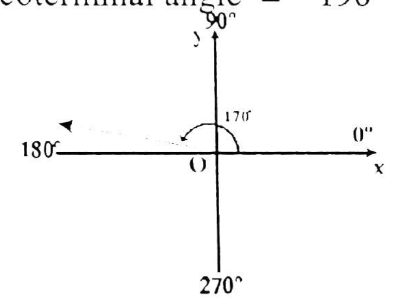
EXERCISE 7.3

Q.1. Locate each of the following angles in standard position using a protractor or fair free hand guess. Also find a positive and a negative angle coterminal with each given angle:

(i) 170°

Positive coterminal angle $= 360^\circ + 170^\circ$
$= 530^\circ$

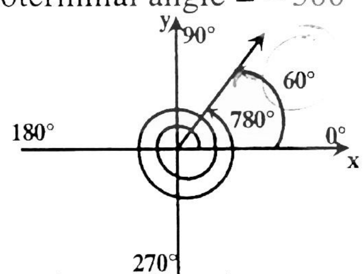
Negative coterminal angle $= -190^\circ$



(ii) **780°**

Positive coterminal angle $780^\circ - 2(360^\circ) = 60^\circ$

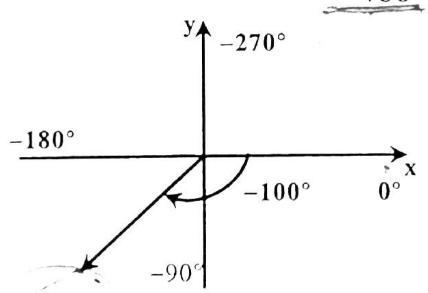
Negative coterminal angle $= -300^\circ$



(iii) **– 100°**

Positive coterminal angle $= 260^\circ$

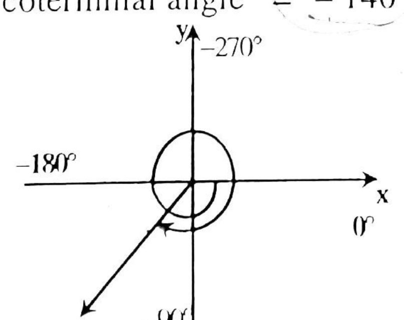
Negative coterminal angle $= -360^\circ - 100^\circ$
$= -460^\circ$



(iv) **– 500°**

Positive coterminal angle $= 220^\circ$

Negative coterminal angle $= -140^\circ$



Q.2. Identify closest quadrantal angles between which the following angles lie.

(i) 156°

Ans: 90° and 180°

(ii) 318°

Ans: 270° and 360°

(iii) 572°

Ans: 540° and 630°

(iv) –330°

Ans: 0° and 90°

Q.3. Write the closest quadrantal angles between which the angles lie. Write your answer in radian measure.

(i) $\frac{\pi}{3}$

Ans: 0 and $\frac{\pi}{2}$

(ii) $\frac{3\pi}{4}$

Ans: $\frac{\pi}{2}$ and π

(iii) $\frac{-\pi}{4}$

Ans: 0 and $-\frac{\pi}{2}$

(iv) $-\frac{3\pi}{4}$

Ans: $-\frac{\pi}{2}$ and $-\pi$

Q.4. In which quadrant θ lies, when

(i) $\sin\theta > 0, \tan\theta < 0$

Ans: II quadrant

(ii) $\cos\theta < 0, \sin\theta < 0$

Ans: III quadrant

(iii) $\sec\theta > 0, \sin\theta < 0$

Ans: IV quadrant

(iv) $\cos\theta < 0, \tan\theta < 0$

Ans: II quadrant

(v) $\text{cosec}\ \theta > 0, \cos\theta > 0$

Ans: I quadrant

(vi) $\sin\theta < 0, \sec\theta < 0$

Ans: III quadrant

Q.5. Fill in the blanks:

(i) $\cos(-150°) =$ ______________ $\cos 150°$

(ii) $\sin(-310°) =$ ______________ $\sin 310°$

(iii) $\tan(-210°) =$ ______________ $\tan 210°$

(iv) $\cot(-45°) =$ ______________ $\cot 45°$

(v) $\sec(-60°) =$ ______________ $\sec 60°$

(vi) $\text{cosec}(-137°) =$ ______________ $\text{cosec}137°$

Answers:

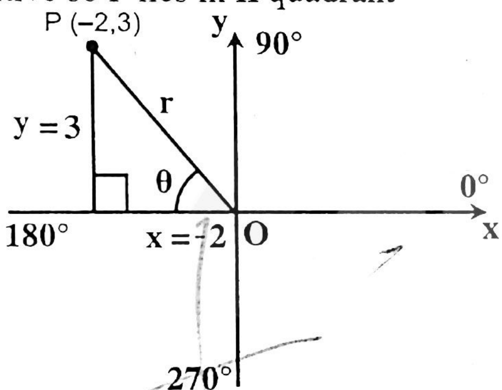
(i) +ve	(ii) −ve	(iii) −ve
(iv) −ve	(v) +ve	(vi) −ve

Q.6. The given point P lies on the terminal side of θ. Find quadrant of θ and all six trigonometric ratios.

(i) (−2, 3)

Solution: $P(x, y) = P(-2, 3)$

As x −coordinate is negative and y− coordinate is positive so P lies in II quadrant



The point P can be shown in II quadrant.

By Pythagorean theorem

$$r^2 = x^2 + y^2$$

$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{(-2)^2 + (3)^2}$$

$$r = \sqrt{4+9}$$

$$r = \sqrt{13}$$

Now, $\sin\theta = \dfrac{y}{r} = \dfrac{3}{\sqrt{13}}$

$$\text{cosec}\,\theta = \frac{r}{y} = \frac{\sqrt{13}}{3}$$

$$\cos\theta = \frac{x}{r} = -\frac{2}{\sqrt{13}}$$

$$\sec\theta = \frac{r}{x} = \frac{-\sqrt{13}}{2}$$

$$\tan\theta = \frac{y}{x} = -\frac{3}{2}$$

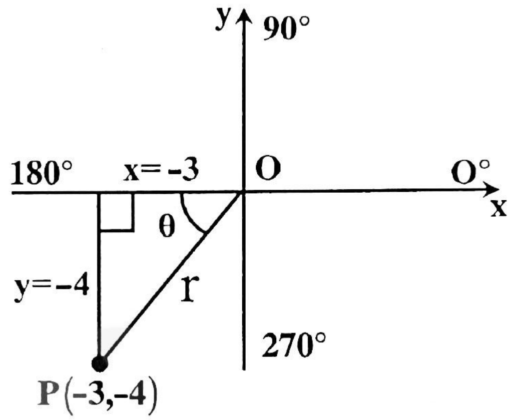
$$\cot\theta = \frac{x}{y} = -\frac{2}{3}$$

(ii) (− 3, − 4)

Solution: $P(x, y) = P(-3, -4)$

As x and y both coordinates are negative , so 'P' lies in III quadrant.

The point P can be shown in III quadrant.



By Pythagorean, theorem

$$r^2 = x^2 + y^2$$

$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{(-3)^2 + (-4)^2}$$

$$r = \sqrt{9+16}$$

$$r = \sqrt{25}$$

$$r = 5$$

Now $\sin\theta = \dfrac{y}{r} = \dfrac{-4}{5}$

$$\text{cosec}\,\theta = \frac{r}{y} = \frac{-5}{4}$$

$$\cos\theta = \frac{x}{r} = \frac{-3}{5}$$

$$\sec\theta = \frac{r}{x} = \frac{-5}{3}$$

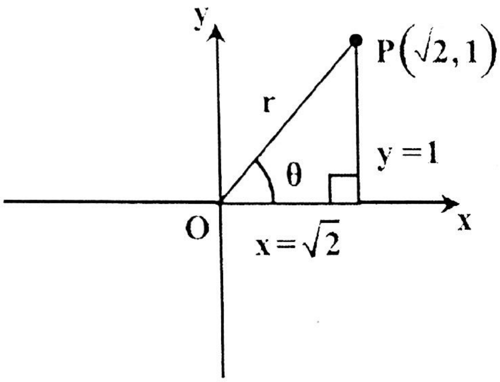
$$\tan\theta = \frac{y}{x} = \frac{-4}{-3} = \frac{4}{3}$$

$$\cot\theta = \frac{x}{y} = \frac{-3}{-4} = \frac{3}{4}$$

(iii) $\left(\sqrt{2},1\right)$

Solution: $P(x, y) = P\left(\sqrt{2},1\right)$

As x and y both coordinates are positive, so P lies in I quadrant.



