Unit 07 INTRODUCTION TO TRIGONOMETRY

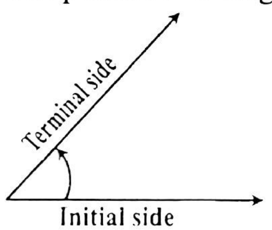
Measurement of an Angle

Angle:

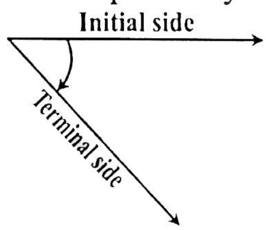
An angle is defined as the union of two non–collinear rays with some common end point. The rays are called arms of the angle and the common end point is known as vertex of the angle.

Clockwise or Anti–clockwise angle:

It is easy if we make an angle by rotating a ray from one position to another. When we form an angle in this way, the original position of the ray is called initial side and final position of the ray is called the terminal side of the angle. If the rotation of the ray is anti–clockwise or clockwise, the angle has positive or negative measure respectively.



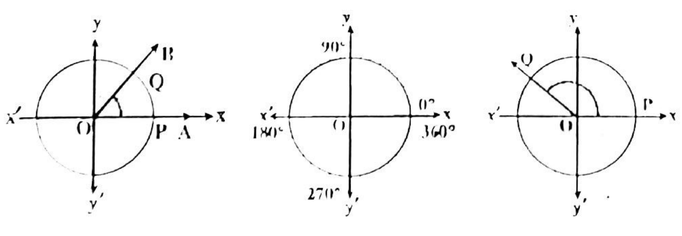
Anti - clockwise rotation
positive measure



Clockwise rotation
Negative measure

Measurement of an angle in sexagesimal system (degree, minute and second)

Degree: We divide the circumference of a circle into 360 equal arcs. The angle subtended at the centre of the circle by one arc is called one degree and is denoted by 1°.



The symbols 1°, 1′ and 1″ are used to denote a degree, a minute and a second respectively.

Thus 60 seconds (60″) make one minute (1′)

60 minutes (60′) make one degree (1°)

90 degrees (90°) make one right angle.

360 degrees (360°) make 4 right angles.

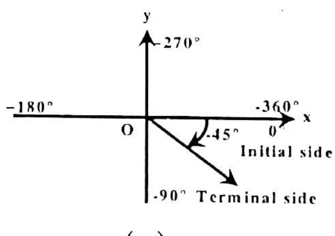
An angle of 360° denotes a complete circle or one revolution. We use coordinate system to locate any angle to a standard position, where its initial side is the positive x–axis and its vertex is the origin.

Example: Locate

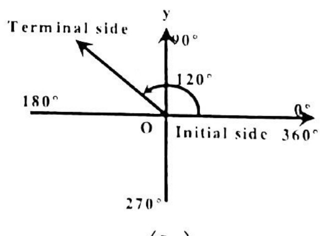
(a) – 45° (b) 120°

(c) 45° (d) –270°

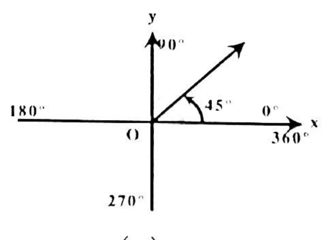
Solution: The angles are shown in figure.



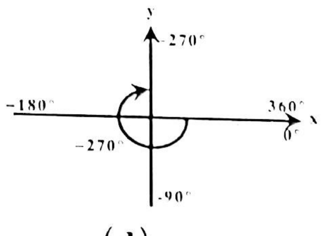
(a)



(b)



(c)



(d)

Conversion of an angle given in D°M′S″ form into decimal form and vice versa.

Example 1:

(i). Convert 25°30′ to decimal degrees.

(ii) Convert 32.25° to D°M′S″ form.

Solution:

(i) $25°30' = 25° + \left(\frac{30}{60}\right)^{o} = 25° + 0.5° = 25.5°$

(ii) $32.25° = 32° + 0.25° = 32° + \left(\frac{25}{100}\right)^{o}$

$= 32° + \frac{1°}{4} = 32° + \left(\frac{1}{4} \times 60\right)'$

$= 32°15'$

Example 2:

Convert 12°23′35″ to decimal degrees correct to three decimal places.

