EXERCISE 4.3

Resolve into partial fractions.

Q.1 $\frac{3x-11}{(x+3)(x^2+1)}$

Solution: $\frac{3x-11}{(x+3)(x^2+1)}$

Let $\frac{3x-11}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1}$ ... (i)

Multiplying both sides $(x+3)(x^2+1)$, we get

$3x - 11 = A(x^2+1) + (Bx+C)(x+3)$... (ii)

$3x - 11 = A(x^2+1) + Bx(x+3) + C(x+3)$... (iii)

Putting $x + 3 = 0$ i.e $x = -3$ in (ii), we get

$$3(-3) - 11 = A[(-3)^2 + 1]$$
$$-9 - 11 = A(9+1)$$
$$-20 = 10A$$
$$A = \frac{-20}{10}$$
$$\Rightarrow \boxed{A = -2}$$

Now equating the coefficients of x^2 and x we get from equation (iii)

$A + B = 0$	$3B + C = 3$
$-2 + B = 0$	$3(2) + C = 3$
$B = 2$	
$\Rightarrow \boxed{B = 2}$	$6 + C = 3$
	$C = 3 - 6$
	$\Rightarrow \boxed{C = -3}$

Putting the value of A, B and C in equation (i) we get required partial fractions.

$$\frac{3x-11}{(x+3)(x^2+1)} = \frac{-2}{x+3} + \frac{2x-3}{x^2+1}$$

Q.2 $\frac{3x+7}{(x^2+1)(x+3)}$

Solution: $\frac{3x+7}{(x^2+1)(x+3)}$

Let $\frac{3x+7}{(x^2+1)(x+3)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+3}$......(i)

Multiplying both sides by $(x^2+1)(x+3)$

$3x + 7 = (Ax + B)(x + 3) + C(x^2+1)$

$3x+7 = Ax(x+3) + B(x+3) + C(x^2+1)$ .....(ii)

Putting $x + 3 = 0$ i.e $x = -3$ in (ii), we get

$$3(-3)+7 = C[(-3)^2+1]$$
$$-9+7 = C(9+1)$$
$$-2 = 10C$$
$$\Rightarrow C = \frac{-2}{10}$$
$$\boxed{C = \frac{-1}{5}}$$

Now equating the coefficients of x^2 and x in equation (iii) we get

$A + C = 0$	$3A + B = 3$
$A + \left(\frac{-1}{5}\right) = 0$	$3\left(\frac{1}{5}\right) + B = 3$
$A - \frac{1}{5} = 0$	$B = 3 - \frac{3}{5}$
	$B = \frac{15-3}{5}$
$\Rightarrow \boxed{A = \frac{1}{5}}$	$\Rightarrow \boxed{B = \frac{12}{5}}$

Putting the value of A, B and C in equation (i) we get required partial fractions.

$$\frac{3x+7}{(x^2+1)(x+3)} = \frac{x+12}{5(x^2+1)} - \frac{1}{5(x+3)}$$

Q.3 $\dfrac{1}{(x+1)(x^2+1)}$

Solution: $\dfrac{1}{(x+1)(x^2+1)}$

Let $\dfrac{1}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$ ....(i)

Multiplying both sides by (x+1) (x^2+1),we get

$$1 = A(x^2+1) + (Bx+C)(x+1)$$

$$1 = A(x^2+1) + Bx(x+1) + C(x+1) \ldots \text{(ii)}$$

Putting x + 1 = 0 i.e x = − 1 in (ii), we get

$$1 = A[(-1)^2+1]$$

$$1 = A(1+1)$$

$$1 = 2A$$

$$\boxed{A = \frac{1}{2}}$$

Equating the coefficients of x^2 and x in equation (ii) we get

$A + B = 0$	$B + C = 0$
$\frac{1}{2} + B = 0$	$-\frac{1}{2} + C = 0$
$\Rightarrow \boxed{B = -\frac{1}{2}}$	$\Rightarrow \boxed{C = \frac{1}{2}}$

Putting the value of A, B and C in equation (i) we get required partial fractions.

$$\frac{1}{(x+1)(x^2+1)} = \frac{1}{2(x+1)} - \frac{x-1}{2(1+x^2)}$$

Q.4 $\dfrac{9x-7}{(x+3)(x^2+1)}$

Solution: $\dfrac{9x-7}{(x+3)(x^2+1)}$

Let $\dfrac{9x-7}{(x+3)(x^2+1)} = \dfrac{A}{x+3} + \dfrac{Bx+C}{x^2+1}$ .....(i)

