EXERCISE 3.4

Q.1 Prove that a : b = c : d , if

(i) $$\frac{4a+5b}{4a-5b}=\frac{4c+5d}{4c-5d}$$

(ii) $$\frac{2a+9b}{2a-9b}=\frac{2c+9d}{2c-9d}$$

(iii) $$\frac{ac^2+bd^2}{ac^2-bd^2}=\frac{c^3+d^3}{c^3-d^3}$$

(iv) $$\frac{a^2c+b^2d}{a^2c-b^2d}=\frac{ac^2+bd^2}{ac^2-bd^2}$$

(v) $pa+qb : pa-qb = pc+qd : pc-qd$

(vi) $$\frac{a+b+c+d}{a+b-c-d}=\frac{a-b+c-d}{a-b-c+d}$$

(vii) $$\frac{2a+3b+2c+3d}{2a+3b-2c-3d}=\frac{2a-3b+2c-3d}{2a-3b-2c+3d}$$

(viii) $$\frac{a^2+b^2}{a^2-b^2}=\frac{ac+bd}{ac-bd}$$

Solutions:

(i) $$\frac{\mathbf{4a+5b}}{\mathbf{4a-5b}}=\frac{\mathbf{4c+5d}}{\mathbf{4c-5d}}$$

Solution:

$$\frac{4a+5b}{4a-5b}=\frac{4c+5d}{4c-5d}$$

By componendo – dividendo theorem

$$\frac{(4a+5b)+(4a-5b)}{(4a+5b)-(4a-5b)}=\frac{(4c+5d)+(4c-5d)}{(4c+5d)-(4c-5d)}$$

$$\frac{4a+\cancel{5b}+4a-\cancel{5b}}{\cancel{4a}+5b-\cancel{4a}+5b}=\frac{4c\cancel{+5d}+4c-\cancel{5d}}{\cancel{4c}+5d-\cancel{4c}+5d}$$

$$\frac{8a}{10b}=\frac{8c}{10d}$$

By multiplying both sides by $\frac{10}{8}$

$$\frac{\cancel{10}}{\cancel{8}}\frac{\cancel{8}}{\cancel{10}}\frac{a}{b}=\frac{\cancel{10}}{\cancel{8}}\frac{\cancel{8}c}{\cancel{10}d}$$

$\frac{a}{b}=\frac{c}{d} \Rightarrow a : b = c : d$ Hence proved

(ii) $\dfrac{2a+9b}{2a-9b}=\dfrac{2c+9d}{2c-9d}$

Solution:

$$\frac{2a+9b}{2a-9b}=\frac{2c+9d}{2c-9d}$$

By componendo – dividendo theorem

$$\frac{(2a+9b)+(2a-9b)}{(2a+9b)-(2a-9b)}=\frac{(2c+9d)+(2c-9d)}{(2c+9d)-(2c-9d)}$$

$$\frac{2a+\cancel{9b}+2a-\cancel{9b}}{\cancel{2a}+9b-\cancel{2a}+9b}=\frac{2c+\cancel{9d}+2c-\cancel{9d}}{\cancel{2c}+9d-\cancel{2c}+9d}$$

$$\frac{4a}{18b}=\frac{4c}{18d}$$

Multiplying both sides by $\dfrac{18}{4}$

$$\frac{\cancel{18}}{\cancel{4}}\times\frac{\cancel{4}\,a}{\cancel{18}\,b}=\frac{\cancel{4}\,c}{\cancel{18}\,d}.\frac{\cancel{18}}{\cancel{4}}$$

$$\frac{a}{b}=\frac{c}{d}$$

$\Rightarrow$ $a:b=c:d$

Hence proved

(iii) $\dfrac{ac^2+bd^2}{ac^2-bd^2}=\dfrac{c^3+d^3}{c^3-d^3}$

Solution:

$$\frac{ac^2+bd^2}{ac^2-bd^2}=\frac{c^3+d^3}{c^3-d^3}$$

By componendo – dividendo theorem

$$\frac{(ac^2+bd^2)+(ac^2-bd^2)}{(ac^2+bd^2)-(ac^2-bd^2)}=\frac{(c^3+d^3)+(c^3-d^3)}{(c^3+d^3)-(c^3-d^3)}$$

