CHAPTER 7

PERMUTATION, COMBINATION AND PROBABILITY

Factorial Notation:

Let n be a positive integer then the product $n(n-1)(n-2)......3.2.1$ is denoted by $n!$ and read as n factorial i.e. $n! = n(n-1)(n-2)...3.2.1$

e.g.

$1! = 1$
$2! = 2.1 = 2$
$3! = 3.2.1 = 6$
$4! = 4.3.2.1 = 24$
$5! = 5.4.3.2.1 = 120$
$6! = 6.5.4.3.2.1 = 720$

Note:
$0! = 1$
$n! = n(n-1)!$

EXERCISE 7.1

Q.1 Evaluate each of the following:

(i) 4!

Solution: $4! = 4\cdot3\cdot2\cdot1 = 24$

(ii) 6!

Solution: $6! = 6\cdot5\cdot4\cdot3\cdot2\cdot1 = 720$

(iii) $\frac{8!}{7!}$

Solution: $\frac{8!}{7!} = \frac{8\cdot7!}{7!}$
$= 8\cdot1$
$= 8$

(iv) $\frac{10!}{7!}$

Solution: $\frac{10!}{7!} = \frac{10\cdot9\cdot8\cdot7!}{7!}$
$= 10\cdot9\cdot8 = 720$

(v) $\frac{11!}{4!7!}$

Solution: $\frac{11!}{4!7!} = \frac{11\cdot10\cdot9\cdot8\cdot7!}{4\cdot3\cdot2\cdot1\cdot7!}$
$= \frac{11\cdot10\cdot9\cdot8}{4\cdot3\cdot2\cdot1}$
$= 11\cdot10\cdot3 = 330$

(vi) $\frac{6!}{3!3!}$

Solution: $\frac{6!}{3!3!} = \frac{6\cdot5\cdot4\cdot3!}{3!3!} = \frac{6\cdot5\cdot4}{3!}$
$= \frac{6\cdot5\cdot4}{3\cdot2\cdot1} = 5\cdot4 = 20$

(vii) $\frac{8!}{4!2!}$

Solution: $\frac{8!}{4!2!} = \frac{8\cdot7\cdot6\cdot5\cdot4!}{4!\cdot2\cdot1}$
$= \frac{8\cdot7\cdot6\cdot5}{2\cdot1} = 8\cdot7\cdot3\cdot5$
$= 840$

(viii) $\frac{11!}{2!4!5!}$

Solution:
$\frac{11!}{2!4!5!} = \frac{11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5!}{2!4!5!}$
$= \frac{11\cdot10\cdot9\cdot8\cdot7\cdot6}{2!4!} = \frac{11\cdot10\cdot9\cdot8\cdot7\cdot6}{2\cdot1\cdot4!}$
$= \frac{11\cdot10\cdot9\cdot8\cdot7\cdot3}{4!} = \frac{11\cdot10\cdot9\cdot8\cdot7\cdot3}{4\cdot3\cdot2\cdot1}$
$= 11\cdot10\cdot9\cdot7 = 6930$

(ix) $\dfrac{9!}{2!(9-2)!}$

Solution: $\dfrac{9!}{2!(9-2)!} = \dfrac{9\cdot8\cdot7!}{2!7!} = \dfrac{9\cdot8}{2!}$

$$= \frac{9\cdot8}{2\cdot1} = 9\cdot4 = 36$$

(x) $\dfrac{15!}{15!(15-15)!}$

Solution: $\dfrac{15!}{15!(15-15)!} = \dfrac{15!}{15!0!}$

$$= \frac{1}{0!} = \frac{1}{1} = 1$$

(xi) $\dfrac{3!}{0!}$

Solution: $\dfrac{3!}{0!} = \dfrac{3\cdot2\cdot1}{1} = 6$

(xii) $4!\cdot0!\cdot1!$

Solution:

$$4!\cdot0!\cdot1! = 4!\cdot1\cdot1 = 4! = 4\cdot3\cdot2\cdot1 = 24$$

Q.2 Write each of the following in the factorial form:

(i) $6\cdot5\cdot4$

Solution: $6\cdot5\cdot4 = \dfrac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1} = \dfrac{6!}{3!}$

(ii) $12\cdot11\cdot10$

Solution:

$$12\cdot11\cdot10 = \frac{12\cdot11\cdot10\cdot9!}{9!} = \frac{12!}{9!}$$

(iii) $20\cdot19\cdot18\cdot17$

Solution: $20\cdot19\cdot18\cdot17$

$$= \frac{20\cdot19\cdot18\cdot17\cdot16!}{16!} = \frac{20!}{16!}$$

(iv) $\dfrac{10\cdot9}{2\cdot1}$ ***(LHR 2022)***

Solution:

$$\frac{10\cdot9}{2\cdot1} = \frac{10\cdot9}{2!} = \frac{10\cdot9\cdot8!}{2!\cdot8!} = \frac{10!}{2!8!}$$

(v) $\dfrac{8\cdot7\cdot6}{3\cdot2\cdot1}$

Solution: $\dfrac{8\cdot7\cdot6}{3\cdot2\cdot1} = \dfrac{8\cdot7\cdot6}{3!}$

$$= \frac{8\cdot7\cdot6\cdot5!}{3!\cdot5!} = \frac{8!}{3!5!}$$

(vi) $\dfrac{52\cdot51\cdot50\cdot49}{4\cdot3\cdot2\cdot1}$

Solution:

$$\frac{52\cdot51\cdot50\cdot49}{4\cdot3\cdot2\cdot1} = \frac{52\cdot51\cdot50\cdot49}{4!}$$

$$= \frac{52\cdot51\cdot50\cdot49\cdot48!}{4!\cdot48!} = \frac{52!}{4!\cdot48!}$$

(vii) $n(n-1)(n-2)$ ***(MTN 2022)***

Solution: $n(n-1)(n-2)$

$$= \frac{n\cdot(n-1)(n-2)\cdot(n-3)!}{(n-3)!}$$

$$= \frac{n!}{(n-3)!}$$

(viii) $(n+2)(n+1)(n)$

(GRW 2021, SGD 2022)

Solution: $(n+2)(n+1)(n)$

$$= \frac{(n+2)(n+1)(n)(n-1)!}{(n-1)!}$$

$$= \frac{(n+2)!}{(n-1)!}$$

(ix) $\dfrac{(n+1)(n)(n-1)}{3\cdot2\cdot1}$ ***(FSD 2021)***

Solution: $\dfrac{(n+1)(n)(n-1)}{3\cdot2\cdot1}$

$$= \frac{(n+1)(n)(n-1)\cdot(n-2)!}{3\cdot2\cdot1\cdot(n-2)!}$$

$$= \frac{(n+1)!}{3!(n-2)!}$$

(x) $n(n-1)(n-2)\ldots(n-r+1)$

(GRW 2022, SWL 2023)

Solution: $n(n-1)(n-2)\ldots(n-r+1)$

$$= \frac{n(n-1)(n-2)---(n-r+1)\cdot(n-r)!}{(n-r)!}$$

$$= \frac{n!}{(n-r)!}$$

Fundamental Principle of Counting:

Suppose **A** and **B** are two events. The first event **A** can occur in **p** different ways. After **A** has occurred, **B** can occur in **q** different ways. The number of ways that the two events can occur is the product **pq**.

Permutation:

An ordering (arrangement) of **n** objects is called permutation of the objects.

A permutation of **n** different objects taken $r(\le n)$ at a time is an arrangement of the **r** objects. Generally it is denoted by ^nP_r or $P(n, r)$.

$$^nP_r = \frac{n!}{(n-r)!} \qquad \therefore {}^nP_r = n(n-1)...(n-r+1)$$

Corollary: If $r = n$. Then

$$^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

EXERCISE 7.2

Q.1 Evaluate the following:

(i) $^{20}P_3$ ***(BWP 2021)***

Solution: $^{20}P_3 = \frac{20!}{(20-3)!} = \frac{20\cdot19\cdot18\cdot17!}{17!} = 20\cdot19\cdot18 = 6840$

(ii) $^{16}P_4$ ***(SGD 2021)***

Solution: $^{16}P_4 = \frac{16!}{(16-4)!} = \frac{16!}{12!} = \frac{16\cdot15\cdot14\cdot13\cdot12!}{12!} = 16\cdot15\cdot14\cdot13 = 43680$

(iii) $^{12}P_5$

Solution: $^{12}P_5 = \frac{12!}{(12-5)!} = \frac{12!}{7!} = \frac{12\cdot11\cdot10\cdot9\cdot8\cdot7!}{7!} = 12\cdot11\cdot10\cdot9\cdot8 = 95040$

(iv) $^{10}P_7$

Solution: $^{10}P_7 = \frac{10!}{(10-7)!} = \frac{10!}{3!} = \frac{10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3!}{3!} = 10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4 = 604800$

(v) $^9P_8$

Solution: $^9P_8 = \frac{9!}{(9-8)!} = \frac{9!}{1!} = \frac{9!}{1} = 9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1 = 362880$

Q.2 Find the value of n when:

Solution:

(i) $^nP_2 = 30$

(LHR 2021, GRW 2022, D.G.K 2022, RWP 2022)

$\frac{n!}{(n-2)!} = 30$

$= \frac{n\cdot(n-1)\cdot(n-2)!}{(n-2)!} = 30$

$= n\cdot(n-1) = 30$

$= n^2 - n = 30$

$= n^2 - n - 30 = 0$

$= n^2 - 6n + 5n - 30 = 0$

$= n(n-6) + 5(n-6) = 0$

or $(n-6)(n+5) = 0$

$\Rightarrow \quad n-6 = 0$ or $n+5 = 0$

$\Rightarrow \quad n = 6 \quad , \quad n = -5$

Ignoring $n = -5$, because n cannot be negative.