The point 'P' can be shown in quadrant I.

By Pythagorean theorem

$$r^2 = x^2 + y^2$$

$$r = \sqrt{x^2+y^2}$$

$$r = \sqrt{\left(\sqrt{2}\right)^2+(1)^2}$$

$$r = \sqrt{2+1}$$

$$r = \sqrt{3}$$

Now $\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{3}}$

$$\text{cosec}\theta = \frac{r}{y} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\cos\theta = \frac{x}{r} = \frac{\sqrt{2}}{\sqrt{3}} = \sqrt{\frac{2}{3}}$$

$$\sec\theta = \frac{r}{x} = \frac{\sqrt{3}}{\sqrt{2}} = \sqrt{\frac{3}{2}}$$

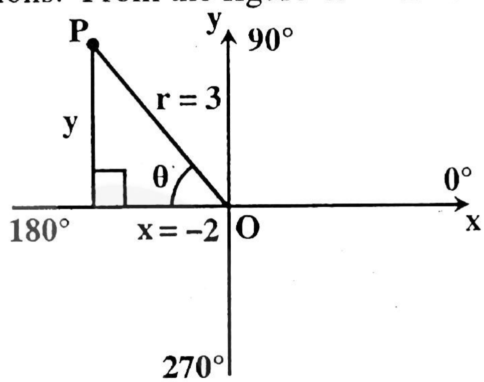
$$\tan\theta = \frac{y}{x} = \frac{1}{\sqrt{2}}$$

$$\cot\theta = \frac{x}{y} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

Q.7. If $\cos\theta = \frac{-2}{3}$ and terminal arm of the angle θ is in quadrant II, find the values of remaining trigonometric functions.

Solution:

As $\text{Cos}\ \theta = \frac{-2}{3}$ and θ is in quadrant II, so we complete the figure according to conditions. From the figure $x = -2$ and $r = 3$



By Pythagorean theorem

$$r^2 = x^2 + y^2$$

$$y^2 = r^2 - x^2$$

$$y = \sqrt{r^2 - x^2}$$

$$y = \sqrt{(3)^2 - (-2)^2}$$

$$y = \sqrt{9-4}$$

$$y = \sqrt{5}$$

Now

$$\sin\theta = \frac{y}{r} = \frac{\sqrt{5}}{3}$$

$$\text{cosec}\ \theta = \frac{r}{y} = \frac{3}{\sqrt{5}}$$

$$\cos\ \theta = \frac{x}{r} = \frac{-2}{3}$$

$$\sec\ \theta = \frac{r}{x} = \frac{-3}{2}$$

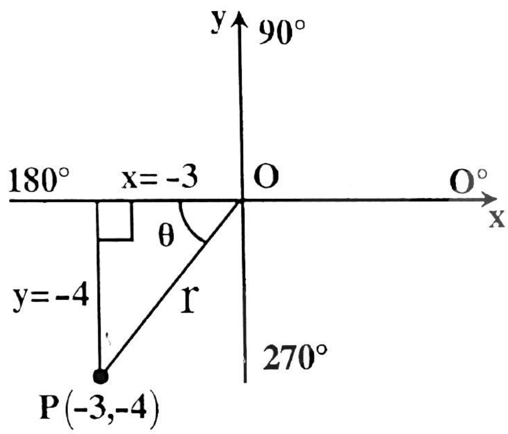
$$\tan\theta = \frac{y}{x} = \frac{-\sqrt{5}}{2}$$

$$\cot\theta = \frac{x}{y} = \frac{-2}{\sqrt{5}}$$

Q.8. If $\tan\theta = \frac{4}{3}$ and $\sin\theta < 0$, find the values of other trigonometric functions at θ.

