CHAPTER 10

TRIGONOMETRIC IDENTITIES

Trigonometric Identities.

Identities involving trigonometric ratios $(\sin\theta, \cos\theta, etc.)$ are called trigonometric identities.

Distance Formula:

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points. If 'd' denotes the distance between them, then

$$d = |PQ| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

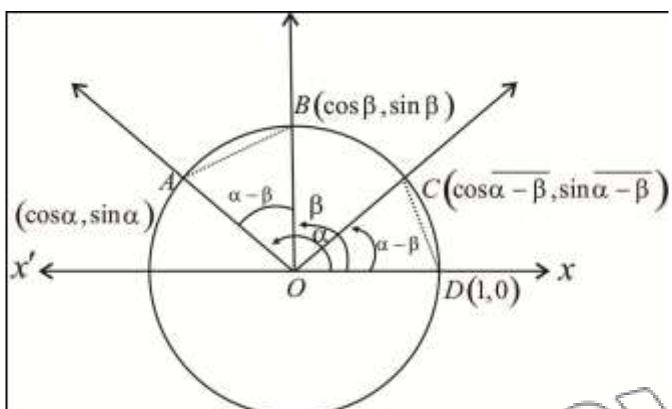
Fundamental law of trigonometry:

Let α and β any two angles (real numbers), then

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Proof:

Let $\alpha > \beta > 0$, consider a unit circle with centre at origin $'O'$.



Let terminal sides of angles α and β cut the unit circle at A and B respectively.

Evidently $\angle AOB = \alpha - \beta$

Take a point $'C'$ on the unit circle so that $\angle XOC = m\angle AOB = \alpha - \beta$

Join A, B and C, D.

Now angles α, β and $\alpha - \beta$ are in standard position.

$\therefore$ the coordinates of A are $(\cos\alpha, \sin\alpha)$

the coordinates of B are $(\cos\beta, \sin\beta)$

the coordinates of C are $(\cos\overline{\alpha - \beta}, \sin\overline{\alpha - \beta})$

and the coordinates of D are $(1, 0)$

Now ΔAOB *and* ΔCOD are congruent by SAS theorem

$\therefore\ |AB| = |CD|$

$\Rightarrow |AB|^2 = |CD|^2$

using distance formula

$$(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = [\cos(\alpha-\beta) - 1]^2 + [\sin(\alpha-\beta) - 0]^2$$

$$\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta + \sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta$$

$$= \cos^2(\alpha-\beta) + 1 - 2\cos(\alpha-\beta) + \sin^2(\alpha-\beta)$$

$$\left(\cos^2\alpha + \sin^2\alpha\right) + \left(\cos^2\beta + \sin^2\beta\right) - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta$$

$$= \cos^2(\alpha-\beta) + \sin^2(\alpha-\beta) + 1 - 2\cos(\alpha-\beta)$$

$$1 + 1 - 2\left(\cos\alpha\cos\beta + \sin\alpha\sin\beta\right) = 1 + 1 - 2\cos(\alpha-\beta)$$

$$2 - 2\left(\cos\alpha\cos\beta + \sin\alpha\sin\beta\right) = 2 - 2\cos(\alpha-\beta)$$

Subtracting "2" from both sides

$-2\left(\cos\alpha\cos\beta + \sin\alpha\sin\beta\right) = -2\cos(\alpha-\beta)$ dividing by '-2' both sides

$$\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta)$$

Hence $\boxed{\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta}$

<u>Note:</u>

It is true for all values of α *and* β

<u>Deductions from Fundamental Law:</u>

(1) We know that $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Putting $\alpha = \frac{\pi}{2}$ in it, we get

$$\cos\left(\frac{\pi}{2} - \beta\right) = \cos\frac{\pi}{2}\cos\beta + \sin\frac{\pi}{2}\sin\beta$$

$$= 0.\cos\beta + 1.\sin\beta \qquad \left(\because \cos\frac{\pi}{2} = 0, \sin\frac{\pi}{2} = 1\right)$$

$$\boxed{\therefore \cos\left(\frac{\pi}{2} - \beta\right) = \sin\beta} \qquad \text{(i)}$$

(2) We know that $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

Putting $\beta=-\frac{\pi}{2}$ in it, we get

$$\cos\left[\alpha-\left(-\frac{\pi}{2}\right)\right]=\cos\alpha\cos\left(-\frac{\pi}{2}\right)+\sin\alpha\sin\left(-\frac{\pi}{2}\right)$$

$$\Rightarrow \cos\left(\alpha+\frac{\pi}{2}\right)=\cos\alpha.0+\sin\alpha(-1)$$

$$\boxed{\therefore \cos\left(\frac{\pi}{2}+\alpha\right)=-\sin\alpha} \quad \text{(ii)}$$

(3) We know that $\cos\left(\frac{\pi}{2}-\beta\right)=\sin\beta$ [(i) above]

Put $\beta=\frac{\pi}{2}+\alpha$ in it, we get

$$\cos\left[\frac{\pi}{2}-\left(\frac{\pi}{2}+\alpha\right)\right]=\sin\left(\frac{\pi}{2}+\alpha\right)$$

$$\Rightarrow \quad \cos(-\alpha)=\sin\left(\frac{\pi}{2}+\alpha\right)$$

$$\Rightarrow \quad \cos\alpha=\sin\left(\frac{\pi}{2}+\alpha\right) \qquad \because \cos(-\alpha)=\cos\alpha$$

$$\boxed{\therefore \sin\left(\frac{\pi}{2}+\alpha\right)=\cos\alpha}$$

(4) We know that $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

Replace β by $-\beta$

we get

$$\cos[\alpha-(-\beta)]=\cos\alpha\cos(-\beta)+\sin\alpha\sin(-\beta)$$

Hence $\boxed{\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta}$

(5) We know that $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

Replace "α" by "$\frac{\pi}{2}+\alpha$" we get

$$\cos\left[\left(\frac{\pi}{2}+\alpha\right)+\beta\right]=\cos\left(\frac{\pi}{2}+\alpha\right)\cos\beta-\sin\left(\frac{\pi}{2}+\alpha\right)\sin\beta$$

$\cos\left[\frac{\pi}{2}+(\alpha+\beta)\right]=-\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$\cos\frac{\pi}{2}\cos(\alpha+\beta)-\sin\frac{\pi}{2}\sin(\alpha+\beta)=-[\sin\alpha\cos\beta+\cos\alpha\sin\beta]$

$0-\sin(\alpha+\beta)=-(\sin\alpha\cos\beta+\cos\alpha\sin\beta)$

$-\sin(\alpha+\beta)=-(\sin\alpha\cos\beta+\cos\alpha\sin\beta)$

multiply both sides by '-1'