So, $n = 6$

(ii) $\quad {}^{11}P_n = 11\cdot 10\cdot 9$

$$\frac{11!}{(11-n)!} = 11\cdot 10\cdot 9$$

$$\frac{11\cdot 10\cdot 9\cdot 8!}{(11-n)!} = 11\cdot 10\cdot 9$$

$$\frac{8!}{(11-n)!} = 1$$

or $\quad 8! = (11-n)!$

$8 = 11-n$

$\Rightarrow \quad n = 11-8$

$\Rightarrow \quad n = 3$

(iii) $\quad {}^{n}P_4 : {}^{n-1}P_3 = 9:1$

$$\frac{{}^{n}P_4}{{}^{n-1}P_3} = \frac{9}{1}$$

$${}^{n}P_4 = 9\cdot {}^{n-1}P_3$$

$$\frac{n!}{(n-4)!} = 9\cdot\frac{(n-1)!}{(n-1-3)!}$$

$$\frac{n!}{(n-4)!} = 9\cdot\frac{(n-1)!}{(n-4)!}$$

$$n! = 9\cdot(n-1)!$$

$$n\cdot(n-1)! = 9\cdot(n-1)!$$

$\Rightarrow \quad n = 9$

Q.3 Prove from the first principle that:

(i) $\mathbf{{}^{n}P_r = n\cdot {}^{n-1}P_{r-1}}$ ***(LHR 2022, MTN 2023)***

Solution:

(i) Let there be r places to be filled by n objects. First place can be filled in n different ways. When first place has been filled, $n-1$ objects are left to fill the second place, so the second place can be filled in $n-1$ different ways. Thus, the first two places can be filled in $n(n-1)$ ways. After filling the second place, there are $n-2$ objects to fill the third place, so the third place can be filled in $n-2$ different ways. Thus, the first three places can be filled in $n(n-1)(n-2)$ ways. Continuing in this way the first r places can be filled in $n(n-1)(n-2)---[n-(r-1)]$ different ways. If ${}^{n}P_r$ denotes the number of ways in which n different objects can be arranged taken r at a time, then

$${}^{n}P_r = n(n-1)(n-2)...[n-(r-1)] \qquad \text{(i)}$$

If $r-1$ places have to be filled by $n-1$ objects then the first place can be filled in $n-1$ different ways. When first place has been filled, $n-2$ objects are left to fill the second place, so the second place can be filled in $(n-2)$ ways. After filling the second place, there are $n-3$ objects to fill the third place, so the third place can be filled in $n-3$ different ways. Thus, the first three places can be filled in $(n-1)(n-2)(n-3)$ ways. Continuing in this way, the first $r-1$ places can be filled in $(n-1)(n-2)---[n-(r-1)]$ different ways. If ${}^{n-1}P_{r-1}$ denotes the number of ways in which $n-1$ different objects can be arranged taken $r-1$ at a time, then

$${}^{n-1}P_{r-1} = (n-1)(n-2)...[n-(r-1)]$$

$$\Rightarrow \quad n\cdot {}^{n-1}P_{r-1} = n(n-1)(n-2)...[n-(r-1)] \qquad \text{(ii)}$$

From equation (i) and equation (ii), we have

$${}^{n}P_r = n\cdot {}^{n-1}P_{r-1}$$

Which is required result.

(ii) $^{n}\mathrm{P}_{\mathrm{r}} = {}^{n-1}\mathrm{P}_{\mathrm{r}} + r \cdot {}^{n-1}\mathrm{P}_{\mathrm{r}-1}$

Let there be r places to be filled by n objects. First place can be filled in different ways. When first place has been filled, $n-1$ objects are left to fill the second place, so the second place can be filled in $n-1$ different ways. Thus, the first two places can be filled in $n(n-1)$ ways. After filling the second place, there are $n-2$ objects to fill the third place, so the third place can be filled in $n-2$ different ways. Thus, the first three places can be filled in $n(n-1)(n-2)$ ways. Continuing in this way, the first r places can be filled in $n(n-1)(n-2)...\left[n-(r-1)\right]$ different ways. If $^{n}\mathrm{P}_{\mathrm{r}}$ denotes the number of ways in which n different objects can be arranged taken r at a time, then

$^{n}\mathrm{P}_{\mathrm{r}} = n(n-1)(n-2)...\left[n-(r-1)\right]$ (i)

If $r-1$ places have to fill by $n-1$ objects, then the first place can be filled in $n-1$ different ways. When first place has been filled, $n-2$ objects are left to fill the second place, so the second place can be filled in $n-2$ different ways. Thus, the first two places can be filled in $n-2$ different ways. Thus, the first two places can be filled in $(n-1)(n-2)$ ways. After filling the second place, there are $n-3$ objects to fill the third place, so the third place can be filled in $n-3$ different ways. Thus the first three places can be filled in $(n-1)(n-2)(n-3)$ ways. Continuing in this way, the first $r-1$ places can be filled in $(n-1)(n-2)...\left[n-(r-1)\right]$ different ways. If $^{n-1}\mathrm{P}_{\mathrm{r}-1}$ denotes the number of ways in which $n-1$ different objects can be arranged taken $r-1$ at a time, then

$^{n-1}\mathrm{P}_{\mathrm{r}-1} = (n-1)(n-2)...\left[n-(\mathrm{r}-1)\right]$

$\Rightarrow$ $r \cdot {}^{n-1}\mathrm{P}_{\mathrm{r}-1} = r(n-1)(n-2)...\left[n-(r-1)\right]$ (ii)

If r places have to fill by $n-1$ objects, then the first place can be filled in $n-1$ different ways. When first place has been filled, $n-2$ objects are left to fill the second place, So the second place can be filled in $n-2$ different ways. Thus, the first two places can be filled in $(n-1)(n-2)$ ways. After filling the second place, there are $n-3$ objects to fill the third place, so the third place can be filled in $n-3$ different ways. Thus, the first three places can be filled in $(n-1)(n-2)(n-3)$ ways. Continuing in this way, the first r places can be filled in $(n-1)(n-2)...(n-r)$ different ways. If $^{n-1}\mathrm{P}_{\mathrm{r}}$ denotes the number of ways in which $n-1$ different objects can be arranged taken r at a time, then

$^{n-1}P_{r} = (n-1)(n-2)...(n-r)$ (iii)

Adding equation (ii) and (iii), we have

$^{n-1}\mathrm{P}_{\mathrm{r}} + r \cdot {}^{n-1}\mathrm{P}_{\mathrm{r}-1} = (n-1)(n-2)...(n-r) + r(n-1)(n-2)...\left[n-(r-1)\right]$

$= (n-1)(n-2)...\left[n-(r-1)\right](n-r) + r(n-1)(n-2)...\left[n-(r-1)\right]$

$= \left[(n-r)+\mathrm{r}\right](n-1)(n-2)...\left[n-(r-1)\right]$

$\Rightarrow {}^{n-1}\mathrm{P}_{\mathrm{r}} + r \cdot {}^{n-1}\mathrm{P}_{\mathrm{r}-1} = n(n-1)(n-2)...\left[n-(r-1)\right]$

(iv)

From equation (i) and (iv), we have

$^{n}\mathrm{P}_{\mathrm{r}} = {}^{n-1}\mathrm{P}_{\mathrm{r}} + r \cdot {}^{n-1}\mathrm{P}_{\mathrm{r}-1}$

Which is required result.

Q.4 How many signals can be given by 5 flags of different colours, using 3 flags at a time? ***(GRW 2021)***

Solution:

The total number of flags $= n = 5$

Using flags at a time $= r = 3$

Required number of signals

$= {}^5P_3$

$= \dfrac{5!}{(5-3)!}$

$= \dfrac{5!}{2!}$

$= \dfrac{5\cdot 4\cdot 3\cdot 2!}{2!}$

$= 5\cdot 4\cdot 3$

$= 60$

Q.5 How many signals can be given by 6 flags of different colours when any number of flags can be used at a time?

Solution:

The total number of flags $= n = 6$

Number of signals using 1 flag $= {}^6P_1 = 6$

Number of signals using 2 flags $= {}^6P_2 = 6\cdot 5 = 30$

Number of signals using 3 flags $= {}^6P_3 = 6\cdot 5\cdot 4 = 120$

Number of signals using 4 flags $= {}^6P_4 = 6\cdot 5\cdot 4\cdot 3 = 360$

Number of signals using 5 flags $= {}^6P_5 = 6\cdot 5\cdot 4\cdot 3\cdot 2 = 720$

Number of signals using 6 flags $= {}^6P_6 = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 720$

Total number of signals $= 6+30+120+360+720+720 = 1956$

Q.6 How many words can be formed from the letters of the following words using all letters when no letter is to be repeated:

Solution:

(i) PLANE ***(MTN 2022, RWP 2023)***

Total number of letters $= n = 5$

Using at a time $= r = 5$

Required number of words

$= {}^5P_5 = 5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$

(ii) OBJECT

Total number of letters $= n = 6$

Using at a time $= r = 6$

Required number of words

$= {}^6P_6 = 6! = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 720$

(iii) FASTING

Total number of letters $= n = 7$

Using at a time $= r = 7$

Required number of words

$= {}^7P_7 = 7! = 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 5040$

Q.7 How many 3-digit numbers can be formed by using each one of the digits 2,3,5,7,9 only once? ***(SWL 2022, MTN 2023)***

Solution:

Given digits are: 2,3,5,7,9

Number of digits $= n = 5$

Using at a time $= r = 3$

Required number of 3-digit numbers

$= {}^5P_3$

$= \dfrac{5!}{(5-3)!}$

$= \dfrac{5!}{2!}$

$= \dfrac{5\cdot 4\cdot 3\cdot 2\cdot 1}{2!}$

$= 60$

Q.8 Find the numbers greater than 23000 that can be formed from the digits 1,2,3,5,6, without repeating any digit.

Solution: Given digits are 1,2,3,5,6.

Numbers greater than 23000 are of the form:

Numbers with 23 on the extreme left $23\otimes\otimes\otimes = {}^3P_3 = 3! = 6$

Numbers with 25 on the extreme left $25\otimes\otimes\otimes = {}^3P_3 = 3! = 6$

Numbers with 26 on the extreme left $26\otimes\otimes\otimes = {}^3P_3 = 3! = 6$

Numbers with 3 on the extreme left $3\otimes\otimes\otimes\otimes = {}^4P_4 = 4! = 24$

Numbers with 5 on the extreme left $5\otimes\otimes\otimes\otimes = {}^4P_4 = 4! = 24$

Numbers with 6 on the extreme left $6\otimes\otimes\otimes\otimes = {}^4P_4 = 4! = 24$

Total Numbers $= 6+6+6+24+24+24 = 90$

Q.9 Find the number of 5-digit numbers that can be formed from the digits 1,2,4,6,8 (when no digit is repeated), but

(i) The digits 2 and 8 are next to each other.

(ii) The digits 2 and 8 are not next to each other.

Solution: Given digits are 1,2,4,6,8

Number of digits $= 5$

(i) When 2 and 8 are next to each other in the form of 28 and 82, we consider them as one digit and their two places as one place.

Required number of permutations $= {}^4P_4 + {}^4P_4$

$= 4! + 4!$

$= 24 + 24 = 48$

(ii) Number of total permutations $= {}^5P_5 = 5! = 120$

Number of permutations when 2 and 8 are not next to each other $= 120 - 48 = 72$

Q.10 How many 6-digit numbers can be formed without repeating any digit from the digits 0,1,2,3,4,5? In how many of them will 0 be at the tens place?

Solution: Given digits are: 0,1,2,3,4,5

To form 6-digit numbers, we consider 6 places.

□□□□□□

Since 0 cannot be placed on the extreme left because in this case the number will be of five digits. So the first place on L.H.S can be filled by five digits (excluding 0) in ${}^5P_1 = 5$ ways.

After filling the first place, remaining five places can be filled by the remaining five digits (including 0) in ${}^5P_5 = 5! = 120$ ways.

Total numbers are 5×120 = 600

Now fixing 0 at tens place, the required numbers are of the form.

□□□□0□

Remaining five places can be filled by remaining five digits in ${}^5P_5 = 5! = 120$ ways.

Q.11 How many 5-digit multiples of 5 can be formed from the digits 2,3,5,7,9 when no digit is repeated.

