number a (except zero) there exists a real number a^{-1} such that $aa^{-1} = 1$.

The number a^{-1} is called the multiplicative inverse of a.

$$\boxed{A^{-1} = \frac{adj\ A}{|A|}},\ |A| \neq 0$$

> If A is a singular matrix then the multiplicative inverse of A does not exist.

> **Remember that:**
>
> (i) Inverse of square matrix A is denoted by A^{-1}.
> (ii) Only non-singular matrices have inverses.
> (iii) Inverse of square matrix A is always unique.
> (iv) Non-square matrices cannot possess inverses.
> (v) $A^{-1} = \dfrac{adj\ A}{|A|}$

Exercise 6.5

1- Find the determinants of the following matrices.

(i) $\begin{bmatrix} u & v \\ x & y \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & 5 \\ 1 & 4 \end{bmatrix}$

(iii) $\begin{bmatrix} -8 & -4 \\ -4 & -2 \end{bmatrix}$ (iv) $\begin{bmatrix} \frac{1}{1} & \frac{3}{8} \\ \frac{1}{8} & \frac{1}{4} \end{bmatrix}$

(i). $A = \begin{bmatrix} u & v \\ x & y \end{bmatrix}$

Sol: Let $|A| = \begin{vmatrix} u & v \\ x & y \end{vmatrix} = uy - vx$

(ii) $P = \begin{bmatrix} -2 & 5 \\ 1 & 4 \end{bmatrix}$

Sol: Let $|P| = \begin{vmatrix} -2 & 5 \\ 1 & 4 \end{vmatrix}$

$= (-2)(4) - (5)(1)$

$= -8 - 5 = -13$

(iii) $B = \begin{bmatrix} -8 & -4 \\ -4 & -2 \end{bmatrix}$

Sol: Let $| B | = \begin{vmatrix} -8 & -4 \\ -4 & -2 \end{vmatrix}$

$= (-8)(-2) - (-4)(-4)$

$= 16 - 16 = 0$

(iv) $A = \begin{bmatrix} \frac{1}{1} & \frac{3}{8} \\ \frac{1}{8} & \frac{1}{4} \end{bmatrix}$

Sol: Let $|A| = \begin{vmatrix} \frac{1}{1} & \frac{3}{8} \\ \frac{1}{8} & \frac{1}{4} \end{vmatrix}$

$$=(1)\left(\frac{1}{4}\right)-\left(\frac{3}{8}\right)\left(\frac{1}{8}\right)$$

$$=\frac{1}{4}-\frac{3}{64}$$

$$=\frac{16-3}{64}=\frac{13}{64}$$

2. Identify the singular and non-singular matrices.

(i) $$\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}$$

Sol: Let $A=\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}$

$$|A|=\begin{vmatrix} -1 & 3 \\ 1 & -3 \end{vmatrix}$$

$$=(-1)(-3)-(3)(1)$$

$$=3-3=0$$

Therefore, $\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}$ is singular matrix

(ii) $$\begin{bmatrix} 3 & 8 \\ 4 & 9 \end{bmatrix}$$

Sol: Let $B=\begin{bmatrix} 3 & 8 \\ 4 & 9 \end{bmatrix}$

$$|B|=\begin{vmatrix} 3 & 8 \\ 4 & 9 \end{vmatrix}$$

$$=(3)(9)-(8)(4)$$

$$= 27 - 32 = -5 \neq 0$$

Therefore, $\begin{bmatrix} 3 & 8 \\ 4 & 9 \end{bmatrix}$ is non-singular matrix.

(iii) $\begin{bmatrix} -a & b \\ a & b \end{bmatrix}$

Sol Let $P = \begin{bmatrix} -a & b \\ a & b \end{bmatrix}$

$$|P| = \begin{vmatrix} -a & b \\ a & b \end{vmatrix}$$

$$= (-a)(b) - (a)(b)$$

$$= -ab - ab$$

$$= -2ab \neq 0$$

Therefore, $\begin{bmatrix} -a & b \\ a & b \end{bmatrix}$ is non-singular matrix.

