EXERCISE 3.3

Q1: Find a third proportional

(i) 6, 12

Solution:

Let third proportional is x then 6,12, x

By proportion.

$$6:12::12:x$$

Product of Extremes = Product of Means

$$6(x) = 12 \times 12$$

$$6(x) = 144$$

$$x = \frac{144}{6} = 24$$

$$\boxed{x = 24}$$

(ii) $a^3, 3a^2$

Solution:

Let 3rd proportional is x then $a^3, 3a^2, x$

By proportion

$$a^3 : 3a^2 :: 3a^2 : x$$

Product of Extremes = Product of Means

$$xa^3 = (3a^2)(3a^2)$$

$$xa^3 = 9a^4$$

$$x = \frac{9a^4}{a^3}$$

$$x = 9a^{4-3}$$

$$\boxed{x = 9a}$$

(iii) $a^2 - b^2,\ a - b$

Solution:

Let 3rd proportional is x then $a^2 - b^2, a - b, x$

By proportion

$$a^2 - b^2 : a - b :: a - b : x$$

Product of Extremes = Product of Means

$$x(a^2 - b^2) = (a-b)(a-b)$$

$$x = \frac{(a-b)(a-b)}{a^2 - b^2}$$

$$x = \frac{(a-b)\cancel{(a-b)}}{(a+b)\cancel{(a-b)}}$$

$$\boxed{x = \frac{a-b}{a+b}}$$

(iv) $(x-y)^2,\ x^3-y^3$

Solution:

Let 3rd proportional is "a" then $(x-y)^2,\ x^3 - y^3, a$

By proportion:

$$(x-y)^2 : x^3 - y^3 :: x^3 - y^3 : a$$

Product of Extremes = Product of Means

$$a(x-y)^2 = (x^3 - y^3)(x^3 - y^3)$$

$$a = \frac{(x^3 - y^3)^2}{(x-y)^2}$$

$$a = \frac{\left[(x-y)(x^2+xy+y^2)\right]^2}{(x-y)^2}$$

$$a = \frac{\cancel{(x-y)^2}(x^2+xy+y^2)^2}{\cancel{(x-y)^2}}$$

$$\boxed{a = (x^2+xy+y^2)^2}$$

(v) $(x+y)^2,\ x^2 - xy - 2y^2$

Solution:

Let 3rd proportional is "a"

Then $(x+y)^2,\ x^2 - xy - 2y^2, a$

By Proportion:

$$(x+y)^2 : x^2 - xy - 2y^2 :: x^2 - xy - 2y^2 : a$$

Product of Extremes = Product of Means:

$$a(x+y)^2 = (x^2 - xy - 2y^2)(x^2 - xy - 2y^2)$$

$$a = \frac{(x^2 - xy - 2y^2)^2}{(x+y)^2}$$

$$a = \frac{(x^2 - xy - y^2 - y^2)^2}{(x+y)^2}$$

$$a = \frac{(x^2 - y^2 - xy - y^2)^2}{(x+y)^2}$$

$$a = \frac{\left[(x+y)(x-y) - y(x+y)\right]^2}{(x+y)^2}$$

$$a = \frac{\left[(x+y)(x-y-y)\right]^2}{(x+y)^2}$$

$$= \frac{\cancel{(x+y)^2}(x-2y)^2}{\cancel{(x+y)^2}}$$

$$\boxed{a = (x-2y)^2}$$

(vi) $\frac{p^2-q^2}{p^3+q^3}, \frac{p-q}{p^2-pq+q^2}$

Solution:

Let 3rd proportional is x

Then $\frac{p^2-q^2}{p^3+q^3}$, $\frac{p-q}{p^2-pq+q^2}$, x

By proportion.

$$\frac{p^2-q^2}{p^3+q^3} : \frac{p-q}{p^2-pq+q^2} :: \frac{p-q}{p^2-pq+q^2} : x$$

Product of Extremes = Product of Means.