Solution: Given digits are: 2,3,5,7,9

To form 5-digit numbers multiples of 5, we fix digit 5 at unit place. Then remaining four places can be filled by remaining four digits in ${}^{4}P_{4} = 4! = 24$ ways.

Q.12 In how many ways can 8 books including 2 on English be arranged on a shelf in such a way that the English books are never together.

Solution:

Total number of books = 8

Number of English books = 2

We denote two English books by E_1 and E_2. When E_1 and E_2 are together in the form of E_1E_2 and E_2E_1, we consider them as one book and their two places as one place.

Number of such permutations $= {}^{7}P_{7} + {}^{7}P_{7} = 7! + 7! = 5040 + 5040 = 10080$

Total number of permutations of 8 books $= {}^{8}P_{8} = 8! = 40320$

Number of permutations, when English books are not together. $= 40320 - 10080 = 30240$

Q.13 Find the number of arrangements of 3 books on English and 5 books on Urdu for placing them on a shelf such that the books on the same subject are together?

Solution:

Number of English books $= 3$

Number of Urdu books $= 5$

When English books are placed first and then Urdu books, then

Number of arrangements $= {}^{3}P_{3} \times {}^{5}P_{5} = 3! \times 5!\ 6 \times 120 = 720$

When Urdu books are placed first and then English books, then

Number of arrangements $= {}^{5}P_{5} \times {}^{3}P_{3} = 5! \times 3! = 120 \times 6 = 720$

Total number of arrangements $= 720 + 720 = 1440$

Q.14 In how many ways can 5 boys and 4 girls be seated on a bench so that the girls and the boys occupy alternate seats?

Solution:

Number of boys $= 5$

Number of girls $= 4$

Places on the bench for boys and girls be of the form.

B_1	G_1	B_2	G_2	B_3	G_3	B_4	G_4	B_5

Now, 5 boys can be seated on a bench to occupy 5 seats in ${}^{5}P_{5} = 5! = 120$ ways

4 girls can be seated on a bench to occupy 4 seats in ${}^{4}P_{4} = 4! = 24$ ways

Total number of ways $= 120 \times 24 = 2880$

Permutation of things not all different:

Suppose that out of **n** things, $\mathbf{n_1}$ are alike (same) of one kind and $\mathbf{n_2}$ are alike of second kind and the rest of them are all different.

Then total arrangements are $\dfrac{\mathbf{n!}}{\mathbf{n_1!n_2!}} = \dbinom{\mathbf{n}}{\mathbf{n_1, n_2}}$

Circular permutation:

The permutation of things which can be represented by the points on a circle are called circular permutation.

For **n** circular objects. **(Non-flipable)** Number of arrangements = $\mathbf{(n-1)!}$

For **n** circular objects **(flipable)** Number of arrangements = $\dfrac{\mathbf{(n-1)!}}{\mathbf{2}}$

EXERCISE 7.3

Q.1 How many arrangements of the letters of the following words, taken all together can be made:

(i) PAKPATTAN ***(GRW 2023)***

Solution:

Number of letters in "PAKPATTAN" $=9$

In PAKPATTAN,

A is repeated 3 times

P is repeated 2 times

T is repeated 2 times

K and N comes only once.

Required number of Permutations

$$=\binom{9}{3,2,2,1,1}$$

$$=\frac{9!}{3!2!2!1!1!}$$

$$=\frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3!}{3!\cdot 2\cdot 1\cdot 2\cdot 1\cdot 1\cdot 1}$$

$$=9\cdot 8\cdot 7\cdot 6\cdot 5=15120$$

(ii) Number of letters in "PAKISTAN" = 8 ***(MTN 2021)***

In PAKISTAN,

A is repeated 2 times

P comes only once

K comes only once

I comes only once

S comes only once

T comes only once

N comes only once

Required number of Permutations

$$=\binom{8}{2,1,1,1,1,1,1}=\frac{8!}{2!1!1!1!1!1!1!}$$

$$=\frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{2!1\cdot 1\cdot 1\cdot 1\cdot 1\cdot 1}=8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot$$

$$=20160 \text{ ways}$$

(iii) Number of letters in "MATHEMATICS" = 11 ***(FSD 2023)***

In MATHEMATICS,

M is repeated 2 times

A is repeated 2 times

T is repeated 2 times

H,E,I,C and S comes only once.

Required number of Permutations

$$=\binom{11}{2,2,2,1,1,1,1,1}$$

$$=\frac{11!}{2!2!2!1!1!1!1!1!}$$

$$=\frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2!}{2!\cdot 2\cdot 1\cdot 2\cdot 1\cdot 1\cdot 1\cdot 1\cdot 1\cdot 1}$$

$$=11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 3$$

$$=4989600 \text{ ways}$$

(iv) Number of letters in "ASSASSINATION" $=13$

In ASSASSINATION,

S is repeated 4 times

A is repeated 3 times

I is repeated 2 times

N is repeated 2 times

T and O comes only once.

Required number of Permutations

$$=\binom{13}{4,3,2,2,1,1}$$

$$=\frac{13!}{4!3!2!2!1!1!}$$

$$=\frac{13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4!}{4!.3\cdot 2\cdot 1\cdot 2\cdot 1\cdot 2\cdot 1\cdot 1\cdot 1}$$

$$=13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 2\cdot 7\cdot 5$$

$$=10810800 \text{ ways}$$

Q.2 How many Permutation of the letters of the word PANAMA can be made, if P is to be the first letter in each arrangement? ***(MTN 2023)***

Solution:

If P is the first letter of each arrangement then number of remaining letters $=5$

A is repeated 3 times

N and M comes only once.

Required number of Permutations

$$=\binom{5}{3,1,1}=\frac{5!}{3!1!1!}$$

$$\frac{5\cdot 4\cdot 3!}{3!\cdot 1\cdot 1}=5\cdot 4=20$$

Q.3 How many arrangements of the letters of the word ATTACKED can be made, if each arrangement begins with C and ends with K?

Solution:

If C is the first letter and K is the last letter of each arrangement, then

Number of remaining letters $= 6$

A is repeated 2 times

T is repeated 2 times

E and D comes only once.

Required number of Permutations

$$= \binom{6}{2,2,1,1}$$

$$= \frac{6!}{2!2!1!1!}$$

$$= \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{2!\cdot2\cdot1\cdot1\cdot1}$$

$$= 6\cdot5\cdot2\cdot3 = 180$$

Q.4 How many numbers greater than 1000,000 can be formed from the digits $0,2,2,2,3,4,4$? ***(RWP 2023)***

Solution:

Given digits are: $0,2,2,2,3,4,4$.

The numbers greater than 1000,000 are of the following forms:

(i) [2][ ][ ][ ][ ][ ][ ]

In this case, we have to fill 6 places by $0,2,2,3,4,4$.

Number of digits $= 6$

2 is repeated 2 times

4 is repeated 2 times

Each 0 and 3 comes only once.

Number of Permutations in this case

$$= \binom{6}{2,2,1,1} = \frac{6!}{2!2!1!1!} = 180$$

(ii) [3][ ][ ][ ][ ][ ][ ]

In this case, we have to fill 6 places by 0,2,2,2,4,4.

Number of digits $= 6$

2 is repeated 3 times

4 is repeated 2 times

0 comes only once.

Number of Permutations in this case

$$= \binom{6}{3,2,1} = \frac{6!}{3!2!1!}$$

$$= \frac{6\cdot5\cdot4\cdot3!}{3!\cdot2\cdot1\cdot1}$$

$$= 6\cdot5\cdot2 = 60$$

(iii) [4][ ][ ][ ][ ][ ][ ]

In this case, we have to fill 6 places by $0,2,2,2,3,4$

Number of digits $= 6$

2 is repeated 3 times

3 comes only once

0 comes only once

4 comes only once.

Number of Permutations in this case

$$= \binom{6}{3,1,1,1}$$

$$= \frac{6!}{3!1!1!1!}$$

$$= \frac{6\cdot5\cdot4\cdot3!}{3!\cdot1\cdot1\cdot1} = 6\cdot5\cdot4 = 120$$

Hence, the required numbers greater than 1000,000 are

$$= 180 + 60 + 120 = 360$$

Q.5 How many 6-digit numbers can be formed from the digits $2,2,3,3,4,4$? How many of them will lie between 400,000 and 430,000?

Solution:

Given digits are: $2,2,3,3,4,4$.

Number of digits $= 6$

2 is repeated 2 times

3 is repeated 2 times

4 is repeated 2 times.

Required number of Permutations

$$= \binom{6}{2,2,2}$$

$$= \frac{6!}{2!2!2!}$$

$$= \frac{6\cdot5\cdot4\cdot3\cdot2!}{2\cdot1\cdot2\cdot1\cdot2!}$$

$$= 6\cdot5\cdot3 = 90$$

The numbers lying between 400,000 and 430,000 are of the form

$\boxed{4}\boxed{2}\square\square\square\square$

In this case, we have to fill four places by $2,3,3,4$.

Number of digits $=4$

3 is repeated 2 times

Each 2 and 4 comes only once.

Required number of Permutations

$$=\binom{4}{2,1,1}=\frac{4!}{2!1!1!}=\frac{4\cdot3\cdot2!}{2!\cdot1\cdot1}$$

$=4\cdot3$

$=12$

Q.6 11 members of a club form 4 committees of $3,4,2,2$ members so that no member is a member of more than one committee. Find the number of committees.

Solution:

Total members of a club $=11$

First committee has 3 members

Second committee has 4 members

Third committee has 2 members

Fourth committee has 2 members

Required number of committees

$$=\binom{11}{3,4,2,2}$$

$$=\frac{11!}{3!4!2!2!}$$

$$=\frac{11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4!}{3\cdot2\cdot1\cdot4!\cdot2\cdot1\cdot2\cdot1}$$

$=11\cdot10\cdot9\cdot2\cdot7\cdot5$

$=69300$

Q.7 The D.C.Os of 11 districts meet to discuss the law and order situation in their districts. In how many ways can they be seated at a round table, when two Particular D.C.Os insist on sitting together?

Solution: Since two Particular D.C.Os insist on sitting together, so consider them as one man for seating. The total number of men now are 10, and they can be seated at a round table in $9!$ ways. Also, two Particular D.C.Os can occupy their seats in $2!$ ways. Therefore, the total number of ways are $9!\times2!=725760$

Q.8 The Governor of the Punjab calls a meeting of 12 officers. In how many ways can they be seated at a round table?

Solution: Number of officers $=12$

Number of ways that they can be seated at a round table, (fixing one seat), are $=11!=11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1=39916800$

Q.9 Fatima invites 14 people to a dinner. There are 9 males and 5 females who are seated at two different tables so that guests of one sex sit at one round table and the guests of the other sex at the second round table. Find the number of ways in which all guests are seated.

Solution: Number of ways that 9 males can be seated at a round table $=8!=40320$ ways.

5 females can be seated at a round table in $4!=24$ ways.

Both males and females can be seated at a round table $=40320\times24=967680$ ways.