3. Find the inverse of each matrix A and show that $A^{-1}A = I$. If the inverse does not exist, give reason.

(i) $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

Sol: $|A| = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$

$$= (1)(3) - (2)(1)$$

$$= 3 - 2 = 1$$

$$A^{-1} = \frac{adj\ A}{|A|}$$

$$= \frac{\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}}{1}$$

$$= \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

Now $A^{-1}A = ?$

$$A^{-1}A = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3(1)+(-2)(1) & 3(2)+(-2)(3) \\ -1(1)+(1)(1) & -1(2)+(1)(3) \end{bmatrix}$$

$$= \begin{bmatrix} 3-2 & 6-6 \\ -1+1 & -2+3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$A^{-1}A = I$ (proved)

(ii) $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$

Sol: $|A| = \begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix} = (2)(3)-(1)(5) = 6-5 = 1$

$$A^{-1} = \frac{adj\ A}{|A|}$$

$$= \frac{\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}}{1}$$

$$= \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

and $A^{-1}A = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 3(2) + (-1)(5) & 3(1) + (-1)(3) \\ -5(2) + (2)(5) & -5(1) + (2)(3) \end{bmatrix}$$

$$= \begin{bmatrix} 6-5 & 3-3 \\ -10+10 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$A^{-1}A = I$ (proved)

(iii) $A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$

Sol: $|A| = (2)(3) - (-1)(0)$

$$|A| = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} = (2)(3) - (-1)(0)$$

$$= 6 - 0 = 6$$

$$A^{-1} = \frac{adj\ A}{|A|}$$

$$A^{-1} = \frac{\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}}{6}$$

$$= \frac{1}{6}\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$$

Now $A^{-1}A = \frac{1}{6}\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$

$$= \frac{1}{6}\begin{bmatrix} (3)(2)+(0)(-1) & (3)(0)+(0)(3) \\ (1)(2)+(2)(-1) & (1)(0)+(2)(3) \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 6+0 & 0+0 \\ 2-2 & 0+6 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{6}{6} & \frac{0}{6} \\ \frac{0}{6} & \frac{6}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$A^{-1}A = I$ (proved)

(iv) $A = \begin{bmatrix} -6 & 4 \\ 3 & -2 \end{bmatrix}$

Sol: $|A| = \begin{vmatrix} -6 & 4 \\ 3 & -2 \end{vmatrix}$

$$= (-6)(-2) - (4)(3)$$

$$= 12 - 12 = 0$$

It is a singular matrix, its inverse is not possible.

(v) $A = \begin{bmatrix} 1 & 3 \\ 2 & 8 \end{bmatrix}$

Sol: $|A| = \begin{vmatrix} 1 & 3 \\ 2 & 8 \end{vmatrix}$

$= (1)(8) - (3)(2)$

$= 8 - 6 = 2$

$A^{-1} = \dfrac{adj\ A}{|A|}$

$= \dfrac{\begin{bmatrix} 8 & -3 \\ -2 & 1 \end{bmatrix}}{2}$

$= \dfrac{1}{2}\begin{bmatrix} 8 & -3 \\ -2 & 1 \end{bmatrix}$

Now $A^{-1}A = \dfrac{1}{2}\begin{bmatrix} 8 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 8 \end{bmatrix}$

$= \dfrac{1}{2}\begin{bmatrix} 8(1) + (-3)(2) & 8(3) + (-3)(8) \\ -2(1) + 1(2) & -2(3) + (1)(8) \end{bmatrix}$

$= \dfrac{1}{2}\begin{bmatrix} 8 - 6 & 24 - 24 \\ -2 + 2 & -6 + 8 \end{bmatrix}$

$= \dfrac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} \frac{2}{2} & \frac{0}{2} \\ \frac{0}{2} & \frac{2}{2} \end{bmatrix}$

$$=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=I$$

$A^{-1}A=I$ (proved)