$$\frac{ac^2+\cancel{bd^2}+ac^2-\cancel{bd^2}}{\cancel{ac^2}+bd^2-\cancel{ac^2}+bd^2}=\frac{c^3+\cancel{d^3}+c^3-\cancel{d^3}}{\cancel{c^3}+d^3-\cancel{c^3}+d^3}$$

$$\frac{\cancel{2}ac^2}{\cancel{2}bd^2}=\frac{\cancel{2}c^3}{\cancel{2}d^3}$$

$$\frac{ac^2}{bd^2}=\frac{c.c^2}{d.d^2}$$

Multiplying both sides by $\dfrac{d^2}{c^2}$

$$\frac{a\cancel{c^2}}{b\cancel{d^2}}\times\frac{\cancel{d^2}}{\cancel{c^2}}=\frac{c\cancel{c^2}}{d\cancel{d^2}}\times\frac{\cancel{d^2}}{\cancel{c^2}}$$

$$\frac{a}{b}=\frac{c}{d}$$

$\Rightarrow$ $a:b=c:d$ Hence proved

(iv) $\dfrac{a^2c+b^2d}{a^2c-b^2d}=\dfrac{ac^2+bd^2}{ac^2-bd^2}$

Solution:

$$\frac{a^2c+b^2d}{a^2c-b^2d}=\frac{ac^2+bd^2}{ac^2-bd^2}$$

By componendo – dividendo theorem

$$\frac{(a^2c+b^2d)+(a^2c-b^2d)}{(a^2c+b^2d)-(a^2c-b^2d)}=\frac{(ac^2+bd^2)+(ac^2-bd^2)}{(ac^2+bd^2)-(ac^2-bd^2)}$$

$$\frac{a^2c+\cancel{b^2d}+a^2c-\cancel{b^2d}}{\cancel{a^2c}+b^2d-\cancel{a^2c}+b^2d}=\frac{ac^2+\cancel{bd^2}+ac^2-\cancel{bd^2}}{\cancel{ac^2}+bd^2-\cancel{ac^2}+bd^2}$$

$$\frac{\cancel{2}a^2c}{\cancel{2}b^2d}=\frac{\cancel{2}ac^2}{\cancel{2}bd^2}$$

$$\frac{a.ac}{b.bd}=\frac{ac.c}{bd.d}$$

Multiplying both sides by $\dfrac{bd}{ac}$

$$\frac{a.\cancel{ac}}{b.\cancel{bd}}\times\frac{\cancel{bd}}{\cancel{ac}}=\frac{\cancel{bd}}{\cancel{ac}}.\frac{\cancel{ac}.c}{\cancel{bd}.d}$$

$$\frac{a}{b}=\frac{c}{d}$$

$\Rightarrow$ $a:b=c:d$ Hence proved

(v) $\mathbf{pa+qb:pa-qb=pc+qd:pc-qd}$

Solution:

$$pa+qb:pa-qb=pc+qd:pc-qd$$

OR

$$\frac{pa+qb}{pa-qb}=\frac{pc+qd}{pc-qd}$$

By componendo– dividendo theorem

$$\frac{(pa+qb)+(pa-qb)}{(pa+qb)-(pa-qb)}=\frac{(pc+qd)+(pc-qd)}{(pc+qd)-(pc-qd)}$$

$$\frac{pa+\cancel{qb}+pa-\cancel{qb}}{\cancel{pa}+qb-\cancel{pa}+qb}=\frac{pc+\cancel{qd}+pc-\cancel{qd}}{\cancel{pc}+qd-\cancel{pc}+qd}$$

(vi) $\frac{p^2-q^2}{p^3+q^3}, \frac{p-q}{p^2-pq+q^2}$

Solution:

Let 3^{rd} proportional is x

Then $\frac{p^2-q^2}{p^3+q^3}$, $\frac{p-q}{p^2-pq+q^2}$, x

By proportion.

$$\frac{p^2-q^2}{p^3+q^3} : \frac{p-q}{p^2-pq+q^2} :: \frac{p-q}{p^2-pq+q^2} : x$$

Product of Extremes = Product of Means.

