CHAPTER 8

MATHEMATICAL INDUCTIONS AND BINOMIAL THEOREM

Introduction:

Francesco Maurolico (1494-1575) devised the method of induction and applied this device first to prove that the **sum of the first n odd positive integers equals n^2.**

We are aware of the fact that even one exception or case to a mathematical formula is enough to prove it to be false. Such a case or exception which fails the mathematical formula or statement is called a **counter example.**

For example, we consider the statement $S(n) = n^2 - n + 41$ is a prime number for every natural number n. The values of the expression $n^2 - n + 41$ for some first natural numbers are given in the table as shown below.

n	1	2	3	4	5	6	7	8	9	10	11
$S(n)$	41	43	47	53	61	71	83	97	113	131	151

From the table, it appears that the statement $S(n)$ has enough chance of being true. If we go on trying for the next natural numbers. We find $n = 41$ as a counter example which fails the claim of the above statement. So we conclude that to derive a general formula without proof from some special cases is not a wise step. This example was discovered by **Euler** (1707-1783)

Principle of Mathematical Induction

The principle of mathematical induction is stated as follows:

If a proposition or statement $S(n)$ for each positive integer n is such that

1. $S(1)$ is true i.e., $S(n)$ is true for $n = 1$.

2. $S(k+1)$ is true whenever $S(k)$ is true for any positive integer k,

Then $S(n)$ is true for all positive integers.

Procedure:

1. Substituting $n = 1$, show that the statement is true for $n = 1$.

2. Assuming that the statement is true for any integer k, then show that it is true for the next higher integer.

M1: Starting with one side of $S(k+1)$, its other side is derived by using $S(k)$.

M2: $S(k+1)$ is established by performing algebraic operations on $S(k)$.

Principle of Extended Mathematical Induction:

Let i be an integer. If a formula or statement for $n \geq i$ is such that

1. $S(i)$ is true and

2. $S(k+1)$ is true whenever $S(k)$ is true for integral values of $n \geq i$.

EXERCISE 8.1

Use the mathematical induction to prove the following formulae for every positive integer n. ***(RWP 2021, LHR 2022)***

Q.1 $\mathbf{1+5+9+.....+(4n-3)=n(2n-1)}$

Solution:

Let S(n) be the give statement , i.e.,

$S(n): 1+5+9+.....+(4n-3)=n(2n-1)$ (i)

(i) when $n=1$, Equation (i) becomes;

$S(1): 4(1)-3=1(2\times1-1)$

$S(1): 1=1$

Thus $S(1)$ is true i.e., condition (I) is satisfied.

(ii) Let us assume that $S(n)$ is true for any $n=k\in N$ i.e.,

$1+5+9+.....+(4k-3)=k(2k-1)$ (A)

The statement for $n=k+1$ becomes;

$1+5+9+....+(4k-3)+(4k+1)=(k+1)(2k+1)$ (B)

Adding $(4k+1)$ on both sides of (A) we get;

$1+5+9+.....+(4k-3)+(4k+1)=k(2k-1)+(4k+1)$

$=2k^2-k+4k+1$

$=2k^2+3k+1$

$=2k^2+2k+k+1$

$=2k(k+1)+1(k+1)$

$=(2k+1)(k+1)$

$=(2k+2-1)(k+1)$

$=[2(k+1)-1](k+1)$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition (II) is satisfied. Since both the condition are satisfied, therefore, $S(n)$ is true for all $n\in N$.

Q.2 $\mathbf{1+3+5+.....+(2n-1)=n^2}$ ***(LHR 2022)***

Solution:

Let $S(n)$ be the given statement, i.e.,

$S(n): 1+3+5+.....+(2n-1)=n^2$ (i)

(i) when $n=1$, equation (i) becomes;

$S(1): 2\times1-1=1^2$

$S(1): 1=1$

Thus $S(1)$ is true that is condition (I) satisfied.

(ii) Let us assume that $S(n)$ is true for any $n=k\in N$, i.e.,

$S(k): 1+3+5+....+(2k-1)=k^2$ (A)

The statement for $n=k+1$ becomes ;

$1+3+5+.....+(2k-1)+(2k+1)=(k+1)^2$ (B)

Adding $(2k+1)$ on both sides of (A) we get;

$1+3+5+...+(2k-1)+(2k+1)=k^2+(2k+1)$

$=k^2+2k+1$

$=(k+1)^2$

Thus $S(k+1)$ is true if $S(k)$ is true. So condition (II) is satisfied.

Since both the conditions are satisfied, therefore, $S(n)$ is true for each positive integer n.

Q.3 $\mathbf{1+4+7+.....+(3n-2)=\frac{n(3n-1)}{2}}$ ***(RWP 2022, MTN 2023)***

Solution:

Let $S(n)$ be the given statement, i.e.,

$S(n):\ 1+4+7+.....+(3n-2)=\frac{n(3n-1)}{2}$ (i)

(i) When $n=1, S$ equation (i) becomes;

$S(1): 3(1)-2=\frac{1(3(1)-1)}{2}$

$S(1):\quad 1=1$

Thus $S(1)$ is true, i.e., condition (I) is satisfied

(ii) Let us assume that $S(n)$ is true for any $n=k\in N$, i.e.,

$S(k): 1+4+7+....+(3k-2)=\frac{k(3k-1)}{2}$ (A)

The statement for $n=k+1$ becomes;

$1+4+7+.....+(3k-2)+(3k+1)=\frac{(k+1)(3k+2)}{2}$ (B)

Adding $(3k+1)$ on both sides of equation (A) we get

$S(k): 1+4+7+....+(3k-2)+(3k+1)=\frac{k(3k-1)}{2}+3k+1$

$=\frac{3k^2-k+6k+2}{2}$

$=\frac{3k^2+5k+2}{2}$

$=\frac{3k^2+3k+2k+2}{2}$

$=\frac{3k(k+1)+2(k+1)}{2}$

$=\frac{(k+1)(3k+2+3-3)}{2}$

$=\frac{(k+1)(3(k+1)-1)}{2}$

Hence $S(k+1)$ is true whenever $S(k)$ is true so condition (II) is satisfied.

Since both the conditions are satisfied, therefore $S(n)$ is true for each positive integer n.

Q.4 $\mathbf{1+2+4+.....+2^{n-1}=2^n-1}$ ***(FSD 2021, MTN 2023, LHR 2023)***

Solution:

Let $S(n)$ be the given statement, i.e.

$S(n):\ 1+2+4+....+2^{n-1}=2^n-1$ (i)

(i) when $n=1$, equation (i) becomes;

$S(1):\ 2^{1-1}=2^1-1$

$S(1):\ 2^0=2-1$

$S(1):\ 1=1$

Thus $S(1)$ is true that is condition (I) is satisfied.

(ii) Let us assume that $S(n)$ is true for any $n=k\in N$, i.e.,

$S(k):1+2+4+.....+2^{k-1}=2^k-1$ (A)

The statement for $n=k+1$ becomes;

$1+2+4+....+2^{(k+1)-1}=2^{k+1}-1$ (B)

Adding $2^{(k+1)-1}$ on both sides of (A) we get;

$1+2+4+.....+2^{k-1}+2^{(k+1)-1}=\left(2^k-1\right)+2^{(k+1)-1}$

$=2^k-1+2^k$

$=2.2^k-1$

$=2^{k+1}-1$

$=2^{(k+1)}-1$

Thus $S(k+1)$ is true if $S(k)$ is true, so the condition (II) is satisfied.

Since both the conditions are satisfied, therefore, $S(n)$ is true for each positive integer n.

Q.5 $\mathbf{1+\frac{1}{2}+\frac{1}{4}+....+\frac{1}{2^{n-1}}=2\left(1-\frac{1}{2^n}\right)}$ ***(FSD 2022,GRW 2023)***

Solution:

Let $S(n)$ be the given statement, i.e.,

$S(n)\ :1+\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2^{n-1}}=2\left(1-\frac{1}{2^n}\right)$ (i)

(i) when $n=1$, equation (i) becomes;

$S(1):\frac{1}{2^{1-1}}=2\left(1-\frac{1}{2^1}\right)$

$S(1):\frac{1}{2^0}=2\left(\frac{1}{2}\right)$

$S(1):\ 1=1$

Thus $S(1)$ is true that is condition (I) is satisfied.

(ii) Let us assume that $S(k)$ is true for any $n=k\in N$ that is

$S(k):1+\frac{1}{2}+\frac{1}{4}+....+\frac{1}{2^{k-1}}=\left(1-\frac{1}{2^k}\right)$ (A)

The given statement for $n=k+1$ becomes;

$$1+\frac{1}{2}+\frac{1}{4}+....+\frac{1}{2^{k-1}}+\frac{1}{2^k}=2\left(1-\frac{1}{2^{k+1}}\right) \quad \text{(B)}$$

Adding $\frac{1}{2^k}$ on both sides of (A) we get;

$$1+\frac{1}{2}+\frac{1}{4}+....+\frac{1}{2^{k-1}}+\frac{1}{2^k}=2\left(1-\frac{1}{2^k}\right)+\frac{1}{2^k}$$

$$=2-\frac{2}{2^k}+\frac{1}{2^k}$$

$$=2-\frac{1}{2^k}$$

$$=2\left(1-\frac{1}{2.2^k}\right)$$

$$=2\left(1-\frac{1}{2^{k+1}}\right)$$

Hence $S(k+1)$ is true whenever $S(k)$ is true.

So condition (II) is satisfied.

Since both conditions are satisfied therefore $S(n)$ is true for each positive integer *n*.

Q.6 $\mathbf{2+4+6+.....+2n=n(n+1)}$ ***(GRW 2022, MTN 2023)***

Solution:

Let $S(n)$ be the given statement, i.e.,

$S(n): 2+4+6+...2n=n(n+1)$ (i)

when $n=1$, equation (i) becomes;

(i) $S(1): 2(1)=1(1+1)$

$S(1):\quad 2=2$

Thus $S(1)$ is true that is condition (I) is satisfied.

(ii) Let us assume that $S(n)$ is true for any $n=k\in N$, i.e.,

$S(k):\ 2+4+6+.....+2k=k(k+1)$ (A)

The given statement for $n=k+1$ becomes

$2+4+6+.....+2k+2(k+1)=(k+1)(k+2)$ (B)

Adding $2(k+1)$ on both sides of (A) we get;

$$2+4+6+.....+2k+2(k+1)=k(k+1)+2(k+1)$$

$$=(k+1)(k+2)$$

Hence $S(k+1)$ true whenever $S(k)$ is true. So condition (II) is satisfied.

Since both conditions are satisfied, therefore $S(n)$ is true $\forall n\in N$.

Q.7 $\mathbf{2+6+18+......+2\times 3^{n-1}=3^n-1}$

Solution:

Let $S(n)$ be the given statement, i.e.,

$S(n):=2+6+18+.....+2\times 3^{n-1}=3^n-1$ (i)

(i) when $n=1, S(1)$ becomes;

$S(1):\quad 2\times 3^{1-1}=3^{1}-1$

$S(1):\quad 2=2$

Thus $S(1)$ is true that is condition (I) is satisfied

(ii) Let us assume that $S(k)$ is true for any $n=k\in N$, i.e.,

$S(k):\ 2+6+18+.....+2\times 3^{k-1}=3^{k}-1$ (A)

The given statement for $n=k+1$ becomes;

$2+6+18+.....+2\times 3^{k-1}+2\times 3^{k}=3^{k+1}-1$ (B)

Adding 2×3^{k} on both sides we have

$2+6+18+.....+2\times 3^{k-1}+2\times 3^{k}=3^{k}-1+2\times 3^{k}$

$=3^{k}+2\times 3^{k}-1$

$=3^{k}(1+2)-1$

$=3.3^{k}-1$

$=3^{k+1}-1$

Hence $S(k+1)$ is true whenever $S(k)$ is true. So condition (II) is satisfied.

There fore both condition are satisfied, so $S(n)$ is true $\forall n\in N$

Q.8 $\mathbf{1\times 3+2\times 5+3\times 7+.....+n\times(2n+1)=\dfrac{n(n+1)(4n+5)}{6}}$

Solution:

Let $S(n)$ be the given statement, i.e.,

$S(n):1\times 3+2\times 5+3\times 7+.....+n\times(2n+1)=\dfrac{n(n+1)(4n+5)}{6}$ (i)

(i) When $n=1$, equation (i) becomes;

$S(1)\quad :1\times(2\times 1+1)=\dfrac{1(1+1)(4\times 1+5)}{6}$

$S(1):\quad 3=3.$ thus $S(1)$ is true that is condition (I) is satisfied.

(ii) Let us assume that $S(n)$ is true for any $n=k\in \mathrm{N}$, i.e.,

$S(k):=1\times 3+2\times 5+3\times 7+.....+k\times(2k+1)=\dfrac{k(k+1)(4k+5)}{6}$ (A)

The given statement for $n=k+1$ becomes;

$1\times 3+2\times 5+3\times 7+.....+k\times(2k+1)+(k+1)\times(2k+3)=\dfrac{(k+1)(k+2)(4k+9)}{6}$ (B)

Adding $(k+1)(2k+3)$ in (A) we get;

$1\times 3+2\times 5+3\times 7+.....+k\times(2k+1)+(k+1)\times(2k+3)=\dfrac{k(k+1)(4k+5)}{6}+(k+1)(2k+3)$

$$=(k+1)\left[\frac{k(4k+5)}{6}+(2k+3)\right]$$

$$1\times3+2\times5+3\times7+.....+k(2k+1)+(k+1)(2k+3)=(k+1)\left[\frac{k(4k+5)+6(2k+3)}{6}\right]$$

$$=(k+1)\left[\frac{4k^2+5k+12k+18}{6}\right]$$

$$=(k+1)\left[\frac{4k^2+17k+18}{6}\right]$$

$$=(k+1)\left[\frac{4k^2+8k+9k+18}{6}\right]$$

$$=(k+1)\left[\frac{4k(k+2)+9(k+2)}{6}\right]$$

$$=\frac{(k+1)(k+2)(4k+9)}{6}$$

Which is same as R.H.S of (B)

Hence $S(k+1)$ is true when $S(k)$ is true so condition (II) is satisfied.

Therefore both conditions are satisfied, so $S(n)$ is true $\forall n\in \mathrm{N}$.

Q.9 $\mathbf{1\times2+2\times3+3\times4+.....+n\times(n+1)=\frac{n(n+1)(n+2)}{3}}$

Solution:

Let $S(n)$ be the given statement, i.e.,

$$S(n):1\times2+2\times3+3\times4+.....+n\times(n+1)=\frac{n(n+1)(n+2)}{3} \quad \text{(i)}$$

1. When $n=1$, equation (i) becomes;

$$S(1):1\times(1+1)=\frac{1(1+1)(1+2)}{3}$$

$$S(1): \quad 2=2$$

So statement is true for $n=1$, that is condition (I) is satisfied.

2. Let us assume that statement is true for $n=k\in N$, i.e.,

$$S(k):1\times2+2\times3+....+k(k+1)=\frac{k(k+1)(k+2)}{3} \quad \text{(A)}$$

Give statement for $n=k+1$ becomes;

$$S(k+1):1\times2+2\times3+3\times4+.....+(k+1)\times(k+2)=\frac{(k+1)(k+2)(k+3)}{3} \quad \text{(B)}$$

Adding $(k+1)(k+2)$ both sides of (A) we get;

$$1\times2+2\times3+.....+k(k+1)+(k+1)(k+2)=\frac{k(k+1)(k+2)}{3}+(k+1)(k+2)$$

$$=(k+1)(k+2)\left(\frac{k}{3}+1\right)$$

$$=\frac{(k+1)(k+2)(k+3)}{3}$$

Which is same as R.H.S of (B)

Hence $S(k+1)$ is true if $S(k)$ is true, so condition (II) is satisfied, so $S(n)$ is true $\forall n\in \mathrm{N}$

Q.10 $\mathbf{1\times2+3\times4+5\times6+...+(2n-1)\times2n=\frac{n(n+1)(4n-1)}{3}}$

Solution:

Let $S(n)$ be the given statement, i.e.,

$$S(n):1\times2+3\times4+5\times6+.....+(2n-1)(2n)=\frac{n(n+1)(4n-1)}{3} \quad \text{(i)}$$

1. when n = 1 , equation (i) becomes;

$$S(1):(2\times1-1)(2\times1)=\frac{1(1+1)(4\times1-1)}{3}$$

$$S(1):2=2$$

So statement is true for $n=1$ so conduction (I) is satisfied.

2. Suppose that statement is true for $n=k$, i.e.,

$$S(k):1\times2+3\times4+5\times6+....+(2k-1)(2k)=\frac{k(k+1)(4k-1)}{3} \quad \text{(A)}$$

Given statement for $n=k+1$ becomes;

$$S(k+1):1\times2+3\times4+5\times6+....+(2k-1)(2k)+(2k+1)(2k+2)=\frac{(k+1)(k+2)(4k+3)}{3} \quad \text{(B)}$$

Adding $(2k+1)(2k+2)$ on both sides of (A) we get :

$$1\times2+3\times4+5\times6+....+(2k-1)(2k)+(2k+1)(2k+2)$$

$$=\frac{k(k+1)(4k-1)}{3}+(2k+1)(2k+2)$$

$$=\frac{k(k+1)(4k-1)}{3}+(2k+1)2(k+1)$$

$$=(k+1)\left[\frac{k(4k-1)}{3}+2(2k+1)\right]$$

$$=(k+1)\left[\frac{4k^2-k+12k+6}{3}\right]$$

$$=(k+1)\left[\frac{4k^2+11k+6}{3}\right]$$

$$=(k+1)\left[\frac{4k^2+8k+3k+6}{3}\right]$$

$$=(k+1)\left[\frac{4k(k+2)+3(k+2)}{3}\right]$$

$$=\frac{(k+1)(k+2)(4k+3)}{3}$$

Hence $S(k+1)$ is true if $S(k)$ is true, so condition (II) is satisfied, so $S(n)$ is true $\forall n \in N$

Q.11 $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+...+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$

Solution:

Let $S(n)$ be the given statement, i.e.,

$S(n):\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+......+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$ (i)

1. when n = 1, equation (i) becomes;

$S(1):\frac{1}{1(1+1)}=1-\frac{1}{1+1}$

$S(1):\frac{1}{2}=\frac{1}{2}$

So statement is true for $n=1$ so condition (I) is satisfied

2. Suppose that statement is true for $n=k$

$S(k):\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+.......+\frac{1}{k(k+1)}=1-\frac{1}{k+1}$ (A)

Given statement for $n=k+1$ becomes

$S(k+1):\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+.....+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=1-\frac{1}{k+2}$ (B)

Adding $\frac{1}{(k+1)(k+2)}$ on both sides of (A) we get ;

$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+.....+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=1-\frac{1}{(k+1)}+\frac{1}{(k+1)(k+2)}$

$=1-\left[\frac{(k+2)-1}{(k+1)(k+2)}\right]$

$=1-\left[\frac{k+1}{(k+1)(k+2)}\right]$

$=1-\frac{1}{k+2}$

Which is same as R.H.S of (B)

Hence $S(k+1)$ is true if $S(k)$ is true, so condition (II) is satisfied, so $S(n)$ is true $\forall n\in \mathrm{N}$

Q.12 $\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+......+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

Solution:

Let $S(n)$ be the given statement, i.e.,

$$S(n):\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+......+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1} \quad \text{(i)}$$

1. when $n=1$, equation (i) becomes,

$$S(1):\frac{1}{(2(1)-1)(2(1)+1)}=\frac{1}{2(1)+1}$$

$$S(1):\frac{1}{3}=\frac{1}{3}$$

Thus statement is true for $n=1$, so condition (I) is satisfied.

2. Suppose that statement is true for $n=k$, i.e.,

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+......+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1} \quad \text{(A)}$$

Given statement for $n=k+1$ becomes

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+......+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}=\frac{(k+1)}{(2k+3)} \quad \text{(B)}$$

Adding $\frac{1}{(2k+1)(2k+3)}$ on both sides of (A) we get ;

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+......+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{k}{(2k+1)}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{1}{(2k+1)}\left[k+\frac{1}{2k+3}\right]$$

$$=\frac{1}{(2k+1)}\left[\frac{2k^2+3k+1}{2k+3}\right]$$

$$2k^2+2k+k+1$$
$$\because 2k(k+1)+1(k+1)$$
$$(k+1)(2k+1)$$

$$=\frac{1}{(2k+1)}\left[\frac{2k(k+1)+1(k+1)}{2k+3}\right]$$

$$=\frac{1}{(2k+1)}\left[\frac{(2k+1)(k+1)}{2k+3}\right]$$

$$=\frac{k+1}{2k+3}$$

Which is same as R.H.S of (B)

Hence $S(k+1)$ is true if $S(k)$ is true, so condition (II) is satisfied, so $S(n)$ is true $\forall n \in \mathbb{N}$

Q.13 $$\frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+......+\frac{1}{(3n-1)(3n+2)}=\frac{n}{2(3n+2)}$$

Solution:

Let $S(n)$ be the given statement, i.e.

$$S(n):\ \frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+......+\frac{1}{(3n-1)(3n+2)}=\frac{n}{2(3n+2)} \quad \text{(i)}$$

1. when $n=1$, equation (i) becomes;

$$S(1):\ \frac{1}{(2)(5)}=\frac{1}{(2)(5)}$$

So $S(1)$ is true, so condition (I) is satisfied

2. Suppose that given statement is true for $n=k$, i.e.,

$$S(k):\frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+......+\frac{1}{(3k-1)(3k+2)}=\frac{k}{2(3k+2)} \quad \text{(A)}$$

Given statement for $n=k+1$ becomes

$$S(k+1):\frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+......+\frac{1}{(3k-1)(3k+2)}+\frac{1}{(3k+2)(3k+5)}=\frac{k+1}{2(3k+5)} \quad \text{(B)}$$

Adding $\frac{1}{(3k+2)(3k+5)}$ on both sides of (A) we get ;

$$\frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+.....+\frac{1}{(3k-1)(3k+2)}+\frac{1}{(3k+2)(3k+5)}$$

$$=\frac{k}{2(3k+2)}+\frac{1}{(3k+2)(3k+5)}$$

$$\frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+.....+\frac{1}{(3k-1)(3k+2)}+\frac{1}{(3k+2)(3k+5)}=\frac{1}{(3k+2)}\left[\frac{k}{2}+\frac{1}{3k+5}\right]$$

$$=\frac{1}{(3k+2)}\left[\frac{k(3k+5)+2}{2(3k+5)}\right]$$

$$=\frac{1}{(3k+2)}\left[\frac{3k^2+5k+2}{2(3k+5)}\right]$$

$$=\frac{1}{(3k+2)}\left[\frac{3k^2+3k+2k+2}{2(3k+5)}\right]$$

$$=\frac{1}{(3k+2)}\left[\frac{3k(k+1)+2(k+1)}{2(3k+5)}\right]$$

$$=\frac{1}{(3k+2)}\left[\frac{(3k+2)(k+1)}{2(3k+5)}\right]$$

$$=\frac{(k+1)}{2(3k+5)}$$

Which is same as R.H.S of (B)

Hence $S(k+1)$ is true if $S(k)$ is true, so condition (II) is satisfied, so $S(n)$ is true $\forall n\in \mathrm{N}$

Q.14 $\mathbf{r+r^2+r^3+......+r^n=\frac{r(1-r^n)}{(1-r)},(r\neq 1)}$

Solution:

Let $S(n)$ be the given statement, i.e.,

$$S(n): r+r^2+r^3+......+r^n=\frac{r(1-r^n)}{(1-r)} \qquad \text{(i)}$$

1. when $n=1$, equation (i) becomes;

$$S(1): r^1=\frac{r(1-r^1)}{(1-r)}$$

$$S(1): r=r$$

Thus S(1) is true, so condition (I) is satisfied.