$$x.\frac{p^2-q^2}{p^3+q^3} = \frac{p-q}{p^2-pq+q^2} \times \frac{p-q}{p^2-pq+q^2}$$

$$x.\frac{p^2-q^2}{p^3+q^3} = \left[\frac{(p-q)}{(p^2-pq+q^2)}\right]^2$$

$$x = \frac{(p-q)^2}{(p^2-pq+q^2)^2} \times \frac{p^3+q^3}{(p^2-q^2)}$$

$$x = \frac{\cancel{(p-q)}(p-q)}{(p^2-pq+q^2)^{\cancel{2}}} \times \frac{\cancel{(p+q)}\cancel{(p^2-pq+q^2)}}{\cancel{(p+q)}\cancel{(p-q)}}$$

$$\boxed{x = \frac{p-q}{p^2-pq+q^2}}$$

Q.2 Find a fourth proportional

(i) 5, 8, 15

Solution:

Let 4th proportional is x then 5, 8, 15, x

By proportion

5 : 8 : : 15 : x

Product of Extremes = Product of Means

$$5(x) = 8(15)$$

$$x = \frac{8(\cancel{15}^{3})}{\cancel{5}_1}$$

$$x = 8(3)$$

$$\boxed{x = 24}$$

(ii) $4x^4, 2x^3, 18x^5$

Solution:

Let 4th proportional is "a" then $4x^4, 2x^3, 18x^5$, a

By proportion

$$4x^4 : 2x^3 : : 18x^5 : a$$

Product of Extreme = Product of Means.

$$a(4x^4) = 2x^3(18x^5)$$

$$a = \frac{36x^8}{4x^4}$$

$$a = 9x^{8-4}$$

$$\boxed{a = 9x^4}$$

(iii) $15a^5b^6, 10a^2b^5, 21a^3b^3$

Solution:

Let 4th proportional is x

then $15a^5b^6, 10a^2b^5, 21a^3b^3$, x

By proportion

$$15a^5b^6 : 10a^2b^5 : : 21a^3b^3 : x$$

Product of Extremes = Product of Means

$$x\left(15a^5b^6\right) = \left(10a^2b^5\right)\left(21a^3b^3\right)$$

$$x = \frac{{}^{14}\cancel{210}\,\cancel{a^5}\,b^8}{\cancel{15}\,\cancel{a^5}\,b^6}$$

$$x = 14b^{8-6}$$

$$\boxed{x = 14b^2}$$

(iv) $x^2-11x+24; (x-3), (5x^4-40x^3)$

Solution:

Let 4th proportional is "a"

$x^2-11x+24, (x-3), (5x^4-40x^3)$, a

By proportion

$$x^2-11x+24 : (x-3) : : 5x^4-40x^3 : a$$

Product of Extremes = Product of Means

$$a(x^2-11x+24) = (x-3)(5x^4-40x^3)$$

$$a = \frac{(x-3)(5x^4-40x^3)}{x^2-11x+24}$$

$$a = \frac{(x-3).5x^3(x-8)}{x^2-3x-8x+24}$$

$$a = \frac{5x^3(x-3)(x-8)}{x(x-3)-8(x-3)}$$

$$a = \frac{5x^3\cancel{(x-3)}\cancel{(x-8)}}{\cancel{(x-3)}\cancel{(x-8)}}$$

$$\boxed{a = 5x^3}$$

(v) $p^3+q^3, p^2-q^2, p^2-pq+q^2$

Solution:

Let 4^{th} proportional is x

$p^3+q^3, p^2-q^2, p^2-pq+q^2, x$

By proportion

$p^3+q^3, p^2-q^2, p^2-pq+q^2 : x$

Product of Extremes = Product of Means.

$x(p^3+q^3) = (p^2-q^2)(p^2-pq+q^2)$

$$x = \frac{(p^2-q^2)(p^2-pq+q^2)}{p^3+q^3}$$

$$x = \frac{(p+q)(p-q)(p^2-pq+q^2)}{(p+q)(p^2-pq+q^2)}$$

$\boxed{x = (p-q)}$

(vi) $(p^2-q^2)(p^2+pq+q^2), p^3+q^3, p^3-q^3$

Solution:

Let 4^{th} proportional is x.