Solution:

As $\tan\theta = \frac{4}{3}$ and $\sin\theta$ is –ve, which is possible in quadrant III only. We complete the figure.



From the figure $x = -3$ and $y = -4$

By pathagorean theorem

$$r^2 = x^2 + y^2$$

$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{(-3)^2 + (-4)^2}$$

$$r = \sqrt{9+6}$$

$$r = \sqrt{25}$$

$$r = 5$$

Now

$$\sin\theta = \frac{y}{r} = \frac{-4}{5}$$

$$\operatorname{cosec}\theta = \frac{r}{y} = \frac{-5}{4}$$

$$\cos\theta = \frac{x}{r} = \frac{-3}{5}$$

$$\sec\theta = \frac{r}{x} = \frac{-5}{3}$$

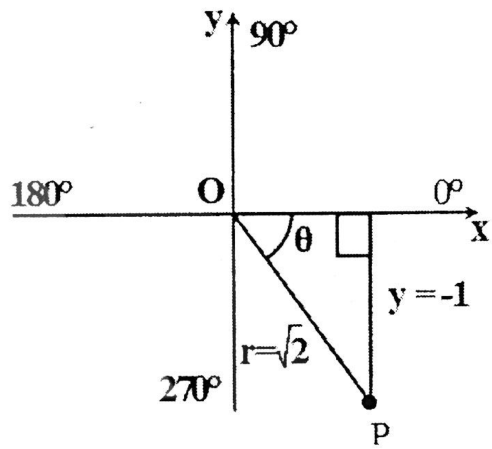
$$\tan\theta = \frac{y}{x} = \frac{4}{3}$$

$$\cot\theta = \frac{x}{y} = \frac{3}{4}$$

Q.9. If $\sin\theta = -\frac{1}{\sqrt{2}}$ and terminal side of the angle is not in quadrant III, find the values of $\tan\theta$, $\sec\theta$ and $\operatorname{cosec}\theta$.

Solution:

As $\sin\theta = \frac{-1}{\sqrt{2}}$ and terminal side of angle is not in III quadrant, so it lies in quadrant IV.



From the figure $y = -1$ and $r = \sqrt{2}$

By Pathagorean theorem

$$r^2 = x^2 + y^2$$

$$x^2 = r^2 - y^2$$

$$x = \sqrt{r^2 - y^2}$$

$$x = \sqrt{(\sqrt{2})^2 - (-1)^2}$$

$$x = \sqrt{2-1}$$

$$x = \sqrt{1}$$

$$x = 1$$

Now

$$\tan\theta = \frac{y}{x} = \frac{-1}{1} = -1$$

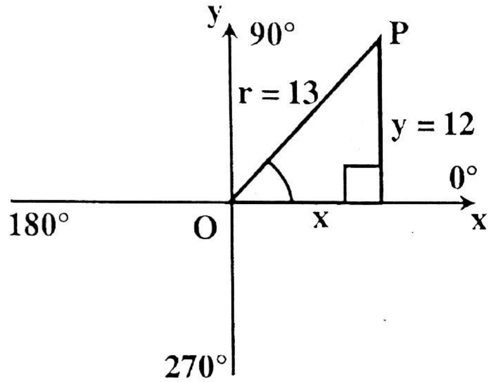
$$\sec\theta = \frac{r}{x} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\operatorname{cosec}\theta = \frac{r}{y} = \frac{\sqrt{2}}{-1} = -\sqrt{2}$$

Q.10. If $\operatorname{cosec}\theta = \frac{13}{12}$ and $\sec\theta > 0$, find the remaining trigonometric functions.

Solution:

As $\operatorname{cosec}\theta = \frac{13}{12}$ and also $\sec\theta$ is +ve, which is only possible in quadrant I.