Q.10 Find the number of ways in which 5 men and 5 women can be seated at a round table in such a way that no persons of the same sex sit together.

Solution: Number of men $=5$

Number of women $=5$

Number of ways the 5 men be seated a round table (fixing one seat) $=4!=24$ ways.

Number of ways the 5 women be seated, each between two men $=5!=120$ ways.

Required number of ways $=24\times120=2880$ ways.

Q.11 In how many ways can 4 keys be arranged on a circular key ring?

(GRW 2021,LHR 2022, D.G.K 2023)

Solution: Number of keys $= 4$

By fixing one of the keys, the number of arrangements of the remaining 3 keys $= 3! = 6$

Among these arrangements, half are the same.

Required number of arrangements $= \frac{6}{2} = 3$ ways

Q.12 How many necklaces can be made from 6 beads of different colours?

Solution: Number of beads $= 6$

By fixing one of the beads, the number of arrangements of the remaining beads $= 5! = 120$ ways.

Among these arrangements, half are the same.

Required number of necklaces $= \frac{120}{2} = 60$

> **Note:**
> See Example # 3 Page # 238,
> ***(BWP 2022)***

Combinations:

The number of **n** different objects taken **r** at a time is denoted by $^{\mathbf{n}}\mathbf{C}_{\mathbf{r}}$ or $\begin{pmatrix}\mathbf{n}\\ \mathbf{r}\end{pmatrix}$ or $\mathbf{C(n,r)}$ and is given by $^{\mathbf{n}}\mathbf{C}_{\mathbf{r}} = \frac{\mathbf{n!}}{\mathbf{r!(n-r)!}}$ a

> **Corollary:**
>
> **(i)** If $r = n$ then
>
> $$^{n}C_{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n!0!} = 1$$
>
> **(ii)** If $r = 0$ then
>
> $$^{n}C_{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{0!n!} = 1$$

Complementary Combination:

(GRW 2023, SWL 23 BWP 2022, D.G.K 2022)

Prove that $^{n}C_{r} = {}^{n}C_{n-r}$

Proof:

$$^{n}C_{n-r} = \frac{n!}{(n-r)!(n-n+r)!}$$

$$= \frac{n!}{(n-r)!\ r!}$$

$$^{n}C_{n-r} = {}^{n}C_{r}$$

EXERCISE 7.4

Q.1 Evaluate the following:

(i) $^{12}\mathbf{C}_{3}$

(ii) $^{20}\mathbf{C}_{17}$

(iii) $^{\mathbf{n}}\mathbf{C}_{4}$

Solution:

(i) $^{12}C_{3} = \frac{12!}{(12-3)!.3!}$ $\because {}^{n}C_{r} = \frac{n!}{(n-r)!r!}$

$$= \frac{12!}{9!3!}$$

$$= \frac{12\cdot 11\cdot 10\cdot 9!}{9!\cdot 3\cdot 2\cdot 1}$$

$$= 2\cdot 11\cdot 10 = 220$$

Hence $^{12}C_{3} = 220$

(ii) $^{20}C_{17} = \frac{20!}{(20-17)!\ 17!}$

$$= \frac{20!}{3!\cdot 17!}$$

$$= \frac{20\cdot 19\cdot 18\cdot 17!}{3\cdot 2\cdot 1\cdot 17!}$$

$$= 20\cdot 19\cdot 3 = 1140$$

Hence $^{20}C_{17} = 1140$

(iii) $${}^nC_4 = \frac{n!}{(n-4)!4!}$$

$$= \frac{n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdot(n-4)!}{(n-4)!\cdot 4!}$$

$$= \frac{n\cdot(n-1)\cdot(n-2)\cdot(n-3)}{4!}$$

Hence $${}^nC_4 = \frac{n.(n-1).(n-2).(n-3)}{4!}$$

Q.2 Find the value of n, when

(i) $\mathbf{{}^nC_5 = {}^nC_4}$

(ii) $\mathbf{{}^nC_{10} = \frac{12\times 11}{2!}}$

(iii) $\mathbf{{}^nC_{12} = {}^nC_6}$

Solution:

(i) $${}^nC_5 = {}^nC_4$$

(GRW 2022 MTN 2023)

$$\Rightarrow \quad {}^nC_{n-5} = {}^nC_4 \quad \because {}^nC_r = {}^nC_{n-r}$$

$$\Rightarrow \quad n-5 = 4$$

$$n = 9$$

(ii) $${}^nC_{10} = \frac{12\times 11}{2!}$$

(GRW 2021 RWP 2023, FSD 2023)

$$= \frac{12\times 11\times 10!}{2!\times 10!}$$

$$= \frac{12!}{2!\times 10!}$$

$$= \frac{12!}{(12-10)!10!}$$

$$\Rightarrow {}^nC_{10} = {}^{12}C_{10}$$

$$\Rightarrow n = 12$$

(iii) $${}^nC_{12} = {}^nC_6$$

(FSD 2022)

> **Note:** See Example # 1
> Page # 241,
> ***(BWP 2021, MTN 2023)***

$${}^nC_{n-12} = {}^nC_6 \quad \because {}^nC_r = {}^nC_{n-r}$$

$$\Rightarrow \quad n-12 = 6$$

$$\Rightarrow \quad n = 18$$

Q.3 Find the values of n and r, when

(i) $\mathbf{{}^nC_r = 35}$ **and** $\mathbf{{}^nP_r = 210}$

(GRW 2022)

(ii) $\mathbf{{}^{n-1}C_{r-1} : {}^nC_r : {}^{n+1}C_{r+1} = 3:6:11}$

Solution:

(i) $${}^nC_r = 35,\ {}^nP_r = 210$$

$$\frac{n!}{(n-r)!r!} = 35 \qquad \text{(i)}$$

$$\frac{n!}{(n-r)!} = 210 \qquad \text{(ii)}$$

Dividing equation (i) by equation (ii), we have

$$\frac{\frac{n!}{(n-r)!r!}}{\frac{n!}{(n-r)!}} = \frac{35}{210}$$

$$\frac{n!}{(n-r)!r!}\times\frac{(n-r)!}{n!} = \frac{35}{210}$$

$$\frac{1}{r!} = \frac{1}{6}$$

$$6 = r!$$

$$3! = r!$$

$$\Rightarrow r = 3$$

Putting $r = 3$ in eq (2), we have

$$\frac{n!}{(n-3)!} = 210$$

$$\frac{n\cdot(n-1)\cdot(n-2)\cdot(n-3)!}{(n-3)!} = 210$$

$$\Rightarrow n\cdot(n-1)\cdot(n-2) = 7\cdot 6\cdot 5$$

$$\Rightarrow n = 7$$

Hence $r = 3$ and $n = 7$

(ii) $${}^{n-1}C_{r-1} : {}^nC_r : {}^{n+1}C_{r+1} = 3:6:11$$

(RWP 2023, SGD 2022)

From the given equation, we consider

$${}^{n-1}C_{r-1} : {}^nC_r = 3:6 \qquad \text{(i)}$$

$${}^nC_r : {}^{n+1}C_{r+1} = 6:11 \qquad \text{(ii)}$$

From equation (i), we have

$$\frac{(n-1)!}{(n-1-r+1)!(r-1)!} \div \frac{n!}{(n-r)!r!} = 3\div 6$$

$$\frac{(n-1)!}{(n-r)!(r-1)!} \times \frac{(n-r)!r!}{n!} = \frac{3}{6}$$

$$\frac{(n-1)!}{(n-r)!\cdot(r-1)!} \times \frac{(n-r)!\cdot r\cdot(r-1)!}{n\cdot(n-1)!} = \frac{1}{2}$$

$$\Rightarrow \frac{r}{n} = \frac{1}{2}$$

or $2r = n$

or $n = 2r$ (iii)

From equation (ii), we have

$$\frac{n!}{(n-r)!r!} \div \frac{(n+1)!}{(n+1-r-1)!(r+1)!} = 6 \div 11$$

$$\frac{n!}{(n-r)!r!} \times \frac{(n-r)!(r+1)!}{(n+1)!} = \frac{6}{11}$$

$$\frac{n!}{(n-r)!r!} \times \frac{(n-r)!\cdot(r+1)\cdot(r!)}{(n+1)\cdot n!} = \frac{6}{11}$$

$$\frac{r+1}{n+1} = \frac{6}{11}$$

$11\cdot(r+1) = 6\cdot(n+1)$

$11r + 11 = 6n + 6$

$11r - 6n = -5$ (iv)

Putting $n = 2r$ in equation (iv)

$11r - 6(2r) = -5$

$11r - 12r = -5$

$-r = -5$

$\Rightarrow r = 5$

Putting $r = 5$ in equation (iii)

$n = 2r = 2(5) = 10$

$\Rightarrow n = 10$

Q.4 How many (a) diagonals and (b) triangles can be formed by joining the vertices of the polygon having:

(i) 5 sides ***(SWL 2023)***

(ii) 8 sides

(SWL 2022, SGD 2022, RWP 2022)

(iii) 12 sides ***(GRW 2022, RWP 2022)***

Solution:

(i) 5 sides

(a) Number of vertices of a 5 sided polygon $= 5$

Joining any two vertices, we have a line segment.

Number of line segments

$$= {}^5C_2 = \frac{5!}{(5-2)!2!} = \frac{5!}{3!2!}$$

$$= \frac{5\cdot4\cdot3!}{3!\cdot2\cdot1} = 5\cdot2 = 10$$

But these line segments include the 5 sides of the figure, which are not the diagonals.

Number of diagonals $= 10 - 5 = 5$

(b) Number of vertices of a 5 sided polygon $= 5$

Joining any three vertices, we have a triangle.

Number of triangles $= {}^5C_3$

$$= \frac{5!}{(5-3)!3!}$$

$$= \frac{5!}{2!3!}$$

$$= \frac{5\cdot4\cdot3!}{2\cdot1\cdot3!}$$

$= 5\cdot2 = 10$

(ii) 8 sides

(a) Number of vertices of a 8 sided polygon $= 8$

Joining any two vertices, we have a line segment.

Number of line segments

$$= {}^8C_2 = \frac{8!}{(8-2)!2!}$$

$$= \frac{8!}{6!2!}$$

$$= \frac{8\cdot7\cdot6!}{6!2!}$$

$$= \frac{8\cdot7\cdot6!}{6!\cdot2\cdot1}$$

$= 4\cdot7 = 28$

But these line segments include the 8 sides of the figure, which are not the diagonals.

Number of diagonals $= 28 - 8 = 20$

(b) Number of vertices of a 8 sided polygon $=8$

Joining any three vertices, we have a triangle.

Number of triangles

$$={}^{8}C_3=\frac{8!}{(8-3)!3!}$$

$$=\frac{8!}{5!3!}$$

$$=\frac{8\cdot7\cdot6\cdot5!}{5!\cdot3\cdot2\cdot1}$$

$$=8\cdot7=56$$

(iii) 12 sides

(a) Number of vertices of a 12 sided polygon $=12$

Joining any two vertices, we have a line segment.