Hence $\boxed{\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta}$

(6) We know that $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

Replace "β" by "$-\beta$"

$\sin(\alpha-\beta)=\sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta)$

Hence $\boxed{\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta}$

(7) We know that $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

Let $\alpha=2\pi$, and $\beta=\theta$

$\therefore\cos(2\pi-\theta)=\cos2\pi\cos\theta+\sin2\pi\sin\theta$

$=1.\cos\theta+0.\sin\theta$

$=\cos\theta$

$\boxed{\cos(2\pi-\theta)=\cos\theta}$

(8) We know that $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

Let $\alpha=2\pi$ and $\beta=\theta$

$\therefore\sin(2\pi-\theta)=\sin2\pi\cos\theta-\cos2\pi.\sin\theta$

$=0.\cos\theta-1.\sin\theta$

$=-\sin\theta$

$\boxed{\sin(2\pi-\theta)=-\sin\theta}$

(9) We know that $\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$

$\tan(\alpha+\beta)=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$

dividing $\cos\alpha\cos\beta$ in numerator and denominator

$$= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$= \frac{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}{1-\dfrac{\sin\alpha}{\cos\alpha}\cdot\dfrac{\sin\beta}{\cos\beta}}$$

$$\boxed{\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}}$$

(10) We know that $\tan(\alpha-\beta)=\dfrac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}$

$$= \frac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\cos\alpha\cos\beta+\sin\alpha\sin\beta}$$

Dividing by $\cos\alpha\cos\beta$ in numerator and denominator

$$= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

$$\boxed{\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}}$$

Allied Angle:

The angles associated with basic angles of measure θ to a right angle or its multiple are called **allied angles**. So, the angles of measure $90^o \pm \theta$, $180^o \pm \theta$, $270^o \pm \theta$, $360^o \pm \theta$, are known as **allied angles**.

$$\begin{cases}\sin\left(\dfrac{\pi}{2}-\theta\right)=\cos\theta, & \cos\left(\dfrac{\pi}{2}-\theta\right)=\sin\theta, \quad \tan\left(\dfrac{\pi}{2}-\theta\right)=\cot\theta \\ \sin\left(\dfrac{\pi}{2}+\theta\right)=\cos\theta, & \cos\left(\dfrac{\pi}{2}+\theta\right)=-\sin\theta, \quad \tan\left(\dfrac{\pi}{2}+\theta\right)=-\cot\theta\end{cases}$$

$$\begin{cases}\sin(\pi-\theta)=\sin\theta, & \cos(\pi-\theta)=-\cos\theta, \quad \tan(\pi-\theta)= =-\tan\theta \\ \sin(\pi+\theta)=-\sin\theta, & \cos(\pi+\theta)=-\cos\theta, \quad \tan(\pi+\theta)= =\tan\theta\end{cases}$$

$$\begin{cases} \sin\left(\frac{3\pi}{2}-\theta\right)=-\cos\theta, & \cos\left(\frac{3\pi}{2}-\theta\right)=-\sin\theta, \quad \tan\left(\frac{3\pi}{2}-\theta\right)= \cot\theta \\ \sin\left(\frac{3\pi}{2}+\theta\right)=-\cos\theta, & \cos\left(\frac{3\pi}{2}+\theta\right)=\sin\theta, \quad \tan\left(\frac{3\pi}{2}+\theta\right)= -\cot\theta \end{cases}$$

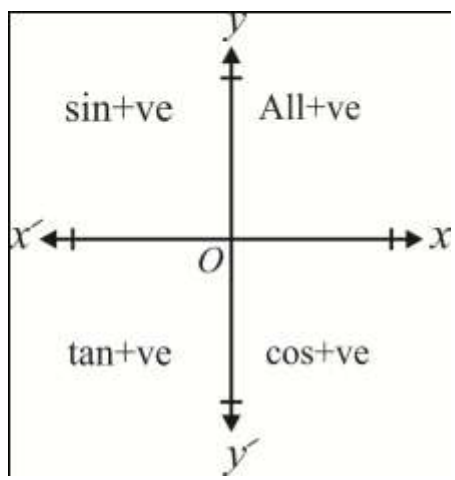
$$\begin{cases} \sin(2\pi-\theta)=-\sin\theta, & \cos(2\pi-\theta)=\cos\theta, \quad \tan(2\pi-\theta)= =-\tan\theta \\ \sin(2\pi+\theta)=+\sin\theta, & \cos(2\pi+\theta)=\cos\theta, \quad \tan(2\pi+\theta)= =\tan\theta \end{cases}$$

(1) If θ is added or subtracted from **odd multiple** of right angle, the trigonometric ratios change into **co-ratios** and vice versa.

$i.e.$ $\sin \rightleftarrows \cos$, $\tan \rightleftarrows \cot$, $\sec \rightleftarrows \operatorname{cosec}$

(2) If θ is added to or subtracted from an even multiple of $\frac{\pi}{2}$, the trigonometric ratios shall remain the same.

Measure of the angle	Quad.
$\frac{\pi}{2}-\theta$ or $2\pi+\theta$	I
$\frac{\pi}{2}+\theta$ or $\pi-\theta$	II
$\pi+\theta$ or $\frac{3\pi}{2}-\theta$	III
$\frac{3\pi}{2}+\theta$ or $2\pi-\theta$	IV



Note:

(1) If basic angle θ is greater than 360^o, we subtract multiples of 360^o to basic angle

For example:

$\sin 750^o = \sin(750^o + (-2)(360^o)) = \sin(750^o - 720^o) = \sin 30^o$

(2) If basic angle θ is between 0^o to 360^o we will reduce it into multiples of 90^o.