2. Suppose that given statement is true for $n=k$, i.e.,

$$S(k): r+r^2+r^3+.....+r^k=\frac{r(1-r^k)}{(1-r)} \qquad \text{(A)}$$

For $n=k+1$ given statement becomes;

$$S(k+1): r+r^2+r^3+.....+r^k+r^{k+1}=\frac{r(1-r^{k+1})}{(1-r)} \qquad \text{(B)}$$

Adding r^{k+1} on both sides of (A) we get ;

$$r+r^2+r^3+...+r^k+r^{k+1}=\frac{r(1-r^k)}{(1-r)}+r^{k+1}=r\left(\frac{(1-r^k)}{1-r}+r^k\right)$$

$$=r\left(\frac{1-r^k+r^k(1-r)}{(1-r)}\right)$$

$$=r\left(\frac{1-r^k+r^k-r^{k+1}}{(1-r)}\right)$$

$$=\frac{r(1-r^{k+1})}{(1-r)}$$

Which is same as R.H.S of (B)

Hence $S(k+1)$ is true if $S(k)$ is true, so condition (II) is satisfied, so $S(n)$ is true $\forall n\in \mathrm{N}$

Q.15 $\mathbf{a+(a+d)+(a+2d)+...+(a+(n-1)d)=\frac{n}{2}[2a+(n-1)d]}$

Solution:

Let $S(n)$ be the given statement, i.e.,

$S(n):\ a+(a+b)+(a+2d)+...+(a+(n-1)d)$

$=\frac{n}{2}\left[2a+(n-1)d\right]$ (i)

1. when $n=1$, equation (i) becomes,

$S(1):\ \left[a+(1-1)d\right]=\frac{1}{2}\left[2a+(1-1)d\right]$

$S(1):a=\frac{1}{2}(2a)$

$S(1):a=a$

Thus S(1) is true, so condition (I) is satisfied.

2. Suppose that given statement is true for $n=k$, i.e.,

$S(k):a+(a+d)+(a+2d)+...+(a+(k-1)d)$

$=\frac{k}{2}\left[2a+(k-1)d\right]$ (A)

For $n=(k+1)$ given statement becomes;

$S(k+1):a+(a+d)+(a+2d)+...+(a+(k-1)d)+(a+kd)$

$=\frac{(k+1)}{2}\left[2a+kd\right]$ (B)

Adding $(a+kd)$ on both sides of (A) we get ;

$a+(a+d)+(a+2d+...+)(a+(k-1)d)+(a+kd)$

$=\frac{k}{2}\left[2a+(k-1)d\right]+(a+kd)$

$a+(a+d)+(a+2d)+...+(a+(k-1)d)+(a+kd)$

$=ka+\frac{k}{2}(k-1)d+a+kd$

$=ka+a+\frac{k}{2}(k-1)d+kd$

$=a(k+1)+kd\left(\frac{(k-1)}{2}+1\right)$

$=a(k+1)+kd\left(\frac{k+1}{2}\right)$

$=\frac{(k+1)}{2}(2a+kd)$

Hence $S(k+1)$ is true if $S(k)$ is true, so condition (II) is satisfied, so $S(n)$ is true $\forall n\in \mathrm{N}$

Q.16 $\mathbf{1|\underline{1}+2|\underline{2}+3|\underline{3}+...+n|\underline{n}=|\underline{n+1}-1}$

Solution:

Let $S(n)$ be the given statement, i.e.,

$S(n): 1.|\underline{1}+2.|\underline{2}+3.|\underline{3}+...+n|\underline{n}=|\underline{n+1}-1$ (i)

1. when $n=1$, equation (i) becomes;

$S(1):\ 1|\underline{1}=|\underline{1+1}-1$

$S(1):\ 1=|\underline{2}-1$

$S(1):\ 1=2-1$ $\because 2!=2$

$S(1):\ 1=1$

Thus S(1) is true, so condition (I) is satisfied.

2. Suppose that given statement is true for $n=k\in N$, i.e.,

$S(k): 1|\underline{1}+2|\underline{2}+3|\underline{3}+...+k|\underline{k}=|\underline{k+1}-1$ (A)

Given statement for $n=k+1$ becomes ;

$S(k+1): 1|\underline{1}+2|\underline{2}+3|\underline{3}+...+k|\underline{k}+(k+1)|\underline{(k+1)}=|\underline{k+2}\ -1$ (B)

Adding $(k+1).|\underline{(k+1)}$ on both sides of (A) we get ;

$1|\underline{1}+2|\underline{2}+3|\underline{3}+...+k|\underline{k}+(k+1)|\underline{(k+1)}=|\underline{k+1}-1+(k+1)|\underline{(k+1)}$

$=|\underline{(k+1)}-1+(k+1)|\underline{(k+1)}$

$=|\underline{(k+1)}\left[1+(k+1)\right]-1$

$=|\underline{(k+1)}(k+2)-1$

$=(k+2).|\underline{(k+1)}-1$

$=|\underline{(k+2)}-1$

Which is Same as R.H.S of (B)

Hence $S(k+1)$ is true if $S(k)$ is true, so condition (II) is satisfied, so $S(n)$ is true $\forall n\in \mathrm{N}$

Q.17 $a_n=a_1+(n-1)d$ when $a_1, a_1+d, a_1+2d, ...$ are in A.P.

Solution:

Let $S(n)$ be the given statement, i.e.,

$S(n):\ a_n=a_1+(n-1)d$ (i)

1. when $n=1$, equation (i) becomes;

$S(1): a_1=a_1+(1-1)d$

$S(1): a_1=a_1$

Thus S(1) is true, so condition (I) is satisfied.

2. Suppose that given statement is true for $n=k\in N$, i.e.,

$S(k): a_k=a_1+(k-1)d$ (A)

So given statement for n = k+1 becomes

$S(k): a_{k+1} = a + kd$ (B)

Adding d on both sides of (A) we get ;

$a_k + d = a_1 + (k-1)d + d$

$= a_1 + kd - d + d$

$= a_1 + kd =$ R.H.S of (B)

Hence $S(k+1)$ is true if $S(k)$ is true, so condition (II) is satisfied, so $S(n)$ is true $\forall n \in N$

Q.18 $a_n = a_1 r^{n-1}$ when $a_1, a_1 r, a_1 r^2, \ldots$ form a G.P.

Solution:

Let $S(n)$ be the given statement, i.e.,

$S(n): \; a_n = a_1 r^{n-1}$ (i)

1. when $n = 1$, equation (i) becomes;

$S(1): a_1 = a_1 r^{1-1}$

$S(1): a_1 = a_1$

Thus S(1) is true, so condition (I) is satisfied.

2. Suppose that given statement is true for $n = k$, i.e.,

$S(k): a_k = a_1 r^{k-1}$ (A)

So given statement for $n = k+1$ becomes

$S(k+1): a_{k+1} = a_1 r^k$ (B)

Multiply r on both sides of (A) we get ;

$r.a_k = a_1 r^{k-1}.r$

$a_{k+1} = ar^k$

Which is right hand side of (B)

Hence $S(k+1)$ is true if $S(k)$ is true, so condition (II) is satisfied, so $S(n)$ is true $\forall n \in N$

Q.19 $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \dfrac{n(4n^2-1)}{3}$

Solution:

Let $S(n)$ be the given statement, i.e.,

$S(n): \; 1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \dfrac{n(4n^2-1)}{3}$ (i)

1. when $n = 1$, equation (i) becomes;

$S(1): (2\times 1 - 1)^2 = \dfrac{1(4\times 1^2 - 1)}{3}$

$S(1): 1 = 1$

Thus S(1) is true, so condition (I) is satisfied.

2. Suppose that given statement is true for $n = k$, i.e.,

$$S(k): 1^2+3^2+5^2+...+(2k-1)^2=\frac{k(4k^2-1)}{3} \quad (A)$$

So given statement for $n=k+1$ becomes

$$S(k+1): 1^2+3^2+5^2+...+(2k-1)^2+(2k+1)^2=\frac{(k+1)\left(4(k+1)^2-1\right)}{3}$$

$$=\frac{(k+1)(4k^2+8k+3)}{3} \quad (B)$$

Adding $(2k+1)^2$ on both sides of (A) we get;

$$1^2+3^2+5^2+...+(2k-1)^2+(2k+1)^2=\frac{k(4k^2-1)}{3}+(2k+1)^2$$

$$=\frac{k(4k^2-1)+3(2k+1)^2}{3}$$

$$=\frac{k((2k-1)(2k+1))}{3}+(2k+1)^2$$

$$=(2k+1)\left[\frac{k(2k-1)}{3}+(2k+1)\right]$$

$$=(2k+1)\left[\frac{(2k-1)+3(2k+1)}{3}\right]$$

$$=(2k+1)\left[\frac{2k^2-k+6k+3}{3}\right]$$

$$=(2k+1)\left[\frac{2k^2+5k+3}{3}\right]$$

$$=(2k+1)\frac{\left[2k^2+2k+3k+3\right]}{3}$$

$$=\frac{(2k+1)(2k+3)(k+1)}{3}$$

$$=\frac{(k+1)\left[4k^2+8k+3\right]}{3}$$

Which is right hand side of (B)

Hence $S(k+1)$ is true if $S(k)$ is true, so condition (II) is satisfied, so $S(n)$ is true $\forall n \in N$

Q.20 $\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+...+\binom{n+2}{3}=\binom{n+3}{4}$

Solution:

Let $S(n)$ be the given statement, i.e.,

$$S(n):\ \binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\ldots+\binom{n+2}{3}=\binom{n+3}{4} \qquad \text{(i)}$$

1. when $n=1$, equation (i) becomes;

$$S(1):\binom{1+2}{3}=\binom{1+3}{4}$$

$$S(1):\binom{3}{3}=\binom{4}{4}$$

$$S(1):1=1$$

Thus S(1) is true, so condition (I) is satisfied.

2. Suppose that given statement is true for $n=k$, i.e.,

$$S(k):\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\ldots+\binom{k+2}{3}=\binom{k+3}{4} \quad \text{(A)}$$

So given statement for n = k+1 becomes

$$S(k+1):\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\ldots+\binom{k+2}{3}+\binom{k+3}{3}=\binom{k+4}{4} \quad \text{(B)}$$

Adding $\binom{k+3}{3}$ on both sides of (A) we get ;

$$\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\ldots+\binom{k+2}{3}+\binom{k+3}{3}=\binom{k+3}{3}+\binom{k+3}{4}$$

$$=\binom{k+4}{4} \qquad \because {}^{n}C_{r-1}+{}^{n}C_{r}={}^{n+1}C_{r}$$

Which is right hand side of (B)

Hence $S(k+1)$ is true if $S(k)$ is true, so condition (II) is satisfied, so $S(n)$ is true $\forall n\in \mathbb{N}$

Q.21 Prove by the mathematical induction that for all positive integral value of n.

(i) n^2+n is divisble by 2.

Solution:

Let $S(n)=n^2+n$ be the given statement, i.e.,

$$S(n):n^2+n \quad \text{(i)}$$

1. when $n=1$, equation (i) becomes;

$S(1)=1^2+1=2$ that is divisible by 2.

Thus S(1) is true, so condition (I) is satisfied.

2. Suppose that given statement is true for $n=k$, i.e.,

$S(k):k^2+k$ is divisible by 2, that is $\frac{k^2+k}{2}=Q$ where Q is Quotient,

i.e., $k^2+k=2Q$ (A)

So given statement for n = k+1 becomes

$$S(k+1): \quad (k+1)^2+(k+1) \quad \text{(B)}$$

$$=k^2+1+2k+k+1$$

$=\left(k^2+k\right)+\left(2k+2\right)$

$=2Q+2\left(k+1\right)$ by using A

$=2\left[Q+\left(k+1\right)\right]$

= Which is divisible by 2.

Hence $S\left(k+1\right)$ is true if $S\left(k\right)$ is true, so condition (II) is satisfied, so $S\left(n\right)$ is true $\forall n\in \mathrm{N}$

(ii) $5^n - 2^n$ is divisible by 3.

Solution:

Let $S(n)$ be the given statement, i.e.,

$S\left(n\right):\ 5^n-2^n$ (i)

1. when $n=1$, equation (i) becomes;

$S(1):\ 5^1-2^1$

$S\left(1\right):\ 3$ that is divisible by 3

Thus $S\left(1\right)$ is true, so condition (I) is satisfied.

2. Suppose that given statement is true for $n=k$, i.e.,

$S\left(k\right):\ 5^k-2^k$ is divisible by 3, so

$\frac{5^k-2^k}{3}=Q$ where Q is Quotient, i.e.,

$5^k-2^k=3Q$ (A)

Next we have to show that statement is also true for $n=k+1$, that is we have to show that $S\left(k+1\right)=5^{k+1}-2^{k+1}$ is also divisible by 3.

So consider

$5^{k+1}-2^{k+1}=5^k.5-2^{k+1}$

$=5\left(3Q+2^k\right)-2^{k+1}$ $\because$ from(A) $5^k=3Q+2^k$

$=15Q+5.2^k-2^{k+1}$

$=15Q+5.2^k-2^k.2$

$=15Q+2^k\left(5-2\right)$

$=15Q+3.2^k$

$=3\left[5Q+2^k\right]$ that is divisible by $3\forall k\in N$.

Thus statement is true for $n=k+1$ when $S\left(k\right)$ is true. So condition (II) is satisfied hence result is true $\forall n\in N$.

(iii) $5^n - 1$ is divisible by 4

Solution

Let the given statement is $S\left(n\right)$ i.e.,

$S\left(n\right):\ 5^n-1$ (i)

1. when n = 1 then equation (i) becomes

$S(1):\ 5^1 - 1 = 4$ which is divisible by 4

So the condition (I) is satisfied

2. Let the statement is true for $n = k$ i.e.,

$S(k):\ 5^k - 1^k$ is divisible by 4 i.e.,

$\dfrac{5^k - 1}{4} = Q$ where Q is the Quotient

$\Rightarrow 5^k - 1 = 4Q$ (A)

Now we have to show that statement is also true for $n = k + 1$ i.e.,

$S(k+1):\quad 5^{k+1} - 1$ is also divisible by 4.

So consider

$5^{k+1} - 1 = 5.5^k - 1$

$= 5(4Q + 1) - 1$

$= 20Q + 5 - 1$ $\quad \because$ *from*(A)

$= 20Q + 4$ $\quad 5^k = 4Q + 1$

$= 4(5Q + 1)$ which is divisible by 4

Thus $S(k+1)$ is true whenever $S(k)$ is true.

Hence result is true $\forall n \in N$.

(iv) $8\times10^n-2$ is divisible by 6.

Solution:

Let the given statement is $S(n)$ i.e.,

$S(n):\ 8\times10^n-2$ (i)

1. For $n=1$, equation (i) becomes

$S(1): 8\times10^1-2=78$ that is divisible by 6.

2. Suppose that given statement is true for $n=k$, i.e.,

$S(k):\ 8\times10^k-2$ is divisible by 6, i.e.,

$\frac{8\times10^k-2}{6}=Q$ where Q is Quotient

$8\times10^k-2=6Q$ (A)

Now we have to show that statement is also true for $n=k+1$, i.e.,

$S(k+1):\quad 8\times10^{k+1}-2$ is also divisible by 6.

So consider

$8\times10^{k+1}-2=8\times10^k\times10-2$

$=10(6Q+2)-2$

$\because$ from(A)

$=60Q+20-2$ $\quad 8\times10^k=6Q+2$

$=60Q+18$

$=6(10Q+3)$ that is divisible by 6.

Thus $S(k+1)$ is true whenever $S(k)$ is true. Hence $S(n)$ is true $\forall n\in N$.

(v) n^3-n is divisible by 6

Solution:

Let $S(n)$ be the given statement i.e.,

$S(n):\ n^3-n$ (i)

1. When $n=1$ then equation (i) will become ;

$S(1):1^3-1=0$ which is divisible by 6.

2. Suppose that given statement is true for $n=k$ i.e.,

$S(k):k^3-k$ is divisible by 6, so

$\frac{k^3-k}{6}=Q$ where Q is quotient

$k^3-k=6Q$ (A)

Now we have to show that statement is also true for $n=k+1$ i.e.,

$S(k+1):(k+1)^3-(k+1)$

So consider

$(k+1)^3-(k+1)=k^3+1+3k^2+3k-k-1$

$= \left(k^3 - k\right) + 3k^2 + 3k$

$= 6Q + 3\left(k^2 + k\right)$ $\quad \because k^3 - k = 6Q$

$\therefore x^2 + x$ is divisible by 2

$= 6Q + 3k\left(k+1\right)$

$= 6Q + 3\left(2p\right)$ $\quad \because k \in N$, so

$= 6Q + P$ $\quad k\left(k+1\right) =$ an even integer

$= 6\left(Q + P\right)$ which is divisible by 6

Thus $S\left(k+1\right)$ is true whenever $S\left(k\right)$ is true so given statement is true $\forall n \in N$.

Q.22 $\quad \mathbf{\frac{1}{3} + \frac{1}{3^2} + ... + \frac{1}{3^n} = \frac{1}{2}\left[1 - \frac{1}{3^n}\right]}$

Solution:

Let S(n) be the given statement i.e.,

$$S\left(n\right): \frac{1}{3} + \frac{1}{3^2} + ... + \frac{1}{3^n} = \frac{1}{2}\left[1 - \frac{1}{3^n}\right] \quad \text{(i)}$$

1. when $n = 1$, equation (i)becomes ;

$$S\left(1\right): \frac{1}{3^1} = \frac{1}{2}\left[1 - \frac{1}{3^1}\right]$$

$$\frac{1}{3} = \frac{1}{2}\left[\frac{2}{3}\right]$$

$$\frac{1}{3} = \frac{1}{3}$$

So S(1) is true so condition (I) is satisfied.

2. Suppose that given statement is true for $n = k$ i.e.,

$$S\left(k\right): \frac{1}{3} + \frac{1}{3^2} + ... + \frac{1}{3^k} = \frac{1}{2}\left[1 - \frac{1}{3^k}\right] \quad \text{(A)}$$

Given statement for $n = k+1$ becomes ;

$$S\left(k+1\right): \frac{1}{3} + \frac{1}{3^2} + ... + \frac{1}{3^k} + \frac{1}{3^{k+1}} = \frac{1}{2}\left[1 - \frac{1}{3^{k+1}}\right] \quad \text{(B)}$$

Adding $\frac{1}{3^{k+1}}$ on both side of (A) , we get :

$$\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + ... + \frac{1}{3^k} + \frac{1}{3^{k+1}} = \frac{1}{2}\left[1 - \frac{1}{3^k}\right] + \frac{1}{3^{k+1}}$$

$$= \frac{1}{2} - \frac{1}{2}.\frac{1}{3^k} + \frac{1}{3}.\frac{1}{3^k}$$

$$= \frac{1}{2} - \left(\frac{1}{2} - \frac{1}{3}\right).\frac{1}{3^k}$$

$$= \frac{1}{2} - \frac{1}{6}.\frac{1}{3^k}$$

$$=\frac{1}{2}-\frac{1}{2}.\frac{1}{3}.\frac{1}{3^k}$$

$$=\frac{1}{2}\left[1-\frac{1}{3^{k+1}}\right]$$

Which is right hand side of (B)

Thus $S(k+1)$ is true whenever $S(k)$ is true. So condition (II) is satisfied, so S(n) is true $\forall n\in N$.

Q.23 $\mathbf{1^2-2^2+3^2-4^2+...+(-1)^{n-1}.n^2=\frac{(-1)^{n-1}.n(n+1)}{2}}$

Solution:

Let the given statement is $S(n)$, i.e.,

$$S(n):1^2-2^2+3^2-4^2+...+(-1)^{n-1}.n^2=\frac{(-1)^{n-1}.n(n+1)}{2} \quad \text{(i)}$$

1. when $n=1$ then equation (i) becomes ;

$$S(1):(-1)^{1-1}.(1)^2=\frac{(-1)^{1-1}.(1)(1+1)}{2}$$

$$S(1):1=2$$

$$S(1):1=1$$

So S(1) is true and condition (I) is satisfied

2. Suppose that given statement is true for n = k i.e.,

$$S(k):1^2-2^2+3^2-4^2+...+(-1)^{k-1}.(k)^2=\frac{(-1)^{k-1}.k(k+1)}{2} \quad \text{(A)}$$

Given statement for $n=k+1$ becomes

$$S(k+1):1^2-2^2+3^2-4^2+...+(-1)^{k}(k+1)^2=\frac{(-1)^{k}.(k+1)(k+2)}{2} \quad \text{(B)}$$

By adding

$(-1)^k(k+1)^2$ on both sides of (A) we get ;

$$1^2-2^2+3^2-4^2+...+(-1)^{k-1}k^2+(-1)^k(k+1)^2=\frac{(-1)^{k-1}.k(k+1)}{2}+(-1)^k(k+1)^2$$

$$=(-1)^k(k+1)\left[\frac{(-1)^{-1}.k}{2}+(k+1)\right]$$

$$=(-1)^k(k+1)\left[\frac{-k}{2}+k+1\right]$$

$$=(-1)^k(k+1)\left[\frac{-k+2k+2}{2}\right]$$

$$=(-1)^k(k+1)\left[\frac{k+2}{2}\right]$$

$$=\frac{(-1)^k(k+1)(k+2)}{2}$$

Which is right hand side of (B)

Thus $S(k+1)$ is true whenever $S(k)$ is true. So condition (II) is satisfied, so $S(n)$ is true $\forall n\in N$.