(vi) $A=\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$

Sol: $|A|=\begin{vmatrix}-1 & 0\\0 & -1\end{vmatrix}$

$=(-1)(-1)-(0)(0)$

$=1-0=1$

$$A^{-1}=\frac{adj\ A}{|A|}$$

$$=\frac{\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}}{1}$$

$$=\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$$

Now $A^{-1}A=\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$

$$A^{-1}A=\begin{bmatrix}(-1)(-1)+(0)(0) & (-1)(0)+(0)(-1)\\0(-1)+(-1)(0) & (0)(0)+(-1)(-1)\end{bmatrix}$$

$$=\begin{bmatrix}1+0 & 0+0\\0+0 & 0+1\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=I$$

$A^{-1}A=I$ (proved)

(vii) $$A = \begin{bmatrix} \frac{3}{5} & \frac{-4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

Sol: $$|A| = \begin{vmatrix} \frac{3}{5} & \frac{-4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{vmatrix}$$

$$= \left(\frac{3}{5}\right)\left(\frac{3}{5}\right) - \left(-\frac{4}{5}\right)\left(\frac{4}{5}\right)$$

$$= \frac{9}{25} + \frac{16}{25}$$

$$= \frac{9+16}{25}$$

$$= \frac{25}{25} = 1$$

$$A^{-1} = \frac{adj\ A}{|A|}$$

$$= \frac{\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}}{1}$$

$$= \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

Now $A^{-1}A = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$

$$= \begin{bmatrix} \left(\frac{3}{5}\right)\left(\frac{3}{5}\right) + \left(\frac{4}{5}\right)\left(\frac{4}{5}\right) & \left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) + \left(\frac{4}{5}\right)\left(\frac{3}{5}\right) \\ \left(-\frac{4}{5}\right)\left(\frac{3}{5}\right) + \left(\frac{3}{5}\right)\left(\frac{4}{5}\right) & \left(-\frac{4}{5}\right)\left(-\frac{4}{5}\right) + \left(\frac{3}{5}\right)\left(\frac{3}{5}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{25} + \frac{16}{25} & -\frac{12}{25} + \frac{12}{25} \\ -\frac{12}{25} + \frac{12}{25} & \frac{16}{25} + \frac{9}{25} \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} \frac{25}{25} & \frac{0}{25} \\ \frac{0}{25} & \frac{25}{25} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$A^{-1}A = I$ Hence proved

4. Let $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(a) Find M^{-1}

(b) Verify that $M^{-1}M = MM^{-1}$

Sol: $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$|M| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

$$= (1)(4) - (2)(3)$$

$$= 4 - 6 = -2$$

$$M^{-1} = \frac{adj\ M}{|M|}$$

$$= \frac{\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}}{-2}$$

$$= \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 4\left(-\frac{1}{2}\right) & (-2)\left(-\frac{1}{2}\right) \\ -3\left(-\frac{1}{2}\right) & (1)\left(-\frac{1}{2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$$M^{-1}M = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} (-2)(1) + (1)(3) & (-2)(2) + (1)(4) \\ \left(\frac{3}{2}\right)(1) + \left(-\frac{1}{2}\right)(3) & \left(\frac{3}{2}\right)(2) + \left(-\frac{1}{2}\right)(4) \end{bmatrix}$$

$$= \begin{bmatrix} -2 + 3 & -4 + 4 \\ \frac{3}{2} - \frac{3}{2} & 3 - 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ldots\ldots\ldots.(i)$$

and now $MM^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

$$= \begin{bmatrix} (1)(-2) + (2)\left(\frac{3}{2}\right) & (1)(1) + (2)\left(-\frac{1}{2}\right) \\ (3)(-2) + (4)\left(\frac{3}{2}\right) & (3)(1) + (4)\left(-\frac{1}{2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} -2+3 & 1-1 \\ -6+6 & 3-2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ldots\ldots (ii)$$