$$x.\ \frac{p^2-q^2}{p^3+q^3} = \frac{p-q}{p^2-pq+q^2} \times \frac{p-q}{p^2-pq+q^2}$$

$$x.\ \frac{p^2-q^2}{p^3+q^3} = \left[\frac{(p-q)}{(p^2-pq+q^2)}\right]^2$$

$$x = \frac{(p-q)^2}{(p^2-pq+q^2)^2} \times \frac{p^3+q^3}{(p^2-q^2)}$$

$$x = \frac{\cancel{(p-q)}(p-q)}{(p^2-pq+q^2)^{\cancel{2}}} \times \frac{\cancel{(p+q)}\ \cancel{(p^2-pq+q^2)}}{\cancel{(p+q)}\ \cancel{(p-q)}}$$

$$\boxed{x = \frac{p-q}{p^2-pq+q^2}}$$

Q.2 Find a fourth proportional

(i) 5, 8, 15

Solution:

Let 4^{th} proportional is x then 5, 8, 15, x

By proportion

5 : 8 : : 15 : x

Product of Extremes = Product of Means

$$5(x) = 8(15)$$

$$x = \frac{8(\cancel{15}^{3})}{\cancel{5}1}$$

$$x = 8(3)$$

$$\boxed{x = 24}$$

(ii) $4x^4, 2x^3, 18x^5$

Solution:

Let 4^{th} proportional is "a" then $4x^4$, $2x^3$, $18x^5$, a

By proportion

$$4x^4 : 2x^3 :: 18x^5 : a$$

Product of Extreme = Product of Means.

$$a(4x^4) = 2x^3(18x^5)$$

$$a = \frac{36x^8}{4x^4}$$

$$a = 9x^{8-4}$$

$$\boxed{a = 9x^4}$$

(iii) $15a^5b^6, 10a^2b^5, 21a^3b^3$

Solution:

Let 4^{th} proportional is x

then $15a^5b^6$, $10a^2b^5$, $21a^3b^3$, x

By proportion

$$15a^5b^6 : 10a^2b^5 :: 21a^3b^3 : x$$

Product of Extremes = Product of Means

$$x(15a^5b^6) = (10a^2b^5)(21a^3b^3)$$

$$x = \frac{^{14}\cancel{210}\ \cancel{a^5}\ b^8}{\cancel{15}\ \cancel{a^5}\ b^6}$$

$$x = 14b^{8-6}$$

$$\boxed{x = 14b^2}$$

(iv) $x^2-11x+24; (x-3), (5x^4-40x^3)$

Solution:

Let 4^{th} proportional is "a"

$x^2-11x+24$, $(x-3)$, $(5x^4-40x^3)$, a

By proportion

$$x^2-11x+24 : (x-3) :: 5x^4-40x^3 : a$$

Product of Extremes = Product of Means

$$a(x^2-11x+24) = (x-3)(5x^4-40x^3)$$

$$a = \frac{(x-3)(5x^4-40x^3)}{x^2-11x+24}$$

$$a = \frac{(x-3).5x^3(x-8)}{x^2-3x-8x+24}$$

$$a = \frac{5x^3(x-3)(x-8)}{x(x-3)-8(x-3)}$$

$$a = \frac{5x^3\cancel{(x-3)}\ \cancel{(x-8)}}{\cancel{(x-3)}\ \cancel{(x-8)}}$$

$$\boxed{a = 5x^3}$$

$$\frac{(a^2+b^2)+(a^2-b^2)}{(a^2+b^2)-(a^2-b^2)}=\frac{(ac+bd)+(ac-bd)}{(ac+bd)-(ac-bd)}$$

$$\frac{a^2+\cancel{b^2}+a^2-\cancel{b^2}}{\cancel{a^2}+b^2-\cancel{a^2}+b^2}=\frac{ac+\cancel{bd}+ac-\cancel{bd}}{\cancel{ac}+bd-\cancel{ac}+bd}$$

$$\frac{\cancel{2}a^2}{\cancel{2}b^2}=\frac{\cancel{2}ac}{\cancel{2}bd}$$

$$\frac{a.a}{b.a}=\frac{ac}{bd}$$

Multiplying both sides by "$\frac{b}{a}$"

$$\frac{\cancel{b}}{\cancel{a}}\cdot\frac{\cancel{a}.a}{\cancel{b}.a}=\frac{\cancel{b}}{\cancel{a}}\frac{\cancel{a}c}{\cancel{b}d}$$

$$\frac{a}{b}=\frac{c}{d}$$

$\Rightarrow$ $a:b=c:d$ Hence proved

Q.2 Use componendo–dividendo theorem to find the values of the following.