Then $(p^2-q^2)(p^2+pq+q^2), p^3+q^3, p^3-q^3, x$

By proportion:

$(p^2-q^2)(p^2+pq+q^2) : p^3+q^3 :: p^3-q^3 : x$

Product of Extremes = Product of Means:

$x\ (p^2-q^2)(p^2+pq+q^2) = (p^3+q^3)(p^3-q^3)$

$$x = \frac{(p^3+q^3)(p^3-q^3)}{(p^2-q^2)(p^2+pq+q^2)}$$

$$x = \frac{(p+q)(p^2-pq+q^2)(p-q)(p^2+pq+q^2)}{(p+q)(p-q)(p^2+pq+q^2)}$$

$\boxed{x = (p^2-pq+q^2)}$

Q.3: Find mean proportional:

(i) 20, 45

Solution:

Let mean proportional is m

then 20, m, 45

By proportion

$20 : m :: m : 45$

Product of Means = Product of Extremes

$m.m = 20 \times 45$

$m^2 = 900$

Taking square root

$\sqrt{m^2} = \pm\sqrt{900}$

$\boxed{m = \pm 30}$

(ii) $20x^3y^5,\ 5x^7y$

Solution:

Let mean proportional is m

then $20x^3y^5,\ m,\ 5x^7y$

By proportion,

$20x^3y^5 : m :: m : 5x^7y$

Product of Means = Product of Extremes

$m.m = (20x^3y^5)(5x^7y)$

$m^2 = 100x^{10}y^6$

Taking square root of both sides

$\sqrt{m^2} = \pm\sqrt{100x^{10}y^6}$

$m = \pm\sqrt{100}.\sqrt{x^{10}}.\sqrt{y^6}$

$m = \pm\sqrt{100}.\sqrt{x^{10}}.\sqrt{y^6}$

$m = \pm 10x^{10\times\frac{1}{2}}.y^{6\times\frac{1}{2}}$

$\boxed{m = \pm 10x^5y^3}$

(iii) $15p^4qr^3, 135q^5r^7$

Solution:

Let mean proportional is m

then $15p^4qr^3,\ m,\ 135q^5r^7$

By proportion

$15p^4qr^3 : m :: m : 135q^5r^7$

Product of Means = Product of Extremes

$m.m = (15p^4qr^3)(135q^5r^7)$

$m^2 = 2025p^4q^6r^{10}$

Taking square root

$\sqrt{m^2} = \pm\sqrt{2025p^4q^6r^{10}}$

$m = \pm\sqrt{2025}\sqrt{p^4}.\sqrt{q^6}.\sqrt{r^{10}}$

$m = \pm 45p^{4\times\frac{1}{2}}.q^{6\times\frac{1}{2}}.r^{10\times\frac{1}{2}}$

$\boxed{m = \pm 45p^2.q^3.r^5}$

(iv) $x^2-y^2, \frac{x-y}{x+y}$

Solution:

Let mean proportional is m.

then $x^2-y^2, m, \frac{x-y}{x+y}$

By proportion

$$x^2-y^2 : m :: m : \frac{x-y}{x+y}$$

Product of Means = Product of Extremes

$$m.m = (x^2-y^2)\frac{(x-y)}{x+y}$$

$$m^2 = \frac{(x+y)(x-y)(x-y)}{(x+y)}$$

$$m^2 = (x-y)^2$$

Taking square root

$$\sqrt{m^2} = \pm\sqrt{(x-y)^2}$$

$$\boxed{m = \pm(x-y)}$$

Q.4 Find the values of the letter involved in the following continued proportions

(i) 5, p, 45

Solution:

By continued proportion

$$5 : p :: p : 45$$

Product of Means = Product of Extremes

$$p.p = 5\times 45$$

$$p^2 = 225$$

Taking square root of both sides

$$\sqrt{p^2} = \pm\sqrt{225}$$

$$\boxed{p = \pm 15}$$

(ii) 8, x, 18

Solution:

By continued proportion

$$8 : x :: x : 18$$

Product of Means = Product of Extremes

$$x.x = 8\times 18$$

$$x^2 = 144$$

Taking square root

$$\sqrt{x^2} = \pm\sqrt{144}$$

$$\boxed{x = \pm 12}$$

(iii) 12, 3p – 6, 27

Solution:

By continued proportion

$$12 : 3p-6 :: 3p-6 : 27$$

Product of Means = Product of Extremes.

$$(3p-6)(3p-6) = 12\times 27$$

$$(3p-6)^2 = 324$$

Taking square root of both sides

$$\sqrt{(3p-6)^2} = \pm\sqrt{324}$$

$$3p-6 = \pm 18$$

$$3p-6 = 18 \quad \text{or} \quad 3p-6 = -18$$

$$3p = 18+6 \quad \text{or} \quad 3p = -18+6$$

$$3p = 24 \quad \text{or} \quad 3p = -12$$

$$p = \frac{24}{3} \quad \text{or} \quad p = \frac{-12}{3}$$

$$p = 8 \quad \text{or} \quad p = -4$$

(iv) 7, m – 3, 28

Solution:

By continued proportion

$$7 : m-3 :: m-3 : 28$$

Product of Means = Product of Extremes

$$(m-3)(m-3) = 7\times 28$$

$$(m-3)^2 = 196$$

Taking square root of both sides

$$\sqrt{(m-3)^2} = \pm\sqrt{196}$$

$$m-3 = \pm 14$$

$$m-3 = 14 \quad \text{or} \quad m-3 = -14$$

$$m = 14+3 \quad \text{or} \quad m = -14+3$$

$$m = 17 \quad \text{or} \quad m = -11$$

Theorems on Proportions

If four quantities a, b, c and d form a proportion, then many other useful properties may be deduced by the properties of fractions.

(1) Theorem of Invertendo

If $a : b = c : d$, then $b : a = d : c$ is called invertendo theorem.

Example 1: If $3m : 2n = p : 2q$, then $2n : 3m = 2q : p$

Solution: Since $3m : 2n = p : 2q$

$$\therefore \quad \frac{3m}{2n} = \frac{p}{2q}$$

By invertendo theorem

$$\frac{2n}{3m} = \frac{2q}{p}$$

i.e $2n : 3m = 2q : p$

(2) Theorem of Alternando

If $a : b = c : d$, then $a : c = b : d$ is called alternando theorem.

Example 2: If $3p + 1 : 2q = 5r : 7s$, then prove that $3p + 1 : 5r = 2q : 7s$

Solution:

Given that $3p + 1 : 2q = 5r : 7s$

then $\frac{3p+1}{2q} = \frac{5r}{7s}$

By alternando theorem

$$\frac{3p+1}{5r} = \frac{2q}{7s}$$

Thus, $3p + 1 : 5r = 2q : 7s$

(3) Theorem of Componendo

If $a : b = c : d$ then componendo theorem is written as.

(i) $a + b : b = c + d : d$

and (ii) $a : a + b = c : c + d$

Example 3: If $m + 3 : n = p : q - 2$, then prove that $m + n + 3 : n = p + q - 2 : q - 2$

Solution: Since $m + 3 : n = p : q - 2$

$$\therefore \quad \frac{m+3}{n} = \frac{p}{q-2}$$

By componendo theorem

$$\frac{(m+3)+n}{n} = \frac{p+(q-2)}{q-2}$$

or $$\frac{m+n+3}{n} = \frac{p+q-2}{q-2}$$

Thus $m + n + 3 : n = p + q - 2 : q - 2$

(4) Theorem of Dividendo

If $a : b = c : d$, then dividendo theorem is written as:

(i) $a - b : b = c - d : d$

and (ii) $a : a - b = c : c - d$

Example 4: If $m + 1 : n - 2 = 2p + 3 : 3q + 1$ Then $m - n + 3 : n - 2 = 2p - 3q + 2 : 3q + 1$

Solution:

Given that $m + 1 : n - 2 = 2p + 3 : 3q + 1$

Then $\frac{m+1}{n-2} = \frac{2p+3}{3q+1}$

By didvidendo theorem

$$\frac{(m+1)-(n-2)}{n-2} = \frac{(2p+3)-(3q+1)}{3q+1}$$

$$\frac{(m+1-n+2}{n-2} = \frac{2p+3-3q-1}{3q+1}$$

$$\frac{m-n+3}{n-2} = \frac{2p-3q+2}{3q+1}$$

Thus $m - n + 3 : n - 2 = 2p - 3q + 2 : 3q + 1$

(5) Theorem of componendo–dividendo

If $a : b = c : d$, then componendo–dividendo theorem is written as:

(i) $a + b : a - b = c + d : c - d$

(ii) $a - b : a + b = c - d : c + d$

Example 5: If $m : n = p : q$ then prove that $3m + 7n : 3m - 7n = 3p + 7q : 3p - 7q$

Solution: Since $m : n = p : q$

or $$\frac{m}{n} = \frac{p}{q}$$

Multiplying both sides by $\frac{3}{7}$, we get

$$\frac{3m}{7n} = \frac{3p}{7q}$$

Then using componendo– dividendo theorem

$$\frac{3m+7n}{3m-7n} = \frac{3p+7q}{3p-7q}$$

or $3m + 7n : 3m - 7n = 3p + 7q : 3p - 7q$

Example 6:

If 5m + 3n : 5m – 3n = 5p + 3q : 5p – 3q, then show that m : n = p : q

Solution: Given that

$$5m + 3n : 5m - 3n = 5p + 3q : 5p - 3q$$

Or $\frac{5m+3n}{5m-3n} = \frac{5p+3q}{5p-3q}$

By componendo–dividendo theorem

$$\frac{(5m+3n)+(5m-3n)}{(5m+3n)-(5m-3n)} = \frac{(5p+3q)+(5p-3q)}{(5p+3q)-(5p-3q)}$$

$$\frac{5m+3n+5m-3n}{5m+3n-5m+3n} = \frac{5p+3q+5p-3q}{5p+3q-5p+3q}$$

$$\frac{10m}{6n} = \frac{10p}{6q}$$

Multiplying both sides by $\frac{6}{10}$

$$\frac{6}{10}\times\frac{10m}{6n} = \frac{6}{10}\times\frac{10p}{6q}$$

$$\frac{m}{n} = \frac{p}{q}$$

i.e $m : n = p : q$

Example 7:

Using theorem of componendo – dividend, find the value of

$$\frac{m+3p}{m-3p}+\frac{m+2q}{m-2q}, \text{ if } m = \frac{6pq}{p+q}$$

Solution: Since $m = \frac{6pq}{p+q}$ or

$$m = \frac{(3p)(2q)}{p+q} \text{..................(i)}$$

$$\therefore \frac{m}{3p} = \frac{2q}{p+q}$$

By componendo– dividend theorem

$$\frac{m+3p}{m-3p} = \frac{2q+(p+q)}{2q-(p+q)} = \frac{2q+p+q}{2q-p-q}$$

$$\frac{m+3p}{m-3p} = \frac{p+3q}{q-p} \text{...............(ii)}$$

Again from eq. (i), we have

$$\frac{m}{2q} = \frac{3p}{p+q}$$

By componendo–dividendo theorem

$$\frac{m+2q}{m-2q} = \frac{3p+(p+q)}{3p-(p+q)} = \frac{3p+p+q}{3p-p-q}$$

$$\frac{m+2q}{m-2q} = \frac{4p+q}{2p-q} \text{............(iii)}$$

Adding (ii) and (iii)

$$\frac{m+3p}{m-3p}+\frac{m+2q}{m-2q} = \frac{p+3q}{q-p}+\frac{4p+q}{2p-q}$$

$$= -\frac{p+3q}{p-q}+\frac{4p+q}{2p-q}$$

$$= \frac{-(p+3q)(2p-q)+(p-q)(4p+q)}{(p-q)(2p-q)}$$

$$= \frac{-[2p^2-pq+6pq-3q^2]+[4p^2+pq-4pq-q^2]}{(p-q)(2p-q)}$$

$$= \frac{-[2p^2+5pq-3q^2]+[4p^2-3pq-q^2]}{(p-q)(2p-q)}$$

$$= \frac{-2p^2-5pq+3q^2+4p^2-3pq-q^2}{(p-q)(2p-q)}$$

$$= \frac{2p^2-8pq+2q^2}{(p-q)(2p-q)} = \boxed{\frac{2(p^2-4pq+q^2)}{(p-q)(2p-q)}}$$