Number of line segments

$$={}^{12}C_2=\frac{12!}{(12-2)!2!}$$

$$=\frac{12!}{10!2!}$$

$$=\frac{12\cdot11\cdot10!}{10!\cdot2\cdot1}$$

$$=6\cdot11=66$$

But these line segments include 12 sides of the figure, which are not the diagonals.

Number of diagonals $=66-12=54$.

(b) Number of vertices of a 12 sided polygon $=12$

Joining any three vertices, we have a triangle

Number of triangles

$$={}^{12}C_3=\frac{12!}{(12-3)!3!}=\frac{12!}{9!3!}$$

$$=\frac{12\cdot11\cdot10\cdot9!}{9!\cdot3\cdot2\cdot1}=2\cdot11\cdot10=220$$

Q.5 The members of a club are 12 boys and 8 girls. In how many ways can a committee of 3 boys and 2 girls be formed?

Solution:

Committees of 3 boys out of 12 boys and 2 girls out of 8 girls are to be formed.

Number of such committees are

$$={}^{12}C_3\times{}^{8}C_2$$

$$=\frac{12!}{(12-3)!3!}\times\frac{8!}{(8-2)!2!}$$

$$=\frac{12!}{9!\cdot3!}\times\frac{8!}{6!2!}$$

$$=\frac{12\cdot11\cdot10\cdot9!}{9!\cdot3\cdot2\cdot1}\times\frac{8\cdot7\cdot6!}{6!\cdot2\cdot1}$$

$$=2\cdot11\cdot10\times4\cdot7=220\times28=6160$$

Q.6 How many committees of 5 members can be chosen from a group of 8 persons when each committee must include 2 particular persons?

Solution:

Since each committee must include 2 particular persons, so remaining 3 members are to be chosen out of remaining 6 persons.

Number of such committees

$$={}^{6}C_3$$

$$=\frac{6!}{(6-3)!3!}$$

$$=\frac{6!}{3!\cdot3!}$$

$$=\frac{6\cdot5\cdot4\cdot3!}{3!\cdot3\cdot2\cdot1}$$

$$=5\cdot4=20$$

Q.7 In how many ways can a hockey team of 11 players be selected out of 15 players? How many of them will include a particular player?

Solution:

A hockey team of 11 players out of 15 players is to be selected.

Number of teams

$$={}^{15}C_{11}=\frac{15!}{(15-11)!11!}$$

$$=\frac{15!}{4!11!}$$

$$=\frac{15\cdot 14\cdot 13\cdot 12\cdot 11!}{4\cdot 3\cdot 2\cdot 1\cdot 11!}$$

$$15\cdot 7\cdot 13=1365$$

Since each team must include a particular player, so remaining 10 players are to be selected out of remaining 14 players.

Number of such teams

$$={}^{14}C_{10}=\frac{14!}{(14-10)!10!}$$

$$=\frac{14!}{4!10!}$$

$$=\frac{14\cdot 13\cdot 12\cdot 11\cdot 10!}{4\cdot 3\cdot 2\cdot 1\cdot 10!}$$

$$=7\cdot 13\cdot 11$$

$$=1001$$

Q.8 Show that ${}^{16}C_{11}+{}^{16}C_{10}={}^{17}C_{11}$. ***(BWP 2023)***

Solution: L.H.S $={}^{16}C_{11}+{}^{16}C_{10}$

$$=\frac{16!}{(16-11)!\cdot 11!}+\frac{16!}{(16-10)!\cdot 10!}$$

$$=\frac{16!}{5!11!}+\frac{16!}{6!10!}$$

$$=\frac{16!}{5!\cdot 11\cdot 10!}+\frac{16!}{6\cdot 5!\cdot 10!}$$

$$=\frac{16!}{5!\cdot 10!}\left[\frac{1}{11}+\frac{1}{6}\right]$$

$$=\frac{16!}{5!10!}\left[\frac{6+11}{11\cdot 6}\right]$$

$$=\frac{16!}{5!10!}\left[\frac{17}{11\cdot 6}\right]$$

$$=\frac{17\cdot 16!}{6\cdot 5!\cdot 11\cdot 10!}=\frac{17!}{6!11!}=\frac{17!}{(17-11)!11!}$$

$$={}^{17}C_{11}=\text{R.H.S}$$

Q.9 There are 8 men and 10 women members of a club. How many committees of Seven can be formed, having:

(i) 4 women

(ii) at the most 4 women

(iii) at least 4 women?

Solution: Total number of men $=8$

Total number of women $=10$

(i) 4 women:

We have to form combinations of 4 women out of 10 and 3 men out of 8.

Number of such committees are

$$={}^{10}C_4\times {}^{8}C_3=\frac{10!}{(10-4)!4!}\times\frac{8!}{(8-3)!3!}$$

$$=\frac{10!}{6!4!}\times\frac{8!}{5!3!}$$

$$=\frac{10\cdot 9\cdot 8\cdot 7\cdot 6!}{6!\cdot 4\cdot 3\cdot 2\cdot 1}\times\frac{8\cdot 7\cdot 6\cdot 5!}{5!\cdot 3\cdot 2\cdot 1}$$

$$=10\cdot 3\cdot 7\times 8\cdot 7=210\times 56=11760$$

(ii) At the most 4 women:

In this case, womens are less than or equal to 4, which implies the following possibilities.

$(0W,7M)(1W,6M),(2W,5M),(3W,4M),(4W,3M)$

Number of such committees are:

$={}^{10}C_0\times{}^{8}C_7+{}^{10}C_1\times{}^{8}C_6+{}^{10}C_2\times{}^{8}C_5+{}^{10}C_3\times{}^{8}C_4+{}^{10}C_4\times{}^{8}C_3$

$={}^{8}C_1+{}^{10}C_1\times{}^{8}C_2+{}^{10}C_2\times{}^{8}C_3+{}^{10}C_3\times{}^{8}C_4+{}^{10}C_4\times{}^{8}C_3$ $\quad\therefore {}^{n}C_r={}^{n}C_{n-r}$

$=\frac{10}{1}\times\frac{8\cdot7}{2\cdot1}+\frac{10\cdot9}{2\cdot1}\times\frac{8\cdot7\cdot6}{3\cdot2\cdot1}+\frac{10\cdot9\cdot8}{3\cdot2\cdot1}\times\frac{8\cdot7\cdot6\cdot5}{4\cdot3\cdot2\cdot1}+\frac{10\cdot9\cdot8\cdot7}{4\cdot3\cdot2\cdot1}\times\frac{8\cdot7\cdot6}{3\cdot2\cdot1}+\frac{8}{1}$

$=280+2520+8400+11760+8=22968$

(iii) At least 4 women:

In this case, womens are greater than or equal to 4, which implies the following possibilities.

$(4W,3M),(5W,2M),(6W,1M),(7W,0M)$

Number of such committees are:

$={}^{10}C_4\times{}^{8}C_3+{}^{10}C_5\times{}^{8}C_2+{}^{10}C_6\times{}^{8}C_1+{}^{10}C_7\times{}^{8}C_0$

$={}^{10}C_4\times{}^{8}C_3+{}^{10}C_5\times{}^{8}C_2+{}^{10}C_4\times{}^{8}C_1+{}^{10}C_3$ $\quad\therefore {}^{n}C_r={}^{n}C_{n-r}$

$=\frac{10\cdot9\cdot8\cdot7}{4\cdot3\cdot2\cdot1}\times\frac{8\cdot7\cdot6}{3\cdot2\cdot1}+\frac{10\cdot9\cdot8\cdot7\cdot6}{5\cdot4\cdot3\cdot2\cdot1}\times\frac{8\cdot7}{2\cdot1}+\frac{10\cdot9\cdot8\cdot7}{4\cdot3\cdot2\cdot1}\times\frac{8}{1}+\frac{10\cdot9\cdot8}{3\cdot2\cdot1}$

$=11760+7056+1680+120=20616$

Q.10 Prove that ${}^{n}C_r+{}^{n}C_{r-1}={}^{n+1}C_r$ ***(LHR 2022, MTN 2022)***

Proof:

L.H.S:

${}^{n}C_r+{}^{n}C_{r-1}$

$=\frac{n!}{r!(n-r)!}+\frac{n!}{(r-1)![n-(r-1)]!}$

$=\frac{n!}{r.(r-1)!(n-r)!}+\frac{n!}{(r-1)!(n-r+1)!}$

$=\frac{n!}{r.(r-1)!(n-r)!}+\frac{n!}{(r-1)!(n-r+1)(n-r)!}$

$=\frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{r}+\frac{1}{n-r+1}\right]$

$=\frac{n!}{(r-1)!(n-r)!}\left[\frac{n-r+1+r}{r(n-r+1)}\right]$

$=\frac{n!}{(r-1)!(n-r)!}\left[\frac{n+1}{r(n-r+1)}\right]$

$=\frac{(n+1)n!}{r(r-1)!(n-r+1)!(n-r)!}$

$=\frac{(n+1)n!}{r!(n-r+1)!}$

$=\frac{(n+1)!}{r!((n+1)-r)!}$

$={}^{n+1}C_r=\text{R.H.S}$

Hence L.H.S = R.H.S

Note: See Example # 3 Page # 241, ***(GRW 2022, FSD 2023, RWP 2022, D.G.K 2023)***

Sample Space:

The set consisting of all possible outcomes of a given experiment is called the **sample space.** ***(RWP 2021, LHR 2022)***

Event: *(RWP 2021, LHR 2022)*

A particular outcome is called an **event** and usually denoted by **E**.
e.g. In tossing a fair coin, the possible outcomes are Head **(H)** or a Tail **(T)** and it is written as $\mathbf{S = \{H, T\}}$

Mutually Exclusive Events:

Two events are mutually exclusive events if they cannot occur together. If we toss a balanced coin so occurrence of head and occurrence of tail are mutually exclusive events.

Equally Likely Events:

If two events **A** and **B** occur in an experiment, then **A** and **B** are said to be **equally likely events** if each one of them has equal number of chances of occurrence.

Probability:

Probability is the numerical evaluation of a chance that a particular event would occur. The probability of the occurrence of the event **E** is denoted by **P(E)**

Such that: $\mathbf{P(E) = \frac{n(E)}{n(S)} = \frac{\text{number of ways in which event occurs}}{\text{number of the elements of the sample space}}}$

Note:

(i) $0 \le P(E) \le 1$

(ii) If $\mathbf{P(E) = 0}$, event **E** cannot occur and **E** is called an impossible event.

(iii) If $\mathbf{P(E) = 1}$, event **E** is sure to occur and **E** is called a certain event.

Probability That an Event Does Not Occur:

If a sample space **S** is such that $\mathbf{n(S) = N}$ and out of the **N** equally likely events an event **E** occurs **R** times, then , evidently, **E** does not occur $\mathbf{N - R}$ times.

The non-occurrence of the event **E** is denoted as $\mathbf{\overline{E}}$.