Multiplying both sides by (x+3)(x^2+1) we get

$$9x - 7 = A(x^2+1) + (Bx+C)(x+3)$$

$$9x - 7 = A(x^2+1) + Bx(x+3) + C(x+3) \ldots \text{(ii)}$$

Putting x+3 = 0 i.e x = − 3 in (ii), we get

$$9(-3) - 7 = A[(-3)^2+1]$$

$$-27 - 7 = A(9+1)$$

$$-34 = 10A$$

$$\Rightarrow A = \frac{-34}{10} \Rightarrow \boxed{A = \frac{-17}{5}}$$

Equating coefficients of x^2 and x in equation (ii) we get

$A + B = 0$	$3B + C = 9$
$\frac{-17}{5} + B = 0$	$3\left(\frac{17}{5}\right) + C = 9$
$\Rightarrow \boxed{B = \frac{17}{5}}$	$\frac{51}{5} + C = 9$
	$C = 9 - \frac{51}{5}$
	$C = \frac{45-51}{5}$
	$\Rightarrow \boxed{C = \frac{-6}{5}}$

Putting the value of A, B and C in equation (i) we get required partial fractions.

$$\frac{9x-7}{(x+3)(x^2+1)} = \frac{-17}{5(x+3)} + \frac{17x-6}{5(x^2+1)}$$

Q.5 $\dfrac{3x+7}{(x+3)(x^2+4)}$

Solution: $\dfrac{3x+7}{(x+3)(x^2+4)}$

Let $\dfrac{3x+7}{(x+3)(x^2+4)} = \dfrac{A}{x+3} + \dfrac{Bx+C}{x^2+4}$ ... (i)

Multiplying both sides by $(x+3)(x^2+4)$ we get

$$3x + 7 = A(x^2+4) + (Bx+C)(x+3)$$

$$3x + 7 = A(x^2+4) + Bx(x+3) + C(x+3) \ldots \text{(ii)}$$

Putting $x + 3 = 0$ i.e $x = -3$ in (ii) we get

$$3(-3)+7 = A((-3)^2+4)$$
$$-9+7 = A(9+4)$$
$$-2 = 13A$$

$\Rightarrow$ $\boxed{A = \dfrac{-2}{13}}$

Equating the coefficients of x^2 and x in equation (ii) we get

$$A + B = 0$$
$$\frac{-2}{13} + B = 0$$
$$B = \frac{2}{13}$$

$\Rightarrow$ $\boxed{B = \dfrac{2}{13}}$

$$3B + C = 3$$
$$3\left(\frac{2}{13}\right) + C = 3$$
$$\frac{6}{13} + C = 3$$
$$C = 3 - \frac{6}{13}$$
$$C = \frac{39-6}{13}$$

$\Rightarrow$ $\boxed{C = \dfrac{33}{13}}$

Putting the value of A, B and C in equation (i) we get required partial fractions.

$$\frac{3x+7}{(x+3)(x^2+4)} = \frac{-2}{13(x+3)} + \frac{2x+33}{13(x^2+4)}$$

Q.6 $\dfrac{x^2}{(x+2)(x^2+4)}$

Solution: $\dfrac{x^2}{(x+2)(x^2+4)}$

Let $\dfrac{x^2}{(x+2)(x^2+4)} = \dfrac{A}{x+2} + \dfrac{Bx+C}{x^2+4}$ ....(i)

Multiplying both sides by $(x+2)(x^2+4)$ we get

$$x^2 = A(x^2+4) + (Bx+C)(x+2)$$

$$x^2 = A(x^2+4) + Bx(x+2) + C(x+2) \ldots \text{(ii)}$$

Putting $x + 2 = 0$ i.e $x = -2$ in (ii) we get

$$(-2)^2 = A[(-2)^2+4]$$
$$4 = A(4+4)$$
$$4 = 8A$$

$\Rightarrow A = \dfrac{4}{8}$

$\boxed{A = \dfrac{1}{2}}$

Equating the coefficients of x^2 and x in equation (ii) we get

$$A + B = 1$$
$$\frac{1}{2} + B = 1$$
$$B = 1 - \frac{1}{2}$$

$\Rightarrow$ $\boxed{B = \dfrac{1}{2}}$

$$2B + C = 0$$
$$\cancel{2}\left(\frac{1}{\cancel{2}}\right) + C = 0$$
$$1 + C = 0$$

$\Rightarrow$ $\boxed{C = -1}$

Putting the value of A, B and C in equation (i) we get required partial fractions.

$$\frac{x^2}{(x+2)(x^2+4)} = \frac{1}{2(x+2)} + \frac{x-2}{2(x^2+4)}$$

Q.7 $\frac{1}{x^3+1}$

Solution: $\frac{1}{x^3+1}$

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)}$$

Let $\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$ .....(i)