Example 8:

Using theorem of componendo – dividendo solve the equation

$$\frac{\sqrt{x+3}+\sqrt{x-3}}{\sqrt{x+3}-\sqrt{x-3}} = \frac{4}{3}$$

Solution: Given equation is

$$\frac{\sqrt{x+3}+\sqrt{x-3}}{\sqrt{x+3}-\sqrt{x-3}} = \frac{4}{3}$$

By componendo– dividendo theorem

$$\frac{(\sqrt{x+3}+\sqrt{x-3})+(\sqrt{x+3}-\sqrt{x-3})}{(\sqrt{x+3}+\sqrt{x-3})-(\sqrt{x+3}-\sqrt{x-3})} = \frac{4+3}{4-3}$$

$$\frac{\sqrt{x+3}+\sqrt{x-3}+\sqrt{x+3}-\sqrt{x-3}}{\sqrt{x+3}+\sqrt{x-3}-\sqrt{x+3}+\sqrt{x-3}} = \frac{7}{1}$$

$$\frac{2\sqrt{x+3}}{2\sqrt{x-3}} = \frac{7}{1} \Rightarrow \sqrt{\frac{x+3}{x-3}} = 7$$

Squaring both sides

$$\left(\sqrt{\frac{x+3}{x-3}}\right)^2 = (7)^2$$

$$\frac{x+3}{x-3} = 49$$

$$x + 3 = 49\ (x - 3)$$

$$\Rightarrow \quad x + 3 = 49x - 147$$

$$\Rightarrow \quad x - 49x = -147 - 3$$

$$-48x = -150$$

$$\Rightarrow \quad 48x = 150$$

$$\Rightarrow \quad x = \frac{150}{48} = \frac{25}{8} \quad \Rightarrow \quad \boxed{x = \frac{25}{8}}$$

Example 9:

Using componendo–dividendo theorem, solve the equation $\frac{(x+3)^2-(x-5)^2}{(x+3)^2+(x-5)^2} = \frac{4}{5}$

Solution: Given equation is

$$\frac{(x+3)^2-(x-5)^2}{(x+3)^2+(x-5)^2} = \frac{4}{5}$$

By componendo–dividendo theorem

$$\frac{[(x+3)^2-(x-5)^2]+[(x+3)^2+(x-5)^2]}{[(x+3)^2-(x-5)^2]-[(x+3)^2+(x-5)^2]} = \frac{4+5}{4-5}$$

$$\frac{(x+3)^2-\cancel{(x-5)^2}+(x+3)^2+\cancel{(x-5)^2}}{\cancel{(x+3)^2}-(x-5)^2-\cancel{(x+3)^2}-(x-5)^2} = \frac{4+5}{4-5}$$

$$\frac{\cancel{2}(x+3)^2}{\cancel{-}\cancel{2}(x-5)^2} = \frac{9}{\cancel{-}1}$$

$$\Rightarrow \quad \left(\frac{x+3}{x-5}\right)^2 = (3)^2$$

Taking square root

$$\frac{x+3}{x-5} = \pm 3$$

$$\frac{x+3}{x-5} = 3 \quad \text{or} \quad \frac{x+3}{x-5} = -3$$

$$x + 3 = 3\ (x - 5) \quad \text{or} \quad x + 3 = -3\ (x - 5)$$

$$x + 3 = 3x - 15 \quad \text{or} \quad x + 3 = -3x + 15$$

$$3 + 15 = 3x - x \quad \text{or} \quad x + 3x = 15 - 3$$

$$18 = 2x \quad \text{or} \quad 4x = 12$$

$$\frac{18}{2} = x \quad \text{or} \quad x = \frac{12}{4}$$

$$9 = x \quad \text{or} \quad x = 3$$

$$\Rightarrow \quad x = 9$$

The solution set is $\{3, 9\}$