Q.24 $\mathbf{1^3+3^3+5^3+...+(2n-1)^3=n^2(2n^2-1)}$

Solution:

Let the given statement is $S(n)$, i.e.,

$$S(n)=1^3+3^3+5^3+...+(2n-1)^3=n^2(2n^2-1) \quad \text{(i)}$$

1. When n = 1then equation (i) becomes

$S(1):\quad (2(1)-1)^3=1^2(2(1)^2-1)$

$S(1):\qquad 1=1$

Thus S(1) is true, so condition (I) is satisfied

2. Suppose that statement is true for n = k i.e.,

$$S(k):1^3+3^3+5^3+...+(2k-1)^3=k^2(2k^2-1) \quad \text{(A)}$$

Given statement for $n=k+1$ becomes ;

$$S(k+1):1^3+3^5+5^3+...+(2k-1)^3+(2k+1)^3=(k+1)^2(2(k+1)^2-1)$$

$$=(k+1)^2(2k^2+4k+1) \quad \text{(B)}$$

Adding $(2k+1)^3$ on both sides of (A) we get;

$$1^3+3^3+5^3+...+(2k-1)^3+(2k+1)^3=k^2(2k^2-1)+(2k+1)^3$$

$$=2k^4-k^2+(8k^3+1+12k^2+6k)$$

$$=2k^4+8k^3+11k^2+6k+1$$

$$=2k^4+2k^3+6k^3+6k^2+5k^2+5k+k+1$$

$$=2k^3(k+1)+6k^2(k+1)+5k(k+1)+(k+1)$$

$$=(k+1)[2k^3+6k^2+5k+1]$$

$$=(k+1)[2k^3+2k^2+4k^2+4k+k+1]$$

$$=(k+1)[2k^2(k+1)+4k(k+1)+(k+1)]$$

$$=(k+1)^2[2k^2+4k+1]$$

Which is right hand side of (B)

Thus $S(k+1)$ is true whenever $S(k)$ is true. So condition (II) is satisfied

Hence S(n) is true $\forall n\in N$.

Q.25 $\mathbf{(x+1)}$ **is factor of** $\mathbf{x^{2n}-1;(x\neq -1)}$

Solution:

Let $S(n)$ be the given statement i.e.,

$S(n):\ x^{2n}-1$ (i)

1. When n = 1, then equation (i) becomes

$S(1): x^{2(1)}-1 = x^2-1$ which is divisible by $x+1$

So (1) is true, so condition (I) is satisfied.

2. Suppose that statement is true for n = k that is S(k) is divisible by $(x+1)$. So

$\frac{x^{2k}-1}{x+1}=Q$ where Q is Quotient

$x^{2k}-1=Q(x+1)$ (A)

Now we show that statement is also true for $n=k+1$ i.e.,

$S(k+1)$ is also divisible by $(x+1)$.

consider $S(k+1): x^{2(k+1)}-1$

So

$x^{2k+2}-1 = x^{2k}.x^2-1$

$= x^2\left[Q(x+1)+1\right]-1$ $\because$ from(A) $x^{2k}=Q(x+1)+1$

$= x^2.Q.(x+1)+(x^2-1)$

$= x^2.Q.(x+1)+(x-1)(x+1)$

$=(x+1)\left[x^2.Q+(x-1)\right]$

Which is divisible by $(x+1)$.

Thus $S(k+1)$ is true whenever $S(k)$ is true, so condition (II) is satisfied. Hence $S(n)$ is true $\forall n \in N$.

Q.26 $(\mathbf{x-y})$ is a factor of $\mathbf{x^n - y^n};(\mathbf{x \neq y})$

Solution:

Let $S(n)$ be the given statement i.e.,

$S(n): x^n - y^n$ (i)

1. When n = 1 then equation (i) becomes

$S(1): x^1 - y^1$ which is divisible by $(x-y)$.

So $S(1)$ is true and condition (I) is satisfied.

2. Suppose that statement is true $n=k$ i.e.,

$S(k): x^k - y^k$ is divisible by $x-y$

i.e.,

$\frac{x^k-y^k}{x-y}=Q$ where Q is Quotient

$x^k - y^k = Q(x-y)$ (A)

Now we show that statement is also true for $n=k+1$ i.e.,

$S(k+1)$ is also divisible by $(x+1)$

So consider

$$S(k+1): x^{k+1}-y^{k+1} = x^k.x-y^{k+1}$$

$$= x\left[Q(x-y)+y^k\right]-y^{k+1} \quad \because \text{from (A)} \quad x^k = Q(x-y)+y^k$$

$$= x.Q(x-y)+x.y^k-y^k.y$$

$$= x.Q(x-y)+y^k(x-y)$$

$$= (x-y)\left[x.Q+y^k\right]$$

Which is divisible by $(x-y)$. thus $S(k+1)$ is true. Whenever $S(k)$ is true. So condition (II) is satisfied, so the given statement is true $\forall n \in N$.

Q.27 $\mathbf{(x+y)}$ **is a factor of** $\mathbf{x^{2n-1}+y^{2n-1}(x \neq -y)}$

Solution:

Let $S(n)$ be the given statement i.e.,

$S(n): (x+y)$ is a factor of $x^{2n-1}+y^{2n-1}$

1. When $n=1$ then $S(n)$ becomes

$S(1)$: $x^{2(1)-1}+y^{2(1)-1}=x+y$ so which is divisible by $x+y$ so $S(1)$ is true and condition (I) is satisfied.

2. Suppose that statement is true for $n=k$ that is

$$\frac{x^{2k-1}+y^{2k-1}}{x+y}=Q$$

$$x^{2k-1}+y^{2k-1}=Q(x+y) \qquad \text{(A)}$$

Now we show that statement is also true for $n=k+1$ i.e.,

So consider

$$x^{2k+1}+y^{2k+1}=x^{2k-1}.x^2+y^{2k-1}.y^2$$

$$=x^2\left[Q(x+y)-y^{2k-1}\right]+y^{2k-1}.y^2 \qquad \because \text{ from (A)} \quad x^{2k-1}=Q(x+y)-y^{2k-1}$$

$$=x^2.Q(x+y)-x^2.y^{2k-1}+y^{2k-1}y^2$$

$$=x^2Q(x+y)-y^{2k-1}\left[x^2-y^2\right]$$

$$=x^2.Q(x+y)-y^{2k-1}(x-y)(x+y)$$

$$=(x+y)\left[x^2.Q-(x-y).y^{2k-1}\right]$$

Which is divisible by $(x+y)$ thus $S(k+1)$ is true whenever S(k) is true so condition (II) is satisfied.

Hence $S(n)$ is true $\forall n \in N$.

Q.28 Use mathematical induction to show that

$\mathbf{1+2+2^2+...+2^n=2^{n+1}-1}$ **for all non-negative integers n.**

Solution:

Let $S(n)$ be the given statement i.e.,

$S(n): 1+2+2^2+...+2^n = 2^{n+1}-1$ (i)

1. When $n=0$ then equation (i)becomes

$S(0): 2^0 = 2^{0+1}-1$ Here $\because n \in W$

$S(0): 1 = 2^1 - 1$

$S(0): 1 = 1$

So $S(0)$ is true, so condition (I) is satisfied

2. Suppose that statement is true for n = k i.e.,

$S(k): 1+2+2^2+2^3+...+2^k = 2^{k+1}-1$ (A)

Now we show that statement is also true for $n=k+1$ i.e.,

$S(k+1): 1+2+2^2+2^3+...+2^k+2^{k+1} = 2^{k+2}-1$ (B)

In order to prove (B) we add 2^{k+1} on both sides of (A) we get

$1+2+2^2+2^3+...+2^k+2^{k+1} = 2^{k+1}-1+2^{k+1}$

$= 2.2^{k+1}-1$

$= 2^{k+2}-1$

Which is right hand side of (B)

Thus $S(k+1)$ is true whenever S(k) is true so condition (II) is satisfied.

Hence $S(n)$ is true $\forall n \in N$.

Q.29 If A and B are square matrices and $AB = BA$, then show by mathematical Induction that $AB^n = B^nA$ for any positive integer n.

Solution:

Let $S(n)$ be the given statement i.e.,

$S(n): AB^n = B^nA$ (i)

1. When $n=1$ then $S(n)$ becomes

$S(1): AB^1 = B^1A$

$S(1): AB = BA$

So $S(1)$ is true, so condition (I) is satisfied

2. Suppose that statement is true for $n = k$ i.e.,

$S(k): AB^k = B^kA$ (A)

Now we show that statement is also true for $n=k+1$ i.e.,

$S(k+1): AB^{k+1} = B^{k+1}A$ (B)

Multiply matrix B from left side with (A) we get

$B.(AB^k) = B.B^kA$

$(BA)B^k = B^{k+1}A$ By associative law

$(AB)B^k = B^{k+1}A$ AB= BA (Given)

$A(BB^k) = B^{k+1}A$ By associative law

$AB^{k+1} = B^{k+1}A$

So $S(k+1)$ is true whenever $S(k)$ is true, so condition (II) is satisfied, hence $S(n)$ is true $\forall n \in N$.

Q.30 Prove by the principle of mathematical induction that n^2-1 is divisible by 8, when n an odd integer.

Solution:

Let $S(n)$ be the given statement i.e.,

$S(n): n^2-1$ is divisible by 8 (i)

1. When $n=1$ then equation (i) becomes

$S(1) = 1^2 - 1$

$S(1) = 0$ which is divisible by 8

So $S(1)$ is true, so condition (I) is satisfied

2. Suppose that statement is true for $n = k$ i.e.,

$S(k): \dfrac{k^2-1}{8} = Q \Rightarrow k^2 - 1 = 8Q$ (A)

Where Q is quotient.

Now we show that statement is also true for $n = k+2$ i.e.,

$S(k+2): (k+2)^2 - 1$ (B)

So consider,

$$(k+2)^2 - 1 = k^2 + 4k + 4 - 1$$
$$= (k^2-1) + (4k+4)$$
$$= 8Q + 4(k+1)$$
$$= 8Q + 4(2p) \qquad \therefore k \in O,\ p \in N$$
$$= 8[Q+p]$$

Which is divisible by 8

So $S(k+2)$ is true whenever $S(k)$ is true, so condition (II) is satisfied, hence $S(n)$ is true $\forall n \in N$.

Q.31 Use the principle of mathematical induction to prove that $\ln x^n = n \ln x$ for any positive integer $n \geq 0$ if x is positive integer.

Solution:

Let $S(n)$ be the given statement i.e.,

$S(n): \ln x^n = n \ln x$ (i)

1. When $n=0$ then equation (i) becomes

$S(0): \ln x^0 = 0.\ln x$

$S(0): \ln 1 = 0$

$S(0): 0 = 0$

So $S(0)$ is true, so condition (I) is satisfied

2. Suppose that statement is true for $n = k$ i.e.,

$S(k): \ln x^k = k.\ln x$ (A)

Now we show that statement is also true for $n = k+1$ i.e.,

$S(k+1): \ln x^{k+1} = (k+1).\ln x$ (B)

So in order to prove (B) adding $\ln x$ on both sides of (A) we get;

$\ln x^k + \ln x = k.\ln x + \ln x$

$\ln(x^k.x) = (k+1)\ln x$

$\ln x^{k+1} = (k+1)\ln x$

So $S(k+1)$ is true whenever $S(k)$ is true, so condition (II) is satisfied, hence $S(n)$ is true $\forall n \in N$.

Use the Principle of extended mathematical Induction to prove that

Q.32 $n! > 2^n - 1$ **for integral values of** $n \geq 4$

Solution:

Let $S(n)$ be the given statement i.e.,

$S(n): n! > 2^n - 1$ (i)

1. When $n = 4$ then $S(n)$ becomes

$S(4): 4! > 2^4 - 1$

$S(4): 24 > 15$

So $S(4)$ is true, so condition (I) is satisfied

2. Suppose that statement is true for $n = k$ i.e.,

$S(k): \; k! > 2^k - 1$ (A)

Now we show that statement is also true for $n = k+1$ i.e.,

$S(k+1): (k+1)! > 2^{k+1} - 1$ (B)

Multiply $(k+1)$ on both sides of (A) we get ;

$(k+1).k! > (k+1)(2^k - 1)$

$(k+1)! > 2(2^k - 1) \qquad \therefore k > 4$

$(k+1)! > 2^{k+1} - 2 \qquad \because k+1 > 2$

$(k+1)! > (2^{k+1} - 1) - 1$

$(k+1)! > 2^{k+1} - 1$

So $S(k+1)$ is true whenever $S(k)$ is true, so condition (II) is satisfied, hence $S(n)$ is true $\forall n \geq 4. n \in N$.

Q.33 $\mathbf{n^2 > n + 3}$ **for integral values of** $n \geq 3$

Solution:

Let $S(n)$ be the given statement i.e.,

$S(n): n^2 > n + 3$ (i)

1. When $n = 3$ then $S(n)$ becomes

$S(3): 3^2 > 3 + 3$

$S(3): 9 > 6$

So $S(3)$ is true, so condition (I) is satisfied

2. Suppose that statement is true for $n = k$ i.e.,

$S(k): \; k^2 > k + 3$ where $k \geq 3$ (A)

Now we show that statement is also true for $n = k+1$ i.e.,

$S(k+1): (k+1)^2 > k + 4$ (B)

Adding $2k+1$ in (A) on both sides

$k^2 + 2k + 1 > k + 3 + 2k + 1$

$$(k+1)^2 > (k+4) + 2k$$

$$(k+1)^2 > (k+4) \qquad \therefore k \geq 3$$

So 2k is positive integer, so by neglecting 2k L.H.S. become more larger.

So $S(k+1)$ is true whenever $S(k)$ is true, so condition (II) is satisfied, hence $S(n)$ is true $\forall n \geq 3, n \in N$.

Q.34 $4^n > 3^n + 2^{n-1}$ for integral values of $n \geq 2$.

Solution:

Let $S(n)$ be the given statement i.e.,

$$S(n): 4^n > 3^n + 2^{n-1} \qquad \text{(i)}$$

1. When $n = 2$ then $S(n)$ becomes

$$S(2): 4^2 > 3^2 + 2^{2-1}$$

$$S(2): 16 > 9 + 2$$

$$S(2): 16 > 11$$

So $S(2)$ is true, so condition (I) is satisfied

2. Suppose that statement is true for $n = k$ i.e.,

$$S(k):\ 4^k > 3^k + 2^{k-1} \qquad \text{(A)}$$

Now we show that statement is also true for $n = k+1$ i.e.,

$$S(k+1): 4^{k+1} > 3^{k+1} + 2^{k+1-1}$$

$$S(k+1): 4^{k+1} > 3^{k+1} + 2^k \qquad \text{(B)}$$

In order to prove (2) we multiply (A) by 4 on both sides we get ;

$$4.4^k > 4\left(3^k + 2^{k-1}\right)$$

$$4^{k+1} > 4.3^k + 4.2^{k-1}$$

$$4^{k+1} > (3+1).3^k + (2+2)2^{k-1}$$

$$4^{k+1} > \left(3.3^k + 2^k\right) + \left(3^k + 2^k\right)$$

$$4^{k+1} > 3^{k+1} + 2^k$$

$3^k + 2^k$ is always positive so by neglecting it, L.H.S become more larger.

So $S(k+1)$ is true whenever $S(k)$ is true, so condition (II) is satisfied, hence $S(n)$ is true $\forall n \geq 2, n \in N$.

Q.35 $3^n < n!$ for integral values of $n > 6$.

Solution:

Let $S(n)$ be the given statement i.e.,

$$S(n): 3^n < n! \qquad \text{(i)}$$

1. When $n > 6$, suppose then for $n = 7$ then $S(n)$ becomes

$$S(7): 3^7 < 7!$$

$$S(7): 2187 < 5040$$

So $S(7)$ is true, so condition (I) is satisfied

2. Suppose that statement is true for $n = k$ i.e.,

$S(k):\ 3^k < k!$ (A)

Now we show that statement is also true for $n = k+1$ i.e.,

$S(k+1): 3^{k+1} < (k+1)!\ \forall\ k > 6$ (B)

In order to prove (1) we multiply (k+1) on both sides of (A) we get ;

$(k+1)3^k < (k+1)k! \qquad \because k+1 > 0$

$3.3^k < (k+1)3^k < (k+1)! \qquad \because k+1 > 3$

$\Rightarrow 3^{k+1} < (k+1)!$

Which is required as in the equation B

So $S(k+1)$ is true whenever $S(k)$ is true, so condition (II) is satisfied, hence $S(n)$ is true $\forall n > 6.$ where $n \in N.$

Q.36 $n! > n^2$ for integral value of $n \geq 4.$

Solution:

Let $S(n)$ be the given statement i.e.,

$S(n): n! > n^2$ (i)

1. When $n = 4$ then $S(n)$ becomes

$S(4): 4! > 4^2 \Rightarrow 24 > 16$

So $S(4)$ is true, so condition (I) is satisfied

2. Suppose that statement is true for $n = k$ i.e.,

$S(k):\ k! > k^2$ (A)

Now we show that statement is also true for $n = k+1$ i.e.,

$S(k+1): (k+1)! > (k+1)^2$ (B)

In order to prove (2) we multiple $(k+1)$ on both sides of (A), we get;

$(k+1)k! > (k+1)k^2 \quad \therefore k^2 > k+1 \quad \forall k \geq 4$

$(k+1)! > (k+1)(k+1)$

$(k+1)! > (k+1)^2$

So $S(k+1)$ is true whenever $S(k)$ is true, so condition (II) is satisfied, hence $S(n)$ is true $\forall n \geq 4.$ where $n \in N.$

Q.37 $3+5+7+...+(2n+5) = (n+2)(n+4)$ for integral values of $n \geq -1.$

Solution: Let $S(n)$ be the given statement i.e.,

$S(n): 3+5+7+...+(2n+5) = (n+2)(n+4)$ (i)

1. When $n = -1$ then equation (i) becomes

$S(-1): 2(-1)+5 = (-1+2)(-1+4)$

$S(-1): 3 = (1)(3)$

$S(-1): 3 = 3$

So $S(-1)$ is true, so condition (I) is satisfied

2. Suppose that statement is true for $n = k$ i.e.

$S(k):\ 3+5+7+...+(2k+5)=(k+2)(k+4)$ (A)

Now we show that statement is also true for $n=k+1$ i.e.,

$S(k+1): 3+5+7+...+(2k+5)+(2k+7)=(k+3)(k+5)$ (B)

In order to prove (B) we add $(2k+7)$ on both sides of (A) we get ;

$$3+5+7+...+(2k+5)+(2k+7)=(k+2)(k+4)+(2k+7)$$
$$=k^2+6k+8+2k+7$$
$$=k^2+8k+15$$
$$=k^2+5k+3k+15$$
$$=k(k+5)+3(k+5)$$
$$=(k+3)(k+5)$$

So $S(k+1)$ is true whenever $S(k)$ is true, so condition (II) is satisfied, hence $S(n)$ is true $\forall n \geq -1,$ where $n \in Z$.

Q.38 $1+nx \leq (1+x)^n$ **for** $n \geq 2$ **and** $x > -1$**.**

Solution: Let $S(n)$ be the given statement i.e.,

$S(n): 1+nx \leq (1+x)^n$ (i)

1. When $n \geq 2$ then for $n=2$, $S(n)$ becomes

$S(2): 1+2x \leq (1+x)^2$

$S(2): 1+2x \leq 1+x^2+2x$

So $S(2)$ is true, so condition (I) is satisfied

2. Suppose that statement is true for $n = k$ i.e.,

$S(k):\ 1+kx \leq (1+x)^2$ (A)

Now we show that statement is also true for $n=k+1$ i.e.,

$S(k+1): 1+(k+1)x \leq (1+x)^{k+1}$ (B)

In order to prove (B) we multiply $(x+1)$ on both sides of (A) we get ;

$(1+kx)(1+x) \leq (1+x)(1+x)^k$

$1+x+kx+kx^2 \leq (1+x)^{k+1}$

$1+(k+1)x+kx^2 \leq (1+x)^{k+1}$

$1+(k+1)x \leq (1+x)^{k+1}$

So $S(k+1)$ is true whenever $S(k)$ is true, so condition (II) is satisfied, hence $S(n)$ is true $\forall n \geq 2.$ where $n \in N$.

Binomial Theorem:

An algebraic expression consisting of two terms such as $a+x,\ x-2y,\ ax+b$ etc. is called a binomial or a binomial expression e.g.

$(a+x)^2 = a^2 + 2ax + x^2$ (i)

$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3$ (ii)

The right side of (i) and (ii) are called binomial expansions of binomial $a+x$ for the indices 2 and 3 respectively.

In general,

$$(a+x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \dots + \binom{n}{r-1}a^{n-(r-1)}x^{r-1} + \binom{n}{r}a^{n-r}x^r + \dots + \binom{n}{n-1}ax^{n-1} + \binom{n}{n}x^n$$

Or

$$(a+x)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}x^r$$

Where a and x are real numbers

In the expansion of $(a+x)^n$ following points can be observed.