From the figure $y = 12$ and $r = 13$

By Pythagorean theorem

$r^2 = x^2 + y^2$

$x^2 = r^2 - y^2$

$x = \sqrt{r^2 - y^2}$

$x = \sqrt{(13)^2 - (12)^2}$

$x = \sqrt{169 - 144}$

$x = \sqrt{25}$

$x = 5$

Now

$\sin\theta = \frac{y}{r} = \frac{12}{13}$

$\operatorname{cosec}\theta = \frac{r}{y} = \frac{13}{12}$

$\cos\theta = \frac{x}{r} = \frac{5}{13}$

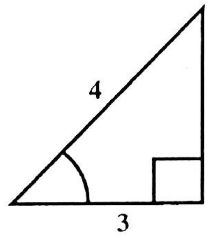
$\sec\theta = \frac{r}{x} = \frac{13}{5}$

$\tan\theta = \frac{y}{x} = \frac{12}{5}$

$\cot\theta = \frac{x}{y} = \frac{5}{12}$

Q.11. Find the values of trigonometric functions at the indicated angle θ in the right triangle.

(i)



From the figure Hypotenuse = 4 and Base = 3

By Pythagorean theorem we can find perpendicular

$(\text{Per.})^2 + (\text{Base})^2 = (\text{Hyp.})^2$

$(\text{Per.})^2 + (3)^2 = (4)^2$

$(\text{Per.})^2 = 16 - 9$

$(\text{Per.})^2 = 7$

$\text{Perpendicular} = \sqrt{7}$

Now $\sin\theta = \frac{\text{Per.}}{\text{Hyp.}} = \frac{\sqrt{7}}{4}$

$\operatorname{cosec}\theta = \frac{\text{Hyp.}}{\text{Per.}} = \frac{4}{\sqrt{7}}$

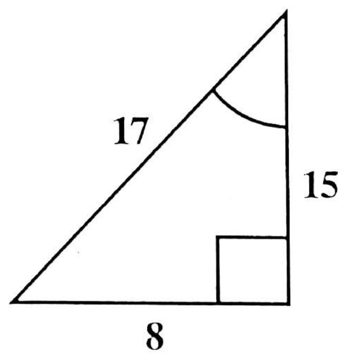
$\cos\theta = \frac{\text{Base}}{\text{Hyp.}} = \frac{3}{4}$

$\sec\theta = \frac{\text{Hyp.}}{\text{Base}} = \frac{4}{3}$

$\tan\theta = \frac{\text{Per.}}{\text{Base}} = \frac{\sqrt{7}}{3}$

$\cot\theta = \frac{\text{Base}}{\text{Per.}} = \frac{3}{\sqrt{7}}$

(ii)



From the figure

Hypotenuse = 17

Perpendicular = 8

Base = 15

Now

$$\sin\theta = \frac{\text{Per.}}{\text{Hyp.}} = \frac{8}{17}$$

$$\text{cosec}\ \theta = \frac{\text{Hyp.}}{\text{Per.}} = \frac{17}{8}$$

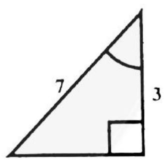
$$\cos\theta = \frac{\text{Base}}{\text{Hyp.}} = \frac{15}{17}$$

$$\sec\theta = \frac{\text{Hyp.}}{\text{Base}} = \frac{17}{15}$$

$$\tan = \frac{\text{Per.}}{\text{Base}} = \frac{8}{15}$$

$$\cot\theta = \frac{\text{Base}}{\text{Per.}} = \frac{15}{8}$$

(iii)



From the figure Hypotenuse = 7, Base = 3
We can find perpendicular by Pythagorean theorem.

$$(\text{Base})^2 + (\text{Per.})^2 = (\text{Hyp.})^2$$
$$(\text{Per.})^2 + (3)^2 = (7)^2$$
$$(\text{Per.})^2 = 49 - 9$$
$$(\text{Per.})^2 = 40$$
$$\text{Per.} = \sqrt{40}$$
$$\text{Per.} = \sqrt{4\times10}$$
$$\text{Per.} = 2\sqrt{10}$$

Now, $\sin\theta = \frac{\text{Per.}}{\text{Hyp.}} = \frac{2\sqrt{10}}{7}$

$$\text{cosec}\theta = \frac{\text{Hyp.}}{\text{Per.}} = \frac{7}{2\sqrt{10}}$$

$$\cos\theta = \frac{\text{Base}}{\text{Hyp.}} = \frac{3}{7}$$

$$\sec\theta = \frac{\text{Hyp.}}{\text{Base}} = \frac{7}{3}$$

$$\tan\theta = \frac{\text{Per.}}{\text{Base}} = \frac{2\sqrt{10}}{3}$$

$$\cot\theta = \frac{\text{Base}}{\text{Per.}} = \frac{3}{2\sqrt{10}}$$

Q.12. Find the values of the trigonometric functions. Do not use trigonometric table or calculator.