$\sin 240^o = \sin(270^o - 30^o) = \sin(3\times 90^o - 30^o) = -\cos 30^o$

Exercise 10.1

Q.1 Without using tables, find the values of:

(i) $\sin(-780^\circ)$

Solution:

$\sin(-780^\circ)$

$=\sin(-780^\circ+2(360^\circ))$

$=\sin(-780^\circ+720^\circ)$

$=\sin(-60^\circ)$

$=-\sin 60^\circ$

$=-\dfrac{\sqrt{3}}{2}$

(ii) $\cot(-855^\circ)$

Solution:

$\cot(-855^\circ)$

$=\cot(-855^\circ+2(360^\circ))$

$=\cot(-855^\circ+720^\circ)$

$=\cot(-135^\circ)$

$=-\cot 135^\circ$

$=-\cot(90^\circ+45^\circ)$

$=-(-\tan 45^\circ)$

$=1$

(iii) $\operatorname{cosec}(2040^\circ)$

Solution:

Here $\operatorname{cosec}(2040^\circ)$

$=\operatorname{cosec}(2040^\circ+(-5)(360^\circ))$

$=\operatorname{cosec}(2040^\circ-1800^\circ)$

$=\operatorname{cosec}(240^\circ)$

$=\operatorname{cosec}(180^\circ+60^\circ)$

$=-\operatorname{cosec}60^\circ$

$=-\dfrac{2}{\sqrt{3}}$

(iv) $\sec(-960^\circ)$

Solution:

$\sec(-960^\circ)$

$=\sec(-960^\circ+3(360^\circ))$

$=\sec(-960^\circ+1080^\circ)$

$=\sec 120^\circ$

$=\sec(90^\circ+30^\circ)$

$=-\operatorname{cosec}30^\circ$

$=-2$

(v) $\tan(1110^\circ)$

Solution:

Here $\tan(1110^\circ)$

$=\tan(1110^\circ+(-3)(360^\circ))$

$=\tan(1110^\circ-1080^\circ)$

$=\tan 30^\circ$

$=\dfrac{1}{\sqrt{3}}$

(vi) $\sin(-300^\circ)$

Solution:

Here $\sin(-300^\circ)$

$=\sin(-300^\circ+360^\circ)$

$=\sin 60^\circ$

$=\dfrac{\sqrt{3}}{2}$

Q.2 Express each of the following as a trigonometric function of an angle of positive degree measure of less then $45°$.

(i) $\sin(196°)$

Solution:

$\sin 196°$

$= \sin(180° + 16°)$

$= \sin(90° \times 2 + 16°)$

$= -\sin 16°$

(ii) $\cos 147°$

Solution:

$\cos 147°$

$= \cos(180° - 33°)$

$= \cos(2 \times 90° - 33°)$

$= -\cos 33°$

(iii) $\sin 319°$

Solution:

$\sin 319°$

$= \sin(360° - 41°)$

$= \sin(4 \times 90° - 41°)$

$= -\sin 41°$

(iv) $\cos 254°$

Solution:

$\cos 254°$

$= \cos(270° - 16°)$

$= \cos(3 \times 90° - 16°)$

$= -\sin 16°$

(v) $\tan 294°$

Solution:

$\tan 294°$

$= \tan(270° + 24°)$

$= \tan(3 \times 90° + 24°)$

$= -\cot 24°$

(vi) $\cos 728°$

Solution:

$\cos 728°$

$= \cos(728° + (-2)(360°))$

$= \cos(728° - 720°)$

$= \cos 8°$

(vii) $\sin(-625°)$

Solution:

$\sin(-625°)$

$= \sin(-625° + 2(360°))$

$= \sin(-625° + 720°)$

$= \sin 95°$

$= \sin(90° + 5°)$

$= \cos 5°$

(viii) $\cos(-435°)$

Solution:

$\cos(-435°)$

$= \cos(-435° + 360°)$

$= \cos(-75°)$

$= \cos 75°$

$= \cos(90° - 15°)$

$= \sin 15°$

(ix) $\sin 150°$

Solution:

$\sin 150°$

$= \sin(180° - 30°)$

$= \sin(2 \times 90° - 30°)$

$= \sin 30°$

Q.3 Prove the following:

(i) $\sin(180°+\alpha)\sin(90°-\alpha) = -\sin\alpha\cos\alpha$

Solution:

$$\begin{aligned}\text{L.H.S} &= \sin(180°+\alpha)\sin(90°-\alpha)\\ &= (-\sin\alpha)(\cos\alpha)\\ &= -\sin\alpha\cos\alpha\\ &= \text{R.H.S}\end{aligned}$$

(ii) $\sin 780°\sin 480° + \cos 120°\sin 30° = \frac{1}{2}$

Solution:

$$\begin{aligned}\text{L.H.S} &= \sin 780°\sin 480° + \cos 120°\sin 30°\\ &= \sin(780° + (-2)(360°))\sin(480° + (-1)(360°)) + \cos(90°+30°)\sin 30°\\ &= \sin(780°-720°)\sin(480°-360°) + (-\sin 30°)\sin 30°\\ &= \sin 60°\sin 120° - \frac{1}{2}.\frac{1}{2}\\ &= \frac{\sqrt{3}}{2}.\sin(90°+30°) - \frac{1}{4}\\ &= \frac{\sqrt{3}}{2}.(\cos 30°) - \frac{1}{4}\\ &= \frac{\sqrt{3}}{2}.\frac{\sqrt{3}}{2} - \frac{1}{4}\\ &= \frac{3}{4} - \frac{1}{4}\\ &= \frac{3-1}{4}\\ &= \frac{2}{4}\\ &= \frac{1}{2} = \text{R.H.S}\end{aligned}$$

(iii) $\cos 306° + \cos 234° + \cos 162° + \cos 18° = 0$

Solution:

$$\begin{aligned}\text{L.H.S} &= \cos 306° + \cos 234° + \cos 162° + \cos 18°\\ &= \cos(360°-54°) + \cos(180°+54°) + \cos(180°-18°) + \cos 18°\\ &= \cos(4\times 90°-54°) + \cos(2\times 90°+54°) + \cos(2\times 90°-18°) + \cos 18°\\ &= \cos 54° - \cos 54° - \cos 18° + \cos 18°\\ &= 0 = \text{R.H.S}\end{aligned}$$

(iv) $\cos 330^\circ \sin 600^\circ + \cos 120^\circ \sin 150^\circ = 1$

Solution:

$$\text{L.H.S} = \cos 330^\circ \sin 600^\circ + \cos 120^\circ \sin 150^\circ$$

$$= \cos(270^\circ + 60^\circ)\sin(600^\circ + (-2)(360^\circ)) + \cos(90^\circ + 30^\circ)\sin(90^\circ + 60^\circ)$$

$$= \cos(3\times 90^\circ + 60^\circ)\sin(600^\circ - 720^\circ) + (-\sin 30^\circ)(\cos 60^\circ)$$

$$= \sin 60^\circ \sin(-120^\circ) - \sin 30^\circ \cos 60^\circ$$

$$= -\sin 60^\circ \sin(90^\circ + 30^\circ) - \frac{1}{2}\cdot\frac{1}{2}$$

$$= -\frac{\sqrt{3}}{2}\cdot\cos 30^\circ - \frac{1}{4}$$

$$= -\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{1}{4}$$

$$= \frac{-3}{4} - \frac{1}{4}$$

$$= \frac{-4}{4}$$

$$= -1 = \text{R.H.S}$$

Q.4 Prove that:

(i) $$\frac{\sin^2(\pi+\theta)\tan\left(\frac{3\pi}{2}+\theta\right)}{\cot^2\left(\frac{3\pi}{2}-\theta\right)\cos^2(\pi-\theta)\operatorname{cosec}(2\pi-\theta)} = \cos\theta$$

Solution:

$$\text{L.H.S} = \frac{\sin^2(\pi+\theta)\tan\left(\frac{3\pi}{2}+\theta\right)}{\cot^2\left(\frac{3\pi}{2}-\theta\right)\cos^2(\pi-\theta)\operatorname{cosec}(2\pi-\theta)}$$

$$= \frac{\left[\sin(\pi+\theta)\right]^2 \tan\left(\frac{3\pi}{2}+\theta\right)}{\left[\cot\left(\frac{3\pi}{2}-\theta\right)\right]^2\left[\cos(\pi-\theta)\right]^2 .\operatorname{cosec}(2\pi-\theta)}$$

$$= \frac{(-\sin\theta)^2.(-\cot\theta)}{(\tan\theta)^2(-\cos\theta)^2.(-\operatorname{cosec}\theta)}$$

$$= \frac{\sin^2\theta(-\cot\theta)}{\tan^2\theta.\cos^2\theta(-\text{cosec}\,\theta)}$$

$$= \frac{\left(\sin^2\theta\right)\left(-\frac{\cos\theta}{\sin\theta}\right)}{\left(\frac{\sin^2\theta}{\cos^2\theta}\right).\cos^2\theta\left(-\frac{1}{\sin\theta}\right)}$$

$$= \frac{-\sin\theta\cos\theta}{-\sin\theta}$$

$= \cos\theta = \text{R.H.S}$

(ii) $\dfrac{\cos(90°+\theta)\sec(-\theta)\tan(180°-\theta)}{\sec(360°-\theta)\sin(180°+\theta)\cot(90°-\theta)} = -1$

Solution:

L.H.S $= \dfrac{\cos(90°+\theta)\sec(-\theta)\tan(180°-\theta)}{\sec(360°-\theta)\sin(180°+\theta)\cot(90°-\theta)}$

$$= \frac{(-\sin\theta)(\sec\theta)(-\tan\theta)}{(\sec\theta)(-\sin\theta)(\tan\theta)}$$

$= -1 = \text{R.H.S}$

Q.5 If α,β,γ are the angles of a triangle ABC, then prove that

(i) $\sin(\alpha+\beta) = \sin\gamma$

Solution:

Since α,β,γ are the angles of triangle ABC

So $\alpha+\beta+\gamma = 180°$

$\Rightarrow \alpha+\beta = 180°-\gamma$

$\Rightarrow \sin(\alpha+\beta) = \sin(180°-\gamma)$

$\Rightarrow \sin(\alpha+\beta) = \sin\gamma$

(ii) $\cos\left(\dfrac{\alpha+\beta}{2}\right) = \sin\dfrac{\gamma}{2}$

Solution:

Here $\alpha+\beta+\gamma = 180°$

$\Rightarrow \alpha+\beta = 180°-\gamma$

$\Rightarrow \dfrac{\alpha+\beta}{2} = \dfrac{180°-\gamma}{2}$

$\Rightarrow \dfrac{\alpha+\beta}{2} = 90°-\dfrac{\gamma}{2}$

$\Rightarrow \cos\left(\dfrac{\alpha+\beta}{2}\right) = \cos\left(90°-\dfrac{\gamma}{2}\right)$

$\Rightarrow \cos\left(\dfrac{\alpha+\beta}{2}\right) = \sin\dfrac{\gamma}{2}$

(iii) $\cos(\alpha+\beta) = -\cos\gamma$

Solution:

Here $\alpha+\beta+\gamma = 180°$

$\Rightarrow \alpha+\beta = 180°-\gamma$

$\Rightarrow \cos(\alpha+\beta) = \cos(180°-\gamma)$

$\Rightarrow \cos(\alpha+\beta) = -\cos\gamma$

(iv) $\tan(\alpha+\beta)+\tan\gamma = 0$

Solution:

Here $\alpha+\beta+\gamma = 180°$

$\Rightarrow \alpha+\beta = 180°-\gamma$

$\Rightarrow \tan(\alpha+\beta) = \tan(180°-\gamma)$

$\Rightarrow \tan(\alpha+\beta) = -\tan\gamma$

$\Rightarrow \tan(\alpha+\beta)+\tan\gamma = 0$

Exercise 10.2

Q.1 Prove that

(i) $\sin(180°+\theta)=-\sin\theta$

Solution:

L.H.S. $=\sin(180°+\theta)$

$=\sin 180°\cos\theta+\cos 180°\sin\theta$

$=(0)(\cos\theta)+(-1)(\sin\theta)$

$=-\sin\theta$ = R.H.S

(ii) $\cos(180°+\theta)=-\cos\theta$

Solution:

L.H.S $=\cos(180°+\theta)$

$=\cos 180°\cos\theta-\sin 180°\sin\theta$

$=(-1)(\cos\theta)-(0)(\sin\theta)$

$=-\cos\theta$ = R.H.S

(iii) $\tan(270°-\theta)=\cot\theta$

Solution:

L.H.S $=\tan(270°-\theta)$

$$=\frac{\sin(270°-\theta)}{\cos(270°-\theta)}$$

$$=\frac{(\sin 270°)(\cos\theta)-(\cos 270°)(\sin\theta)}{(\cos 270°)(\cos\theta)+(\sin 270)(\sin\theta)}$$

$$=\frac{(-1).\cos\theta-0.\sin\theta}{0.\cos\theta+(-1).\sin\theta}$$

$$=\frac{(-1)(\cos\theta)}{(-1)(\sin\theta)}$$

$$=\frac{-\cos\theta}{-\sin\theta}$$

$$=\frac{\cos\theta}{\sin\theta}$$

$=\cot\theta=$ R.H.S

(iv) $\cos(\theta-180°)=-\cos\theta$

Solution:

L.H.S $=\cos(\theta-180°)$

$=\cos\theta\cos(180°)+\sin\theta\sin(180°)$

$=(\cos\theta)(-1)+(\sin\theta)(0)$

$=-\cos\theta$ = R.H.S

(v) $\cos(270°+\theta)=\sin\theta$

Solution:

L.H.S $=\cos(270°+\theta)$

$=(\cos 270°)(\cos\theta)-(\sin 270°)(\sin\theta)$

$=0-(-1)\sin\theta$

$=\sin\theta=$ R.H.S

(vi) $\sin(\theta+270°)=-\cos\theta$

Solution:

L.H.S $=\sin(\theta+270°)$

$=(\sin\theta)(\cos 270°)+(\cos\theta)(\sin 270°)$

$=(\sin\theta)(0)+(\cos\theta)(-1)$

$=-\cos\theta=$ R.H.S

(vii) $\tan(180°+\theta)=\tan\theta$

Solution:

L.H.S $=\tan(180°+\theta)$

$$=\frac{\tan 180°+\tan\theta}{1-\tan 180°\tan\theta}$$

$$=\frac{0+\tan\theta}{1-(0)\tan\theta}$$

$=\tan\theta$ = R.H.S

(viii) $\cos(360°-\theta)=\cos\theta$

Solution:

L.H.S $=\cos(360°-\theta)$

$=\cos 360°.\cos\theta+\sin 360°.\sin\theta$

$=(1)(\cos\theta)+(0)(\sin\theta)$

$=\cos\theta=$ R.H.S

Q.2 Find the values of the following:

(i) $\sin 15°$

Solution:

$$\sin 15°$$
$$= \sin(45° - 30°)$$
$$= \sin 45° \cos 30° - \cos 45° \sin 30°$$
$$= \left(\frac{1}{\sqrt{2}}\right) \times \left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right) \times \left(\frac{1}{2}\right)$$
$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$
$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

(ii) $\cos 15°$

Solution:

$$\cos 15°$$
$$= \cos(45° - 30°)$$
$$= \cos 45° \cos 30° + \sin 45° \sin 30°$$
$$= \left(\frac{1}{\sqrt{2}}\right) \times \left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}}\right) \times \left(\frac{1}{2}\right)$$
$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$
$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(iii) $\tan 15°$

Solution:

$$\tan 15° = \tan(45° - 30°)$$
$$= \frac{\tan 45° - \tan 30°}{1 + (\tan 45°)(\tan 30°)}$$
$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}$$
$$= \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}}$$
$$= \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

(iv) $\sin 105°$

Solution:

$$\sin 105°$$
$$= \sin(60° + 45°)$$
$$= \sin 60° \cos 45° + \cos 60° \sin 45°$$
$$= \left(\frac{\sqrt{3}}{2}\right) \times \left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2}\right) \times \left(\frac{1}{\sqrt{2}}\right)$$
$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$
$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(v) $\cos 105°$

Solution:

$$\cos 105°$$
$$= \cos(60° + 45°)$$
$$= \cos 60° \cos 45° - \sin 60° \sin 45°$$
$$= \left(\frac{1}{2}\right) \times \left(\frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2}\right) \times \left(\frac{1}{\sqrt{2}}\right)$$
$$= \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}}$$
$$= \frac{1-\sqrt{3}}{2\sqrt{2}}$$

(vi) $\tan 105°$

Solution:

$$\tan 105°$$
$$= \tan(60° + 45°)$$
$$= \frac{\tan 60° + \tan 45°}{1 - \tan 60° \tan 45°}$$
$$= \frac{\sqrt{3}+1}{1-\sqrt{3}}$$

Q.3 Prove that

(i) $\sin(45^\circ+\alpha)=\frac{1}{\sqrt{2}}(\sin\alpha+\cos\alpha)$

Solution:

$$\text{L.H.S} = \sin(45^\circ+\alpha)$$
$$= \sin 45^\circ \cos\alpha + \cos 45^\circ \sin\alpha$$
$$= \frac{1}{\sqrt{2}}\cos\alpha + \frac{1}{\sqrt{2}}\sin\alpha$$
$$= \frac{1}{\sqrt{2}}(\sin\alpha+\cos\alpha) = \text{R.H.S}$$

(ii) $\cos(\alpha+45^\circ)=\frac{1}{\sqrt{2}}(\cos\alpha-\sin\alpha)$

Solution:

$$\text{L.H.S} = \cos(\alpha+45^\circ)$$
$$= \cos\alpha\cos 45^\circ - \sin\alpha\sin 45^\circ$$
$$= (\cos\alpha)\left(\frac{1}{\sqrt{2}}\right) - (\sin\alpha)\left(\frac{1}{\sqrt{2}}\right)$$
$$= \frac{1}{\sqrt{2}}(\cos\alpha-\sin\alpha) = \text{R.H.S}$$

Q.4 Prove that

(i) $\tan(45^\circ+A)\tan(45^\circ-A)=1$

Solution:

$$\text{L.H.S} = \tan(45^\circ+A)\tan(45^\circ-A)$$
$$= \left(\frac{\tan 45^\circ+\tan A}{1-\tan 45^\circ\tan A}\right)\left(\frac{\tan 45^\circ-\tan A}{1+\tan 45^\circ\tan A}\right)$$
$$= \left(\frac{1+\tan A}{1-\tan A}\right)\left(\frac{1-\tan A}{1+\tan A}\right)$$
$$= 1 = \text{R.H.S}$$

(ii) $\tan\left(\frac{\pi}{4}-\theta\right)+\tan\left(\frac{3\pi}{4}+\theta\right)=0$

Solution:

$$\text{L.H.S} = \tan\left(\frac{\pi}{4}-\theta\right)+\tan\left(\frac{3\pi}{4}+\theta\right)$$
$$= \left(\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right)+\left(\frac{\tan\left(\frac{3\pi}{4}\right)+\tan\theta}{1-\tan\left(\frac{3\pi}{4}\right)\tan\theta}\right)$$
$$= \left(\frac{1-\tan\theta}{1+\tan\theta}\right)+\left(\frac{-1+\tan\theta}{1-(-1)\tan\theta}\right)$$
$$= \left(\frac{1-\tan\theta}{1+\tan\theta}\right)+\left(\frac{-1+\tan\theta}{1+\tan\theta}\right)$$
$$= \frac{1-\tan\theta-1+\tan\theta}{1+\tan\theta}$$
$$= \frac{0}{1+\tan\theta}$$
$$= 0 = \text{R.H.S}$$

(iii) $\sin\left(\theta+\frac{\pi}{6}\right)+\cos\left(\theta+\frac{\pi}{3}\right) = \cos\theta$

Solution:

$$\text{L.H.S} = \sin\left(\theta+\frac{\pi}{6}\right)+\cos\left(\theta+\frac{\pi}{3}\right)$$
$$= \left(\sin\theta\cos\frac{\pi}{6}+\cos\theta\sin\frac{\pi}{6}\right)+\left(\cos\theta\cos\frac{\pi}{3}-\sin\theta\sin\frac{\pi}{3}\right)$$
$$= \sin\theta\frac{\sqrt{3}}{2}+\cos\theta\frac{1}{2}+\cos\theta\frac{1}{2}-\sin\theta\frac{\sqrt{3}}{2}$$
$$= \frac{1}{2}\cos\theta+\frac{1}{2}.\cos\theta$$
$$= 2\times\frac{1}{2}\cos\theta$$
$$= \cos\theta = \text{R.H.S}$$