1. The number of terms in the expansion is one greater than its index.
2. The sum of exponents of a and x in each term of the expansion is equal to its index.
3. The exponent of a decreases from index to zero.
4. The exponent of x increases from zero to index.
5. The coefficients of the terms equidistant from beginning and end of the expansion are equal as $\binom{n}{r} = \binom{n}{n-r}$
6. The $(r+1)^{th}$ term in the expansion is $\binom{n}{r}a^{n-r}x^r$ and we denote it as T_{r+1}

i.e.,

$$T_{r+1} = \binom{n}{r}a^{n-r}x^r$$

Middle term in the expansion of $(a+x)^n$

In the expansion of $(a+x)^n$, the total number of terms are $n+1$.

Case-I

(n is even)

If n is even then n+1 is odd,

So $\left(\frac{n}{2}+1\right)^{th}$ term will be the only middle term in the expansion.

Case-II

(n is odd)

If n is odd then $n+1$ is even,

So $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms of the expansion will be the two middle terms.

Note:

The sum of coefficients in the expansion of $(1+x)^n$ is 2^n.

The sum of odd coefficients of binomial expansion = The sum of its even coefficients of binomial expansion = 2^{n-1}.

EXERCISE 8.2

Q.1 Using binomial theorem to expand the following

(i) $(\mathbf{a}+\mathbf{2b})^5$

Solution:

$$= {}^5C_0(a)^5(2b)^0 + {}^5C_1(a)^4(2b)^1 + {}^5C_2(a)^3(2b)^2 + {}^5C_3(a)^2(2b)^3 + {}^5C_4(a)^1(2b)^4 + {}^5C_5(a)^0(2b)^5$$

$$= a^5 + 5.a^4.2b + \frac{20}{2}.a^3.4b^2 + 10.a^2.8b^3 + 5.a^1.16b^4 + 32b^5$$

$$= a^5 + 10a^4b + 40a^3b^2 + 80a^2b^3 + 80ab^4 + 32b^5$$

(ii) $\left(\frac{\mathbf{x}}{\mathbf{2}} - \frac{\mathbf{2}}{\mathbf{x^2}}\right)^6$

Solution:

$$= {}^6C_0\left(\frac{x}{2}\right)^6\left(-\frac{2}{x^2}\right)^0 + {}^6C_1\left(\frac{x}{2}\right)^5\left(-\frac{2}{x^2}\right)^1 + {}^6C_2\left(\frac{x}{2}\right)^4\left(-\frac{2}{x^2}\right)^2 + {}^6C_3\left(\frac{x}{2}\right)^3\left(-\frac{2}{x^2}\right)^3$$

$$+ {}^6C_4\left(\frac{x}{2}\right)^2\left(-\frac{2}{x^2}\right)^4 + {}^6C_5\left(\frac{x}{2}\right)^1\left(-\frac{2}{x^2}\right)^5 + {}^6C_6\left(\frac{x}{2}\right)^0\left(-\frac{2}{x^2}\right)^6$$

$$= \frac{x^6}{64} + 6\left(\frac{x^5}{32}\right)\left(-\frac{2}{x^2}\right) + 15\left(\frac{x^4}{16}\right)\left(\frac{4}{x^4}\right) + 6\left(\frac{x}{2}\right)\left(\frac{-32}{x^{10}}\right) + \left(\frac{64}{x^2}\right) + 20\left(\frac{x^3}{8}\right)\left(-\frac{8}{x^6}\right) + 15\left(\frac{x^2}{4}\right)\left(\frac{16}{x^8}\right)$$

$$= \frac{x^6}{64} - \frac{3}{8}x^3 + \frac{15}{4} - \frac{20}{x^3} + \frac{60}{x^6} - \frac{96}{x^9} + \frac{64}{x^{12}}$$

(iii) $\left(\mathbf{3a} - \frac{\mathbf{x}}{\mathbf{3a}}\right)^4$

Solution:

$$= {}^6C_0(3a)^4\left(\frac{-x}{3a}\right)^0 + {}^4C_1(3a)^3\left(\frac{-x}{3a}\right)^1 + {}^4C_2(3a)^2\left(\frac{-x}{3a}\right)^2 + {}^4C_3(3a)^1\left(\frac{-x}{3a}\right)^3 + {}^4C_3(3a)^0\left(\frac{-x}{3a}\right)^4$$

$$= (81a^4) + 4(27a^3)\left(\frac{-x}{3a}\right) + 6(9a^2)\left(\frac{x^2}{9a^2}\right) + 4(3a)\left(\frac{-x^3}{27a^3}\right) + \frac{x^4}{81a^4}$$

$$= 81a^4 - 36a^2x + 6x^2 - \frac{4x^3}{9a^2} + \frac{x^4}{81a^4}$$

(iv) $\left(\frac{\mathbf{x}}{\mathbf{2y}} - \frac{\mathbf{2y}}{\mathbf{x}}\right)^8$

Solution:

$$= {}^8C_0\left(\frac{x}{2y}\right)^8\left(\frac{-2y}{x}\right)^0 + {}^8C_1\left(\frac{x}{2y}\right)^7\left(\frac{-2y}{x}\right)^1 + {}^8C_2\left(\frac{x}{2y}\right)^6\left(\frac{-2y}{x}\right)^2 + {}^8C_3\left(\frac{x}{2y}\right)^5\left(\frac{-2y}{x}\right)^3$$

$$+ {}^8C_4\left(\frac{x}{2y}\right)^4\left(\frac{-2y}{x}\right)^4 + {}^8C_5\left(\frac{x}{2y}\right)^3\left(\frac{-2y}{x}\right)^5 + {}^8C_6\left(\frac{x}{2y}\right)^2\left(\frac{-2y}{x}\right)^6 + {}^8C_7\left(\frac{x}{2y}\right)^1\left(\frac{-2y}{x}\right)^7 + {}^8C_8\left(\frac{x}{2y}\right)^0\left(\frac{-2y}{x}\right)^8$$

$$=\frac{x^8}{256y^8}+8\left(\frac{x^7}{128y^7}\right)\left(\frac{-2y}{x}\right)+28\left(\frac{x^6}{64y^6}\right)\left(\frac{4y^2}{x^2}\right)+56\left(\frac{x^5}{32y^5}\right)\left(\frac{-8y^3}{x^3}\right)+28\left(\frac{x^2}{4y^2}\right)\left(\frac{64y^6}{x^6}\right)$$

$$+8\left(\frac{x}{2y}\right)\left(\frac{-128y^7}{x^7}\right)+\left(\frac{256y^8}{x^8}\right)$$

$$=\frac{x^8}{256y^8}-\frac{x^6}{8y^6}+\frac{7x^4}{4y^4}-\frac{14x^2}{y^2}+70-224\frac{y^2}{x^2}+448\frac{y^4}{x4}-512\frac{y^6}{x^6}+\frac{256y^8}{x^8}$$

$$=\frac{x^8}{256y^8}-\frac{x^6}{8y^6}+\frac{7x^4}{4y^4}-\frac{14x^2}{y^2}+70-224\frac{y^2}{x^2}+448\frac{y^4}{x^4}-512\frac{y^6}{x^6}+\frac{256y^8}{x^8}$$

(v) $\left(\sqrt{\frac{\mathbf{a}}{\mathbf{x}}}-\sqrt{\frac{\mathbf{x}}{\mathbf{a}}}\right)^{6}$

Solution:

$$={}^6C_0\left(\sqrt{\frac{a}{x}}\right)^6\left(-\sqrt{\frac{x}{a}}\right)^0+{}^6C_1\left(\sqrt{\frac{a}{x}}\right)^5\left(-\sqrt{\frac{x}{a}}\right)^1+{}^6C_2\left(\sqrt{\frac{a}{x}}\right)^4\left(-\sqrt{\frac{x}{a}}\right)^2+{}^6C_3\left(\sqrt{\frac{a}{x}}\right)^3\left(\sqrt{\frac{x}{a}}\right)^3$$

$$+{}^6C_4\left(\sqrt{\frac{a}{x}}\right)^2\left(-\sqrt{\frac{x}{a}}\right)^4+{}^6C_5\left(\sqrt{\frac{a}{x}}\right)^1\left(-\sqrt{\frac{x}{a}}\right)^5+{}^6C_6\left(\sqrt{\frac{a}{x}}\right)^0\left(-\sqrt{\frac{x}{a}}\right)^6$$

$$=\frac{a^3}{x^3}+6\left(\frac{a^{\frac{5}{2}}}{x^{\frac{5}{2}}}\right)-\left(\frac{x^2}{a^{\frac{5}{2}}}\right)+15\left(\frac{a}{x}\right)^2\left(\frac{x}{a}\right)^1+20\left(\frac{a}{x}\right)^{\frac{3}{2}}\left(\frac{-x}{3}\right)^{\frac{3}{2}}+15\left(\frac{a}{x}\right)\left(\frac{x}{a}\right)^2+6\left(\frac{a}{x}\right)^{\frac{1}{2}}\left(\frac{-x}{a}\right)^{\frac{5}{2}}+\left(\frac{x}{a}\right)^3$$

$$=\frac{a^3}{x^3}-\frac{6a^2}{x^2}+15\frac{a}{x}-20+15\frac{x}{a}-6\frac{x^2}{a^2}+\frac{x^3}{a^3}$$

Q.2 Calculate the following by means of binomial theorem:

(i) $(0.97)^3$

Solution:

$(0.97)^3$

$=(1-0.03)^1$

$={}^3C_0(1)^3(-0.03)^0+{}^3C_1(1)^2(-0.03)^1+{}^3C_2(1)(-0.03)^2+{}^3C_3(1)^0(-0.03)^3$

$=1-.09+.0027+.000027$

$=1.0027$

(ii) $(2.02)^4$

Solution:

$(2.02)^4$

$=(2+0.02)^4$

$= {}^4C_0(2)^4(0.02)^0 + {}^4C_1(2)^3(0.02)^1 + {}^4C_2(2)^2(0.02)^2 + {}^4C_3(2)^1(0.02)^3 + {}^4C_4(2)^0(0.02)^4$

$= 1\times 16 + 4(8)(0.02) + 6(4)(0.0004) + 4(2)(0.000008) + 1(0.00000016)$

$= 16 + 0.64 + 0.0096 + 0.000064 + 0.00000016$

$= 16.64966416$

(iii) $(9.98)^4$

Solution:

$(9.98)^4$

$= (10-0.02)^4$

$= {}^4C_0(10)^4(-0.02)^0 + {}^4C_1(10)^3(-0.02)^1 + {}^4C_2(10)^2(-0.02)^2 + {}^4C_3(10)^1(-0.02)^3 + {}^4C_4(10)^0(-0.02)^4$

$= 1\times 10000 + 4(1000)(-0.02) + 6(100)(0.0004) - 4(10)(0.000008) + 1\times(0.00000016)$

$= 10000 - 80 + 0.24 + 0.00032 + 0.00000016$

$= 9920.228968$

(iv) $(21)^5$

Solution:

$(21)^5$

$= (20+1)^5$

$= {}^5C_0(20)^5(1) + {}^5C_1(20)^4(1)^1 + {}^5C_2(20)^3(1)^2 + {}^5C_3(20)^2(1)^3 + {}^5C_4(20)^1(1)^4 + {}^5C_5(20)^0(1)^5$

$= (3200000) + 5(160000) + 10(8000) + 10(400) + 5(20) + 1\times 1$

$= 3200000 + 800000 + 80000 + 4000 + 100 + 1$

$= 4084101$

Q.3 Expand and simply the followings.

(i) $(a+\sqrt{2}x)^4 + (a-\sqrt{2}x)^4$

Solution:

$(a+\sqrt{2}x)^4 = {}^4C_0 a^4(\sqrt{2x})^0 + {}^4C_1(a^3)(\sqrt{2x})^1 + {}^4C_2(a^2)(\sqrt{2x})^2 + {}^4C_3(a)(\sqrt{2x})^3 + {}^4C_4(a)^0(\sqrt{2x})^4$ (i)

Similarly,

$(a-\sqrt{2x})^4 = {}^4C_o a^4(\sqrt{2x})^0 - {}^4C_1(a^3)(\sqrt{2x})^1 + {}^4C_2(a^2)(\sqrt{2x})^2 - {}^4C_3(a)(\sqrt{2x})^3 + {}^4C_4(a)^0(\sqrt{2x})^4$ (ii)

Adding (i) & (ii) we get;

$(a+\sqrt{2}x)^4 + (a-\sqrt{2}x)^4 = 2({}^4C_0 a^4(\sqrt{2}x)^0 + {}^4C_2(a^2)(\sqrt{2}x)^2 + {}^4C_4(a)^0(\sqrt{2}x^4))$

$= 2a^4 + 24a^2x^2 + 8x^4$

(ii) $\mathbf{(2+\sqrt{3})^5 + (2-\sqrt{3})^5}$

Solution:

$$\left(2+\sqrt{3}\right)^5 = {}^5C_0\left(2\right)^5\left(\sqrt{3}\right)^0 + {}^5C_1\left(2\right)^4\left(\sqrt{3}\right)^1 + {}^5C_2\left(2\right)^3\left(\sqrt{3}\right)^2$$
$$+{}^5C_3\left(2\right)^2\left(\sqrt{3}\right)^3 + {}^5C_4\left(2\right)^1\left(\sqrt{3}\right)^4 + {}^5C_5\left(2\right)^0\left(\sqrt{3}\right)^5 \quad \text{(i)}$$

Similarly,

$$\left(2-\sqrt{3}\right)^5 = {}^5C_0\left(2\right)^5\left(\sqrt{3}\right)^0 - {}^5C_1\left(2\right)^4\left(\sqrt{3}\right)^1 + {}^5C_2\left(2\right)^3\left(\sqrt{3}\right)^2$$
$$-{}^5C_3\left(2\right)^2\left(\sqrt{3}\right)^3 + {}^5C_4\left(2\right)^1\left(\sqrt{3}\right)^4 - {}^5C_5\left(2\right)^0\left(\sqrt{3}\right)^5 \quad \text{(ii)}$$

Adding (i) and (ii) we get;

$$\left(2+\sqrt{3}\right)^5 + \left(2-\sqrt{3}\right)^5 = 2\left({}^5C_0\left(2\right)^5\left(3\right)^0 + {}^5C_2\left(2\right)^3\left(\sqrt{3}\right)^2 + {}^5C_4\left(2\right)^1\left(\sqrt{3}\right)^4\right.$$
$$= 2\left(32+10\times8\times3+5\times2\times9\right)$$
$$= 64+480+180$$
$$= 724$$

(iii) $\boldsymbol{\left(2+i\right)^5-\left(2-i\right)^5}$

Solution:

$$\left(2+i\right)^5 = {}^5C_0\left(2^5\right)\left(i\right)^0 + {}^5C_1\left(2^4\right)\left(i\right) + {}^5C_2\left(2\right)\left(i\right)^2$$
$$+{}^5C_3\left(2\right)^2\left(i\right)^3 + {}^5C_4\left(2\right)^1\left(i\right)^4 + {}^5C_5\left(2\right)^0\left(i\right)^5 \quad \text{(i)}$$

Similarly;

$$\left(2-i\right)^5 = {}^5C_0\left(2\right)^5\left(i\right)^0 - {}^5C_1\left(2\right)^4\left(i\right) + {}^5C_2\left(2\right)^3\left(i\right)^2$$
$$-{}^5C_3\left(4\right)\left(i\right)^3 + {}^5C_4\left(2\right)\left(i\right)^4 + {}^5C_5\left(2\right)^0\left(i\right)^5$$

Subtracting (ii) from (i)

$$\left(2+i\right)^5 - \left(2-i\right)^5 = 2\left({}^5C_1\left(2\right)^4\left(i\right) + {}^5C_3\left(2\right)^2\left(i\right)^3 + {}^5C_5\left(2\right)^0\left(i\right)^5\right)$$
$$= 2\left(5\times16i+10\times4\left(-i\right)+\left(+i\right)\right)$$
$$= 2\left(80i-40i+i\right)$$
$$= 2\left(41i\right)$$
$$= 82i$$

(iv) $\boldsymbol{\left(x+\sqrt{x^2-1}\right)^3+\left(x-\sqrt{x^2-1}\right)^3}$

Solution:

$$\left(x+\sqrt{x2-1}\right)^3$$
$$= {}^3C_0\,x^3\left(\sqrt{x^2-1}\right)^0 + {}^3C_1\left(x\right)^2\left(\sqrt{x-1}\right)^1 + {}^3C_2\left(x\right)^1\left(\sqrt{x^2-1}\right)^2 + {}^3c_3\left(x\right)^0\left(\sqrt{x^2-1}\right)^3 \quad \text{(i)}$$

Similarly,

$$\left(x-\sqrt{x^2}-1\right)^3$$

$$= {}^3C_0 x^3 \left(\sqrt{x^2-1}\right)^0 + {}^3C_1 (x)^2 \left(\sqrt{x^2-1}\right)^1 + {}^3C_2 (x)^1 \left(\sqrt{x^2-1}\right)^2 - {}^3C_3 (x)^0 \left(\sqrt{x^2}-1\right)^3 \quad \text{(ii)}$$

Adding (i) and (ii) we get;

$$\left(x+\sqrt{x^2-1}\right)^3 + \left(x-\sqrt{x^2-1}\right)^3 = 2\left({}^3C_0\, x^3 \left(\sqrt{x^2-1}\right)^0 + {}^3C_2 \left(x^1\right)\left(\sqrt{x^2}-1\right)^2\right)$$

$$= 2\left(x^3 + 3x^1\left(x^2-1\right)\right)$$

$$= 2\left(x^3 + 3x^3 - 3x\right)$$

$$= 2\left(4x^3 - 3x\right)$$

$$= 2x\left(4x^2 - 3\right)$$

Q.4 Expand the following in ascending power of x.

(i) $\left(2+x-x^2\right)^4$

Solution:

$\left(2+x-x^2\right)^4$

Put $2+x=a$

then

$$\left(a-x^2\right)^4 = {}^4C_0 (a)^4 \left(-x^2\right)^0 + {}^4C_1 (a)^3 \left(-x^2\right)^1 + {}^4C_2 (a)^2 \left(-x^2\right)^2 + {}^4C_3 (a)^1 \left(-x^2\right)^3 + {}^4C_4 (a)^0 \left(-x^2\right)^4$$

$$= a^4 - 4a^3x^2 + 6a^2x^4 - 4ax^6 + x^8$$

Put $a = 2+x$ back we get;

$$\left(2+x-x^2\right)^4 = (2-x)^4 - 4x^2(2+x)^3 + 6x^4(2+x)^2 - 4x^6(2+x) + x^8$$

$$= \left[{}^4C_0 2^4 + {}^4C_1 (2)^3 x + {}^4C_2 (2)^2 x^2 + {}^4C_3 (2)^1 x^3 + {}^4C_4{}^0 x^4\right]$$

$$-4x^2\left[{}^3C_0 2^3 + {}^3C_1\left(2^2\right)(x) + {}^3C_2\left(2^1\right)\left(x^2\right) + {}^3C_3 x^3\right] + 6x^4\left(4+4x+x^2\right) - 4x^6(2+x) + x^8$$

$$= \left(16+32x+24x^2+8x^3+x^4\right) - 4x^2\left(8+12x+6x^2+x^3\right)$$

$$+6x^4\left(4+4x+x^2\right) - 4x^6(2+x) + x^8$$

$$= 16+32x-8x^2-40x^3+x^4+20x^5-2x^6-4x^7+x^8$$

(ii) $\mathbf{\left(1-x+x^2\right)^4}$

Solution:

Let $(1-x)=a$ then

$$\left(1-x+x^2\right)^4 = \left(a+x^2\right)^4$$

$$\left(1-x+x^2\right)^4 = \left(a+x^2\right)^4$$

$$= {}^4C_0 (a)^4 + {}^4C_1 (a)^3 \left(x^2\right)^1 + {}^4C_2 (a)^2 \left(x^2\right)^2 + {}^4C_3 (a)^1 \left(x^2\right)^3 + {}^4C_4 (a)^0 \left(x^2\right)^4$$

$$= a^4 + 4a^3x^2 + 6a^2x^4 + 4ax^6 + x^8$$

Put $a = 1-x$ Back, we get;

$$= (1-x)^4 + 4x^2(1-x)^3 + 6x^4(1-x)^2 + 4x^6(1-x) + 8x$$

$=1-4x+6x^2-4x^3+x^4+4x^2-12x^3+12x^4-4x^5+6x^4-12x^5+6x^6+4x^6-4x^5+x^8-16x^5$

$=\{1-4x+6x^2-4x^3+x^4\}+4x^2\{1-3x+3x^2-x^3\}+6x^4(1-2x+x^2)+4x^6(1-x)+x^8$

$=1-4x+6x^2+4x^2-4x^3-12x^3+x^4+12x^4+6x^4-4x^5-12x^5+6x^6+4x^6-4x^7+x^8$

$=1-4x+10x^2-16x^3+19x^4-16x^5+10x^6-4x^7+x^8$

(iii) $(1-x-x^2)^4$

Solution:

Let $(1-x)=a$ then

$(1-x-x^2)^4=(a-x^2)^4$

$=a^4-4a^3x^2+6a^2x^4-4a^1x^6+x^8$

Put back $a=1-x$ we get;

$=(1-x)^4+4x^2(1-x)^3+6x^4(1-x)^2-4x^6(1-x)+x^8$

$\left(1+\binom{4}{1}1.(-x)+\binom{4}{2}1.(-x)^2+\binom{4}{3}1.(-x)^3+\binom{4}{4}1.(-x)^4\right)$

$+4x^2\left(1+\binom{3}{1}1.(-x)^1+\binom{3}{2}1.(-x)^2+\binom{3}{3}1.(-x)^3\right)+6x^4(1-2x+x^2)-4x^6+4x^7+x^8$

$=1-4x+6x^2-4x^3+x^4+4x^2-12x^3+12x^4-4x^5+6x^4-2x^5+6x^6+4x^6-4x^7+x^8$

$=1-4x+10x^2-16x^3+19x^4-16x^5+10x^6-4x^7+x^8$

Q.5 Expand the following in descending powers of x.