Solution:

We know that $2k\pi + \theta = \theta$, where $k \in Z$

(i) tan 30°

$$30^\circ = 30\ \frac{\pi}{180}\ \text{radian} = \frac{\pi}{6}\ \text{radian}$$

$$\tan 30^\circ = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

(ii) tan 330°

$$\tan 330^\circ = \tan(360^\circ - 30^\circ)$$
$$= \tan\ 2\pi - \frac{\pi}{6}$$
$$= \tan\ -\frac{\pi}{6}$$
$$= -\tan\frac{\pi}{6}$$
$$= -\frac{1}{\sqrt{3}}$$

(iii) sec 330°

$$\sec 330^\circ = \sec(360^\circ - 30^\circ)$$
$$= \sec\ 2\pi - \frac{\pi}{6}$$
$$= \sec\ -\frac{\pi}{6}$$
$$= \sec\frac{\pi}{6}$$
$$= \frac{2}{\sqrt{3}}$$

(iv) $\mathbf{\cot\frac{\ }{4}}$

$$\cot\frac{\pi}{4} = 1$$

(v) $\mathbf{\cos\frac{2}{3}}$

$$\cos\frac{2\pi}{3} = -\cos\frac{\pi}{3} = -\frac{1}{2}$$

(vi) $\text{cosec}\ \frac{2\pi}{3}$

$\text{cosec}\frac{2\pi}{3} = \text{cosec}\ \frac{\pi}{3} = \frac{2}{\sqrt{3}}$

(vii) $\cos(-450°)$

$\cos(-450°) = \cos(-360° - 90°)$

$= \cos \quad -2\pi - \frac{\pi}{2}$

$= \cos \quad 2(-1)\pi - \frac{\pi}{2}$

$= \cos \quad -\frac{\pi}{2}$

$= \cos \quad \frac{\pi}{2} \quad = 0$

(viii) $\tan(-9\pi)$

$\tan(-9\pi) = \tan(-8\pi - \pi)$

$= \tan[2(-4)\pi - \pi]$

$= \tan(-\pi)$

$= -\tan\pi$

$= -(0) = 0$

(ix) $\cos \frac{-5\pi}{6}$

$\cos \quad -\frac{5\pi}{6} \quad = -\cos \quad \frac{\pi}{6}$

$= -\frac{\sqrt{3}}{2}$

(x) $\sin 7\frac{}{6}$

$\sin\frac{7\pi}{6} = \sin \quad +\frac{}{6}$

$= -\sin\frac{\pi}{6} = -\frac{1}{2}$

(xi) $\cot\frac{7}{6}$

$\cot\frac{7\pi}{6} = \cot \quad \pi + \frac{\pi}{6}$

$= \cot\frac{\pi}{6} = \sqrt{3}$

(xii) $\cos 225°$

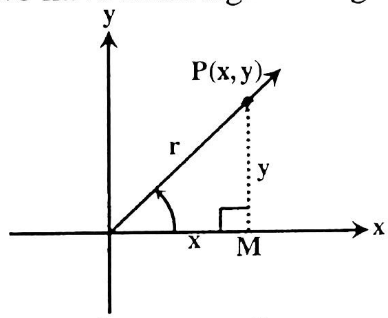
$\cos 225° = \cos(180° + 45°)$

$= \cos \quad \pi + \frac{\pi}{4}$

$= -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

Trigonometric Identities

Consider an angle $\angle MOP = \theta$ radian in standard position. Let point P (x, y) be on the terminal side of the angle. By Pythagorean theorem, we have from right triangle OMP.