(i) Find the value of

$$\frac{x+2y}{x-2y}+\frac{x+2z}{x-2z}\text{, if } x=\frac{4yz}{y+z}$$

Solution: $x=\frac{4yz}{y+z}$ .... (i)

Dividing the equation (i) by "2y"

$$\frac{x}{2y}=\frac{4yz}{(y+z).2y}$$

$$\frac{x}{2y}=\frac{2z}{y+z}$$

By componendo – dividendo theorem

$$\frac{x+2y}{x-2y}=\frac{2z+(y+z)}{2z-(y+z)}$$

$$\frac{x+2y}{x-2y}=\frac{2z+y+z}{2z-y-z}$$

$$\frac{x+2y}{x-2y}=\frac{3z+y}{z-y}$$ .... (ii)

Now dividing the eq (i) by "2z"

$$\frac{x}{2z}=\frac{4yz}{(y+z).2z}$$

$$\frac{x}{2z}=\frac{2y}{y+z}$$

By componendo – dividendo theorem

$$\frac{x+2z}{x-2z}=\frac{2y+(y+z)}{2y-(y+z)}$$

$$\frac{x+2z}{x-2z}=\frac{2y+y+z}{2y-y-z}$$

$$\frac{x+2z}{x-2z}=\frac{3y+z}{y-z}$$ .... (iii)

Adding eq. (ii) and (iii)

$$\frac{x+2y}{x-2y}+\frac{x+2z}{x-2z}=\frac{3z+y}{z-y}+\frac{3y+z}{y-z}$$

$$=\frac{3z+y}{-1(y-z)}+\frac{3y+z}{y-z}$$

$$=\frac{-1(3z+y)}{y-z}+\frac{3y+z}{y-z}$$

$$=\frac{-3z-y+3y+z}{y-z}$$

$$=\frac{2y-2z}{y-z}$$

$$=\frac{2\cancel{(y-z)}}{\cancel{(y-z)}}$$

$$\frac{x+2y}{x-2y}+\frac{x+2z}{x-2z}=2$$

(ii) Find the value of

$$\frac{m+5n}{m-5n}+\frac{m+5p}{m-5p}\text{, if } m=\frac{10np}{n+p}$$

Solution: $m=\frac{10np}{n+p}$ .... (i)

Dividing equation (i) by "5n"

$$\frac{m}{5n}=\frac{10np}{(n+p)5n}$$

$$\frac{m}{5n}=\frac{2p}{n+p}$$

By componendo – dividendo theorem

$$\frac{m+5n}{m-5n}=\frac{2p+(n+p)}{2p-(n+p)}$$

$$\frac{m+5n}{m-5n}=\frac{2p+n+p}{2p-n-p}$$

$$\frac{m+5n}{m-5n}=\frac{3p+n}{p-n} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Now, dividing equation (i) by "5p"

$$\frac{m}{5p}=\frac{10np}{(n+p)5p}$$

$$\frac{m}{5p}=\frac{2n}{n+p}$$

By componendo– dividendo theorem

$$\frac{m+5p}{m-5p}=\frac{2n+(n+p)}{2n-(n+p)}$$

$$\frac{m+5p}{m-5p}=\frac{2n+n+p}{2n-n-p}$$

$$\frac{m+5p}{m-5p}=\frac{3n+p}{n-p} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

Adding equation (ii) and (iii)

$$\frac{m+5n}{m-5n}+\frac{m+5p}{m-5p}=\frac{3p+n}{p-n}+\frac{3n+p}{n-p}$$

$$=\frac{3p+n}{-1(n-p)}+\frac{3n+p}{n-p}$$

$$=\frac{-(3p+n)}{n-p}+\frac{3n+p}{n-p}$$

$$=\frac{-3p-n+3n+p}{n-p}$$

$$=\frac{2n-2p}{n-p}$$

$$=\frac{2\cancel{(n-p)}}{\cancel{(n-p)}}$$

$$\frac{m+5n}{m-5n}+\frac{m+5p}{m-5p}=2$$

(iii) Find the value of

$$\mathbf{\frac{x-6a}{x+6a}-\frac{x+6b}{x-6b} \quad \text{if} \quad x=\frac{12ab}{a-b}}$$