(i) $(x^2+x-1)^3$

Solution:

Let $x-1=a$ them

$(x^2+x-1)^3=(x^2+a)^3$

$={}^3C_0(x^2)^3(a)^0+{}^3C_1(x^2)^2(a)^1+{}^3C_2(x^2)^1(a)^2+{}^3C_3(x^2)^0a^3$

pulling back $x-1=a$ we get;

$=x^6+3x^4(x-1)+3x^2(x-1)^2+(x-1)^3$

$=x^6+3x^5-3x^4+3x^2(x^2-2x+1)+(x^3-3x^2+3x-1)$

$=x^6+3x^5-3x^4+3x^4-6x^3+3x^2+x^3-3x^2+3x-1$

$=x^6+3x^5-5x^3+3x-1$

(ii) $\left(x-1-\frac{1}{x}\right)^3$

Solution:

Let $x-1=a$ then

$\left(x-1-\frac{1}{x}\right)^3=\left(a-\frac{1}{x}\right)^3$

$$= {}^3C_0(a)^3\left(\frac{-1}{x}\right)^0 + {}^3C_1(a)^2\left(\frac{-1}{x}\right) + {}^3C_2(a)^1\left(\frac{-1}{x}\right)^2 + {}^3C_3(a)^0\left(\frac{-1}{x}\right)^3$$

$$= a^3 - 3a^2\left(\frac{1}{x}\right) + 3a\left(\frac{1}{x^2}\right) - \frac{1}{x^3}$$

Put a $a = x-1$ back we get;

$$= (x-1)^3 - 3\left(\frac{1}{x}\right)(x-1)^2 + 3\left(\frac{1}{x^2}\right)(x-1) - \frac{1}{x^3}$$

$$= (x^3 - 3x^2 + 3x - 1) - \frac{3}{x}(x^2 - 2x + 1) + \frac{3}{x^2}(x-1) - \frac{1}{x^3}$$

$$= x^3 - 3x^2 + \cancel{3x} - 1 - \cancel{3x} + \frac{\cancel{-3}}{\cancel{x}} + \frac{\cancel{3}}{\cancel{x}} - \frac{3}{x^2} - \frac{1}{x^3}$$

$$= x^3 - 3x^2 + 5 - \frac{3}{x^2} - \frac{1}{x^3}$$

Q.6 Find the term involving:

(i) x^4 in the expansion of $(3-2x)^7$

Solution:

As we know that $(r+1)^{th}$ term in the expansion of $(a+b)^n$ is

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

Here $n = 7, a = 3, b = 2x$ so we get;

$$T_{r+1} = {}^7C_r(3)^{7-r}(-2x)^r$$

For the term involving x^4, put exponent of x equal to 4 we get $r = 4$

So

$$T_{4+1} = {}^7C_4(3)^{7-4}(-2x)^4$$

$$T_5 = 35(27)(16)x^4$$

$$T_5 = 15120x^4$$

(ii) x^{-2} in the expansion of $\left(x - \frac{2}{x^2}\right)^{13}$

Solution:

As we know that $(r+1)^{th}$ term in the expansion of $(a+b)^n$ is

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

Here $n = 13, a = x, b = \frac{-2}{x^2}$ we get;

$$T_{r+1} = {}^{13}C_r(x)^{13-r}\left(\frac{-2}{x^2}\right)^r$$

$$T_{r+1} = {}^{13}C_r x^{13-r}(-2)^r x^{-2r}$$

$$= {}^{13}C_r(-2)^r x^{13-3r}$$

For the term involving x^{-2} put the exponent of x equal to -2 we get;

$13-3r=-2 \Rightarrow 15=3r$

$r=5$ we get;

$$T_{5+1} = {}^{13}C_5 (-2)^5 (x)^{-2}$$

$$=(1287)(-32)x^{-2}$$

$$T_6 = \frac{-41184}{x^2}$$

(iii) a^4 **in the expansion of** $\left(\frac{2}{x}-a\right)^9$

Solution:

As we know that $(r+1)^{th}$ term in the expansion of $(a+b)^n$ is

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

Here $a = \frac{2}{x}, b=-a, n=9$ we get;

$$T_{r+1} = {}^9C_r \left(\frac{2}{x}\right)^{9-r} (-a)^r$$

For the term involving a^4 put exponent of $\boldsymbol{a}$ equal to 4 i.e., $r=4$

So

$$T_{4+1} = {}^9C_4 \left(\frac{2}{x}\right)^5 (-a)^4$$

$$T_5 = \frac{4032a^4}{x^5}$$

(iv) y^3 **in the expansion of** $\left(x-\sqrt{y}\right)^{11}$

Solution:

As we know that $(r+1)^{th}$ term in the expansion of $(a+b)^n$ is

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

Here $n=11, a=x, b=-\sqrt{y}$

So

$T_{r+1} = {}^{11}C_r (x)^{11-r} \left(-\sqrt{y}\right)^r$, Suppose y occures in T_{r+1} i.e.,

$$y^3 = y^{\frac{r}{2}} \Rightarrow \frac{r}{2} = 3$$

$r=6$

Now

$$T_{6+1} = {}^{11}C_6 (x)^{11-6}\left(-y^{r/2}\right)^6$$

$$T_{6+1} = \binom{11}{6} x^5 y^3$$

$$T_7 = 462x^5y^3$$

Q.7 Find the coefficient of,

(i) x^5 **in the expansion of** $\left(x^2 - \dfrac{3}{2x}\right)^{10}$

Solution:

As we know that $(r+1)^{th}$ term in the expansion of $(a+b)^n$ is

$$T_{r+1} = {}^nC_r \; a^{n-r}b^r$$

Here $a = x, b = -\dfrac{3}{2x}, n = 10$

So

$$T_{r+1} = {}^{10}C_r \left(x^2\right)^{10-r}\left(\frac{-3}{2x}\right)^r$$

$$= {}^{10}C_r x^{20-2r}\left(\frac{-3}{2}\right)^r x^{-r}$$

$$= {}^{10}C_r \left(\frac{-3}{2}\right)^r x^{20-2r-r}$$

$$= {}^{10}C_2 \left(-\frac{3}{2}\right)^r (x)^{20-3r}$$

For the term involving x^5, put $20-3r=5$ we get; i.e.,

$$20-3r=5$$

$$15=3r$$

$$5=r$$

$$T_{5+1} = {}^{10}C_5\left(\frac{-3}{2}\right)^5 x^5$$

$$T_6 = \frac{-15309}{8}x^5$$

Thus coefficient of x^5 is $\dfrac{-15309}{8} = -1913.625$

(ii) x^n **in the expansion of** $\left(x^2 - \dfrac{1}{x}\right)^{2n}$

Solution:

As we know that $(r+1)^{th}$ term in the expansion of $(a+b)^n$ is

$$T_{r+1} = {}^nC_r a^{n-r}b^r$$

Here $n = 2n, a = x^2, b = -\frac{1}{x}$ we get;

$$T_{r+1} = {}^{2n}C_r \left(x^2\right)^{2n-r} \left(\frac{-1}{x}\right)^r$$

$$= {}^{2n}C_r \left(x\right)^{4n-2r} \left(-1\right)^r x^{-r}$$

$$= {}^{2n}C_r x^{4n-3r} \left(-1\right)^r$$

For the term involving x^n put the exponent of x equal to n, so

$$4n - 3r = n$$

$$3n = 3$$

$$r = n$$

Thus

$$T_{n+1} = {}^{2n}C_n x^n \left(-1\right)^n$$

$$T_{n+1} = \frac{(2n)!}{n! \times n!}(-1)^n x^n$$

So the coefficient of x^n is $\frac{(-1)^n (2n)!}{(n!)^2}$

Q.8 Find the 6th term in the expansion of $\left(x^2 - \frac{3}{2x}\right)^{10}$

Solution:

As we know that $(r+1)^{th}$ term in the expansion of $(a+b)^n$ is

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

Here $a = x^2, b = \frac{-3}{2x}, n = 10$

So, we get

$$T_{r+1} = {}^{10}C_r \left(x^2\right)^{10-r} \left(\frac{-3}{2x}\right)^r$$

for the 6th term put $r = 5$ we get;

$$T_{5+1} = {}^{10}C_5 \left(x^2\right)^{10-5} \left(\frac{-3}{2x}\right)^5$$

$$= 252x^{10} \times \left(\frac{-243}{32}\right) \times \frac{1}{x^5}$$

$$T_6 = \frac{-15309}{8} x^5 = -1913.625x^5$$

Q.9 Find the term independent of x in the following expansions.

(i) $\left(x - \frac{2}{x}\right)^{10}$

Solution:

As we know that $(r+1)^{th}$ term in the expansion of $(a+b)^n$ is

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

Here $n = 10, a = x, b = \frac{-2}{x}$

$$T_{r+1} = {}^{10}C_r (x)^{10-r} \left(\frac{-2}{x}\right)^r$$

$$= {}^{10}C_r \left(x^{10-r}\right)\left(x^{-r}\right)(-2)^r$$

$$= {}^{10}C_r x^{10-2r} (-2)^r$$

For the term involving x^0 (term independent from x) put exponent of x equal to zero i.e.,

$10 - 2r = 0$

$r = 5$

Thus

$$T_{5+1} = {}^{10}C_5 x^0 (-2)^5$$

$$= \frac{10\times 9\times 8\times 7\times 6}{5\times 4\times 3\times 2\times 1}(-32)$$

$$T_6 = -8064$$

(ii) $\left(\sqrt{x} + \frac{1}{2x^2}\right)^{10}$

Solution:

As we know that $(r+1)^{th}$ term in the expansion of $(a+b)^n$ is

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

Here $n = 10, a = \sqrt{x}, b = \frac{1}{2x^2}$

$$T_{r+1} = {}^{10}C_r \left(\sqrt{x}\right)^{10-r} \left(\frac{1}{2x^2}\right)^r$$

$$= {}^{10}C_r (x)^{5-\frac{r}{2}} \left(\frac{1}{2}\right)^r x^{-2r}$$

$$= {}^{10}C_r \left(\frac{1}{2}\right)^r x^{5-\frac{r}{2}-2r} \qquad \because 5 - \frac{r}{2} - 2r$$

$$T_{r+1} = {}^{10}C_r \left(\frac{1}{2}\right)^r x^{5-\frac{5r}{2}} \qquad \because \frac{10 - r - 4r}{2} = \frac{10-5r}{2} = 5 - \frac{5}{2}r$$

For the term involving x^0 (term independent from x) put exponent of x equal to zero i.e.,

Put $5 - \frac{5r}{2} = 0$

$\frac{5r}{2} = 5$

$r = 2$

Put $r=2$ in the last expansion we get;

$$T_3 = {}^{10}C_2\left(\frac{1}{2}\right)^2 x^0$$

$$= \frac{10\times 9}{2}\times\frac{1}{4}$$

$$T_3 = \frac{45}{4}$$

(iii) $\left(1+x^2\right)^3\left(1+\frac{1}{x^2}\right)^4$

Solution:

$$\left(1+x^2\right)^3\left(\frac{x^2+1}{x^2}\right)^4$$

$$=\left(1+x^2\right)^3\frac{\left(1+x^2\right)^4}{x^8}$$

$$=x^{-8}\left(1+x^2\right)^7$$

$(r+1)^{th}$ term in the expansion of $\left(1+x^2\right)^7$ is

$$T_{r+1} = {}^7C_r\left(1\right)^{7-r}\left(x^2\right)^r$$

$$= {}^7C_r x^{2r}$$

Thus

$$=x^{-8}\times {}^7C_r x^{2r}$$

$$= {}^7C_r x^{2r-8}$$

For the term involving x^0 (term independent from x) put exponent of x equal to zero i.e.,
Put $2r-8=0 \Rightarrow r=4$
So required term independent from x is

$$T_{4+1} = {}^7C_4 x^0$$

$$=35$$

Q.10 Determine the middle term in the following expansions:

(i) $\left(\frac{1}{x}-\frac{x^2}{2}\right)^{12}$

Solution:

Here $n=12\left(Even\right)$ so that middle term is $\left(\frac{n}{2}+1\right)^{th}$ term i.e.,

$$\left(\frac{n}{2}+1\right)^{th}=\left(\frac{12}{2}+1\right)^{th}=7^{th} \text{ term}$$

Here $a=\frac{1}{x}, b=\frac{-x^2}{2}, \text{n}=12, r=6$

So

$$T_{r+1} = {}^{n}C_r a^{n-r} b^r$$

$$T_{6+1} = {}^{12}C_6 \left(\frac{1}{x}\right)^6 \left(\frac{-x^2}{2}\right)^6$$

$$= 924 \left(\frac{1}{x^6}\right)\left(\frac{x^{12}}{64}\right)$$

$$T_7 = \frac{231}{16} x^6$$

(ii) $\left(\frac{3}{2}x - \frac{1}{3x}\right)^{11}$

Solution:

Here $n = 11(odd)$, so the middle terms are $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$

So

$$\left(\frac{n+1}{2}\right)^{th} = \left(\frac{11+3}{2}\right)^{th} = 6^{th} \text{ term}$$

$$\left(\frac{n+3}{2}\right)^{th} = \left(\frac{11+3}{2}\right)^{th} = 7^{th} \text{ term}$$

Here $a = \frac{3}{2}, b = -\frac{1}{3x}, n = 11$

For 6th term:

$r = 5$

$$T_{r+1} = {}^{n}C_r a^{n-r} b^r$$

$$T_6 = {}^{11}C_5 \left(\frac{3}{2}x\right)^6 \left(\frac{-1}{3x}\right)^5$$

$$T_6 = \frac{-693}{32} x$$

For 7th term:

$r = 6$

$$T_{r+1} = {}^{n}C_r a^{n-r} b^r$$

$$T_7 = {}^{11}C_6 \left(\frac{3}{2}x\right)^5 \left(\frac{-1}{3x}\right)^6$$

$$= 462 \left(\frac{3^5 x^5}{2^5}\right)\left(\frac{1}{3^6 x^6}\right)$$

$$= 462 \left(\frac{1}{32 \times 3x}\right)$$

(iii) $\left(2x-\dfrac{1}{2x}\right)^{2m+1}$

Solution:

Here $2m+1=$ odd so the middle terms are

$\left(\dfrac{n+1}{2}\right)^{th}$ and $\left(\dfrac{n+3}{2}\right)^{th}$ terms

So

$$\left(\frac{n+1}{2}\right)^{th}=\left(\frac{2m+1+1}{2}\right)^{th}=(m+1)^{th}\text{ term}$$

$$\left(\frac{n+3}{2}\right)^{th}=\left(\frac{2m+1+3}{2}\right)^{th}=(m+2)^{th}\text{ term}$$

Here $a=2x, b=-\dfrac{1}{2x}, n=2m+1, r=m$

For $(m+1)^{th}$ **term**:

$r=m$

$$T_{r+1}={}^{n}C_{r}a^{n-r}b^{r}$$

$$T_{m+1}={}^{n}C_{m}(2x)^{n-m}\left(\frac{-1}{2x}\right)^{m}$$

$$={}^{2m+1}C_{m}(2x)^{2m+1-m}\left(\frac{-1}{2x}\right)^{m}$$

$$=\frac{(2m+1)!}{m!(m+1)!}(2x)^{m+1}(-1)^{m}\times\frac{1}{(2x)^{m}}$$

$$=\frac{(2m+1)!}{m!(m+1)!}(-1)^{m}\,2x$$

For $(m+2)^{th}$ **term**:

$r=m+1$

$$T_{r+1}={}^{n}C_{r}a^{n-r}b^{r}$$

$$T_{m+2}={}^{n}C_{m+1}(2x)^{n-(m+1)}\left(\frac{-1}{2x}\right)^{m+1}$$

$$T_{m+2}={}^{2m+1}C_{m+1}(2x)^{2m+1-m-1}\left(\frac{-1}{2x}\right)^{m+1}$$

$$=\frac{(2m+1)!}{(m+1)!(2m+1-m-1)!}(2x)^{m}\left(\frac{-1}{2x}\right)^{m+1}$$

$$=\frac{(2m+1)!}{(m+1)!(m)!}(-1)^{m+1}\cdot\frac{(2x)^{m}}{(2x)^{m+1}}$$

$$=\frac{(2m+1)!}{m!(m+1)!}(-1)^{m+1}\frac{1}{2x}$$

Q.11 Find $(2n+1)^{th}$ term from the end in the expansion of $\left(x-\frac{1}{2x}\right)^{3n}$

Solution:

$(2n+1)^{th}$ term from the end in the expansion of $\left(x-\frac{1}{2x}\right)$ is $(2n+1)^{th}$ term from the beginning in the expansion of $\left(\frac{-1}{2x}+x\right)^{3n}$

As $(r+1)^{th}$ term of $(a+b)^{n}$ is

$$T_{r+1}={}^{n}C_{r}a^{n-r}b^{r}$$

Put $r=2n, a=\frac{-1}{2x}, b=x,\ n=3n$ we get;

$$T_{2n+1}={}^{3n}C_{2n}\left(\frac{-1}{2x}\right)^{3n-2n}(x)^{2n}$$

$$=\frac{(3n)!}{(3n-2n)!\times(2n)!}\times\left(\frac{-1}{2}\right)^{n}\times\frac{1}{x^{n}}\times x^{2n}$$

$$T_{2n+1}=\frac{(-1)^{n}}{2^{n}}\times\frac{(3n)!}{(2n)!\times n!}x^{n}$$

Q.12 Show that middle term of $(1+x)^{2n}$ is $\frac{1.3.5....(2n-1)}{n!}2^{n}x^{n}$

Proof:

Here $2n=$ even so the middle term is $\left(\frac{n}{2}+1\right)^{th}=\left(\frac{2n}{2}+1\right)^{th}=(n+1)^{th}$ term.

Thus

$$T_{r+1}={}^{n}C_{r}a^{n-r}b^{r}$$

Here $r=n, a=1, b=x,\ n=2n$

So

$$T_{n+1}={}^{2n}C_{n}(1)^{2n-n}(x)^{n}$$

$$=\frac{(2n)!}{(2n-n)!\times n!}x^{n}$$

$$=\frac{(2n)!}{n!\times n!}x^{n}$$

$$=\frac{(2n)(2n-1)(2n-2)(2n-3)(2n-4)....4.3.2.1}{n!\times n!}\times x^{n}$$

$$=\frac{\left[(2n)(2n-2)(2n-4).....4.2\ \right]\left[(2n-1)(2n-3)2n-5....3.1\ \right]}{n!\times n!}\times x^n$$

$$=\frac{2^n\left\{n(n-1)(n-2)......2.1\right\}\left\{(2n-1)(2n-3)(2n-5)....3.1\right\}}{n!\times n!}\times x^n$$

$$=\frac{2^n\times n!\times\left\{(2n-1)(2n-3)(2n-5).....3.1\right\}x^n}{n!\times n!}$$

$$=\frac{2^n\times\left\{(2n-1)(2n-3)(2n-5).....5.3.1\right\}x^n}{n!}$$

$$=\frac{\left\{1.3.5....(2n-1)\right\}2^n x^n}{n!}$$

$$=\frac{1.3.5....(2n-1)}{n!}2^n x^n$$

Hence the proof.

Q.13 Show that

$$\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+......\binom{n}{n-1}=2^{n-1}$$

Proof:

We know that

$$(1+x)^n=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+......+\binom{n}{n}x^n \qquad \text{(i)}$$

Put $x = 1$ in (i) we get;

$$(1+1)^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+.....+\binom{n}{n}$$

$$2^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+......+\binom{n}{n} \qquad \text{(ii)}$$

Put $x=-1$ in (i) we get;

$$(1+(-1))^n=\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+......+(-1)^{n-1}\binom{n}{n-1}+(-1)^n\binom{n}{n}$$

$$0=\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+.......+(-1)^{n-1}\binom{n}{n-1}+(-1)^n\binom{n}{n}$$

Assume that here n is even

$$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+...+\binom{n}{n}=\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+...+\binom{n}{n-1} \qquad \text{(iii)}$$

$$2^n=\left\{\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+......+\binom{n}{n}\right\}+\left\{\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+......+\binom{n}{n-1}\right\}$$

Using (iii) we get;

$$2^n=\left\{\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+...+\binom{n}{n-1}\right\}+\left\{\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+...+\binom{n}{n-1}\right\}$$

$$2^n = 2\left\{\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+.....+\binom{n}{n-1}\right\}$$

$$\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+.....+\binom{n}{n-1}=2^{n-1}$$

Hence the proof

Q.14 Show that

$$\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\frac{1}{4}\binom{n}{3}+........+\frac{1}{n+1}\binom{n}{n}=\frac{2^{n+1}-1}{n+1}$$

Proof:

L.H.S = $\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\frac{1}{4}\binom{n}{3}+.......+\frac{1}{n+1}\binom{n}{n}$

$$=\frac{n!}{0!(n-0)!}+\frac{1}{2}\times\frac{n!}{1!(n-1)!}+\frac{1}{3}\frac{n!}{2!(n-2)!}+\frac{1}{4}\frac{n!}{3!(n-3)}+.....+\frac{1}{n+1}\frac{n!}{n!(n-n)!}$$

$$=1+\frac{n}{2!}+\frac{n(n-1)}{3!}+\frac{n(n-1)(n-2)}{4!}+.....+\frac{1}{(n+1)}$$

Taking common $\frac{1}{n+1}$ we get;

$$=\frac{1}{n+1}\left[(n+1)+\frac{(n+1)n}{2!}+\frac{(n+1)n(n-1)}{3!}+\frac{(n+1)n(n-1)(n-2)}{4!}+......+1\right]$$

Above expression can be written as;

$$=\frac{1}{n+1}\left[\binom{n+1}{1}+\binom{n+1}{2}+\binom{n+1}{3}+......+\binom{n+1}{n+1}\right]$$

$$=\frac{1}{n+1}\left[\binom{n+1}{0}+\binom{n+1}{1}+\binom{n+1}{2}+\binom{n+1}{3}+...+\binom{n+1}{n+1}-\binom{n+1}{0}\right]$$

$$=\frac{1}{n+1}\left[2^{n+1}-1\right]$$

= **R.H.S**

The Binomial Theorem when the index n is a negative integer or a fraction.

When n is negative integer or a fraction, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots$$

Provide $|x|<1$

The series of the type

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

Is called the binomial series

Note:

(1) The proof of this theorem is beyond the scope of this book.

(2) Symbols $\binom{n}{0}\binom{n}{1}\binom{n}{2}$ etc are meaningless when n is negative integer or a fraction.