Solution:

$12°23'35'' = 12° + \frac{23°}{60} + \frac{35°}{60 \times 60}$

$= 12° + \frac{23°}{60} + \frac{35°}{3600}$

$= 12° + 0.3833° + 0.00972°$

$= 12.3930°$

$= 12.393°$

Example 3:

Convert 45.36° to D°M′S″ form.

Solution: $(45.36)°$

$= 45° + (0.36)°$

$= 45° + \left(\frac{36}{100} \times 60\right)'$

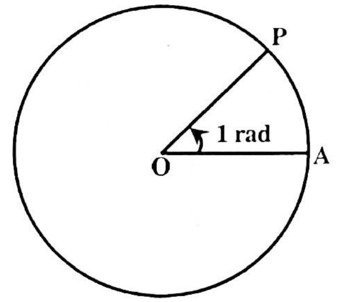
$= 45° + 21.6'$

$= 45° + 21' + (0.6 \times 60)''$

$= 45°21'36''$

Radian measure of an angle (circular system)

Radian: The angle subtended at the centre of the circle by an arc, whose length is equal to the radius of the circle is called one Radian.

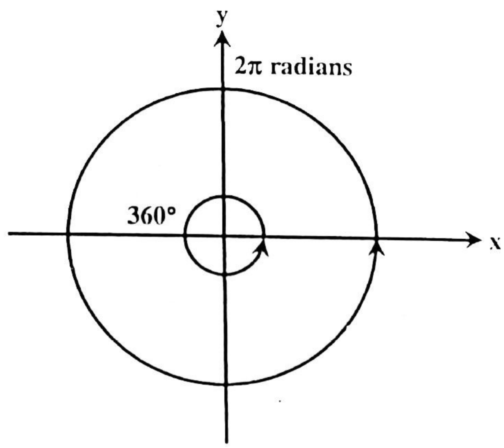


Consider a circle of radius r whose centre is O. From any point A on the circle cut off an arc AP whose length is equal to the radius of the circle. Join O with A and O with P. The ∠AOP is one radian. This means that when

Length of arc AP = length of radius $\overline{OA}$ then $m\angle AOP = 1$ radian

Relationship between radians and degrees

We know that circumference of a circle is $2\pi r$ where r is the radius of the circle. Since a circle is an arc whose length is $2\pi r$. The radian measure of an angle that form a complete circle is $\frac{2\pi r}{r} = 2\pi$



From this we see that 360° = 2π radians

or 180° = π radians............ (i)

Using this relation we can convert degrees into radians and radians into degrees as follows:

$$180^\circ = \pi \text{ radian}$$

$$1^\circ = \frac{\pi}{180} \text{ radian,}$$

$$x^\circ = x \,.\, 1^\circ$$

$$x^\circ = x \frac{\pi}{180} \text{ radian.......(ii)}$$

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^\circ$$

$$y \text{ radian} = y \frac{180}{\pi} \text{ degrees.....(iii)}$$

Some special angles in degree and radians

$$180^\circ = 1(180^\circ) = \pi \text{ radians}$$

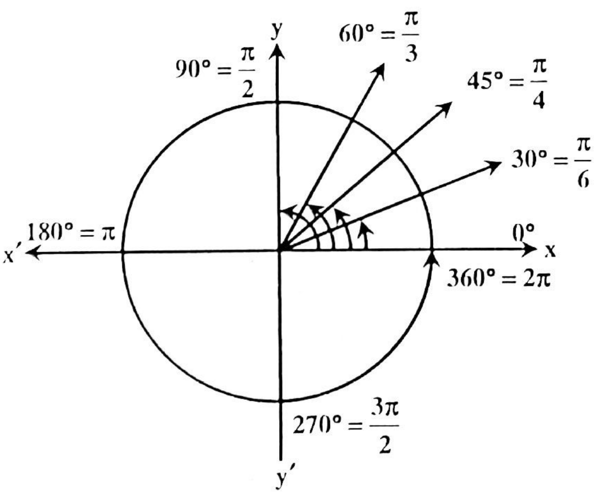
$$90^\circ = \frac{1}{2}(180^\circ) = \frac{\pi}{2} \text{ radians}$$