Multiplying both sides by $(x+1)(x^2-x+1)$, we get

$$1 = A(x^2 - x + 1) + (Bx + C)(x + 1)$$

$$1 = A(x^2 - x + 1) + Bx(x + 1) + C(x + 1) \ldots(ii)$$

Putting $x+1=0$ i.e $x = -1$ in (ii) we get

$$1 = A\,[(-1)^2 - (-1) + 1]$$

$$1 = A(1 + 1 + 1)$$

$$1 = 3A$$

$\Rightarrow$ $\boxed{A = \frac{1}{3}}$

Comparing the coefficients of x^2 and x in equation (ii) we get

$A + B = 0$	$-A + B + C = 0$
$\frac{1}{3} + B = 0$	$-\frac{1}{3} - \frac{1}{3} + C = 0$
$\Rightarrow B = \frac{-1}{3}$	$\frac{-2}{3} + C = 0$
$\boxed{B = \frac{-1}{3}}$	$\Rightarrow \boxed{C = \frac{2}{3}}$

Putting the value of A, B and C in equation (i) we get required partial fractions.

$$\frac{1}{(x+1)(x^2-x+1)} = \frac{1}{3(x+1)} - \frac{x-2}{3(x^2-x+1)}$$

Q.8 $\frac{x^2+1}{x^3+1}$

Solution: $\frac{x^2+1}{x^3+1}$

$$\frac{x^2+1}{x^3+1} = \frac{x^2+1}{(x+1)(x^2-x+1)}$$

Let $\frac{x^2+1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$ .....(i)

Multiplying both sides by $(x+1)(x^2-x+1)$, we get

$$x^2 + 1 = A(x^2 - x + 1) + (Bx + C)(x + 1)$$

$$x^2 + 1 = A(x^2 - x + 1) + Bx(x + 1) + C(x + 1) \ldots(ii)$$

Putting $x + 1 = 0$ i.e $x = -1$ in (ii) we get

$$(-1)^2 + 1 = A\,[(-1)^2 - (-1) + 1]$$

$$1 + 1 = A(1 + 1 + 1)$$

$$2 = 3A$$

$\Rightarrow$ $\boxed{A = \frac{2}{3}}$

Equating the coefficients of x^2 and x in equation (ii) we get

$A + B = 1$	$-A + B + C = 0$
$\frac{2}{3} + B = 1$	$\frac{-2}{3} + \frac{1}{3} + C = 0$
$B = 1 - \frac{2}{3}$	$\frac{-1}{3} + C = 0$
$\Rightarrow \boxed{B = \frac{1}{3}}$	$\Rightarrow \boxed{C = \frac{1}{3}}$

Putting the value of A, B and C in equation (i) we get required partial fractions.

$$\frac{x^2+1}{x^3+1} = \frac{2}{3(x+1)} + \frac{x+1}{3(x^2-x+1)}$$

Rule IV:

Resolution of a fraction when D(x) has repeated irreducible quadratic factors:

If a quadratic factor $(ax^2 + bx + c)$ with $a \neq 0$, occurs twice in the denominator, the corresponding partial fractions are

$$\frac{Ax+B}{(ax^2+bx+c)}+\frac{Cx+D}{(ax^2+bx+c)^2}$$

The constants A, B, C and D are found in the usual way.

Example 1:

Resolve $\frac{x^3-2x^2-2}{(x^2+1)^2}$ into partial fractions.

Solution: $\frac{x^3-2x^2-2}{(x^2+1)^2}$ is a proper fraction as degree of numerator is less than the degree of denominator.

Let, $$\frac{x^3-2x^2-2}{(x^2+1)^2} = \frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}$$

Multiplying both the sides by $(x^2+1)^2$, we get

$$x^3 - 2x^2 - 2 = (Ax + B)(x^2+1) + Cx + D$$

$$x^3-2x^2 - 2 = A(x^3 + x)+B(x^2 + 1)+Cx + D \ldots \text{(i)}$$

Equating the coefficients of x^3, x^2, x and constant on both the sides of (i)

Coefficients of x^3: $A = 1$

Coefficients of x^2: $B = -2$

Coefficients of x: $A + C = 0$

$C= -A$

$\Rightarrow$ $C = -1$

$\boxed{C=-1}$

Constants: $B + D = -2$

$D = -2 - B$

$D = -2-(-2)$

$D = -2 + 2$

$\boxed{D=0}$

Putting the value of A, B C and D, we get required partial fractions.