(3) The general term in the expansion is $T_{r+1} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r$

Some particular cases of the expansion of $(1+x)^n$, $n<0$

(i) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots + (-1)^r x^r + \ldots$

(ii) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^2 + \ldots + (-1)^r (r+1)x^r + \ldots$

(iii) $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \ldots + (-1)^r \frac{(r+1)(r+2)}{2}x^2 + \ldots$

(iv) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots + x^r + \ldots$

(v) $(1-x)^2 = 1 + 2x = 3x^2 + 4x^3 + \ldots + (r=1)x^r + \ldots$

(vi) $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots + \frac{(r+1)(r+2)}{2}x^r + \ldots$

EXERCISE 8.3

Q.1 Expand the following upto 4 terms, taking the value of x such that the expansion in each case is valid.

(i) $(1-x)^{\frac{1}{2}}$

Solution:

$$(1-x)^{\frac{1}{2}}=1+\left(\frac{1}{2}\right)(-x)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(-x)^2+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(-x)^3+\ldots.$$

$$=1-\frac{1}{2}x+\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\frac{1}{2}x^2+\frac{1}{2}\times\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\frac{1}{6}\left(-x^3\right)\ldots\ldots$$

$$=1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3\ldots\ldots$$

The expansion of $(1-x)^{\frac{1}{2}}$ is valid if $|x|<1$

(ii) $(1+2x)^{-1}$

Solution:

$$(1+2x)^{-1}=1+(-1)(2x)+\frac{(-1)(-1-1)}{2!}(2x)^2+\frac{(-1)(-1-1)(-1-2)}{3!}(2x)^3+\ldots\ldots.$$

$$=1-2x+4x^2-8x^3+\ldots.$$

The expansion of $(1+2x)^{-1}$ is valid if $|2x|<1 \quad \Rightarrow |x|<\frac{1}{2}$

(iii) $(1+x)^{\frac{-1}{3}}$

Solution:

$$(1+x)^{\frac{-1}{3}}=1+\left(\frac{-1}{3}\right)x+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2!}x^2+\frac{\left(\frac{-1}{3}\right)\left(\frac{-1}{3}-7\right)\left(\frac{-1}{3}-2\right)}{3!}x^3+\ldots\ldots$$

$$=1-\frac{1}{3}x+\frac{\left(\frac{-1}{3}\right)\left(\frac{-4}{3}\right)}{2}x^2+\frac{\left(\frac{-1}{3}\right)\left(\frac{-4}{3}\right)\left(\frac{-7}{3}\right)}{6}x^3+\ldots\ldots.$$

$$=1-\frac{1}{3}x+\frac{2}{9}x^2\,\frac{4\times 7}{27\times 3\times 2}x^3+\ldots\ldots\ldots$$

$$=1-\frac{1}{3}x+\frac{2}{9}x^2-\frac{14}{81}x^3+\ldots..$$

The expansion of $(1+x)^{-\frac{1}{3}}$ is valid if $|x|<1$

(iv) $(4-3x)^{\frac{1}{2}}$

Solution:

$$(4-3x)^{\frac{1}{2}}=\left[4\left(1-\frac{3}{4}x\right)\right]^{\frac{1}{2}}$$

$$=4^{\frac{1}{2}}\left[1-\frac{3}{4}x\right]^{\frac{1}{2}}$$

$$=2\left[1+\frac{1}{2}\left(\frac{-3}{4}x\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{-3}{4}x\right)^2+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(\frac{-3}{4}x\right)^3+.........\right]$$

$$=2\left[1-\frac{3}{8}x+\frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}{2}\frac{9x^2}{16}+\frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{6}\left(\frac{-27}{64}x^3\right)+........\right]$$

$$=2\left[1-\frac{3}{8}x-\frac{1}{8}\times\frac{9}{16}x^2-\frac{1}{16}\times\frac{27}{64}x^3+........\right]$$

$$=2-\frac{3}{4}x-\frac{9}{64}x^2-\frac{27}{512}x^3+.......$$

The expansion of $(4-3x)^{\frac{1}{2}}$ is valid if $\left|\frac{3}{4}x\right|<1 \quad\Rightarrow\quad |x|<\frac{4}{3}$

(v) $(8-2x)^{-1}$

Solution:

$$(8-2x)^{-1}=8^{-1}\left(1-\frac{x}{4}\right)^{-1}$$

$$=\frac{1}{8}\left[1+(-1)\left(\frac{-x}{4}\right)+\frac{(-1)(-1-1)}{2!}\left(\frac{-x}{4}\right)^2+\frac{(-1)(-1-1)(-1-2)}{3!}\left(\frac{-x}{4}\right)^3+.....\right]$$

$$=\frac{1}{8}\left[1+\frac{x}{4}+\frac{1\times2}{2\times1}\left(\frac{x^2}{16}\right)-\frac{2\times3}{6}\left(\frac{-x^3}{64}\right)......\right]$$

$$=\frac{1}{8}\left[1+\frac{x}{4}+\frac{x^2}{16}+\frac{x^3}{64}+.....\right]$$

$$=\frac{1}{8}+\frac{1}{32}x+\frac{1}{128}x^2+\frac{1}{512}x^3+......$$

The expansion of $(8-2x)^{-1}$ is valid if $\left|\frac{x}{4}\right|<1 \Rightarrow |x|<4$

(vi) $(2-3x)^{-2}$

Solution:

$$(2-3x)^{-2}=\left[2\left(1-\frac{3}{2}x\right)\right]^{-2}$$

$$=(2)^{-2}\left(1-\frac{3}{2}x\right)^{-2}$$

$$=\frac{1}{4}\left[1+(-2)\left(\frac{-3}{2}x\right)+\frac{(-2)(-2-1)}{2!}\left(\frac{-3}{2}x\right)^2+\frac{(-2)(-2-1)(-2-2)}{3!})\left(\frac{-3}{2}x\right)^3+......\right]$$

$$=\frac{1}{4}\left[1+3x+\frac{(-2)(-3)}{2}\left(\frac{9}{4}x^2\right)+\frac{(-2)(-3)(-4)}{6}\left(\frac{-27}{8}x^3\right)+......\right]$$

$$=\frac{1}{4}\left[1+3x+\frac{27}{4}x^2+\frac{27}{2}x^3+......\right]$$

$$=\frac{1}{4}+\frac{3}{4}x+\frac{27}{16}x^2+\frac{27}{8}x^3+......$$

The expansion of $(2-3x)^{-2}$ is valid if $\left|\frac{3}{2}x\right|<1 \quad \Rightarrow |x|<\frac{2}{3}$

(vii) $\frac{(1-x)^{-1}}{(1+x)^2}$

Solution:

$$\frac{(1-x)^{-1}}{(1+x)^2}=(1-x)^{-1}(1+x)^{-2}$$

$$=\left[(1-x)^{-1}(1+x)^{-2}\right]$$

$$=\left[1+(-1)(-x)+\frac{(-1)(-1-1)}{2!}(-x)^2+\frac{(-1)(-1-1)(-1-2)}{3!}(-x)^3+......\right]$$

$$\times\left[1+(-2)(x)+\frac{(-2)(-2-1)}{2!}x^2+\frac{(-2)(-2-1)(-1-2)}{3!}x^3+......\right]$$

$$=\left[1+x+x^2+x^3+.....\right]\left[1-2x+3x^2-4x^3+......\right]$$

$$=1-2x+3x^2-4x^2+x-2x^2+3x^3-4x^3+.....$$

Neglect x^4 and higher powers of x

$$=1-x+2x^2-2x^3+.......$$

The expansion of $(1-x)^{-1}$ and $(1+x)^{-2}$ are valid if $|x|<1$

(viii) $\frac{\sqrt{1+2x}}{1-x}$

Solution:

$$\frac{\sqrt{1+2x}}{1-x}=(1+2x)^{\frac{1}{2}}(1-x)^{-1}$$

$$=\left[1+\left(\frac{1}{2}\right)(2x)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(2x)^2+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(2x)^3+......\right]$$

$$\times\left[1+(-1)(-x)+\frac{(-1)(-1-1)}{2!}x^2+\frac{(-1)(-1-1)(-1-2)}{3!}(-x)^3+......\right]$$

$$=\left[1+x+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}4x^2+\frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{6}8x^3+\ldots\ldots\right]\times\left[1+x+x^2+x^3+\ldots.\right]$$

$$=\left(1+x-\frac{1}{2}x^2+\frac{1}{2}x^3+\ldots..\right)\left(1+x+x^2+x^3+\ldots\right)$$

$$=1+x+x^2+x^3+x+x^3+x^3-\frac{1}{2}x^2-\frac{1}{2}x^3+\frac{1}{2}x^3+..$$

$$=1+2x+\frac{3}{2}x^2+2x^3+\ldots\ldots$$

The expansion of $(1+2x)^{\frac{1}{2}}$ is valid $|2x|<1 \Rightarrow |x|<\frac{1}{2}$

and the expansion of $(1-x)^{\frac{1}{2}}$ is valid if $|x|<1$

So, the expansion of $\frac{\sqrt{1+2\boldsymbol{x}}}{1-\boldsymbol{x}}$ is valid if $|x|<\frac{1}{2}$

(ix) $\frac{(4+2x)^{\frac{1}{2}}}{2-x}$

Solution:

$$\frac{(4+2\text{x})^{\frac{1}{2}}}{2-\text{x}}=\frac{\left(4\left(1+\frac{2}{4}x\right)\right)^{\frac{1}{2}}}{2\left(1-\frac{x}{2}\right)}=\frac{4^{\frac{1}{2}}}{2}\left(1+\frac{x}{2}\right)^{\frac{1}{2}}\left(1-\frac{x}{2}\right)^{-1}$$

$$=\left(1+\frac{x}{2}\right)^{\frac{1}{2}}\left(1-\frac{x}{2}\right)^{-1}$$

$$=\left[1+\frac{1}{2}\left(\frac{x}{2}\right)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}\left(\frac{x}{2}\right)^2+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(\frac{x}{2}\right)^3+\ldots.\right]$$

$$\times\left[1+(-1)\left(-\frac{x}{2}\right)+\frac{(-1)(-1-1)}{2!}\left(-\frac{x}{2}\right)^2+\frac{(-1)(-1-1)(-1-2)}{3!}\left(-\frac{x}{2}\right)^3+\ldots.\right]$$

$$=\left[1+\frac{x}{4}+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}\frac{x^3}{8}+\ldots.\right]\times\left[1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\ldots..\right]$$

$$=\left[1+\frac{x}{4}-\frac{1}{32}x^2+\frac{1}{128}x^2\right]\left[1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\ldots..\right]$$

$$=\left(1+\frac{x}{4}-\frac{1}{32}x^2+\frac{1}{128}x^3+....\right)\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+...\right)$$

$$=1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\frac{x}{4}+\frac{x^2}{8}+\frac{x^3}{16}-\frac{1}{32}x^2-\frac{1}{64}x^3+\frac{1}{128}x^3+....$$

$$=1+\frac{3}{4}x+\frac{11}{32}x^2+\frac{23}{128}x^3+....$$

The expansion of $\left(1+\frac{x}{2}\right)^{\frac{1}{2}}$ and $\left(1-\frac{x}{2}\right)^{-1}$ is valid if $\left|\frac{x}{2}\right|<1 \Rightarrow |x|<2$

(x) $\quad \left(\mathbf{1+x-2x^2}\right)^{\frac{1}{2}}$

Solution:

$$\left(1+x-2x^2\right)^{\frac{1}{2}}=\left[1+\left(x-2x^2\right)\right]^{\frac{1}{2}}$$

$$=1+\frac{1}{2}\left(x-2x^2\right)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}\left(x-2x^2\right)^2+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(x-2x^2\right)^3+......$$

$$=1+\frac{1}{2}\left(x-2x^2\right)+\frac{\left(\frac{-1}{4}\right)}{2}\left(x-2x^2\right)+\frac{\frac{3}{8}}{6}\left(x-2x^2\right)^3+.....$$

$$=1+\frac{1}{2}\left(x-2x^2\right)-\frac{1}{8}\left(x^2-4x^3+4x^4\right)+\frac{1}{16}\left(x^3-6x^4+12x^5-8x^6\right)+....$$

$$=1+\frac{1}{2}x+\left(-1-\frac{1}{8}\right)x^2+\left(\frac{1}{2}+\frac{1}{16}\right)x^3+.....$$

$$=1+\frac{1}{2}x-\frac{9}{8}x^2+\frac{9}{16}x^3+.....$$

The expansion is valid if $\left|x-2x^2\right|<1$

$+\left(x-2x^2\right)<1$

$x-2x^2<1$

$x-2x^2-1<0$

$2x^2-x+1>0 \qquad$ (i)

The inequality i) is not satisfied by any real value of x

$-\left(x-2x^2\right)<1$

$x-2x^2>-1$

$x-2x^2+1>0$

$2x^2-x-1<0 \qquad$ (ii)

$2x^2-2x+x-1<0$

$2x\left(x-1\right)+1\left(x-1\right)<0$

$\left(x-1\right)\left(2x+1\right)<0$

Case1

$x-1<0,\ 2x+1>0$

$x<1, x>\frac{-1}{2}$

Case 2

$x-1>0,\ 2x+1<0$

$x>1,\ x<\frac{-1}{2}$

Which is not possible

Thus the expansion of $\left(1+x-2x^2\right)^{\frac{-1}{2}}$ is valid if $x\in\left(\frac{-1}{2},1\right)$ *or* $\frac{-1}{2}<x<1$

(xi) $\left(1-2x+3x^2\right)^{\frac{1}{2}}$

Solution:

$$\left(1-2x+3x^2\right)^{\frac{1}{2}}$$

$$=\left(1-\left(2x-3x^2\right)\right)^{\frac{1}{2}}$$

$$=1+\left(\frac{-1}{3}\right)\left(-\left(2x-3x^2\right)\right)+\frac{\left(\frac{-1}{3}\right)\left(\frac{-1}{3}-1\right)\left(-\left(2x-3x^2\right)\right)^2}{2!}+\frac{\left(\frac{-1}{3}\right)\left(\frac{-1}{3}-1\right)\left(\frac{-1}{3}-2\right)}{3!}\left(-\left(2x-3x^2\right)\right)^3+....$$

$$=1+\frac{1}{3}\left(2x-3x^2\right)+\frac{4}{9}\times\frac{1}{2}\left(2x-3x^2\right)^2+\ \frac{-28}{27}\times\frac{1}{6}\left(-\left(2x-3x^2\right)^3\right)....$$

$$=1+\frac{1}{3}\left(2x-3x^2\right)+\frac{2}{9}\left(4x^2-12x^3+9x^4\right)\ +\frac{14}{81}\left(8x^3-36x^4+54x^5-27x^6\right)+.......$$

$$=1+\frac{2}{3}x+\left(-1+\frac{8}{9}\right)x^2+\left(\frac{-8}{3}\right)+\frac{112}{81}x^3+....$$

$$=1+\frac{2}{3}x-\frac{1}{9}x^2-\frac{164}{81}x^3+......$$

The expansion is valid if $\left|2x-3x^2\right|<1$

$$+\left(2x-3x^2\right)<1$$
$$2x-3x^2<1$$
$$3x^2-2x+1>0 \qquad \text{(i)}$$

$$-\left(2x-3x^2\right)<1$$
$$-2x+3x^2<1$$
$$3x^2-2x-1>0 \qquad \text{(ii)}$$
$$3x^2-3x+x-1<0$$
$$3x\left(x-1\right)+1\left(x-1\right)<0$$
$$\left(x-1\right)\left(3x-1\right)<0$$

The inequality i) is not satisfied by any real value of x

Case1

$x-1>0,\ 3x+1<0$

$x>1, x<\frac{-1}{3}$

Which is not possible

Case 2

$x-1<0,\ 3x+1>0$

$x<1,\ x>\frac{-1}{3}$

Thus the expansion of $\left(1-2x+3x^2\right)^{\frac{1}{2}}$ is valid if $x\in\left(\frac{-1}{3},1\right)$ or if $\frac{-1}{3}<x<1$

Q.2 Using binomial theorem, find the value of the following to three places of decimals.

(i) $\sqrt{99}$

Solution:

$$\sqrt{99}=(100-1)^{\frac{1}{2}}$$

$$=\left[100\left(1-\frac{1}{100}\right)\right]^{\frac{1}{2}}$$

$$=(100)^{\frac{1}{2}}\left(1-\frac{1}{100}\right)^{\frac{1}{2}}$$

$$=10\left[1+\frac{1}{2}\left(\frac{-1}{100}\right)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}\left(\frac{-1}{100}\right)^{2}+...\right]$$

$$=10\left[1-\frac{1}{2}(0.01)-\frac{1}{8}(.0001)+.....\right]$$

$$=10[1-0.005-0.0000125+.....]$$

$$\approx 10[1-0.0050125]$$

$$\approx 10[0.9949875]$$

$$\approx 9.949875$$

≈ 9.950 convert to the three decimal

(ii) $(\mathbf{0.98})^{\frac{1}{2}}$

Solution:

$$(0.98)^{\frac{1}{2}}=(1-0.02)^{\frac{1}{2}}$$

$$=1+\frac{1}{2}(-0.02)+\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\frac{1}{2}(-0.02)^{2}+\frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{3!}(-0.02)^{3}+.....$$

$$=1-0.1-\frac{1}{8}(0.0004)-\frac{1}{16}(0.000008)+......$$

$$\approx 1-0.1-0.00005-0.0000005$$

$$\approx 1-0.0100505$$

$$\approx 0.9899595$$

$$\approx 0.990$$

(iii) $(\mathbf{1.03})^{\frac{1}{3}}$

Solution:

$$(1.03)^{\frac{1}{3}}=(1+0.03)^{\frac{1}{3}}$$

$$=1+\frac{1}{3}(0.03)+\frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)}{2!}(0.03)^{2}+\frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}(0.03)^{3}+.....$$

$$=1+0.01+\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\frac{1}{2}(0.0009)+\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)\frac{1}{6}(0.000027)+....$$

$$=1+0.01-\frac{1}{9}(.0009)+\frac{5}{81}(0.00027)+....$$

$$\approx 1+0.1-.0001+0.0001$$

$$\approx 1.010$$

(iv) $\sqrt[3]{65}$

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Solution:

$$\sqrt[3]{65} = (64+1)^{\frac{1}{3}}$$

$$= \left[64\left(1+\frac{1}{64}\right)\right]^{\frac{1}{3}}$$

$$= (64)^{\frac{1}{3}}\left(1+\frac{1}{64}\right)^{\frac{1}{3}}$$

$$= 4\left(1+\frac{1}{64}\right)^{\frac{1}{3}}$$

$$= 4\left[1+\frac{1}{3}\left(\frac{1}{64}\right)+\frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)}{2!}\left(\frac{1}{64}\right)^2+....\right]$$

$$= 4\left[1+\frac{1}{3}(0.015625)-\frac{1}{9}(0.015625)^2+....\right]$$

$$\approx 4[1+0.005208-0.000027]$$

$$\approx 4(1.005181)$$

$$\approx 4.020724$$

$$\approx 4.021$$

(v) $\sqrt[4]{17}$

Solution:

$$\sqrt[4]{17} = (17)^{\frac{1}{4}}$$

$$= (16+1)^{\frac{1}{4}}$$

$$= \left[16\left(1+\frac{1}{16}\right)\right]^{\frac{1}{4}}$$

$$= (16)^{\frac{1}{4}}\left(1+\frac{1}{16}\right)^{\frac{1}{4}}$$

$$= 2\left(1+\frac{1}{16}\right)^{\frac{1}{4}}$$

$$= 2\left[1+\frac{1}{4}\left(\frac{1}{16}\right)+\frac{\left(\frac{1}{4}\right)\left(\frac{1}{4}-1\right)}{2!}\left(\frac{1}{16}\right)^2+.....\right]$$

$$= 2\left[1+\frac{1}{64}+\left(\frac{1}{4}\right)\left(\frac{-3}{4}\right)\frac{1}{2}\left(\frac{1}{16}\right)^2+.....\right]$$

$$= 2\left[1+\frac{1}{64}-\frac{3}{2}\times\left(\frac{1}{64}\right)^2 ....\right]$$

$$\approx 2[1+0.015625-0.000366]$$

$$\approx 2[1.015259]$$

$$\approx 2.030518$$

$$\approx 2.031$$

(vi) $\sqrt[5]{31}$

Solution:

$$\sqrt[5]{31} = (31)^{\frac{1}{5}}$$

$$= (32-1)^{\frac{1}{5}}$$

$$= \left[32\left(1-\frac{1}{32}\right)\right]^{\frac{1}{5}}$$

$$= (32)^{\frac{1}{5}}\left(1-\frac{1}{32}\right)^{\frac{1}{5}}$$

$$= 2\left(1-\frac{1}{32}\right)^{\frac{1}{5}}$$

$$= 2\left[1+\frac{1}{5}\left(\frac{-1}{32}\right)+\frac{\left(\frac{1}{5}\right)\left(\frac{1}{5}-1\right)}{2!}\left(\frac{-1}{32}\right)^2+.....\right]$$

$$= 2\left[1-\frac{1}{5\times 32}+\frac{1}{5}\times\frac{-4}{5}\times\frac{1}{2}\times\left(\frac{1}{32}\right)^2+....\right]$$

$$= 2\left[1-\frac{1}{10}\times\frac{1}{16}-2\left(\frac{1}{10}\times\frac{1}{16}\right)^2+....\right]$$

$$= 2\left[1-0.00625-2(0.0000390625)+.....\right]$$

$$\approx 2(1-0.00625-0.000078)$$

$$\approx 2(0.993672)$$

$$\approx 1.987344$$

$$\approx 1.987$$

(vii) $\dfrac{1}{\sqrt[3]{998}}$

Solution:

$$\frac{1}{\sqrt[3]{998}} = (998)^{\frac{-1}{3}}$$

$$= (1000-2)^{\frac{-1}{3}}$$

$$= \left[(1000)\left(1-\frac{2}{1000}\right)\right]^{\frac{-1}{3}}$$

$$= (1000)^{\frac{-1}{3}}\left(1-\frac{1}{500}\right)^{\frac{-1}{3}}$$

$$= (10)^{-1}\left[1+\left(\frac{-1}{3}\right)\left(\frac{-1}{500}\right)+\frac{\left(\frac{1}{3}\right)\left(\frac{-1}{3}-1\right)}{2!}\left(-\frac{1}{500}\right)^2+.....\right]$$

$$\approx \frac{1}{10}\left[1+0.0006667+0.00000080\right]$$

$$\approx \frac{1}{10}(1.0006675)$$

$$\approx 0.10006675$$

$$\approx 0.100$$

(viii) $\mathbf{\frac{1}{\sqrt[5]{252}}}$

Solution:

$$\frac{1}{\sqrt[5]{252}} = (252)^{\frac{-1}{5}}$$

$$= (243+9)^{\frac{-1}{5}}$$

$$= \left[243\left(1+\frac{9}{243}\right)\right]^{\frac{-1}{5}}$$

$$= (243)^{\frac{-1}{5}}\left(1+\frac{1}{27}\right)^{\frac{-1}{5}}$$

$$= \frac{1}{3}\left[1+\left(\frac{-1}{5}\right)\left(\frac{1}{27}\right)+\frac{\left(\frac{-1}{5}\right)\left(\frac{-1}{5}-1\right)}{2!}\left(\frac{1}{27}\right)^2+.....\right]$$

$$= \frac{1}{3}\left[1-\frac{1}{5\times 27}+\left(\frac{-1}{5}\right)\left(\frac{-6}{5}\right)\frac{1}{2}\left(\frac{1}{27}\right)^2+......\right]$$

$$= \frac{1}{3}\left[1-\frac{1}{5\times 27}+3\left(\frac{1}{5\times 27}\right)^2+.....\right]$$