Solution: $x=\frac{12ab}{a-b}$ .... (i)

Dividing equation (i) by 6a

$$\frac{x}{6a}=\frac{12ab}{(a-b).6a}$$

$$\frac{x}{6a}=\frac{2b}{a-b}$$

By componendo– dividendo theorem

$$\frac{x+6a}{x-6a}=\frac{2b+(a-b)}{2b-(a-b)}$$

$$\frac{x+6a}{x-6a}=\frac{2b+a-b}{2b-a+b}$$

$$\frac{x+6a}{x-6a}=\frac{a+b}{3b-a}$$

By invertendo theorem

$$\frac{x-6a}{x+6a}=\frac{3b-a}{a+b} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Now, dividing the equation (i) by 6b.

$$\frac{x}{6b}=\frac{12ab}{(a-b).6b}$$

$$\frac{x}{6b}=\frac{2a}{a-b}$$

By componendo–dividendo theorem

$$\frac{x+6b}{x-6b}=\frac{2a+(a-b)}{2a-(a-b)}$$

$$\frac{x+6b}{x-6b}=\frac{2a+a-b}{2a-a+b}$$

$$\frac{x+6b}{x-6b}=\frac{3a-b}{a+b} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Subtracting equation (iii) from (ii)

$$\frac{x-6a}{x+6a}-\frac{x+6b}{x-6b}=\frac{3b-a}{a+b}-\frac{3a-b}{a+b}$$

$$=\frac{(3b-a)-(3a-b)}{a+b}$$

$$=\frac{3b-a-3a+b}{a+b}$$

$$= \frac{4b-4a}{a+b}$$

$$= \frac{4(b-a)}{(a+b)}$$

(iv) **Find the value of**

$$\frac{x-3y}{x+3y} - \frac{x+3z}{x-3z}, \text{ if } x = \frac{3yz}{y-z}$$

Solution: $x = \dfrac{3yz}{y-z}$ .... (i)

Dividing equation (i) by "3y"

$$\frac{x}{3y} = \frac{3yz}{(y-z).3y}$$

$$\frac{x}{3y} = \frac{z}{y-z}$$

By componendo – dividendo theorem

$$\frac{x+3y}{x-3y} = \frac{z+(y-z)}{z-(y-z)}$$

$$\frac{x+3y}{x-3y} = \frac{\cancel{z}+y-\cancel{z}}{z-y+z}$$

$$\frac{x+3y}{x-3y} = \frac{y}{2z-y}$$

By invertendo theorem

$$\frac{x-3y}{x+3y} = \frac{2z-y}{y} \quad \text{................ (ii)}$$

Now dividing equation (i) by "3z"

$$\frac{x}{3z} = \frac{3yz}{(y-z).3z}$$

$$\frac{x}{3z} = \frac{y}{y-z}$$

By componendo–dividendo theorem

$$\frac{x+3z}{x-3z} = \frac{y+(y-z)}{y-(y-z)}$$

$$\frac{x+3z}{x-3z} = \frac{y+y-z}{\cancel{y}-\cancel{y}+z}$$

$$\frac{x+3z}{x-3z} = \frac{2y-z}{z} \quad \text{................ (iii)}$$

Subtracting equation (iii) from (ii)

$$\frac{x-3y}{x+3y} - \frac{x+3z}{x-3z} = \frac{2z-y}{y} - \frac{2y-z}{z}$$

$$= \frac{z(2z-y)-y(2y-z)}{yz}$$

$$= \frac{2z^2-\cancel{yz}-2y^2+\cancel{yz}}{yz}$$

$$= \frac{2z^2-2y^2}{yz} = \frac{2(z^2-y^2)}{yz}$$

(v) **Find the value of**

$$\frac{s-3p}{s+3p} + \frac{s+3q}{s-3q}, \text{ if } s = \frac{6pq}{p-q}$$

Solution:

$s = \dfrac{6pq}{p-q}$ ....(i)