$$60^\circ = \frac{1}{3}(180^\circ) = \frac{\pi}{3} \text{ radians}$$

$$45^\circ = \frac{1}{4}(180^\circ) = \frac{\pi}{4} \text{ radians}$$

$$30^\circ = \frac{1}{6}(180^\circ) = \frac{\pi}{6} \text{ radians}$$

$$270^\circ = \frac{3}{2}(180^\circ) = \frac{3\pi}{2} \text{ radians}$$



Example 4:

Convert the following angles into radian measure:

(a) 15° (b) $124^\circ 22'$

Solution:

(a) $15^\circ = 15 \frac{\pi}{180}$ rads by using (i)

$$15^\circ = \frac{\pi}{12} \text{ radians}$$

(b) $124^\circ 22' = \left(124 + \frac{22}{60}\right)^\circ$

$$124^\circ 22' = (124.3666) \frac{\pi}{180} \text{ radians}$$

$$124^\circ 22' \approx 2.171 \text{ radians}$$

Example 5:

Express the following into degree.

(a) $\frac{2\pi}{3}$ radians (b) 6.1 radians

Solution:

(a) $\frac{2\pi}{3}$ radians $= \frac{2\pi}{3} \frac{180}{\pi}$ degrees

$$= 120^\circ$$

(b) 6.1 radians $= (6.1) \frac{180}{\pi}$ degrees

$$= 6.1\,(57.295779)$$

$$= 349.5043 \text{ degrees}$$

Remember that:

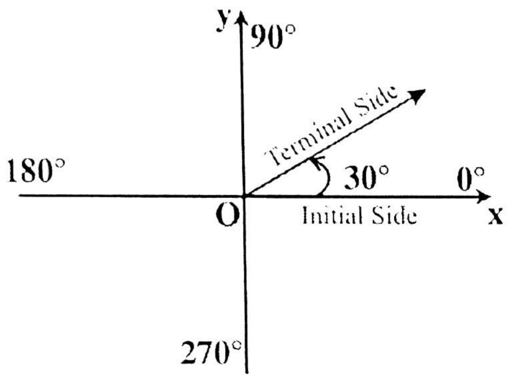
$$1 \text{ radian} \approx \left(\frac{180}{3.1416}\right)^\circ \approx 57.295795^\circ \approx 57^\circ 17' 45''$$

$$1^\circ \approx \frac{3.1416}{180} \approx 0.0175 \text{ radians}$$

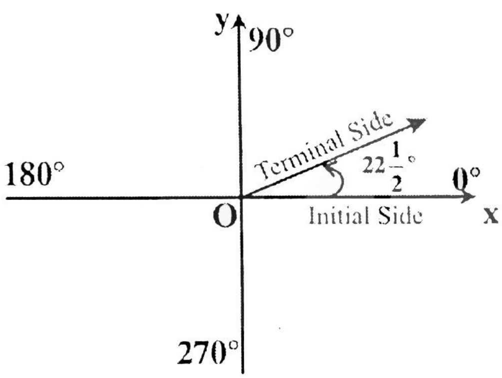
EXERCISE 7.1

Q.1. Locate the following angles:

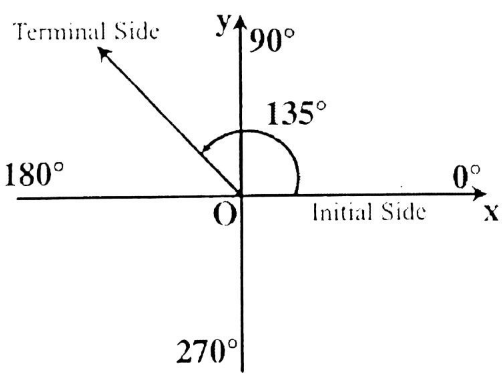
(i) 30°



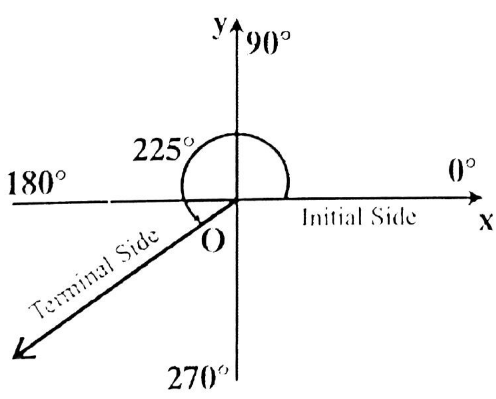
(ii) $22\frac{1}{2}^\circ$



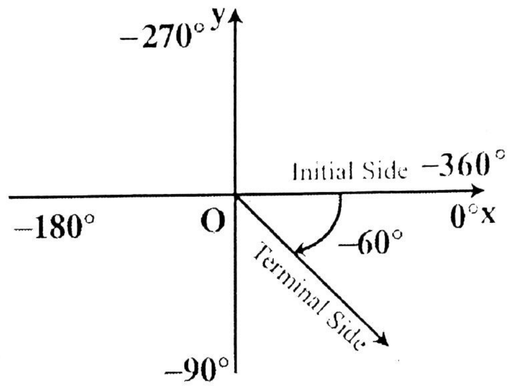
(iii) 135°



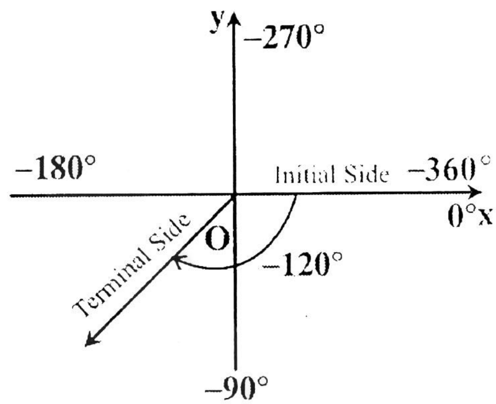
(iv) 225°



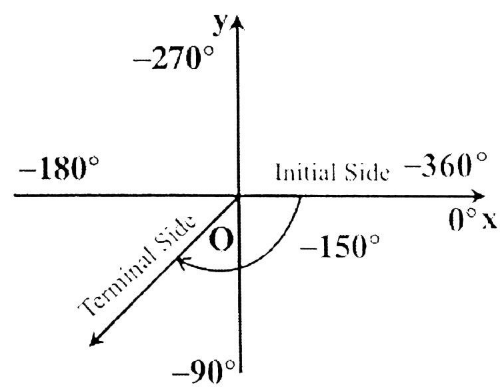
(v) -60°



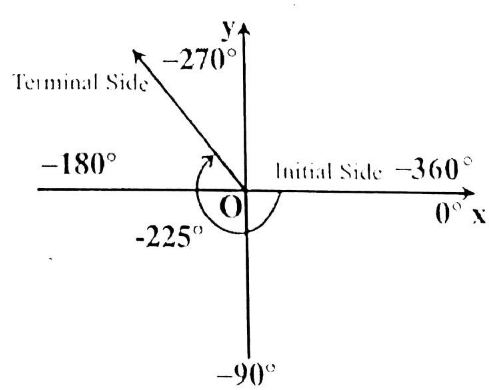
(vi) -120°



(vii) -150°



(viii) -225°



Q.2. Express the following sexagesiml measures of angles in decimal form.

