{1}{\beta}$

Sum $S = \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta+\alpha}{\alpha\beta} = (\alpha+\beta).\frac{1}{\alpha\beta}$

$$S = \frac{3}{2}.\left(-\frac{2}{5}\right) \qquad \left(\because \alpha\beta = -\frac{5}{2}\right)$$

$$\boxed{S = -\frac{3}{5}}$$

Product $P = \frac{1}{\alpha}.\frac{1}{\beta} = \frac{1}{\alpha\beta} = -\frac{2}{5} \Rightarrow \boxed{P = \frac{-2}{5}}$

Using $x^2 - Sx + p = 0$, we have

$$x^2 - \left(-\frac{3}{5}\right)x + \left(-\frac{2}{5}\right) = 0$$

$$\Rightarrow \quad 5x^2 + 3x - 2 = 0$$

(iv) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

Sum $S = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta}$

$$S = \left[(\alpha+\beta)^2 - 2\,\alpha\beta\right].\frac{1}{\alpha\beta}$$

$$S = \left[\left(\frac{3}{2}\right)^2 - 2\left(-\frac{5}{2}\right)\right] \times \left(-\frac{2}{5}\right)$$

$$S = \left(\frac{9}{4} + 5\right) \times \left(-\frac{2}{5}\right) = \left(\frac{9+20}{4}\right) \times \left(\frac{-2}{5}\right)$$

$$S = \frac{29}{4} \times \left(-\frac{2}{5}\right) = -\frac{29}{10} \Rightarrow \boxed{S = \frac{-29}{10}}$$

Product $P = \frac{\alpha}{\beta}.\frac{\beta}{\alpha} = 1$

Using $x^2 - Sx + P = 0$, we have

$$x^2 - \left(\frac{-29}{10}\right)x + 1 = 0$$

$$\Rightarrow 10x^2 + 29x + 10 = 0$$

(v) $\alpha + \beta, \frac{1}{\alpha} + \frac{1}{\beta}$

Sum of the roots

$$S = \alpha + \beta + \frac{1}{\alpha} + \frac{1}{\beta}$$

$$S = \alpha + \beta + \frac{\beta + \alpha}{\alpha\beta}$$

$$S = (\alpha + \beta)\left(1 + \frac{1}{\alpha\beta}\right)$$

$$S = \frac{3}{2}\left(1 - \frac{2}{5}\right) = \frac{3}{2} \times \left(\frac{5-2}{5}\right) = \frac{3}{2} \times \frac{3}{5} \Rightarrow \boxed{S = \frac{9}{10}}$$

Product of the roots

$$P = (\alpha + \beta).\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$$

$$P = (\alpha + \beta)\left(\frac{\beta+\alpha}{\alpha\beta}\right)$$

$$P = (\alpha+\beta)^2 \times \frac{1}{\alpha\beta} = \left(\frac{3}{2}\right)^2 \times \left(\frac{-2}{5}\right)$$

$$P = \frac{9}{4} \times \left(\frac{-2}{5}\right) = \left(\frac{-9}{10}\right) \Rightarrow \boxed{P = \frac{-9}{10}}$$

Using $x^2 - Sx + P = 0$, we have

$$x^2 - \frac{9}{10}x + \left(\frac{-9}{10}\right) = 0$$

$$\Rightarrow 10x^2 - 9x - 9 = 0$$

Example 3: If α, β are the roots of the equation $x^2 - 7x + 9 = 0$
Form an equation whose roots are 2α and 2β.
Solution:

Since α, β are the roots of the equation $x^2 - 7x + 9 = 0$, therefore,

$$\alpha + \beta = \frac{-b}{a} = -\left(\frac{-7}{1}\right) = 7$$

And $$\alpha\beta = \frac{c}{a} = \frac{9}{1} = 9$$

The roots of the required equation are 2α, 2β

Sum of roots

$$S = 2\alpha + 2\beta = 2(\alpha+\beta) = 2(7) = 14$$

Product of roots

$$P = (2\alpha)(2\beta) = 4\alpha\beta = 4(9) = 36$$

Thus the required quadratic equation will be

$x^2 - Sx + P = 0$, that is,

$x^2 - 14x + 36 = 0$