Now $$P(E) = \frac{n(E)}{n(S)} = \frac{R}{N}$$

and $$P(\overline{E}) = \frac{n(\overline{E})}{n(S)} = \frac{N-R}{N} = \frac{N}{N} - \frac{R}{N} = 1 - \frac{R}{N}$$

$$\mathbf{P(\overline{E}) = 1 - P(E).}$$

EXERCISE 7.5

For the following experiments, find the Probability in each case:

Q.1 Experiment:
From a box containing orange flavoured sweets, Bilal takes out one sweet without looking.
Events Happening:
(i) The sweet is orange-flavoured.
(ii) The sweet is lemon-flavoured.

Solution:

(i) The sweet is orange-flavoured:

Total possible outcomes $= n(S) = 1$

Let A be the event that the sweet is orange-flavoured.

Since the box contains just orange-flavoured sweets, so all the possible outcomes are favourable i.e. $n(A) = 1$

The required Probability is

$$P(A) = \frac{n(A)}{n(S)} = \frac{1}{1} = 1$$

(ii) The sweet is lemon-flavoured:

Total Possible outcomes $= n(S) = 1$

Let B the event that the sweet is lemon-flavoured.

Since the box contains just orange-flavoured sweets, so all the possible outcomes are not favourable, i.e. $n(B) = 0$

The required Probability is

$$P(B) = \frac{n(B)}{n(S)} = \frac{0}{1} = 0$$

Q.2 Experiment:

Pakistan and India Play a cricket match.

The result is:

Events Happening:

(i) Pakistan wins

(ii) India does not lose.

Solution:

Since there are three Possible results of the match, win, lose or the match is tied. So, the total possible outcomes are $n(S) = 3$

(i) Pakistan wins:

Let A be the event that Pakistan wins, then $n(A) = 1$

The required Probability that Pakistan wins the match is

$$P(A) = \frac{n(A)}{n(S)} = \frac{1}{3}$$

(ii) India does not lose:

Let B be the event that India does not lose the match, then $n(B) = 2$

The required Probability that India does not lose the match is

$$P(B) = \frac{n(B)}{n(S)} = \frac{2}{3}$$

Q.3 Experiment:

There are 5 green and 3 red balls in a box, one ball is taken out.

Events happening:

(i) The ball is green ***(GRW 2021)***

(ii) The ball is red. ***(FSD 2023)***

Solution: Number of green balls $= 5$

Number of red balls $= 3$

Total number of balls $= n(S) = 8$

(i) The ball is green.

Let A be the event that the ball is green, then $n(A) = 5$

The required Probability that the ball is green is

$$P(A) = \frac{n(A)}{n(S)} = \frac{5}{8}$$

(ii) The ball is red.

Let B be the event that the ball is red, then $n(B) = 3$

The required Probability that the ball is red, is

$$P(B) = \frac{n(B)}{n(S)} = \frac{3}{8}$$

Q.4 Experiment:

A fair coin is tossed three times. It shows Events Happening:

(i) One tail

(ii) At least one head

Solution: When a fair coin is tossed three times, the set of possible outcomes is

$S = \{HHH, THH, HTH, HHT, TTH, THT, HTT, TTT\}$

Total possible outcomes are $n(S) = 8$

(i) One tail.

Let A be the event that the coin shows one tail, then the set of favourable outcomes is

$A = \{HHT, HTH, THH\}$ i.e.

$n(A) = 3$

The required Probability that the coin shows one tail is

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{8}$$

(ii) At least one head:

Let B be the event that the coin shows at least one head, then the set of favourable outcomes is

$B=\{HHH, THH, HTH, HHT, TTH, THT, HTT\}$ i.e.

$n(B)=7$

The required Probability that the coin shows at least one head is:

$$P(B)=\frac{n(B)}{n(S)}=\frac{7}{8}$$

Q.5 Experiment:

A die is rolled. The top shows,

Events Happening:

(i) 3 or 4 dots

(ii) dots less than 5

Solution:

The set of Possible outcomes is

$S=\{1,2,3,4,5,6\}$, then $n(S)=6$

(i) 3 or 4 dots:

If A is the event that the top shows 3 or 4 dots, then set of favorable out comes is

$A=\{3,4\}$

i.e $n(A)=2$

$$P(A)=\frac{n(A)}{n(S)}=\frac{2}{6}=\frac{1}{3}$$

(ii) Dots less than 5:

If B is the event that the top shows dots less than 5, then set of favorable out comes is

$B=\{1,2,3,4\}$

i.e $n(B)=4$

$$P(B)=\frac{n(B)}{n(S)}=\frac{4}{6}=\frac{2}{3}$$

Q.6 Experiment:

From a box containing slips numbered 1,2,3,4,5 one slip is picked up.

Events Happening:

(i) The number on the slip is a prime number.

(ii) The number on the slip is a multiple of 3.

Solution:

The set of possible outcomes is

$S=\{1,2,3,4,5\}$, then $n(S)=5$

(i) The number on the slip is a prime number:

If A is the event that the number on the slip is a prime number i.e. $A=\{2,3,5\}$ then $n(A)=3$

So, the Probability of A is

$$P(A)=\frac{n(A)}{n(S)}=\frac{3}{5}$$

(ii) The number on the slip is a multiple of 3:

If B is the event that the number on the slip is a multiple of 3 i.e. $B=\{3\}$ then $n(B)=1$

So, the Probability of B is

$$P(B)=\frac{n(B)}{n(S)}=\frac{1}{5}$$

Q.7 Experiment:

Two dice, one red and the other blue, are rolled simultaneously. The numbers of dots on the tops are added. The total of the two scores is

Events Happening:

(i) 5 (ii) 7 (iii) 11

Solution:

When two dice are rolled, the set of possible outcomes is.

$$S=\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

Then $n(S)=36$

(i) Let A be the event that the total of two scores is 5, then set of favorable out comes is

i.e. $A=\{(1,4),(2,3),(3,2),(4,1)\}$,

then $n(A)=4$

Thus, the required Probability is

$$P(A)=\frac{n(A)}{n(S)}=\frac{4}{36}=\frac{1}{9}$$

(ii) Let B be the event that the total of two scores is 7, then set of favorable out comes is

i.e.

$B=\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

then $n(B)=6$

Thus the required probability is:

$$P(B)=\frac{n(B)}{n(S)}=\frac{6}{36}=\frac{1}{6}$$

(iii) Let C be the event that the total of two scores is 11, then set of favorable out comes is

i.e. $C=\{(5,6),(6,5)\}$

then $n(C)=2$

Thus the required Probability is

$$P(C)=\frac{n(C)}{n(S)}=\frac{2}{36}=\frac{1}{18}$$

Q.8 Experiment:

A bag contains 40 balls out of which 5 are green, 15 are black and the remaining are yellow. A ball is taken out of the bag.

Events Happening:

(i) The ball is black

(ii) The ball is green

(iii) The ball is not green.

Solution: Total number of balls $=40 \Rightarrow n(S)=40$

Number of green balls $=5$

Number of black balls $=15$

Number of yellow balls $=40-(5+15)=20$

(i) The ball is black:

Let A be the event that the ball is black, then $n(A)=15$

The Probability that the black ball comes is

$$P(A)=\frac{n(A)}{n(S)}=\frac{15}{40}=\frac{3}{8}$$

(ii) The ball is green:

Let B be the event that the ball is green then $n(B)=5$

The Probability that the green ball comes is

$$P(B)=\frac{n(B)}{n(S)}=\frac{5}{40}=\frac{1}{8}$$

(iii) The ball is not green:

Let C be the event that the ball is not green, then

$$P(C)=P\left(\overline{B}\right)=1-P(B)=1-\frac{1}{8}=\frac{7}{8}$$

Q.9 Experiment:

One chit out of 30 containing the names of 30 students of a class of 18 boys and 12 girls is taken out at random, for nomination as the monitor of the class.

Events Happening.

(i) The monitor is a boy.

(ii) The monitor is a girl.

Solution: Total number of boys $=18$

Total number of girls $=12$

Total number of students $=18+12=30$

Also, total number in sample space $=n(S)=30$

(i) The monitor is a boy:

Let A be the event that the monitor is a boy, then $n(A)=18$

Therefore, the probability that the monitor is a boy is

$$P(A)=\frac{n(A)}{n(S)}=\frac{18}{30}=\frac{3}{5}$$

(ii) The monitor is a girl:

Let B be the event that the monitor is a girl, then $n(B)=12$

Therefore, the Probability that the monitor is a girl, is

$$P(B)=\frac{n(B)}{n(S)}=\frac{12}{30}=\frac{2}{5}$$

Q.10 Experiment:
A coin is tossed four times. The tops show
Event Happening:
(i) All heads (ii) 2 heads and 2 tails

Solution: When a coin is tossed four times, then the set of possible outcomes is

$$S=\begin{Bmatrix} \text{HHHH},\text{HHHT},\text{HHTH},\text{HTHH},\text{THHH}, \\ \text{HHTT},\text{HTHT},\text{HTTH},\text{THTH},\text{THHT}, \\ \text{TTHH},\text{HTTT},\text{THTT},\text{TTHT},\text{TTTH},\text{TTTT} \end{Bmatrix}$$

$n(S)=16$

(i) All heads:

Let A be the event that the tops shows all heads, then the set of favorable outcomes, that is $\text{A}=\{\text{HHHH}\}$, so $n(A)=1$

Therefore, the Probability that the tops shows all heads is:

$$P(A)=\frac{n(A)}{n(S)}=\frac{1}{16}$$

(ii) 2 heads and 2 tails:

Let B be the event that the tops shows 2 heads and 2 tails, then the set of possible favourable outcomes is

$B=\{HHTT, HTHT, HTTH, THHT, THTH, \text{TTHH}\}$

Then $n(B)=6$

Therefore, the Probability that the tops shows 2 heads and 2 tails is:

$$P(B)=\frac{n(B)}{n(S)}=\frac{6}{16}=\frac{3}{8}$$

EXERCISE 7.6

Q.1 A fair coin is tossed 30 times, the result of which is tabulated below. Study the table and answer the questions given below the table.

Event	Tally Marks	Frequency
Head	𝍸 𝍸 ||||	**14**
Tail	𝍸 𝍸 𝍸 |	**16**

(i) How many times does 'head' appear?
(ii) How many times does 'tail' appear?
(iii) Estimate the Probability of the appearance of head?
(iv) Estimate the Probability of the appearance of tail?

Solution:

(i) Heads appear in experiment are 14 times.

(ii) Tails appear in experiment are 16 times.

(iii) If A is the event that head appears, then $n(A)=14$

Thus, the Probability of the event A is

$$P(A)=\frac{n(A)}{n(S)}=\frac{14}{30}=\frac{7}{15}$$

(iv) If B is the event that the tail appears, then $n(B)=16$. Thus, the Probability of the event B is

$$P(B)=\frac{n(B)}{n(S)}=\frac{16}{30}=\frac{8}{15}$$

Q.2 A die is tossed 100 times. The result is tabulated below. Study the table and answer the questions given below the table.