(iv) $\dfrac{\sin\theta-\cos\theta\tan\frac{\theta}{2}}{\cos\theta+\sin\theta\tan\frac{\theta}{2}} = \tan\frac{\theta}{2}$

Solution:

$$\text{L.H.S} = \frac{\sin\theta-\cos\theta\tan\frac{\theta}{2}}{\cos\theta+\sin\theta\tan\frac{\theta}{2}}$$

$$=\frac{\sin\theta-\cos\theta\times\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}{\cos\theta+\sin\theta\times\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}$$

$$=\frac{\frac{\sin\theta\cos\frac{\theta}{2}-\cos\theta.\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}{\frac{\cos\theta\cos\frac{\theta}{2}+\sin\theta\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}$$

$$=\frac{\sin\theta\cos\frac{\theta}{2}-\cos\theta\sin\frac{\theta}{2}}{\cos\theta\cos\frac{\theta}{2}+\sin\theta\sin\frac{\theta}{2}}$$

$$=\frac{\sin\left(\theta-\frac{\theta}{2}\right)}{\cos\left(\theta-\frac{\theta}{2}\right)}$$

$$=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$=\tan\frac{\theta}{2}=\text{ R.H.S}$$

Alternate solution:

$$\text{L.H.S } =\frac{\sin\theta-\cos\theta\tan\frac{\theta}{2}}{\cos\theta+\sin\theta\tan\frac{\theta}{2}}$$

Dividing up and down by $\cos\theta$

$$=\frac{\frac{\sin\theta-\cos\theta\tan\frac{\theta}{2}}{\cos\theta}}{\frac{\cos\theta+\sin\theta\tan\frac{\theta}{2}}{\cos\theta}}$$

$$=\frac{\frac{\sin\theta}{\cos\theta}-\frac{\cos\theta\tan\frac{\theta}{2}}{\cos\theta}}{\frac{\cos\theta}{\cos\theta}+\frac{\sin\theta\tan\frac{\theta}{2}}{\cos\theta}}$$

$$=\frac{\tan\theta-\tan\frac{\theta}{2}}{1+\tan\theta\tan\frac{\theta}{2}}$$

$$=\tan\left(\theta-\frac{\theta}{2}\right)$$

$$=\tan\frac{\theta}{2}=\text{R.H.S}$$

(v) $\mathbf{\dfrac{1-\tan\theta\tan\phi}{1+\tan\theta\tan\phi}=\dfrac{\cos(\theta+\phi)}{\cos(\theta-\phi)}}$

Solution:

$$\text{L.H.S } =\frac{1-\tan\theta\tan\phi}{1+\tan\theta\tan\phi}$$

$$=\left(\frac{1-\frac{\sin\theta}{\cos\theta}\times\frac{\sin\phi}{\cos\phi}}{1+\frac{\sin\theta}{\cos\theta}\times\frac{\sin\phi}{\cos\phi}}\right)$$

$$=\frac{\frac{\cos\theta\cos\phi-\sin\theta\sin\phi}{\cos\theta\cos\phi}}{\frac{\cos\theta\cos\phi+\sin\theta\sin\phi}{\cos\theta\cos\phi}}$$

$$=\frac{\cos\theta\cos\phi-\sin\theta\sin\phi}{\cos\theta\cos\phi+\sin\theta\sin\phi}$$

$$=\frac{\cos(\theta+\phi)}{\cos(\theta-\phi)}\quad=\text{R.H.S}$$

Q.5 Show that $\cos(\alpha+\beta)\cos(\alpha-\beta)=\cos^2\alpha-\sin^2\beta=\cos^2\beta-\sin^2\alpha$

Solution:

Consider

$$\cos(\alpha+\beta)\cos(\alpha-\beta)$$

$$=(\cos\alpha\cos\beta-\sin\alpha\sin\beta)(\cos\alpha\cos\beta+\sin\alpha\sin\beta)$$

$$=(\cos\alpha\cos\beta)^2-(\sin\alpha\sin\beta)^2$$

$$=\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta$$

$$=\cos^2\alpha(1-\sin^2\beta)-(1-\cos^2\alpha)\sin^2\beta$$

$$=\cos^2\alpha-\cos^2\alpha\sin^2\beta-\sin^2\beta+\cos^2\alpha\sin^2\beta$$

$$=\cos^2\alpha-\sin^2\beta$$

Further it can be written as

$$=(1-\sin^2\alpha)-(1-\cos^2\beta)$$

$$=1-\sin^2\alpha-1+\cos^2\beta$$

$$=\cos^2\beta-\sin^2\alpha$$

Q.6 Show that $\dfrac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)}=\tan\alpha$

Solution:

$$\text{L.H.S}=\frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)}$$

$$=\frac{\sin\alpha\cos\beta+\cancel{\cos\alpha\sin\beta}+\sin\alpha\cos\beta-\cancel{\cos\alpha\sin\beta}}{\cos\alpha\cos\beta-\cancel{\sin\alpha\sin\beta}+\cos\alpha\cos\beta+\cancel{\sin\alpha\sin\beta}}$$

$$=\frac{2\sin\alpha\cos\beta}{2\cos\alpha\cos\beta}$$

$$=\frac{\sin\alpha}{\cos\alpha}$$

$$=\tan\alpha=\text{R.H.S}$$

Q.7 Show that:

(i) $\cot(\alpha+\beta)=\dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$

Solution:

L.H.S $=\cot(\alpha+\beta)$

$$=\frac{1}{\tan(\alpha+\beta)}$$

$$=\frac{1}{\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}}$$

$$=\frac{1-\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$$

$$=\frac{1-\dfrac{1}{\cot\alpha\cot\beta}}{\dfrac{1}{\cot\alpha}+\dfrac{1}{\cot\beta}}$$

$$=\frac{\dfrac{\cot\alpha\cot\beta-1}{\cot\alpha\cot\beta}}{\dfrac{\cot\beta+\cot\alpha}{\cot\alpha\cot\beta}}$$

$$=\frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}=\text{R.H.S}$$

(ii) $\cot(\alpha-\beta)=\dfrac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}$

Solution:

L.H.S $=\cot(\alpha-\beta)$

$$=\frac{1}{\tan(\alpha-\beta)}$$

$$=\frac{1}{\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}}$$

$$=\frac{1+\tan\alpha\tan\beta}{\tan\alpha-\tan\beta}$$

$$=\frac{1+\dfrac{1}{\cot\alpha\cot\beta}}{\dfrac{1}{\cot\alpha}-\dfrac{1}{\cot\beta}}$$

$$=\frac{\dfrac{\cot\alpha\cot\beta+1}{\cot\alpha\cot\beta}}{\dfrac{\cot\beta-\cot\alpha}{\cot\alpha\cot\beta}}$$

$$=\frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}=\text{R.H.S.}$$

(iii) $\dfrac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}=\dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$

Solution:

L.H.S $=\dfrac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}$

$$=\frac{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}{\dfrac{\sin\alpha}{\cos\alpha}-\dfrac{\sin\beta}{\cos\beta}}$$

$$=\frac{\dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha\cos\beta-\cos\alpha\sin\beta}$$

$$=\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\text{R.H.S}$$

Q.8 **If $\sin\alpha=\frac{4}{5}$ and $\cos\beta=\frac{40}{41}$ where $0<\alpha<\frac{\pi}{2}$ and $0<\beta<\frac{\pi}{2}$. Show that $\sin(\alpha-\beta)=\frac{133}{205}$.**

Solution:

Given that $\sin\alpha=\frac{4}{5}$, $\cos\beta=\frac{40}{41}$

$$\begin{aligned}
\text{L.H.S} &= \sin(\alpha-\beta)\\
&=\sin\alpha\cos\beta-\cos\alpha\sin\beta\\
&=\sin\alpha\cos\beta-\sqrt{1-\sin^2\alpha}\sqrt{1-\cos^2\beta} \quad \text{Here } \cos\alpha=+\sqrt{1-\sin^2\alpha}\ \because 0<\alpha<\frac{\pi}{2}\\
&=\left(\frac{4}{5}\right)\left(\frac{40}{41}\right)-\sqrt{1-\frac{16}{25}}\sqrt{1-\frac{1600}{1681}} \quad \sin\beta=+\sqrt{1-\cos^2\beta}\ \because 0<\beta<\frac{\pi}{2}\\
&=\frac{160}{205}-\sqrt{\frac{25-16}{25}}.\sqrt{\frac{1681-1600}{1681}}\\
&=\frac{160}{205}-\sqrt{\frac{9}{25}}\sqrt{\frac{81}{1681}}\\
&=\frac{160}{205}-\left(\frac{3}{5}\right)\left(\frac{9}{41}\right)\\
&=\frac{160}{205}-\frac{27}{205}\\
&=\frac{160-27}{205}\\
&=\frac{133}{205}
\end{aligned}$$

Q.9 **If $\sin\alpha=\frac{4}{5}$ and $\sin\beta=\frac{12}{13}$, where $\frac{\pi}{2}<\alpha<\pi$ and $\frac{\pi}{2}<\beta<\pi$. Find**

(i) **$\sin(\alpha+\beta)$**

Solution:

Given that $\sin\alpha=\frac{4}{5}$

$$\begin{aligned}
\Rightarrow \cos\alpha &= -\sqrt{1-\sin^2\alpha} \quad \because \frac{\pi}{2}<\alpha<\pi\\
&=-\sqrt{1-\frac{16}{25}}\\
&=-\sqrt{\frac{25-16}{25}}
\end{aligned}$$

$$= -\sqrt{\frac{9}{25}}$$

$$\cos\alpha = -\frac{3}{5}$$

and $\sin\beta = \frac{4}{5}$

$$\Rightarrow \cos\beta = -\sqrt{1-\sin^2\beta} \qquad Q\ \frac{\pi}{2} < \beta < \pi$$

$$= -\sqrt{1-\frac{144}{169}}$$

$$= -\sqrt{\frac{169-144}{169}}$$

$$= -\sqrt{\frac{25}{169}}$$

$$\cos\beta = -\frac{5}{13}$$

Using $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \left(\frac{4}{5}\right)\left(\frac{-5}{13}\right) + \left(\frac{-3}{5}\right)\left(\frac{12}{13}\right)$$

$$= -\frac{20}{65} - \frac{36}{25}$$

$$= \frac{-20-36}{65}$$

$$\sin(\alpha+\beta) = \frac{-56}{65}$$

(ii) $\cos(\alpha+\beta)$

Solution:

Using $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= \left(\frac{-3}{5}\right)\left(\frac{-5}{13}\right) - \left(\frac{4}{5}\right)\left(\frac{12}{13}\right)$$

$$= \frac{15}{65} - \frac{48}{65}$$

$$= \frac{15-48}{65}$$

$$\cos(\alpha+\beta) = \frac{-33}{65}$$

(iii) $\tan(\alpha+\beta)$

Solution:

Using $\tan(\alpha+\beta)=\dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$

$$=\frac{\frac{-56}{65}}{\frac{-33}{65}}$$

$$\tan(\alpha+\beta)=\frac{56}{33}$$

Note Here $\sin(\alpha+\beta)=\dfrac{-56}{65}$ & $\cos(\alpha+\beta)=\dfrac{-33}{65}$ are both are negative, it means terminal arm of $\alpha+\beta$ lies in III Quadrant

(iv) $\sin(\alpha-\beta)$

Solution:

Using $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$$=\left(\frac{4}{5}\right)\left(\frac{-5}{13}\right)-\left(\frac{-3}{5}\right)\left(\frac{12}{13}\right)$$

$$=-\frac{20}{65}+\frac{36}{65}$$

$$=\frac{-20+36}{65}$$

$$\sin(\alpha-\beta)=\frac{16}{65}$$

(v) $\cos(\alpha-\beta)$

Solution:

Using $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$$=\left(\frac{-3}{5}\right)\left(\frac{-5}{13}\right)+\left(\frac{4}{5}\right)\left(\frac{12}{13}\right)$$

$$=\frac{15}{65}+\frac{48}{65}$$

$$\cos(\alpha-\beta)=\frac{63}{65}$$

(vi) $\tan(\alpha-\beta)$

Solution:

Using $\tan(\alpha-\beta)=\dfrac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}$

$$=\frac{\frac{16}{65}}{\frac{63}{65}}$$

$$\tan(\alpha-\beta)=\frac{16}{63}$$

Note: Here $\tan(\alpha-\beta)=\dfrac{16}{63}$ and $\cos(\alpha-\beta)=\dfrac{63}{65}$ that is both are positive, so terminal arm of $\alpha-\beta$ lies in 1st quadrant.

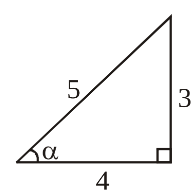
Q.10 Find $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$, given that

(i) $\tan\alpha=\dfrac{3}{4}$, $\cos\beta=\dfrac{5}{13}$, **and neither the terminal side of the angle of measure α nor that of β is in the I quadrant**.