(i) 45°30′

Solution: 45°30′

$$= 45° + \left(\frac{30}{60}\right)^{o}$$

$$= 45° + 0.5°$$

$$= 45.5°$$

(ii) 60°30′30″

Solution: 60°30′30″

$$= 60° + \left(\frac{30}{60}\right)^{o} + \left(\frac{30}{60\times60}\right)^{o}$$

$$= 60° + 0.5° + 0.008°$$

$$= 60.508°$$

(iii) 125°22′50″

Solution: 125°22′50″

$$= 125° + \left(\frac{22}{60}\right)^{o} + \left(\frac{50}{60\times60}\right)^{o}$$

$$= 125° + 0.367° + 0.0139°$$

$$= 125.3808°$$

Q.3. Express the following in D°M′S″:

(i) 47.36°

Solution: 47.36°

$$= 47° + 0.36°$$

$$= 47° + (0.36 \times 60)'$$

$$= 47° + 21.6'$$

$$= 47° + 21' + (0.6 \times 60)''$$

$$= 47° + 21' + 36''$$

$$= 47°\ 21'36''$$

(ii) 125.45°

Solution: 125.45°

$$= 125° + 0.45°$$

$$= 125° + (0.45 \times 60)'$$

$$= 125° + 27'$$

$$= 125°27'0''$$

(iii) 225.75°

Solution: 225.75°

$$= 225° + 0.75°$$

$$= 125° + (0.75 \times 60)'$$

$$= 225° + 45'$$

$$= 225°45'0''$$

(iv) – 22.5°

Solution: – 22.5°

$$= -\left[22° + 0.5°\right]$$

$$= -[22° + (0.5 \times 60)']$$

$$= -[22° + 30']$$

$$= -22°30'$$

(v) – 67.58°

Solution: – (67° + 0.58°)

$$= -[67° + (0.58 \times 60)']$$

$$= -[67° + 34.8']$$

$$= -[67° + 34' + 0.8']$$

$$= -[67° + 34' + (0.8 \times 60)'']$$

$$= -[67° + 34' + 48'']$$

$$= -67°34'48''$$

(vi) 315.18°

Solution: 315.18°

$$= 315° + 0.18°$$

$$= 315° + (0.18 \times 60)'$$

$$= 315° + 10.8'$$

$$= 315° + 10' + 0.8'$$

$$= 315° + 10' + (0.8 \times 60)''$$

$$= 315° + 10' + 48''$$

$$= 315°\ 10'\ 48''$$

Q.4. Express the following angles into radians.

(i) 30°

Solution: 30°

$$= 30\ \frac{\pi}{180}\ radians$$

$$= \cancel{30}\ \frac{\pi}{\cancel{30}\times6}\ radians$$

$$= \frac{\pi}{6}\ \text{radians}$$

(ii) 60°

Solution: 60°

$$= 60\ \frac{\pi}{180}\ radians$$

$$= \cancel{60}\ \frac{\pi}{\cancel{60}\times3}\ radians$$

$$= \frac{\pi}{3}\ \text{radians}$$

(iii) **135°**

Solution: 135°

$$= 135 \ \frac{\pi}{180} \textit{ radians}$$

$$= \cancel{45} \times 3 \ \frac{\pi}{\cancel{45} \times 4} \textit{ radians}$$

$$= \frac{3\pi}{4} \text{ radians}$$

(iv) **225°**

Solution: 225°

$$= 225 \ \frac{\pi}{180} \textit{ radians}$$

$$= \cancel{45} \times 5 \ \frac{\pi}{\cancel{45} \times 4} \textit{ radians}$$

$$= \frac{5\pi}{4} \textit{ radains}$$

(v) **– 150°**

Solution: – 150°

$$= -150 \ \frac{\pi}{180} \textit{ radians}$$

$$= -5 \times \cancel{30} \ \frac{\pi}{\cancel{30} \times 6} \textit{ radians}$$

$$= \frac{-5\pi}{6} \text{ radians}$$

(vi) **– 225°**

Solution: – 225°

$$= -225 \ \frac{\pi}{180} \text{ radians}$$

$$= -5 \times \cancel{45} \ \frac{\pi}{\cancel{45} \times 4} \text{ radians}$$

$$= \frac{-5\pi}{4} \text{ radians}$$

(vii) **300°**

Solution: 300°

$$= 300 \ \frac{\pi}{180} \textit{ radians}$$

$$= \cancel{60} \times 5 \ \frac{\pi}{\cancel{60} \times 3} \textit{ radians}$$

$$= \frac{5\pi}{3} \text{ radians}$$

(viii) **315°**

Solution: 315°

$$= 315 \ \frac{\pi}{180} \textit{ radians}$$

$$= \cancel{45} \times 7 \ \frac{\pi}{\cancel{45} \times 4} \textit{ radians}$$

$$= \frac{7\pi}{4} \text{ radians}$$

Q.5. Convert each of the following to degrees.