Thus $$\frac{x^3-2x^2-2}{(x^2+1)^2} = \frac{x-2}{x^2+1}+\frac{-x+0}{(x^2+1)^2}$$

$$= \frac{x-2}{x^2+1}-\frac{x}{(x^2+1)^2}$$

Example 2:

Resolve $\frac{2x+1}{(x-1)(x^2+1)^2}$ into partial fractions.

Solution: Assume that

$$\frac{2x+1}{(x-1)(x^2+1)^2} = \frac{A}{x-1}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$$

Multiplying both sides by $(x-1)(x^2 + 1)^2$

we get

$$2x + 1 = A(x^2+1)^2 + (Bx+ C)(x - 1)(x^2 + 1) + (Dx + E)(x - 1) \ldots\ldots\text{(i)}$$

Now we use zeros, method. Put $x-1 = 0$

or $x= 1$ in (i), we get

$2(1) +1 = [(1)^2 + 1]^2$

$3 = A(1 + 1)^2$

$3 = A(2)^2$

$3 = 4A$

$\Rightarrow \boxed{A = \frac{3}{4}}$

Now writing terms of (i) in descending order.

$$2x + 1 = A(x^4+2x^2+1) + Bx(x^3- x^2 + x - 1) + C(x^3- x^2 + x -1)+D(x^2 - x) + E(x - 1)$$

or $$2x + 1= A(x^4+2x^2+1) +B(x^4-x^3+x^2 - x) + C(x^3 - x^2+x-1)+D(x^2 - x)+E(x - 1)$$

Equating coefficients of x^4, x^3, x^2, and x on both the sides.

Coefficients of x^4: $A + B = 0$

$$B = -A$$

$$\Rightarrow \quad B = \frac{-3}{4}$$

Coefficients of x^3: $-B + C = 0$

$$C = B$$

$$\Rightarrow \quad C = \frac{-3}{4}$$

Coefficients of x^2 : $2A + B - C + D = 0$

$$2\left(\frac{3}{4}\right) + \left(\frac{-3}{4}\right) - \left(\frac{-3}{4}\right) + D = 0$$

$$\left(\frac{3}{2}\right) - \frac{3}{4} + \frac{3}{4} + D = 0$$

$$\Rightarrow \quad D = \frac{-3}{2}$$

Coefficients of x: $-B + C - D + E = 2$

$$\frac{3}{4} - \frac{3}{4} + \frac{3}{2} + E = 2$$

$$E = 2 - \frac{3}{2} = \frac{4-3}{2}$$

$$\Rightarrow \quad \boxed{E = \frac{1}{2}}$$

Thus required partial fraction are

$$\frac{3}{4(x-1)} + \frac{\frac{-3}{4}x - \frac{3}{4}}{x^2+1} + \frac{\frac{-3}{2}x + \frac{1}{2}}{(x^2+1)^2}$$

$$\therefore \frac{2x+1}{(x-1)(x^2+1)^2} = \frac{3}{4(x-1)} - \frac{3(x+1)}{4(x^2+1)} - \frac{(3x-1)}{2(x^2+1)^2}$$

Equating coefficients of x^4, x^3, x^2, and x on both the sides.

Coefficients of x^4: $A + B = 0$

$$B = -A$$

$$\Rightarrow \quad B = \frac{-3}{4}$$

Coefficients of x^3: $-B + C = 0$

$$C = B$$

$$\Rightarrow \quad C = \frac{-3}{4}$$

Coefficients of x^2 : $2A+B-C+D = 0$

$$\not{2}\left(\frac{3}{\not{4}_2}\right)+\left(\frac{-3}{4}\right)-\left(\frac{-3}{4}\right)+D=0$$

$$\left(\frac{3}{2}\right)-\frac{\not{3}}{\not{4}}+\frac{\not{3}}{\not{4}}+D=0$$

$$\Rightarrow \quad D = \frac{-3}{2}$$

Coefficients of x: $-B + C - D + E = 2$

$$\frac{\not{3}}{\not{4}}-\frac{\not{3}}{\not{4}}+\frac{3}{2}+E=2$$

$$E = 2-\frac{3}{2}=\frac{4-3}{2}$$

$$\Rightarrow \quad \boxed{E=\frac{1}{2}}$$

Thus required partial fraction are

$$\frac{3}{4(x-1)}+\frac{\frac{-3}{4}x-\frac{3}{4}}{x^2+1}+\frac{\frac{-3}{2}x+\frac{1}{2}}{(x^2+1)^2}$$

$$\therefore \frac{2x+1}{(x-1)(x^2+1)^2}=\frac{3}{4(x-1)}-\frac{3(x+1)}{4(x^2+1)}-\frac{(3x-1)}{2(x^2+1)^2}$$