Event	Tally Marks	Frequency
1	卌 卌 ||||	14
2	卌 卌 卌 ||	17
3	卌 卌 卌 卌	20
4	卌 卌 卌 |||	18
5	卌 卌 卌	15
6	卌 卌 卌 |	16

(i) How many times do 3 dots appear?
(ii) How many times do 5 dots appear?
(iii) How many times does an even number of dots appear?
(iv) How many times does a prime number of dots appear?
(v) Find the probability of each one of the above cases.

Solution:

(i) Dots 3 appear 20 times.

(ii) Dots 5 appear 15 times.

(iii) Even dots are 2,4 and 6, appear 17, 18 and 16 times respectively.
So, the total number of even dots are 17+18+16 = 51 times.

(iv) Prime numbers are 2,3 and 5, so the total number of prime dots appearing are $17+20+15=52$ times.

(v) If A is the event that 3 dots appear, then $P(A)=\frac{n(A)}{n(S)}=\frac{20}{100}=\frac{1}{5}$

If B is the event that 5 dots appear, then $P(B)=\frac{n(B)}{n(S)}=\frac{15}{100}=\frac{3}{20}$

If C is the event that the even number of dots appear, then $P(C)=\frac{n(C)}{n(S)}=\frac{51}{100}$

If D is the event that the even number of dots appear, then $P(D)=\frac{n(D)}{n(S)}=\frac{52}{100}=\frac{13}{25}$

Q.3 The eggs supplied by a poultry farm during a week broke during transit as follows: $1\%, 2\%, 1\frac{1}{2}\%, \frac{1}{2}\%, 1\%, 2\%, 1\%$, Find the Probability of the eggs that broke in a day. Calculate the number of eggs that will be broken in transmitting the following number of eggs:

(i) 7,000 (ii) 8,400 (iii) 10,500

Solution: For a week

$1+2+\frac{3}{2}+\frac{1}{2}+1+2+1=3+\frac{4}{2}+4=3+2+4=9$

Number of eggs that are broken in a day $=\frac{9}{7}\%$

Number of eggs that are broken out of 7000 eggs $=7000\times\frac{9}{7}\times\frac{1}{100}=90$

Number of eggs that are broken out of 8400 eggs $=8400\times\frac{9}{7}\times\frac{1}{100}=108$

Number of eggs that are broken out of 10500 eggs $=10500\times\frac{9}{7}\times\frac{1}{100}=135$

Addition of Probabilities:

Let ***A*** and ***B*** are any two events then probability of occurrence of event ***A*** or event ***B*** is given by

$$\mathbf{P(A\cup B)=P(A)+P(B)-P(A\cap B)}$$

When ***A*** and ***B*** are disjoint events then it becomes

$$\mathbf{P(A\cup B)=P(A)+P(B)}$$

EXERCISE 7.7

Q.1 If sample space $=\{1,2,3,...,9\}$, Event A $=\{2,4,6,8\}$ and Event B $=\{1,3,5\}$, find $P(A\cup B)$. ***(GRW 2023, MTN 2023)***

Solution:

Sample Space $=S=\{1,2,3,...,9\}$

$\Rightarrow n(S)=9$

Event A $=\{2,4,6,8\}\Rightarrow n(A)=4$

Event B $=\{1,3,5\}\Rightarrow n(B)=3$

$P(A)=\dfrac{n(A)}{n(S)}=\dfrac{4}{9}$ and $P(B)=\dfrac{n(B)}{n(S)}=\dfrac{3}{9}=\dfrac{1}{3}$

A and B are disjoint sets

So $P(A\cup B)=P(A)+P(B)$

$=\dfrac{4}{9}+\dfrac{1}{3}=\dfrac{4+3}{9}=\dfrac{7}{9}$

Q.2 A box contains 10 red, 30 white and 20 black marbles. A marble is drawn at random. Find the Probability that it is either red or white.

Solution:

Number of red marbles $=10$

Number of white marbles $=30$

Number of black marbles $=20$

Total number of marbles

$=10+30+20=60$

$\Rightarrow n(S)=60$

Let A be the event that a drawn marble is red, then $n(A)=10$

$P(A)=\dfrac{n(A)}{n(S)}=\dfrac{10}{60}=\dfrac{1}{6}$

Let B be the event that a drawn marble is white, then $n(B)=30$

$P(B)=\dfrac{n(B)}{n(S)}=\dfrac{30}{60}=\dfrac{1}{2}$

Since A and B are mutually exclusive events,

So $P(A\cup B)=P(A)+P(B)$

$\dfrac{1}{6}+\dfrac{1}{2}=\dfrac{1+3}{6}=\dfrac{4}{6}=\dfrac{2}{3}$

Q.3 A natural number is chosen out of the first fifty natural numbers. What is the Probability that the chosen number is a multiple of 3 or of 5? ***(D.G.K 2023)***

Solution:

Here, the Sample space is

$S=\{1,2,3,...,50\}\Rightarrow n(S)=50$

Let A be the event that a chosen number is a multiple of 3.

$A=\left\{\begin{matrix}3,6,9,12,15,18,21,24,27,\\30,33,36,39,42,45,48\end{matrix}\right\}$

$\Rightarrow n(A)=16$

$P(A)=\dfrac{n(A)}{n(S)}=\dfrac{16}{50}=\dfrac{8}{25}$

Let B be the event that a chosen number is a multiple of 5.

$B=\{5,10,15,20,25,30,35,40,45,50\}$

$\Rightarrow n(B)=10$

$P(B)=\dfrac{n(B)}{n(S)}=\dfrac{10}{50}=\dfrac{1}{5}$

Now, $A\cap B=\{15,30,45\}$

$\Rightarrow n(A\cap B)=3$

$\therefore P(A\cap B)=\dfrac{n(A\cap B)}{n(S)}=\dfrac{3}{50}$

Since A and B are overlapping events, so

$P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$=\dfrac{8}{25}+\dfrac{1}{5}-\dfrac{3}{50}$

$=\dfrac{16+10-3}{50}=\dfrac{23}{50}$

Q.4 A card is drawn from a deck of 52 playing cards. What is the Probability that it is a diamond card or an ace? ***(D.G.K 2022, RWP 2023)***

Solution:

Total number of cards $=52$

$\Rightarrow n(S)=52$

Let A be the event that a drawn card is diamond card.

$\Rightarrow n(A)=13$

$$P(A)=\frac{n(A)}{n(S)}=\frac{13}{52}=\frac{1}{4}$$

Let B be the event that a drawn card is an ace.

$\Rightarrow n(B)=4$

$$P(B)=\frac{n(B)}{n(S)}=\frac{4}{52}=\frac{1}{13}$$

Since one diamond card is also an ace card, so $n(A\cap B)=1$

$$P(A\cap B)=\frac{n(A\cap B)}{n(S)}=\frac{1}{52}$$

A and B are overlapping events, so

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$=\frac{1}{4}+\frac{1}{13}-\frac{1}{52}$$

$$=\frac{13+4-1}{52}=\frac{16}{52}=\frac{4}{13}$$

Q.5 A die is thrown twice. What is the Probability that the sum of the number of dots shown is 3 or 11?

Solution:

A die is thrown twice. Therefore, sample space is

$$S=\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)\\(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

$\Rightarrow n(S)=36$

Let A be the event that the sum of number of dots shown is 3.

$A=\{(1,2),(2,1)\}\Rightarrow n(A)=2$

$$P(A)=\frac{n(A)}{n(S)}=\frac{2}{36}=\frac{1}{18}$$

Let B be the event that the sum of number of dots shown is 11.

$B=\{(5,6),(6,5)\}\Rightarrow n(B)=2$

$$P(B)=\frac{n(B)}{n(S)}=\frac{2}{36}=\frac{1}{18}$$

Since A and B are disjoint sets.

$$P(A\cup B)=P(A)+P(B)$$

$$=\frac{1}{18}+\frac{1}{18}=\frac{2}{18}=\frac{1}{9}$$

Q.6 Two dice are thrown. What is the Probability that the sum of the number of dots appearing on them is 4 or 6? ***(RWP 2021, BWP 2023)***

Solution:

When two dots are thrown, the set of possible outcomes is

$$S=\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

$\Rightarrow n(S)=36$

Let A be the event that the sum of number of dots shown is 4.

$A=\{(1,3),(2,2),(3,1)\}$

$\Rightarrow n(A)=3$

$$P(A)=\frac{n(A)}{n(S)}=\frac{3}{36}=\frac{1}{12}$$

Let B be the event that the sum of number of dots shown is 6.

$B=\{(1,5),(2,4),(3,3),(4,2),(5,1)\}$

$\Rightarrow n(B)=5$

$$P(B)=\frac{n(B)}{n(S)}=\frac{5}{36}$$

Since A and B are disjoint sets

$P(A \cup B) = P(A) + P(B)$

$= \frac{1}{12} + \frac{5}{36} = \frac{8}{36} = \frac{2}{9}$

Q.7 Two dice are thrown simultaneously. If the event A is that the sum of the number of dots shown is an odd number and the event B is that the number of dots shown on at least one die is 3. Find $P(A \cup B)$.

Solution:

When two dice are thrown the set of possible outcomes is

$$S = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

$\Rightarrow n(S) = 36$

Let A be the event that the sum of number of dots shown is odd.

$$A = \{(1,2),(1,4),(1,6),(2,1),(2,3),(2,5),(3,2),(3,4),(3,6),(4,1),(4,3),(4,5),(5,2),(5,4),(5,6),(6,1),(6,3),(6,5)\}$$

$\Rightarrow n(A) = 18$

$$P(A) = \frac{n(A)}{n(S)} = \frac{18}{36} = \frac{1}{2}$$

Let B be the event that the number of dots on at least one die is 3.

$$B = \left\{\begin{matrix}(1,3),(2,3),(3,1),(3,2),(3,3),\\(3,4),(3,5),(3,6),(4,3),(5,3),(6,3)\end{matrix}\right\}$$

$\Rightarrow n(B) = 11$

$$P(B) = \frac{n(B)}{n(S)} = \frac{11}{36}$$

Now $A \cap B = \left\{\begin{matrix}(2,3),(3,2),(3,4),\\(3,6),(4,3),(6,3)\end{matrix}\right\}$

$\Rightarrow n(A \cap B) = 6$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

Since, events A and B are overlapping, so

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{2} + \frac{11}{36} - \frac{1}{6}$

$= \frac{18 + 11 - 6}{36} = \frac{23}{36}$

Q.8 There are 10 girls and 20 boys in a class. Half of the boys and half of the girls have blue eyes. Find the Probability that one student chosen as monitor is either a girl or has blue eyes. ***(D.G.K 2022)***

Solution: Number of girls $= 10$

Number of boys $= 20$

Total number of students in a class $= 30$

$\Rightarrow n(S) = 30$

Number of boys having blue eyes $= 10$

Number of girls having blue eyes $= 5$

Total number of students having blue eyes $= 10 + 5 = 15$

One student of the class in chosen as a monitor.