$M^{-1}M = MM^{-1}$ from (i) and (ii)

5. If $A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Sol: L.H.S = $(AB)^{-1}$

$$AB = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} (5)(4) + (2)(3) & (5)(2) + (2)(-1) \\ (2)(4) + (1)(3) & (2)(2) + (1)(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 20+6 & 10-2 \\ 8+3 & 4-1 \end{bmatrix}$$

$$= \begin{bmatrix} 26 & 8 \\ 11 & 3 \end{bmatrix}$$

Now $|AB| = \begin{vmatrix} 26 & 8 \\ 11 & 3 \end{vmatrix}$

$= (26)(3) - (11)(8)$

$= 78 - 88$

$= -10$

$$(AB)^{-1} = \frac{adj\,(AB)}{|AB|}$$

$$= \frac{\begin{bmatrix} 3 & -8 \\ -11 & 26 \end{bmatrix}}{-10} = -\frac{1}{10}\begin{bmatrix} 3 & -8 \\ -11 & 26 \end{bmatrix}$$

$$= \begin{bmatrix} (3)\left(-\frac{1}{10}\right) & (-8)\left(-\frac{1}{10}\right) \\ (-11)\left(-\frac{1}{10}\right) & (26)\left(-\frac{1}{10}\right) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{10} & \frac{4}{5} \\ \frac{11}{10} & -\frac{13}{5} \end{bmatrix} \text{.............}(i)$$

R.H.S = $B^{-1}A^{-1}$

$B^{-1}A^{-1} = ?$

$B = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$

$|B| = \begin{vmatrix} 4 & 2 \\ 3 & -1 \end{vmatrix}$

$= (4)(-1) - (2)(3)$

$= -4 - 6$

$= -10$

$$B^{-1} = \frac{adj\ B}{|B|}$$

$$B^{-1} = \frac{\begin{bmatrix} -1 & -2 \\ -3 & 4 \end{bmatrix}}{-10}$$

$$= -\frac{1}{10}\begin{bmatrix} -1 & -2 \\ -3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)\left(-\frac{1}{10}\right) & (-2)\left(-\frac{1}{10}\right) \\ (-3)\left(-\frac{1}{10}\right) & (4)\left(-\frac{1}{10}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{10} & \frac{1}{5} \\ \frac{3}{10} & -\frac{2}{5} \end{bmatrix}$$

and $A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 5 & 2 \\ 2 & 1 \end{vmatrix}$$

$= (5)(1) - (2)(2)$

$= 5 - 4$

$= 1$

$$A^{-1} = \frac{adj\ A}{|A|}$$

$$= \frac{\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}}{1} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$$

Now $B^{-1}A^{-1} = \begin{bmatrix} \frac{1}{10} & \frac{1}{5} \\ \frac{3}{10} & -\frac{2}{5} \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$

$$= \begin{bmatrix} \left(\frac{1}{10}\right)(1) + \left(\frac{1}{5}\right)(-2) & \left(\frac{1}{10}\right)(-2) + \left(\frac{1}{5}\right)(5) \\ \left(\frac{3}{10}\right)(1) + \left(-\frac{2}{5}\right)(-2) & \left(\frac{3}{10}\right)(-2) + \left(-\frac{2}{5}\right)(5) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{10} - \frac{2}{5} & -\frac{1}{5} + 1 \\ \frac{3}{10} + \frac{4}{5} & -\frac{3}{5} - 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1-4}{10} & \frac{-1+5}{5} \\ \frac{3+8}{10} & \frac{-3-10}{5} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{10} & \frac{4}{5} \\ \frac{11}{10} & -\frac{13}{5} \end{bmatrix} \text{.......}(ii)$$

from (i) and (ii)

$(AB)^{-1} = B^{-1}A^{-1}$

SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS

To determine the value of two variables, we need a pair of equations. Such a pair of equations is called a system of simultaneous linear equations.