$\approx \frac{1}{3}\left[1-0.0074074+3(0.00005487)\right]$

$\approx \frac{1}{3}(0.992757)$

≈ 0.330919

≈ 0.331

(ix) $\sqrt{\frac{7}{8}}$

Solution:

$\sqrt{\frac{7}{8}} = \left(1-\frac{1}{8}\right)^{\frac{1}{2}}$

$= 1+\frac{1}{2}\left(\frac{-1}{8}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{-1}{8}\right)^2+.....$

$= 1-\frac{1}{16}+\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)+\frac{1}{2}\times\frac{1}{64}+.....$

$= 1-\frac{1}{16}-\frac{1}{8\times 64}+.....$

$\approx 1-0.0625-0.0193$

≈ 0.935448

≈ 0.935

(x) $(0.998)^{\frac{-1}{3}}$

Solution:

$(0.998)^{\frac{-1}{3}} = (1-0.002)^{\frac{-1}{3}}$

$= 1+\left(-\frac{1}{3}\right)(-0.002)+\frac{\left(\frac{-1}{3}\right)\left(\frac{-1}{3}-1\right)}{2!}(-0.002)^2+....$

$= 1+\frac{1}{3}(0.002)+\frac{2}{9}(0.000004)+....$

$\approx 1+0.000666$

≈ 1.001

(xi) $\frac{1}{\sqrt[6]{486}}$

Solution:

$\frac{1}{\sqrt[6]{486}} = (486)^{\frac{-1}{6}} = (729-243)^{\frac{-1}{6}}$

$= \left[729\left(1-\frac{243}{729}\right)\right]^{\frac{-1}{6}}$

$$=\left(3^6\right)^{\frac{-1}{6}}\left[1+\left(\frac{-1}{6}\right)\left(-\frac{1}{3}\right)+\frac{\left(\frac{-1}{6}\right)\left(\frac{-7}{6}\right)}{2!}\left(\frac{1}{3}\right)^2+\ldots\right]$$

$$=3^{-1}\left[1+\frac{1}{18}+\frac{7}{36}\cdot\frac{1}{2}\cdot\frac{1}{9}+\ldots\ldots\right]$$

$$=\frac{1}{3}\left[1+0.05556+\frac{7}{2}(0.003086)+\ldots.\right]$$

$$\approx\frac{1}{3}[1+0.05556+0.0108]$$

$$\approx\frac{1}{3}[1.06896]$$

$$\approx 0.35632$$

$$\approx 0.3563$$

(xii) $(1280)^{\frac{1}{4}}$

Solution:

$$(1280)^{\frac{1}{4}}=(1296-16)^{\frac{1}{4}}$$

$$=\left[1296\left(1-\frac{16}{1296}\right)\right]^{\frac{1}{4}}$$

$$=(1296)^{\frac{1}{4}}\left[1-\frac{1}{81}\right]^{\frac{1}{4}}$$

$$=6\left[1+\frac{1}{4}\left(\frac{-1}{81}\right)+\frac{\left(\frac{1}{4}\right)\left(\frac{1}{4}-1\right)}{2!}\left(-\frac{1}{81}\right)^2+\ldots.\right]$$

$$\approx 6\left[1-0.003086-\frac{3}{2}(0.0000095)\right]$$

$$\approx 6[1-0.0031]$$

$$\approx 6(0.9969)$$

$$\approx 5.981$$

$$\approx 5.981$$

Q.3 **Find the coefficient of x^n in the expansion of**

(i) $\dfrac{1+x^2}{(1+x)^2}$

Solution:

$$\frac{1+x^2}{(1+x)^2}=(1+x^2)(1+x)^{-2}$$

From $(1+x)^{2}$ firstly we find the coefficient of x^{n-2} and x^n

As we know that $(r+1)^{th}$ term of $(1+x)^n$ is

$$T_{r+1}=\frac{n(n-1)(n-2)(n-3)....(n-(r-1))x^r}{r!}$$

For x^n put $n=-2, r=n$ we get

$$T_{n+1}=\frac{-2(-2-1)(-2-2)(-2-3)....(-2-(n-1))x^n}{n!}$$

$$=\frac{(-2)(-3)(-4)...(-n-1)x^n}{n!}$$

$$=\frac{(-1)^n(2)(3)(4)....(n+1)x^n}{n!}$$

$$=(-1)^n\frac{(n+1)!}{n!}x^n$$

$$T_{n+1}=(-1)^n(n+1)x^n$$

So in $(1+x)^{-2}$ coefficient of x^n is $(-1)^n(n+1)$

So coefficient of x^{n-2} is $(-1)^{n-2}(n-1)x^{n-2}$

$$=(-1)^n(n+1)x^n+(-1)^{n-2}(n-1)x^2\times x^{n-2}$$

$$=(-1)^n\left\{(n+1)+(-1)^{-2}(n-1)\right\}x^n$$

$$=(-1)^n(2n)x^n$$

Hence coefficient of x^n $is(-1)^n(2n)$

(ii) $\dfrac{(1+x)^2}{(1-x)^2}$

Solution:

$$\frac{(1+x)^2}{(1-x)^2}=(1+x)^2(1-x)^{-2}$$

$$=(1+2x+x^2)(1-x)^{-2}$$

From $(1-x)^{-2}$ firstly we find the coefficient of x^{n-2}, x^{n-1} *and* x^n

As we know that $(r+1)^{th}$ term of $(1+x)^n$ is

$$T_{r+1}=\frac{n(n-1)(n-2)(n-3)....(n-(r-1))x^r}{r!}$$

For x^n put $n=-2, r=n$, $x=-x$ we get

$$T_{n+1}=\frac{(-2)(-2-1)(-2-2)(-2-3).....(-2-(n-1))(-x)^n}{n!}$$

$$T_{n+1}=\frac{(-2)(-3)(-4)(-5)....(-1-n)(-x)^n}{n!}$$

There are n factors in the numerators, so taking -1 from each factors we get;

$$T_{n+1}=\frac{(-1)^n(2\times3\times4\times5\times......\times(n+1))x^n(-1)^n}{n!}$$

$$T_{n+1}=(-1)^{n+n}\frac{(n+1)!}{n!}x^n$$

$$T_{n+1}=(-1)^{2n}\times\frac{(n+1)n!}{n!}x^n$$

So in $(1-x)^{-2}$ coefficient of x^n is $(n+1)$.

So coefficient of x^{n-1} is n

Coefficient of x^{n-2} *is* $(n-1)$

Now in $(1+2x+x^2)(1-x)^{-2}$ term involving x^n is

$$=(n+1)x^n+2\times n\times x^{n-1}\times x+1\times(n-1)x^{n-2}\times x^2$$

$$=x^n\{n+1+2n+n-1\}$$

$=\{4n\}x^n$ so coefficient of x^n is $1+n$

(iii) $\dfrac{(\mathbf{1+x})^3}{(\mathbf{1-x})^2}$

Solution:

$$\frac{(1+x)^3}{(1-x)^2}=(1+x)^3(1-x)^2$$

$$=(1+3x^2+3x+x^3)(1-x)^{-2} \text{ -----(i)}$$

In order to find coefficient of x^n in $\dfrac{(1+x)^3}{(1-x)^2}$

We have to need coefficient of $x^n, x^{n-1}, x^{n-2}, x^{n-3}$ from $(1-x)^2$

As we know that $(r+1)^{th}$ term of $(1+x)^n$ is

$$T_{r+1}=\frac{n(n-1)(n-2)(n-3).....(n-(r-1))x^r}{r!}$$

So put $n=-2, x=-x, r=n$ we get

$$T_{n+1}=\frac{(-2)(-2-1)(-2-2)(-2-3).....(-2-(n-1))(-x)^n}{n!}$$

$$T_{n+1}=\frac{(-2)(-3)(-4)(-5).....(-1-n)x^n\times(-1)^n}{n!}$$

$$=\frac{(-1)^n\left(2\times3\times4\times....(n+1)\right)x^n\times(-1)^n}{n!}$$

$$=\left((-1)^n\right)^2\frac{(n+1)!}{n!}x^n$$

$$T_{n+1}=(n+1)x^n$$

So coefficient of x^{n-1} is n

Coefficient of x^{x-2} is $n-1$

Coefficient of x^{n-3} is $n-2$

Now the term Involving x^n in $\frac{(1+x)^3}{(1-x)^2}$ is

$$=1\times(n+1)x^n+3x^2\times(n-1)x^{n-2}+3x\times(n)x^{n-1}\ +(n-2)x^{x-3}\times x^3$$

$$=(n+1)x^n+(3n-3)x^n+3nx^n+(n-2)x$$

$$=(8n-4)x^n$$

$$=4(2n-1)x^n$$

Hence coefficient of x^n is $4(2n-1)$

(iv) $\frac{(1+x)^2}{(1-x)^3}$

Solution:

$$\frac{(1+x)^2}{(1-x)^3}=(1+x)^2(1-x)^{-3}$$

$$=(1+2x+x^2)(1-x)^{-3}$$

From $(1-x)^{-3}$ firstly we find the coefficient of x^{n-2}, x^{n-1} *and* x^n

As we know that $(r+1)^{th}$ term of $(1+x)^n$ is

$$T_{r+1}=\frac{n(n-1)(n-2)(n-3)....(n-(r-1))x^r}{r!}$$

For x^n put $n=-3, r=n$ and $x=-x$ we get

$$T_{n+1}=\frac{-3(-3-1)(-3-2)(-3-3)....(-3-(n-1))(-x)^n}{n!}$$

$$T_{n+1}=\frac{(-3)(-4)(-5)....(-3-n+1)(-x)^n}{n!}$$

$$T_{n+1}=\frac{(-1)^n(2)(3)(4)(5)....(n+2)x^n(-1)^n}{2n!}$$

$$T_{n+1}=\frac{(-1)^{2n}(n+2)!x^n}{2n!}$$

$$T_{n+1}=\frac{(n+2)(n+1)n!x^n}{2n!}$$

$$T_{n+1}=(n+2)(n+1)x^n$$

So coefficient of x^n is $\frac{(n+2)(n+1)}{2}$

Coefficient of x^{n-1} is $\frac{(n+1)n}{2}$ and coefficient of x^{n-2} is $\frac{n(n-1)}{2}$

Now the term Involving x^n in $(1+2x+x^2)(1-x)^{-3}$ is

$$=\frac{(n+2)(n+1)}{2}x^n+\frac{2(n+1)n}{2}x.x^{n-1}+\frac{n(n-1)}{2}x^2.x^{n-2}$$

$$=\left\{\frac{(n+2)(n+1)}{2}+\frac{2(n+1)n}{2}+\frac{n(n-1)}{2}\right\}x^n$$

$$=\frac{1}{2}\left\{n^2+3n+2+2n^2+2n+n^2-n\right\}x^n$$

$$=\frac{1}{2}\left\{4n^2+4n+2\right\}x^n$$

$$=(2n^2+2n+1)x^n$$

Thus coefficient of x^n is $2n^2+2n+1$

(v) $\mathbf{(1-x+x^2-x^3+.....)^2}$

Solution:

As we know that

$$1-x+x^2-x^3+...=(1+x)^{-1}$$

$$\Rightarrow(1-x+x^2-x^3+......)^2=\left((1+x)^{-1}\right)^2=(1+x)^{-2}$$

Now we find the coefficient of x^n in $(1+x)^{-2}$ by using formula

$$T_{r+1}=\frac{n(n-1)(n-2)(n-3).....(n-(r-1))x^r}{r!}$$

Put $n=-2$, $r=n$ we get

$$T_{n+1}=\frac{(-2)(-2-1)(-2-2)(-2-3).....(-2-(n-1))x^n}{n!}$$

$$=\frac{(-2)(-3)(-4)(-5)......(-1-n)x^n}{n!}$$

$$=\frac{(-1)^n\left[2\times3\times4\times5\times......\times(n+1)\right]x^n}{n!}$$

$$=\frac{(-1)^n(n+1)!}{n!}x^n$$

$$=(-1)^n(n+1)x^n$$

Thus coefficient of x^n is $(-1)^n(n+1)$

Q.4 If x is so small that its square and higher powers can be neglected then show that

(i) $\dfrac{\mathbf{1-x}}{\sqrt{\mathbf{1+x}}}\approx \mathbf{1-\frac{3}{2}x}$

Solution:

L.H.S $=\dfrac{1-x}{\sqrt{1+x}}=(1-x)(1+x)^{\frac{-1}{2}}$

$=(1-x)\left\{1+\left(\dfrac{-1}{2}\right)x\right\}$ by neglecting x^2 and highest power of x.

$\approx 1-\dfrac{1}{2}x-x$ by neglecting x^2

$\approx 1-\dfrac{3}{2}x$

$\approx$ R.H.S

(ii) $\dfrac{\sqrt{\mathbf{1+2x}}}{\mathbf{1-x}}\approx \mathbf{1+\frac{3}{2}x}$

Solution:

L.H.S $=\dfrac{\sqrt{1+2x}}{1-x}$

$=(1+2x)^{\frac{1}{2}}(1-x)^{\frac{-1}{2}}$

$=\left(1+\dfrac{1}{2}(2x)\right)\left(1-\left(\dfrac{-1}{2}\right)x\right)$ by neglecting x^2 and higher powers of x.

$\approx(1+x)\left(1+\dfrac{x}{2}\right)$

$\approx 1+\dfrac{x}{2}+x$ by neglecting x^2

$\approx 1+\dfrac{3}{2}x$

$\approx$ R.H.S

(iii) $\dfrac{(\mathbf{9+7x})^{\frac{1}{2}}-(\mathbf{16+3x})^{\frac{1}{2}}}{\mathbf{4+5x}}\approx\mathbf{\frac{1}{4}-\frac{17}{384}x}$

Solution:

$$\text{L.H.S} = \frac{(9+7x)^{\frac{1}{2}} - (16+3x)^{\frac{1}{4}}}{(4+5x)}$$

$$= \left\{ 9^{\frac{1}{2}}\left(1+\frac{7}{9}x\right)^{\frac{1}{2}} - (16)^{\frac{1}{4}}\left(1+\frac{3x}{16}\right)^{\frac{1}{4}} \right\}(4+5x)^{-1}$$

$$= \left\{ 3\left(1+\frac{7}{9}x\right)^{\frac{1}{2}} - 2\left(1+\frac{3x}{16}\right)^{\frac{1}{4}} \right\} 4^{-1}\left\{1+\frac{5}{4}x\right\}^{-1}$$

$$= \left\{ 3\left(1+\frac{7}{9}x.\frac{1}{2}\right) - 2\left(1+\frac{1}{4}\left(\frac{3x}{16}\right)\right)\right\} \frac{\left(1-\frac{5}{4}x\right)}{4}$$

By neglecting x^2 and higher powers of x.

$$\approx \left\{3+\frac{7x}{6}-2-\frac{3x}{32}\right\}\frac{\left(1-\frac{5}{4}x\right)}{4}$$

$$\approx \frac{\left\{1+\frac{103}{96}x\right\}\left\{1-\frac{5}{4}x\right\}}{4}$$

$$\approx \frac{1-\frac{5}{4}x+\frac{103}{96}x}{4} \qquad \text{by neglecting } x^2$$

$$\approx \frac{1-\frac{17}{96}x}{4}$$

$$\approx \frac{1}{4}-\frac{17}{384}x$$

$$\approx \text{R.H.S}$$

(iv) $\dfrac{\sqrt{4+x}}{(1-x)^3} \approx 2 + \dfrac{25}{4}x$

Solution:

$$\text{L.H.S} = \frac{\sqrt{4+x}}{(1-x)^3}$$

$$= (4+x)^{\frac{1}{2}}(1-x)^{-3}$$

$$= 4^{\frac{1}{2}}\left(1+\frac{x}{4}\right)^{\frac{1}{2}}(1-x)^{-3}$$

$$= 2\left(1+\frac{1}{2}\left(\frac{x}{4}\right)\right)(1+3x) \text{ by neglecting } x^2 \text{ and highest power of } x$$

$$\approx 2\left(1+\frac{x}{8}\right)(1+3x)$$

$$\approx 2\left(1+3x+\frac{x}{8}\right) \text{ by neglecting } x^2$$

$$\approx 2+\frac{25}{4}x$$

$$\approx \text{R.H.S}$$

(v) $\dfrac{(1+x)^{\frac{1}{2}}(4-3x)^{\frac{3}{2}}}{(8+5x)^{\frac{1}{3}}} \approx 4\left(1-\dfrac{5x}{6}\right)$

Solution:

$$\text{L.H.S} = \frac{(1+x)^{\frac{1}{2}}(4-3x)^{\frac{3}{2}}}{(8+5x)^{\frac{1}{3}}}$$

$$= \left\{(1+x)^{\frac{1}{2}}(4-3x)^{\frac{3}{2}}\right\}(8+5x)^{\frac{-1}{3}}$$

$$= \left\{(1+x)^{\frac{1}{2}}.(4)^{\frac{3}{2}}\left(1-\frac{3}{4}x\right)^{\frac{3}{2}}\right\}\times(8)^{\frac{-1}{3}}\left(1+\frac{5}{8}x\right)^{\frac{-1}{3}}$$

$$= \left\{\left(1+\frac{1}{2}x\right).2^3\times\left(1-\frac{3}{2}\left(\frac{3}{4}x\right)\right)\right\}\times\left(2^3\right)^{\frac{-1}{3}}\left(1-\frac{1}{3}\left(\frac{5}{8}x\right)\right) \text{ by neglecting } x^2, x^3, ...$$

$$\approx 8\left\{\left(1+\frac{1}{2}x\right)\left(1-\frac{9}{8}x\right)\right\}\times 2^{-1}\left(1-\frac{5}{24}x\right)$$

$$\approx \frac{8}{2}\left\{1-\frac{9}{8}x+\frac{1}{2}x\right\}\left(1-\frac{5}{24}\right) \text{ by neglecting } x^2$$

$$\approx 4\left\{1-\frac{5}{8}x\right\}\left(1-\frac{5}{24}x\right)$$

$$\approx 4\left(1-\frac{5}{24}x-\frac{5}{8}x\right) \text{ by neglecting } x^2$$

$$\approx 4\left(1-\frac{5}{6}x\right)$$

$$\approx \text{R.H.S}$$

(vi) $\dfrac{(1-x)^{\frac{1}{2}}(9-4x)^{\frac{1}{2}}}{(8+3x)^{\frac{1}{3}}} \approx \dfrac{3}{2}-\dfrac{61}{48}x$

Solution:

$$\text{L.H.S} = \frac{(1-x)^{\frac{1}{2}}(9-4x)^{\frac{1}{2}}}{(8+3x)^{\frac{1}{3}}}$$

$$=\left\{(1-x)^{\frac{1}{2}}.9^{\frac{1}{2}}\left(1-\frac{4}{9}x\right)^{\frac{1}{2}}\right\}(8+3x)^{\frac{-1}{3}}$$

$$=3\left(1-\frac{1}{2}x+.....\right)\left(1-\frac{1}{2}\left(\frac{4}{9}x\right)+.....\right)\times 8^{\frac{-1}{3}}\left(1+\frac{3x}{8}\right)^{\frac{-1}{3}} \text{ by neglecting } x^2, x^3, ...$$

$$\approx 3\left(1-\frac{1}{2}x\right)\left(1-\frac{2}{9}x\right)\times 2^{-1}\left(1-\frac{1}{3}\times\frac{3x}{8}+......\right)$$

$$\approx \frac{3}{2}\left(1-\frac{2}{9}x-\frac{1}{2}x\right)\left(1-\frac{x}{8}\right) \text{ by neglecting } x^2.$$

$$\approx \frac{3}{2}\left(1-\frac{13}{18}x\right)\left(1-\frac{x}{8}\right)$$

$$\approx \frac{3}{2}\left(1-\frac{x}{8}-\frac{13}{18}x\right) \text{ by neglecting } x^2$$

$$\approx \frac{3}{2}\left(1-\frac{61}{72}x\right)$$

$$\approx \frac{3}{2}-\frac{61}{48}x$$

$$\approx \text{R.H.S}$$

(vii) $\dfrac{\sqrt{4-x}+(8-x)^{\frac{1}{3}}}{(8-x)^{\frac{1}{3}}} \approx 2-\dfrac{1}{2}x$

Solution:

$$\text{L.H.S} = \frac{(4-x)^{\frac{1}{2}}+(8-x)^{\frac{1}{3}}}{(8-x)^{\frac{1}{3}}}$$

$$=\frac{(4-x)^{\frac{1}{2}}}{(8-x)^{\frac{1}{3}}}+\frac{(8-x)^{\frac{1}{3}}}{(8-x)^{\frac{1}{3}}}$$

$$=(4-x)^{\frac{1}{2}}(8-x)^{\frac{-1}{3}}+1$$

$$=4^{\frac{1}{2}}\left(1-\frac{x}{4}\right)^{\frac{1}{2}}\times 8^{\frac{-1}{3}}\left(1-\frac{x}{8}\right)^{\frac{-1}{3}}+1$$

$$=2\left(1-\frac{1}{2}\left(\frac{x}{4}\right)+......\right)\times 2^{-1}\left(1+\frac{1}{3}\left(\frac{x}{8}\right)+......\right)+1 \text{ by neglecting } x^2, x^3, ...$$

$$\approx \frac{\left(2-\frac{x}{4}\right)\left(1+\frac{x}{24}\right)}{2}+1$$

$$\approx \frac{2+\frac{x}{12}-\frac{x}{4}}{2}+1 \qquad \text{By neglecting } x^2$$

$$\approx \frac{2-\frac{1}{6}x}{2}+1$$

$$\approx 1-\frac{1}{12}x+1$$

$$\approx 2-\frac{1}{12}x$$

$$\approx \text{R.H.S}$$

Q.5 If x is so small that its cube and higher power can be neglected, then show that

(i) $\sqrt{1-x-2x^2} \approx 1-\frac{1}{2}x-\frac{9}{8}x^2$

Solution:

L.H.S = $\sqrt{1-x-2x^2}$

$=\left(1-\left(x+2x^2\right)\right)^{\frac{1}{2}}$

$=1-\frac{1}{2}\left(x+2x^2\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(x+2x^2\right)^2}{2!}$ by neglecting $x^3, x^4, \ldots\ldots$

$=1-\frac{1}{2}x-x^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(x^2\right)}{2}$ by neglecting $x^3, x^4 \ldots$

$\approx 1-\frac{1}{2}x-x^2-\frac{1}{8}x^2$

$\approx 1-\frac{1}{2}x-\frac{9}{8}x^2$

$\approx$ R.H.S

(ii) $\sqrt{\frac{1+x}{1-x}} \approx 1+x+\frac{1}{2}x^2$

Solution:

L.H.S $=\sqrt{\frac{(1+x)}{(1-x)}\times\frac{1+x}{1+x}}$

$=(1+x)\left(1-x^2\right)^{\frac{-1}{2}}$

$=(1+x)\left(1+\frac{1}{2}x^2\right)$ by neglecting $x^3, x^4 \ldots$

$\approx (1+x)\left(1+\frac{1}{2}x^2\right)$

$\approx 1+x+\frac{1}{2}x^2$ by neglecting x^3

$\approx 1+x+\frac{1}{2}x^2$

$\approx$ R.H.S

Q.6 If x is nearly equal to 1, then prove that $\mathbf{px^p - qx^q \approx (p-q)x^{p+q}}$

Proof: As x is nearly equal to 1, so

Let $x = 1 + h$ where h is so small such that $h^2, h^3, \ldots$ are neglected.