(i) $\frac{3\pi}{4}$

Solution: $\frac{3\pi}{4}$ radians

$$= \frac{3\pi}{4} \ \frac{180}{\pi} \text{ degrees}$$

$$= \frac{3\cancel{\pi}}{\cancel{4}} \ \frac{\overset{45}{\cancel{180}}}{\cancel{\pi}} \text{ degrees}$$

$$= 3 \times 45 \text{ degrees}$$

$$= 135°$$

(ii) $\frac{5\pi}{6}$

Solution: $\frac{5\pi}{6}$

$$= \frac{5\pi}{6} \ \frac{180}{\pi} \text{ degrees}$$

$$= \frac{5\cancel{\pi}}{\cancel{6}} \ \frac{\overset{30}{\cancel{180}}}{\cancel{\pi}} \text{ degrees}$$

$$= 5 \times 30 \text{ degrees}$$

$$= 150°$$

(iii) $\frac{7\pi}{8}$

Solution: $\frac{7\pi}{8}$ radians

$$= \frac{7\pi}{8} \ \frac{180}{\pi} \text{ degrees}$$

$$= \frac{7\cancel{\pi}}{8} \ \frac{180}{\cancel{\pi}} \text{ degrees}$$

$$= \frac{7 \times 180}{8} \text{ degrees}$$

$$= \frac{1260}{8} \text{ degrees}$$

$$= 157.5°$$

(iv) $\frac{13\pi}{16}$

Solution: $\frac{13\pi}{16}$ radians

$= \frac{13\pi}{16} \quad \frac{180}{\pi}$ degrees

$= \frac{13\not{\pi}}{16} \quad \frac{180}{\not{\pi}}$ degrees

$= \frac{13 \times 180}{16}$ degrees

$= \frac{2340}{16}$ degrees

$= 146.25°$

(v) 3 radians

Solution: 3 radians

$= 3 \quad \frac{180}{\pi}$ degrees

$= \frac{540}{\pi}$ degrees

$= 171.887°$

(vi) **4.5**

Solution: 4.5 radians

$= 4.5 \quad \frac{180}{\pi}$ degrees

$= \frac{810}{\pi}$ degrees

$= 257.831°$

(vii) $-\frac{7\pi}{8}$

Solution: $-\frac{7\pi}{8}$ radians

$= -\frac{7\not{\pi}}{8} \quad \frac{180}{\not{\pi}}$ degrees

$= \frac{-1260}{8}$ degrees

$= -157.5°$

(viii) $-\frac{13}{16}\pi$

Solution: $-\frac{13}{16}\pi$ radians

$= -\frac{13\not{\pi}}{16} \quad \frac{180}{\not{\pi}}$ degrees

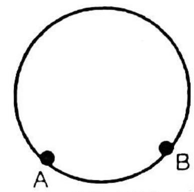
$= \frac{-2340}{16}$

$= -146.25°$

Sector of a Circle

(i) Arc of a Circle

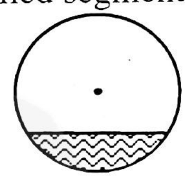
A part of the circumference of a circle is called an arc.



an arc

(ii) Segment of a circle

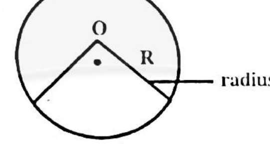
A part of the circular region bounded by an arc and a chord is called segment of a circle.



a segment

(iii) Sector of a circle

A part of the circular region bounded by the two radii and an arc is called sector of the circle.

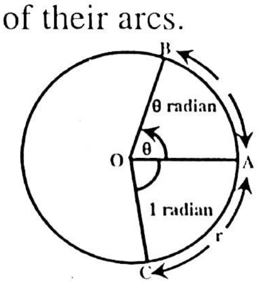


a sector

To establish the rule = r θ:

(where r is the radius of the circle, || the length of circular arc and θ the central angle measured in radians).