Dividing equation (i) by "3P"

$$\frac{s}{3p} = \frac{6pq}{(p-q).3p}$$

$$\frac{s}{3p} = \frac{2q}{p-q}$$

By componendo–dividendo theorem

$$\frac{s+3p}{s-3p} = \frac{2q+(p-q)}{2q-(p-q)}$$

$$\frac{s+3p}{s-3p} = \frac{2q+p-q}{2q-p+q}$$

$$\frac{s+3p}{s-3p} = \frac{p+q}{3q-p}$$

By invertendo theorem

$$\frac{s-3p}{s+3p} = \frac{3q-p}{p+q} \quad \text{................ (ii)}$$

Now dividing equation (i) by "3q"

$$\frac{s}{3q} = \frac{2\cancel{6}p\cancel{q}}{(p-q).\cancel{3}\cancel{q}}$$

$$\frac{s}{3q} = \frac{2p}{p-q}$$

By componendo – dividendo theorem

$$\frac{s+3q}{s-3q} = \frac{2p+(p-q)}{2p-(p-q)}$$

$$\frac{s+3q}{s-3q} = \frac{2p+p-q}{2p-p+q}$$

$$\frac{s+3q}{s-3q}=\frac{3p-q}{p+q}\text{.......................}(iii)$$

Adding equation (ii) and (iii)

$$\frac{s-3p}{s+3p}+\frac{s+3q}{s-3q}=\frac{3q-p}{p+q}+\frac{3p-q}{p+q}$$

$$\frac{s-3p}{s+3p}+\frac{s+3q}{s-3q}=\frac{3q-p+3p-q}{(p+q)}$$

$$\frac{s-3p}{s+3p}+\frac{s+3q}{s-3q}=\frac{2p+2q}{(p+q)}$$

$$\frac{s-3p}{s+3p}+\frac{s+3q}{s-3q}=\frac{2\cancel{(p+q)}}{\cancel{(p+q)}}$$

$$\frac{s-3p}{s+3p}+\frac{s+3q}{s-3q}=2$$

(vi) Solve $\frac{(x-2)^2-(x-4)^2}{(x-2)^2+(x-4)^2}=\frac{12}{13}$

Solution:

$$\frac{(x-2)^2-(x-4)^2}{(x-2)^2+(x-4)^2}=\frac{12}{13}$$

By componendo–dividendo theorem

$$\frac{[(x-2)^2-(x-4)^2]+[(x-2)^2+(x-4)^2]}{[(x-2)^2-(x-4)^2]-[(x-2)^2+(x-4)^2]}=\frac{12+13}{12-13}$$

$$\frac{(x-2)^2-\cancel{(x-4)^2}+(x-2)^2+\cancel{(x-4)^2}}{\cancel{(x-2)^2}-(x-4)^2-\cancel{(x-2)^2}-(x-4)^2}=\frac{25}{-1}$$

$$\frac{\cancel{2}(x-2)^2}{\cancel{-}\cancel{2}(x-4)^2}=\cancel{-}25$$

$$\left(\frac{x-2}{x-4}\right)^2=25$$

Taking square root of both side

$$\sqrt{\left(\frac{x-2}{x-4}\right)^2}=\pm\sqrt{25}$$

$$\frac{x-2}{x-4}=\pm5$$

$\frac{x-2}{x-4}=5$ or $\frac{x-2}{x-4}=-5$

$x-2=5(x-4)$ or $x-2=-5(x-4)$

$x-2=5x-20$ or $x-2=-5x+20$

$20-2=5x-x$ or $x+5x=20+2$

$18=4x$ or $6x=22$

$\frac{\cancel{18}^{9}}{\cancel{4}_{2}}=x$ or $x=\frac{22}{6}$

$x=\frac{9}{2}$ or $x=\frac{11}{3}$

$$S.S=\left\{\frac{9}{2},\frac{11}{3}\right\}$$

(vii) Solve $\frac{\sqrt{x^2+2}+\sqrt{x^2-2}}{\sqrt{x^2+2}-\sqrt{x^2-2}}=\frac{2}{1}$