Let A be the event that monitor of class is a girl, so $n(A) = 10$

$$P(A) = \frac{n(A)}{n(S)} = \frac{10}{30} = \frac{1}{3}$$

Let B be the event that monitor of class has blue eyes, so $n(B) = 15$

$$P(B) = \frac{n(B)}{n(S)} = \frac{15}{30} = \frac{1}{2}$$

Now, $A \cap B$ = set of girls with blue eyes

$n(A \cap B) = 5$

And $P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{5}{30} = \frac{1}{6}$

Events are overlapping, so

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{3} + \frac{1}{2} - \frac{1}{6} = \frac{2 + 3 - 1}{6} = \frac{2}{3}$

Independent Events:

Two events **A** and **B** are said to be **independent**. If the occurrence of any one of them does not influence the occurrence of the other event.

Theorem:

Let **A** and **B** the two independent events, then their probability is calculated by:

$$\mathbf{P(A \cap B) = P(A).P(B)}$$

Note:

The formula $P(A \cap B) = P(A).P(B)$ can be generalized as.

$$P(A_1 \cap A_2 \cap A_3 ... \cap A_n) = P(A_1).P(A_2)...P(A_n)$$

Where $A_1, A_2, A_3 ... A_n$ are independent events.

EXERCISE 7.8

Q.1 The Probability that a person A will be alive 15 years hence is $\frac{5}{7}$ and the Probability that another person B will be alive 15 years hence is $\frac{7}{9}$. Find the Probability that both will be alive 15 years hence.

Solution:

Let E_1 be the event that a person A will alive 15 years hence.

$$P(E_1) = \frac{5}{7}$$

Let E_2 be the event that a person will alive 15 years hence.

$$P(E_2) = \frac{7}{9}$$

Events E_1 and E_2 are independent events

Probability that both will alive 15 years hence is

$$P(E_1 \cap E_2) = P(E_1).P(E_2)$$

$$= \frac{5}{7}.\frac{7}{9} = \frac{5}{9}$$

Q.2 A die is rolled twice: Event E_1 is the appearance of even number of dots and Event E_2 is the appearance of more than 4 dots. Prove that:

$$\mathbf{P(E_1 \cap E_2) = P(E_1).P(E_2)}$$

(BWP 2022)

Solution:

When a die is rolled twice, the set of possible outcomes is

$$S = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),$$
$$(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),$$
$$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),$$
$$(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),$$
$$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),$$
$$(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

$$\Rightarrow n(S) = 36$$

E_1 is the event that even number of dots appear.

$$E_1 = \left\{\begin{matrix}(2,2),(2,4),(2,6),(4,2), \\ (4,4),(4,6),(6,2),(6,4),(6,6)\end{matrix}\right\}$$

$$\Rightarrow n(E_1) = 9$$

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{9}{36} = \frac{1}{4}$$

E_2 is the event that more than 4 dots appear.

$$E_2 = \{(5,5),(5,6),(6,5),(6,6)\}$$

$$\Rightarrow n(E_2) = 4$$

$$P(E_2) = \frac{n(E_2)}{n(S)} = \frac{4}{36} = \frac{1}{9}$$

$$P(E_1) \cdot P(E_2) = \frac{1}{4} \cdot \frac{1}{9} = \frac{1}{36} \quad \text{(i)}$$

Now,

$$E_1 \cap E_2 = \{(6,6)\}$$

$$\Rightarrow n(E_1 \cap E_2) = 1$$

$$P(E_1 \cap E_2) = \frac{n(E_1 \cap E_2)}{n(S)} = \frac{1}{36} \quad \text{(ii)}$$

From eq (1) and eq (2), we have

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$$

Hence proved.

Q.3 Determine the Probability of getting 2 heads in two successive tosses of a balanced coin. ***(RWP 2021)***

Solution:

When two coins are tossed, then the set of possible outcomes is

$$S = \{HH, HT, TH, TT\}$$

$$\Rightarrow n(S) = 4$$

Let A be the event of getting two heads, then $n(A) = 1$ Thus, the required Probability is

$$P(A) = \frac{n(A)}{n(S)} = \frac{1}{4}$$

Q.4 Two coins are tossed twice each. Find the Probability that the head appears on the first toss and the same faces appear in the two tosses. ***(SWL 2022)***

Solution:

Two coins are tossed twice each.

For first toss, sample space is

$$S = \{HH, HT, TH, TT\}$$

$$\Rightarrow n(S) = 4$$

Let A be the event that the head appears in first toss.

$$A = \{HH, HT\} \qquad \Rightarrow n(A) = 2$$

and $P(A) = \frac{n(A)}{n(S)} = \frac{2}{4} = \frac{1}{2}$

For second toss, sample space is same.

Let B be the event that the same faces appear in second toss.

$$B = \{HH, TT\} \Rightarrow n(B) = 2$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{2}{4} = \frac{1}{2}$$

A and B are independent events

$$P(A \cap B) = P(A) \cdot P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Q.5 Two cards are drawn form a deck of 52 playing cards. If one card is drawn and replaced before drawing the second card, find the Probability that both the cards are aces.

Solution:

Since there are 52 playing cards, so $n(S) = 52$

Let A be the event that first card is an ace card, then $n(A) = 4$

$$P(A) = \frac{n(A)}{n(S)} = \frac{4}{52} = \frac{1}{13}$$

Let B the event that the second card is also an ace card so $n(B) = 4$

$$P(B) = \frac{n(B)}{n(S)} = \frac{4}{52} = \frac{1}{13}$$

Since A and B are independent events.

$$P(A \cap B) = P(A) \cdot P(B)$$

$$= \frac{1}{13} \cdot \frac{1}{13} = \frac{1}{169}$$

Q.6 Two cards from a deck of 52 playing cards are drawn in such a way that the card is replaced after the first draw. Find the Probabilities in the following cases:

(i) First card is king and the second is a queen.

(ii) Both the cards are faced cards i.e. king, queen, jack.

Solution:

Since there are 52 playing card, so $n(S) = 52$ ($\because$ for first draw)

(i) Let A be the event that the first card is king, then $n(A) = 4$

$$P(A) = \frac{n(A)}{n(S)} = \frac{4}{52} = \frac{1}{13}$$

For second draw, $n(S) = 52$

Let B be the event that the second card is queen, then $n(B) = 4$

$$P(B)=\frac{n(B)}{n(S)}=\frac{4}{52}=\frac{1}{13}$$

A and B are independent events

$$P(A\cap B)=P(A)\cdot P(B)=\frac{1}{13}\cdot\frac{1}{13}=\frac{1}{169}$$

(ii) For first draw, $n(S)=52$

Let A be the event that the first card is faced card, then $n(A)=12$

$$P(A)=\frac{n(A)}{n(S)}=\frac{12}{52}=\frac{3}{13}$$

For second draw, $n(S)=52$

Let B be the event that second card is also faced card, then $n(B)=12$

$$P(B)=\frac{n(B)}{n(S)}=\frac{12}{52}=\frac{3}{13}$$

A and B are independent events

$$P(A\cap B)=P(A)\cdot P(B)$$

$$=\frac{3}{13}\cdot\frac{3}{13}=\frac{9}{169}$$

Q.7 Two dice are thrown twice. What is the Probability that sum of the dots shown in the first throw is 7 and that of the second throw is 11?
(FSD 2022, SWL 2023, GRW 2023)

Solution: Two dice are thrown twice.

For first throw, sample space is

$$S=\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

$$\Rightarrow n(S)=36$$

Let A be the event that sum of dots in first throw is 7.

$$A=\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$

$$\Rightarrow n(A)=6$$

$$P(A)=\frac{n(A)}{n(S)}=\frac{6}{36}=\frac{1}{6}$$

For second throw, sample space is same.

i.e. $n(S)=36$

Let B be the event that sum of dots in the second throw is 11.

$$B=\{(5,6),(6,5)\}$$

$$\Rightarrow n(B)=2$$

$$P(B)=\frac{n(B)}{n(S)}=\frac{2}{36}=\frac{1}{18}$$

A and B are independent events

$$P(A\cap B)=P(A)\cdot P(B)$$

$$=\frac{1}{6}\cdot\frac{1}{18}=\frac{1}{108}$$

Q.8 Find the Probability that the sum of dots appearing in two successive throws of two dice is every time 7.

Solution: Two dice are thrown twice.

For first throw, sample space is

$$S=\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

$$\Rightarrow n(S)=36$$

Let A be the event that sum of dots in first throw is 7.

$$A=\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$

$$\Rightarrow n(A)=6$$

$$P(A)=\frac{n(A)}{n(S)}=\frac{6}{36}=\frac{1}{6}$$

For second throw, sample space is same

i.e. $n(S)=36$

Let B be the event that sum of dots in the second throw is also 7.

$$B=\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$

$$\Rightarrow n(B)=6$$

$$P(B)=\frac{n(B)}{n(S)}=\frac{6}{36}=\frac{1}{6}$$

A and B are independent events

$$P(A\cap B)=P(A)\cdot P(B)$$

$$=\frac{1}{6}.\frac{1}{6}=\frac{1}{36}$$

Q.9 A fair die is thrown twice. Find the Probability that a prime number of dots appear in the first throw and the number of dots in the second throw is less than 5. ***(BWP 2023)***

Solution: A die is thrown twice.

For first throw, sample space is:

$S=\{1,2,3,4,5,6\} \quad \Rightarrow n(S)=6$

Let A be the event that prime number of dots appear in the first throw.

$A=\{2,3,5\} \quad \Rightarrow n(A)=3$

$$P(A)=\frac{n(A)}{n(S)}=\frac{3}{6}=\frac{1}{2}$$

For second throw, sample space is same.

Let B be the event that number of dots in the second throw is less than 5.

$B=\{1,2,3,4\} \quad \Rightarrow n(B)=4$

$$P(B)=\frac{n(B)}{n(S)}=\frac{4}{6}=\frac{2}{3}$$

A and B are independent events

$P(A\cap B)=P(A)\cdot P(B)$

$$=\frac{1}{2}\cdot\frac{2}{3}=\frac{1}{3}$$

Q.10 A bag contains 8 red, 5 white, and 7 black balls. 3 balls are drawn from the bag. What is the Probability that the first ball is red, the second ball is white and the third ball is black, when every time the ball is replaced?

Solution: Number of red balls $=8$

Number of white balls $=5$

Number of black balls $=7$

Total number of balls

$=8+5+7=20$

For first draw, $n(S)=20$

Let A be the event that first ball is red, then $n(A)=8$

$$P(A)=\frac{n(A)}{n(S)}=\frac{8}{20}=\frac{2}{5}$$

For second draw, $n(S)=20$

Let B be the event that second ball is white, then $n(B)=5$

$$P(B)=\frac{n(B)}{n(S)}=\frac{5}{20}=\frac{1}{4}$$

For third draw, $n(S)=20$

Let C be the event that third ball is black, then $n(C)=7$

$$P(C)=\frac{n(C)}{n(S)}=\frac{7}{20}$$

Events A, B and C are independent events, so

$P(A\cap B\cap C)=P(A)\cdot P(B)\cdot P(C)$

$$\frac{2}{5}\cdot\frac{1}{4}\cdot\frac{7}{20}=\frac{7}{200}$$