Let an arc AB denoted by l subtends an angle θ radian at the centre of the circle. It is a fact of plane geometry that measure of central angles of the arcs of a circle are proportional to the lengths of their arcs.



$$\frac{m\angle AOB}{m\angle AOC} = \frac{m\overline{AB}}{m\overline{AC}}$$

$$\frac{\theta \text{ radian}}{1 \text{ radian}} = \frac{—}{r}$$

$$\frac{—}{r} = \theta \qquad \text{or} \qquad l = r\theta$$

Example 1: **In a circle of radius 10m,**

(a) **Find the length of an arc intercepted by a central angle of 1.6 radian.**

(b) **Find the length of an arc intercepted by a central angle of 60°.**

Solution:

(a) Here $\theta = 1.6$ radian , $r = 10$ m and l ? =

Since

$$l = r\theta \qquad l = 10 \times 1.6 = 16 \text{ m}$$

(b) $$\theta = 60° = 60 \times \frac{\pi}{180} = \frac{\pi}{3} \text{ rad}$$

$$\therefore \; l = r\theta = 10 \times \frac{\pi}{3} = \frac{10\pi}{3} \text{ m}$$

Example 2: **Find the distance travelled by a cyclist moving on a circle of radius 15m, if he makes 3.5 revolutions.**

Solution: $\theta = 1$ revolution $= 2\pi$ radians

3.5 revolution $= 2\pi \times 3.5$ radian

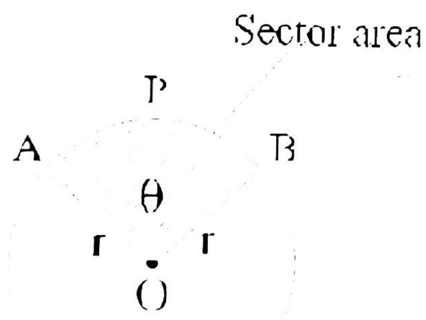
Distance traveled $= l = r\theta$

$= 15\text{m} \times 2\pi \times 3.5$

$= 105\ \pi\text{m}$

Area of circular sector

Consider a circle of radius r units and an arc of length l units, subtending an angle θ at O.



Area of the circle $= \pi r^2$

Angle of the circle $= 2\pi$

Angle of sector $= \theta$ radian

Then by elementary geometry we can use the proportion.

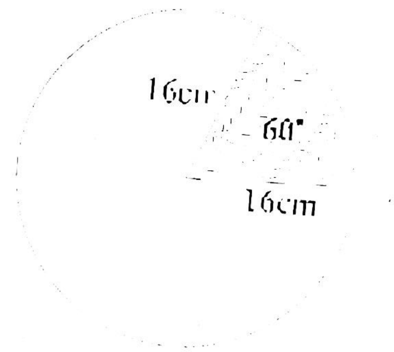
$$\frac{\text{rea of sector AOBP}}{\text{Area of Circle}} = \frac{\text{Angle of sector}}{\text{Angle of circle}}$$

or $$\frac{\text{Area of Sector AOBP}}{\pi r^2} = \frac{\theta}{2\pi}$$

$$\text{Area of Sector AOBP} = \frac{\theta}{2\pi} \times \pi r^2$$

$$\text{Area of Sector AOBP} = \frac{1}{2} r^2 \theta$$

Example 3: **Find area of the sector of a circle of radius 16 cm if the angle at the centre is 60°.**



Solution:

Here central angel = θ = 60°

$$\text{Now } \theta = 60° \times \frac{\pi}{180}$$

$$\theta = \frac{\pi}{3} \text{ rad}$$

$$r = 16\text{cm}$$

$$\text{Area of Sector} = \frac{1}{2} r^2 \theta$$

$$\text{Area of Sector} = \frac{1}{2} (16 \text{ cm})^2 \frac{\pi}{3}$$

$$= \frac{1}{2} (256\text{cm})^2 \times \frac{22}{7 \times 3}$$

$$\approx 1341.1 \text{ cm}^2$$