$(m\overline{OM})^2 = (m\overline{MP})^2 + (m\overline{OP})^2$

$x^2 + y^2 = r^2$ ....(i)

Dividing both sides by r^2, we get

$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$

$\frac{x}{r}^2 \quad \frac{y}{r}^2 + 1$

$\left(\sin\theta = \frac{y}{r},\ \cos\theta = \frac{x}{r},\ \tan\theta = \frac{y}{x}\right)$

$(\cos\theta)^2 + (\sin\theta)^2 = 1$

$\boxed{\cos^2\theta + \sin^2\theta = 1}$ ....(i)

Dividing (i) by x^2, we have

$\frac{x^2}{x^2} \quad \frac{y^2}{x^2} + \frac{r^2}{x^2}$

$1 + \frac{y}{x}^2 = \frac{r}{x}^2$

As $\tan\theta = \frac{y}{x}$ and see$\theta = \frac{r}{x}$

$1 + (\tan\theta)^2 = (\sec\theta)^2$

$1 + \tan^2\theta = (\sec\theta)^2$

$1 + \tan^2\theta = \sec^2\theta$ .... (ii)

or $\sec^2\theta - \tan^2\theta = 1$

Again dividing both sides of (i) by y^2, we get

$\frac{x^2}{y^2} = \frac{y^2}{y^2} + \frac{r^2}{y^2}$

$\frac{x}{y}^2 = 1 + \frac{r}{y}^2$

$\cot\theta = \frac{x}{y}$ and $\text{cosec}\theta = \frac{r}{y}$

$(\cot\theta)^2 + 1 = (\text{cosec}\theta)^2$

$1 + \cot^2\theta = \text{cosec}^2\theta$ .... (iii)

or $\text{cosec}^2\theta - \cot^2\theta = 1$

The identities (1), (2) and (3) are also known as Pythagorean Identities.

Example 1: **Verify that cotθsecθ = cosecθ**

Solution:

L.H.S = cotθsecθ

$$= \frac{\cos\theta}{\sin\theta}\,\frac{1}{\cos\theta}$$

$$= \frac{1}{\sin\theta}$$

= cosecθ

L.H.S = R.H.S

Example 2: **Verify that**

$$\tan^4\theta + \tan^2\theta = \tan^2\theta \sec^2\theta$$

Solution:

L.H.S = $\tan^4\theta + \tan^2\theta$

= $\tan^2\theta(\tan^2\theta + 1)$ $\tan^2\theta + 1 = \sec^2\theta$

= $\tan^2\theta \sec^2\theta$

L.H.S = L.H.S

Example 3:

Show that $\frac{\cot^2}{\text{cosec} \;\; -1} = \text{cosec}\,\alpha + 1$

Solution:

$$\text{L.H.S} = \frac{\cot^2\alpha}{\text{cosec}\,\alpha - 1}$$ $\text{cosec}^2\theta - \cot^2\theta = 1$

$\cot^2\theta = \text{cosec}^2\theta - 1$

$$= \frac{(\text{cosec}^2\alpha - 1)}{\text{cosec}\,\alpha - 1}$$

$$= \frac{(\text{cosec}\,\alpha - 1)(\text{cosec}\,\alpha + 1)}{(\text{cosec}\,\alpha - 1)}$$

= cosec α + 1

L.H.S = R.H.S

Example 4:

Express the trigonometric functions in terms of tanθ.

Solution:

- By using reciprocal identity, we can express cotθ in terms of tanθ .

  i.e $\cot\theta = \frac{1}{\tan\theta}$

- By solving the identity $1+\tan^2\theta = \sec^2\theta$

We have expressed secθ in terms of tanθ

$$\sec\theta = \pm\sqrt{\tan^2\theta + 1}$$

$$\cos\theta = \frac{1}{\sec\theta}$$

$$\cos\theta = \frac{1}{\pm\sqrt{\tan^2\theta + 1}}$$

Because

sinθ = tanθ cosθ, we have

$$\sin\theta = \tan\theta \; \frac{1}{\pm\sqrt{\tan^2\theta + 1}}$$

$$\sin\theta = \frac{\tan\theta}{\pm\sqrt{\tan^2\theta + 1}}$$

$$\text{cosec}\theta = \frac{1}{\sin\theta} = \frac{\pm\sqrt{\tan^2\theta + 1}}{\tan\theta}$$