L.H.S $= px^p - qx^q$

$= p(1+h)^p - q(1+h)^q$

$\approx p(1+ph) - q(1+qh)$ by neglecting $h^2, h^3 \ldots$

$\approx p + p^2h - q - q^2h$

$\approx (p-q) + (p^2 - q^2)h$

$\approx (p-q) + (p-q)(p+q)h$

$\approx (p-q)[1 + (p+q)h]$

$\approx (p-q)(1+h)^{p+q}$

$\approx (p-q)(x)^{p+q}$

$\approx$ R.H.S

Q.7 If $\mathbf{p-q}$ is small when compared with p or q show that

$$\frac{(2n+1)p + (2n-1)q}{(2n-1)p + (2n+1)q} \approx \left(\frac{p+q}{2q}\right)^{\frac{1}{n}}$$

Proof: Let $p - q = h \Rightarrow p = q + h$ where h is very small such that $h^2, h^3, \ldots.$ are neglected

L.H.S $= \dfrac{(2n+1)p + (2n-1)q}{(2n-1)p + (2n+1)q}$

$= \dfrac{(2n+1)(q+h) + (2n-1)q}{(2n-1)(q+h) + (2n+1)q}$

$= \dfrac{2nq + 2nh + q + h + 2nq - q}{2nq + 2nh - q - h + 2nq + q}$

$= \dfrac{4nq + 2nh + h}{4nq + 2nh - h}$

$= \dfrac{4nq + h(2n+1)}{4nq + h(2n-1)}$

$= \dfrac{4nq\left\{1 + \left(\dfrac{2n+1}{4nq}\right)h\right\}}{4nq\left\{1 + \left(\dfrac{2n-1}{4nq}\right)h\right\}}$

$= \left\{1 + \left(\dfrac{2n+1}{4nq}\right)h\right\}\left\{1 + \left(\dfrac{2n-1}{4nq}\right)h\right\}^{-1}$

$$=\left\{1+\left(\frac{2n+1}{4nq}\right)h\right\}\left\{1-\left(\frac{2n-1}{4nq}\right)h\right\} \text{ by neglecting } h^2,h^3,.....$$

$$\approx 1-\left(\frac{2n-1}{4nq}\right)h+\left(\frac{2n+1}{4nq}\right)h \text{ by neglecting } h^2$$

$$\approx 1-h\left(\frac{(2n-1)}{4nq}-\frac{(2n+1)}{4nq}\right)$$

$$\approx 1-\frac{h}{4nq}(2n-1-2n-1)$$

$$\approx 1-\frac{h}{4nq}(-2)$$

$$\approx 1+\frac{h}{2nq}$$

$$\text{R.H.S}=\left(\frac{p+q}{2q}\right)^{\frac{1}{n}}$$

$$\approx\left(\frac{q+h+q}{2q}\right)^{\frac{1}{n}} \qquad \because p=q+h$$

$$\approx\left(\frac{2q+h}{2q}\right)^{\frac{1}{n}}$$

$$\approx\left(1+\frac{h}{2q}\right)^{\frac{1}{n}}$$

$$\approx 1+\frac{h}{2nq} \text{ by neglecting } h^2,h^3,....$$

Hence L.H.S $\approx$ R.H.S

Q.8 Show that

$$\left[\frac{\mathbf{n}}{\mathbf{2(n+N)}}\right]^{\frac{1}{2}}\approx\frac{\mathbf{8n}}{\mathbf{9n-N}}-\frac{\mathbf{n+N}}{\mathbf{4n}} \text{ where n and N are nearly equal.}$$

Proof: Here $N-n=h \Rightarrow N=n+h$ where h is so small, such that $h^2,h^3,.....$ are neglected

$$\text{L.H.S}=\left[\frac{n}{2(n+N)}\right]^{\frac{1}{2}}$$

$$=\left[\frac{n}{2(n+n+h)}\right]^{\frac{1}{2}}$$

$$=\left[\frac{n}{2(2n+h)}\right]^{\frac{1}{2}}$$

$$=\left[\frac{n}{2\times 2n\left(1+\frac{h}{2n}\right)}\right]^{\frac{1}{2}}$$

$$=\left[\frac{1}{4\left(1+\frac{h}{2n}\right)}\right]^{\frac{1}{2}}$$

$$=\frac{1}{2}\left(1+\frac{h}{2n}\right)^{\frac{-1}{2}}$$

$$=\frac{1}{2}\left(1-\frac{1}{2}\left(\frac{h}{2n}\right)\right)\text{by neglecting } h^2,h^3,.....$$

$$\approx\frac{1}{2}-\frac{1}{8}\frac{h}{n}$$

R.H.S $=\dfrac{8n}{9n-N}-\dfrac{n+N}{4n}$

Put $N=n+h$

$$=\frac{8n}{9n-n-h}-\frac{n+n+h}{4n}$$

$$=\frac{8n}{8n-h}-\frac{2n+h}{4n}$$

$$=\frac{\cancel{8n}}{\cancel{8n}\left(1-\frac{h}{8n}\right)}-\frac{2n+h}{4n}$$

$$=\left(1-\frac{h}{8n}\right)^{-1}-\frac{2n}{4n}-\frac{h}{4n}$$

$$=1+\frac{h}{8n}-\frac{1}{2}-\frac{h}{4n}\text{ by neglecting } h^2,h^3,.....$$

$$\approx\frac{1}{2}-\frac{h}{8n}$$

Hence L.H.S $\approx$ R.H.S

Q.9 Identify the following series as binomial expansion and find the sum in each case.

(i) $1-\frac{1}{2}\left(\frac{1}{4}\right)+\frac{1.3}{2!4}\left(\frac{1}{4}\right)^2-\frac{1.3.5}{3!8}\left(\frac{1}{4}\right)+.....$

Solution:

Let $(1+x)^n=1-\frac{1}{2}\left(\frac{1}{4}\right)+\frac{1.3}{2!4}\left(\frac{1}{4}\right)^2-\frac{1.3.5}{3!8}\left(\frac{1}{4}\right)+.....$ (I)

As we know $(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+...$ (II)

Comparing (I) and (II)

$nx=\frac{-1}{2}\left(\frac{1}{4}\right)$

$nx=-\frac{1}{8}$

$x=-\frac{1}{8n}$ (i)

$\frac{n(n-1)x^2}{2!}=\frac{1.3}{2!\times 4}\left(\frac{1}{4}\right)^2$

$n(n-1)x^2=\frac{3}{4}\times\frac{1}{16}$

$n(n-1)\left(-\frac{1}{8n}\right)^2=\frac{3}{64}$ using (i)

$n(n-1)\times\frac{1}{64n^2}=\frac{3}{64}$

$\frac{n-1}{n}=3\Rightarrow n-1=3n\Rightarrow 2n=-1$

$n=\frac{-1}{2}$ Put in (i)

$x=\frac{-1}{8\left(-\frac{1}{2}\right)}\Rightarrow x=\frac{1}{4}$

Now, the sum of the given series = $(1+x)^n=\left(1+\frac{1}{4}\right)^{-\frac{1}{2}}=\left(\frac{5}{4}\right)^{-\frac{1}{2}}=\left(\frac{4}{5}\right)^{-\frac{1}{2}}=\sqrt{\frac{4}{5}}$

(ii) $\mathbf{1-\frac{1}{2}\left(\frac{1}{2}\right)+\frac{1.3}{2.4}\left(\frac{1}{2}\right)^2-\frac{1.3.5}{2.4.6}\left(\frac{1}{2}\right)^3+......}$

Solution:

Let $(1+x)^n=1-\frac{1}{2}\left(\frac{1}{2}\right)+\frac{1.3}{2.4}\left(\frac{1}{2}\right)^2-\frac{1.3.5}{2.4.6}\left(\frac{1}{2}\right)^3+......$ (I)

As we know $(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+...$ (II)

Comparing (I) and (II)

$nx=\frac{-1}{2}\left(\frac{1}{2}\right)$

$nx=-\frac{1}{4}$

$\frac{n(n-1)x^2}{2!}=\frac{1.3}{2\times 4}\left(\frac{1}{2}\right)^2$

$n(n-1)x^2=\frac{3}{4}\times\frac{1}{4}$

$x = -\frac{1}{4n}$ (i)

$n(n-1)\left(-\frac{1}{4n}\right)^2 = \frac{3}{16}$ using (i)

$n(n-1)\times\frac{1}{16n^2} = \frac{3}{16}$

$\frac{n-1}{n} = 3 \Rightarrow n-1 = 3n \Rightarrow 2n = -1$

$n = \frac{-1}{2}$ Put in (i)

$x = -\frac{1}{4\left(-\frac{1}{2}\right)} \Rightarrow x = \frac{1}{2}$

Now, the sum of the given series = $(1+x)^n = \left(1+\frac{1}{2}\right)^{-\frac{1}{2}} = \left(\frac{3}{2}\right)^{-\frac{1}{2}} = \left(\frac{2}{3}\right)^{\frac{1}{2}} = \sqrt{\frac{2}{3}}$

(iii) $\mathbf{1+\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5.7}{4.8.12}+....}$

Solution:

Let $(1+x)^n = 1+\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5.7}{4.8.12}+....$ (I)

As we know $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+...$ (II)

Comparing (I) and (II)

$nx = \frac{3}{4}$

$x = \frac{3}{4n}$ (i)

$\frac{n(n-1)x^2}{2!} = \frac{3.5}{4.8}$

$n(n-1)x^2 = \frac{3.5}{2.8}$

$n(n-1)\left(\frac{3}{4n}\right)^2 = \frac{15}{16}$ using (i)

$n(n-1)\times\frac{9}{16n^2} = \frac{15}{16}$

$\frac{n-1}{n} = \frac{15}{9} \Rightarrow 3n-3 = 5n \Rightarrow 2n = -3$

$n = \frac{-3}{2}$ Put in (i)

$x = \frac{3}{4\left(-\frac{3}{2}\right)} \Rightarrow x = \frac{-1}{2}$

Now, the sum of the given series

$$=(1+x)^n=\left(1-\frac{1}{2}\right)^{-\frac{3}{2}}=\left(\frac{1}{2}\right)^{-\frac{3}{2}}=(2)^{\frac{3}{2}}=\left(2^3\right)^{\frac{1}{2}}=\sqrt{8}=2\sqrt{2}$$

(iv) $\mathbf{1-\frac{1}{2}\left(\frac{1}{3}\right)+\frac{1.3}{2.4}\left(\frac{1}{3}\right)^2-\frac{1.3.5}{2.4.6}\left(\frac{1}{3}\right)^3+\ldots}$

Solution:

Let $(1+x)^n=1-\frac{1}{2}\left(\frac{1}{3}\right)+\frac{1.3}{2.4}\left(\frac{1}{3}\right)^2-\frac{1.3.5}{2.4.6}\left(\frac{1}{3}\right)^3+\ldots..$ (I)

As we know $(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots$ (II)

Comparing (I) and (II)

$$nx=\frac{-1}{2}\left(\frac{1}{3}\right)$$

$$nx=-\frac{1}{6}$$

$$x=-\frac{1}{6n} \quad \text{(i)}$$

$$\frac{n(n-1)x^2}{2!}=\frac{1.3}{2.4}\left(\frac{1}{3}\right)^2$$

$$n(n-1)x^2=\frac{3}{4}\times\frac{1}{9}$$

$$n(n-1)\left(-\frac{1}{6n}\right)^2=\frac{3}{36} \quad \text{using (i)}$$

$$n(n-1)\times\frac{1}{36n^2}=\frac{3}{36}$$

$$\frac{n-1}{n}=3\Rightarrow n-1=3n\Rightarrow 2n=-1$$

$$n=\frac{-1}{2} \text{ Put in (i)}$$

$$x=\frac{-1}{6\left(-\frac{1}{2}\right)}\Rightarrow x=\frac{1}{3}$$

Now, the sum of the given series $=(1+x)^n=\left(1+\frac{1}{3}\right)^{-\frac{1}{2}}=\left(\frac{4}{3}\right)^{-\frac{1}{2}}=\left(\frac{3}{4}\right)^{\frac{1}{2}}=\frac{\sqrt{3}}{2}$

Q.10 Use binomial theorem to show that $\mathbf{1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+\ldots.=\sqrt{2}}$

Proof: Let $(1+x)^n=1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+\ldots..$ (I)

As we know $(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots$ (II)

Comparing (I) and (II)

$$nx=\frac{1}{4}$$

$$\frac{n(n-1)x^2}{2!}=\frac{1.3}{4.8}$$

$x = \frac{1}{4n}$ (i)

$n(n-1)x^2 = \frac{1.3}{2.8}$

$n(n-1)\left(\frac{1}{4n}\right)^2 = \frac{3}{16}$ using (i)

$n(n-1)\times\frac{1}{16n^2} = \frac{3}{16}$

$\frac{n-1}{n} = 3 \Rightarrow n-1 = 3n \Rightarrow 2n = -1$

$n = \frac{-1}{2}$ Put in (i)

$x = \frac{1}{4\left(-\frac{1}{2}\right)} \Rightarrow x = \frac{-1}{2}$

Now, the sum of the given series $= (1+x)^n = \left(1-\frac{1}{2}\right)^{-\frac{1}{2}} = \left(\frac{1}{2}\right)^{-\frac{1}{2}} = (2)^{\frac{1}{2}} = \sqrt{2}$

Hence the proof

Q.11 If $y = \frac{1}{3} + \frac{1.3}{2!}\left(\frac{1}{3}\right)^2 + \frac{1.3.5}{3!}\left(\frac{1}{3}\right)^3 + ...$

Then prove that $y^2 + 2y - 2 = 0$

Proof: Given that

$$y = \frac{1}{3} + \frac{1.3}{2!}\left(\frac{1}{3}\right)^2 + \frac{1.3.5}{3i}\left(\frac{1}{3}\right)^3 + ...$$

Adding 1 on both sides of given series of make it binomial series

$$1+y = 1+\frac{1}{3} + \frac{1.3}{2!}\left(\frac{1}{3}\right)^2 + \frac{1.3.5}{3!}\left(\frac{1}{3}\right)^3 + ... \quad \text{(I)}$$

As we know $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ...$ (II)

Comparing (I) and (II)

$nx = \frac{1}{3}$

$x = \frac{1}{3n}$ (i)

$\frac{n(n-1)x^2}{2!} = \frac{1.3}{2!}\left(\frac{1}{3}\right)^2$

$n(n-1)x^2 = \frac{3}{9}$

$n(n-1)\left(\frac{1}{3n}\right)^2 = \frac{3}{9}$ using (i)

$n(n-1)\times\frac{1}{9n^2} = \frac{3}{9}$

$\frac{n-1}{n} = 3 \Rightarrow n-1 = 3n \Rightarrow 2n = -1$

$n=\frac{-1}{2}$ Put in (i)

$$x=\frac{1}{3\left(-\frac{1}{2}\right)} \Rightarrow x=\frac{-2}{3}$$

From (I) and (II)

$$1+y=(1+x)^n$$

$$1+y=\left(1-\frac{2}{3}\right)^{\frac{-1}{2}}$$

$$1+y=\left(\frac{1}{3}\right)^{\frac{-1}{2}}$$

$$1+y=\sqrt{3}$$

Squaring on both sides

$$1+y^2+2y=3$$

$$y^2+2y-2=0$$

Which is required to prove.

Q.12 **If** $2y = \frac{1}{2^2} + \frac{1.3}{2!}.\frac{1}{2^4} + \frac{1.3.5}{3!}.\frac{1}{2^6} + ...$

Then Prove that $4y^2 + 4y - 1 = 0$

Proof: Given that

$$2y = \frac{1}{2^2} + \frac{1.3}{2!}.\frac{1}{2^4} + \frac{1.3.5}{3!}.\frac{1}{2^6} + ...$$

Adding 1 on both sides to make it binomial series, we get ;

$$1+2y = 1 + \frac{1}{2^2} + \frac{1.3}{2!} \times \frac{1}{2^4} + \frac{1.3.5}{3!} \times \frac{1}{2^6} + ... \qquad \text{(I)}$$

As we know $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ...$ (II)

Comparing (I) and (II)

$$nx = \frac{1}{2^2}$$

$$x = \frac{1}{4n} \qquad \text{(i)}$$

$$\frac{n(n-1)x^2}{2!} = \frac{1.3}{2!}.\frac{1}{2^4}$$

$$n(n-1)x^2 = \frac{3}{16}$$

$$n(n-1)\left(\frac{1}{4n}\right)^2 = \frac{3}{16} \qquad \text{using (i)}$$

$$n(n-1) \times \frac{1}{16n^2} = \frac{3}{16}$$

$$\frac{n-1}{n} = 3 \Rightarrow n-1 = 3n \Rightarrow 2n = -1$$

$$n = \frac{-1}{2} \text{ Put in (i)}$$

$$x = \frac{1}{4\left(-\frac{1}{2}\right)} \Rightarrow x = \frac{-1}{2}$$

From (I) and (II)

$$1+2y = (1+x)^n$$

$$1+2y = \left(1 - \frac{1}{2}\right)^{\frac{-1}{2}}$$

$$1+2y = \left(\frac{1}{2}\right)^{\frac{-1}{2}}$$

$$1+2y = \sqrt{2}$$

Squaring on both sides

$$1 + 4y^2 + 4y = 2$$

$$4y^2 + 4y - 1 = 0$$

Which is required to prove.

Q.13 **If** $y=\frac{2}{5}+\frac{1.3}{2!}\left(\frac{2}{5}\right)^2+\frac{1.3.5}{3!}\left(\frac{2}{5}\right)^3+......$

Then prove that $y^2+2y-4=0$

Proof: Give that

$$y=\frac{2}{5}+\frac{1.3}{2!}\left(\frac{2}{5}\right)^2+\frac{1.3.5}{3!}\left(\frac{2}{5}\right)^3+......$$

Adding 1 on both sides of given series to make it binomial series.

So, $1+y=1+\frac{2}{5}+\frac{1.3}{2!}\left(\frac{2}{5}\right)^2+\frac{1.3.5}{3!}\left(\frac{2}{5}\right)^3+.......$ (I)

As we know $(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+.....$ (II)

Comparing (I) and (II)

$$nx=\frac{2}{5}$$

$$x=\frac{2}{5n} \quad \text{(i)}$$

$$\frac{n(n-1)x^2}{2!}=\frac{1.3}{2!}.\left(\frac{2}{5}\right)^2$$

$$n(n-1)x^2=\frac{12}{25}$$

$$n(n-1)\left(\frac{2}{5n}\right)^2=\frac{12}{25} \quad \text{using (i)}$$

$$n(n-1)\times\frac{4}{25n^2}=\frac{12}{25}$$

$$\frac{n-1}{n}=3\Rightarrow n-1=3n\Rightarrow 2n=-1$$

$$n=\frac{-1}{2} \text{ Put in (i)}$$

$$x=\frac{2}{5\left(-\frac{1}{2}\right)} \Rightarrow x=\frac{-4}{5}$$

From (I) and (II)

$$1+y=(1+x)^n$$

$$1+y=\left(1-\frac{4}{5}\right)^{\frac{-1}{2}}$$

$$1+y=\left(\frac{1}{5}\right)^{\frac{-1}{2}}$$

$$1+y=\sqrt{5}$$

Squaring on both sides

$$1+y^2+2y=5$$

$$y^2+2y-4=0$$

Which is required to prove.