Solution:

$$\frac{\sqrt{x^2+2}+\sqrt{x^2-2}}{\sqrt{x^2+2}-\sqrt{x^2-2}}=\frac{2}{1}$$

By componendo– dividendo theorem

$$\frac{\left(\sqrt{x^2+2}+\sqrt{x^2-2}\right)+\left(\sqrt{x^2+2}-\sqrt{x^2-2}\right)}{\left(\sqrt{x^2+2}+\sqrt{x^2-2}\right)-\left(\sqrt{x^2+2}-\sqrt{x^2-2}\right)}=\frac{2+1}{2-1}$$

$$\frac{\sqrt{x^2+2}+\cancel{\sqrt{x^2-2}}+\sqrt{x^2+2}-\cancel{\sqrt{x^2-2}}}{\cancel{\sqrt{x^2+2}}+\sqrt{x^2-2}-\cancel{\sqrt{x^2+2}}+\sqrt{x^2-2}}=\frac{3}{1}$$

$$\frac{\cancel{2}\sqrt{x^2+2}}{\cancel{2}\sqrt{x^2-2}}=3$$

$$\sqrt{\frac{x^2+2}{x^2-2}}=3$$

Taking square of both sides

$$\left(\sqrt{\frac{x^2+2}{x^2-2}}\right)^{\cancel{2}}=(3)^2$$

$$\frac{x^2+2}{x^2-2}=9$$

$x^2+2=9(x^2-2)$

$x^2+2=9x^2-18$

$2+18=9x^2-x^2$

$20=8x^2$

$$x^2=\frac{\cancel{20}^{5}}{\cancel{8}_{2}}\Rightarrow x^2=\frac{5}{2}$$

Taking square root

$$\sqrt{x^2}=\pm\sqrt{\frac{5}{2}}\Rightarrow x=\pm\sqrt{\frac{5}{2}}$$

$$S.S=\left\{\pm\sqrt{\frac{5}{2}}\right\}$$

(viii) Solve $\dfrac{\sqrt{x^2+8p^2}-\sqrt{x^2-p^2}}{\sqrt{x^2+8p^2}+\sqrt{x^2-p^2}}=\dfrac{1}{3}$

Solution:

$$\frac{\sqrt{x^2+8p^2}-\sqrt{x^2-p^2}}{\sqrt{x^2+8p^2}+\sqrt{x^2-p^2}}=\frac{1}{3}$$

By componendo–dividendo theorem

$$\frac{\left(\sqrt{x^2+8p^2}-\sqrt{x^2-p^2}\right)+\left(\sqrt{x^2+8p^2}+\sqrt{x^2-p^2}\right)}{\left(\sqrt{x^2+8p^2}-\sqrt{x^2-p^2}\right)-\left(\sqrt{x^2+8p^2}+\sqrt{x^2-p^2}\right)}=\frac{1+3}{1-3}$$

$$\frac{\sqrt{x^2+8p^2}-\cancel{\sqrt{x^2-p^2}}+\sqrt{x^2+8p^2}+\cancel{\sqrt{x^2-p^2}}}{\cancel{\sqrt{x^2+8p^2}}-\sqrt{x^2-p^2}-\cancel{\sqrt{x^2+8p^2}}-\sqrt{x^2-p^2}}=\frac{4}{-2}$$

$$\frac{\cancel{2}\sqrt{x^2+8p^2}}{\cancel{-}\cancel{2}\sqrt{x^2-p^2}}=\cancel{-}2$$

$$\sqrt{\frac{x^2+8p^2}{x^2-p^2}}=2$$

Taking square of both sides

$$\left(\sqrt{\frac{x^2+8p^2}{x^2-p^2}}\right)^2=(2)^2$$

$$\frac{x^2+8p^2}{x^2-p^2}=4$$

$x^2 + 8p^2 = 4(x^2 - p^2)$

$x^2 + 8p^2 = 4x^2 - 4p^2$

$8p^2 + 4p^2 = 4x^2 - x^2$

$12p^2 = 3x^2$

$\Rightarrow$ $3x^2 = 12p^2$

$$x^2=\frac{12p^2}{3}$$

$x^2 = 4p^2$

Taking square root

$$\sqrt{x^2}=\pm\sqrt{4p^2}$$

$x = \pm 2p$

S.S = $\{\pm 2p\}$

(ix) Solve $\dfrac{(x+5)^3-(x-3)^3}{(x+5)^3+(x-3)^3}=\dfrac{13}{14}$

Solution:

$$\frac{(x+5)^3-(x-3)^3}{(x+5)^3+(x-3)^3}=\frac{13}{14}$$

By componendo–dividendo theorem

$$\frac{\left[(x+5)^3-(x-3)^3\right]+\left[(x+5)^3+(x-3)^3\right]}{\left[(x+5)^3-(x-3)^3\right]-\left[(x+5)^3+(x-3)^3\right]}=\frac{13+14}{13-14}$$

$$\frac{(x+5)^3-\cancel{(x-3)^3}+(x+5)^3+\cancel{(x-3)^3}}{\cancel{(x+5)^3}-(x-3)^3-\cancel{(x+5)^3}-(x-3)^3}=\frac{27}{-1}$$

$$\frac{\cancel{2}(x+5)^3}{\cancel{-}\cancel{2}(x-3)^3}=\cancel{-}27$$

$$\left(\frac{x+5}{x-3}\right)^3=27$$

Taking cube root

$$\sqrt[3]{\left(\frac{x+5}{x-3}\right)^3}=\sqrt[3]{27}$$

$$\left(\frac{x+5}{x-3}\right)^{\cancel{3}\times\frac{1}{\cancel{3}}}=(3)^{\cancel{3}\times\frac{1}{\cancel{3}}}$$

$$\frac{x+5}{x-3}=3$$

$(x+5) = 3(x - 3)$

$x + 5 = 3x - 9$

$5 + 9 = 3x - x$

$14 = 2x$

$$\frac{14}{2}=x$$

$7 = x$

$\Rightarrow$ $x = 7$

S.S = $\{7\}$

Joint variation

A combination of direct and inverse variations of one or more than one variables forms joint variation. If a variable y varies directly as x and varies inversely as z.

Then $y \propto x$ and $y \propto \frac{1}{z}$

In joint variation, we write it as

$y \propto \frac{x}{z}$ i.e $y = k\frac{x}{z}$

Where $k \neq 0$ is the constant of variation.

For example, by Newton's law of gravitation, if one body attracts another with a force F that varies directly as the product of their masses (m_1), (m_2) and inversely as the square of the distance (d) between them.

i.e $F \propto \frac{m_1 m_2}{d^2}$ or $F = k\frac{m_1 m_2}{d^2}$

Problems related to joint variation

Example 1: **If y varies jointly as x^2 and z and y = 6 when x = 4, z = 9. Write y as a function x and z and determine the value of y, when x = –8 and z = 12.**

Solution: Since y varies jointly as x^2 and z, therefore

$y \propto x^2 z$

i.e $y = kx^2 z$ ....(i)

To find k,

Put y = 6, x = 4, z = 9 in equation (i)

$6 = k(4)^2(9)$

$\frac{6}{16\times 9} = k$

$\Rightarrow k = \frac{6}{144} = \frac{1}{24}$

Put $k = \frac{1}{24}$ in equation (i)

$y = \frac{1}{24}x^2 z$ ....(ii)

To find y,

Now put x = – 8, z = 12 in equation (i)

$y = \frac{1}{24_2}(-8)^2(12^1)$

$y = \frac{1}{2}(64)$ $\Rightarrow$ $\boxed{y = 32}$

Example 2:

p varies jointly as q and r^2 and inversely as s and t^2, p = 40, when q = 8, r = 5, S=3, t =2. Find p in terms of q, r, s and t. Also find the value of p when q = –2, r = 4, s = 3 and t = –1.

Solution: Given that $p \propto \frac{qr^2}{st^2}$

$p = k\frac{qr^2}{st^2}$ ....(i)

To find k,

Put p = 40, q = 8, r = 5, s =3 and t = 2 in equation (i)

$40 = k\frac{(8)(5)^2}{3(2)^2}$

$40 = k\frac{8(25)}{3(4)}$

$\frac{\overset{8}{40}\times 12}{8(\underset{5}{25})} = k$

$\Rightarrow k = \frac{12}{5}$

Put $k = \frac{12}{5}$ in equation (i)

$p = \frac{12}{5}\ \frac{qr^2}{st^2}$..............(ii)

To find P,

Put q= –2, r = 4, s = 3 and t = –1 in equation (i)

$P = \frac{12}{5}\times\frac{(-2)(4)^2}{(3)(-1)^2}$

$P = \frac{12^4}{5}\cdot\frac{(-2)(16)}{3_1}$

$p = \frac{(-8)\times(16)}{5}$

$\boxed{p = \frac{-128}{5}}$