Solution:

It is give that α,β are not in quad.I, it means, α lies in III and β lies in quad.IV.

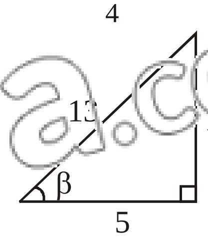
Here $\tan\alpha=\dfrac{3}{4}$ and α lies in quad – III



So $\sin\alpha=\dfrac{-3}{5}$, $\cos\alpha=\dfrac{-4}{5}$

Also $\cos\beta=\dfrac{5}{13}$ and β lies in quad – IV

So $\sin\beta=-\dfrac{12}{13}$



Using $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$$=\left(\frac{-3}{5}\right)\left(\frac{5}{13}\right)+\left(\frac{-4}{5}\right)\left(\frac{-12}{13}\right)$$

$$=\frac{-15}{65}+\frac{48}{65}$$

$$\sin(\alpha+\beta)=\frac{33}{65}$$

$\cos(\alpha+\beta)=?$

Using $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$$=\left(\frac{-4}{5}\right)\left(\frac{5}{13}\right)-\left(\frac{-3}{5}\right)\left(\frac{-12}{13}\right)$$

$$=\frac{-20}{65}-\frac{36}{65}$$

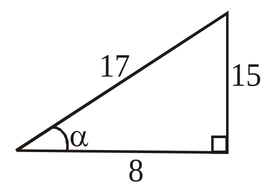
$$\cos(\alpha+\beta)=\frac{-56}{65}$$

(ii) $\tan\alpha=\frac{-15}{8}$ and $\sin\beta=\frac{-7}{25}$ neither the terminal side of the angle of measure α nor that of β is in IV quadrant.

Solution:

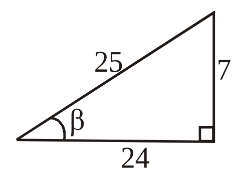
It is give that α,β are not in quad. IV, it means, α lies in quad. II and β lies in quad. III.

Here $\tan\alpha=\frac{-15}{8}$ and α lies in quad – II



So $\sin\alpha=\frac{15}{17}$ $\cos\alpha=\frac{-8}{17}$

And for $\sin\beta=\frac{-7}{25}$ and β lies in quad – III



So $\cos\beta=\frac{-24}{25}$

Using $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$$=\left(\frac{15}{17}\right)\left(\frac{-24}{25}\right)+\left(\frac{-8}{17}\right)\left(\frac{-7}{25}\right)$$

$$=\frac{-360}{425}+\frac{56}{425}$$

$$=\frac{-304}{425}$$

Now

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$=\left(-\frac{8}{17}\right)\left(-\frac{24}{25}\right)-\left(\frac{15}{17}\right)\left(-\frac{7}{25}\right)$$

$$=\frac{192}{425}+\frac{105}{425}$$

$$=\frac{297}{425}$$

Q.11 Prove that $\frac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ + \sin 8^\circ} = \tan 37^\circ$

Solution:

L.H.S $= \frac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ + \sin 8^\circ}$

Dividing numerator and denominator by $\cos 8^\circ$, we get

$$= \frac{1 - \frac{\sin 8^\circ}{\cos 8^\circ}}{1 + \frac{\sin 8^\circ}{\cos 8^\circ}}$$

$$= \frac{1 - \tan 8^\circ}{1 + \tan 8^\circ}$$

$$= \frac{\tan 45^\circ - \tan 8^\circ}{1 + \tan 45^\circ \tan 8^\circ}$$

$$= \tan(45^\circ - 8^\circ)$$

$$= \tan 37^\circ$$

= R.H.S

Alternate Solution:

R.H.S $= \tan 37^\circ$

$$= \tan(45^\circ - 8^\circ)$$

$$= \frac{\tan 45^\circ - \tan 8^\circ}{1 + \tan 45^\circ \tan 8^\circ}$$

$$= \frac{1 - \tan 8^\circ}{1 + \tan 8^\circ}$$

$$= \frac{1 - \frac{\sin 8^\circ}{\cos 8^\circ}}{1 + \frac{\sin 8^\circ}{\cos 8^\circ}}$$

$$= \frac{\frac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ}}{\frac{\cos 8^\circ + \sin 8^\circ}{\cos 8^\circ}}$$

$$= \frac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ + \sin 8^\circ} = \text{L.H.S}$$

Q.12 If α, β, γ are the angles of a triangle ABC, show that

$$\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2} = \cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$$

Solution:

It is given that α, β, γ are the angles of triangle ABC, so

$$\alpha + \beta + \gamma = 180^o$$

$$\Rightarrow \quad \alpha + \beta = 180^o - \gamma$$

$$\Rightarrow \quad \frac{\alpha}{2} + \frac{\beta}{2} = 90^o - \frac{\gamma}{2}$$

$$\Rightarrow \cot\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = \cot\left(90^o - \frac{\gamma}{2}\right)$$

$$\Rightarrow \quad \frac{\cot\frac{\alpha}{2}\cot\frac{\beta}{2} - 1}{\cot\frac{\alpha}{2} + \cot\frac{\beta}{2}} = \tan\frac{\gamma}{2}$$

$\Rightarrow \quad \dfrac{\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}}{\cot\frac{\alpha}{2}\cot\frac{\beta}{2}-1}=\cot\frac{\gamma}{2}$

$\Rightarrow \quad \cot\frac{\alpha}{2}+\cot\frac{\beta}{2}=\cot\frac{\gamma}{2}\left[\cot\frac{\alpha}{2}\cot\frac{\beta}{2}-1\right]$

$\Rightarrow \quad \cot\frac{\alpha}{2}+\cot\frac{\beta}{2}=\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}-\cot\frac{\gamma}{2}$

$\Rightarrow \cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}=\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$

Q.13 If $\alpha+\beta+\gamma=180°$, show that $\cot\alpha\cot\beta+\cot\beta\cot\gamma+\cot\gamma\cot\alpha=1$

Solution:

Here $\alpha+\beta+\gamma=180^o$

$\Rightarrow \quad \alpha+\beta=180^o-\gamma$

$\Rightarrow \quad \cot(\alpha+\beta)=\cot(180^o-\gamma)$

$\Rightarrow \quad \dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}=-\cot\gamma$

$\Rightarrow \quad \cot\alpha\cot\beta-1=-\cot\gamma[\cot\alpha+\cot\beta]$

$\Rightarrow \quad \cot\alpha\cot\beta-1=-\cot\gamma\cot\alpha-\cot\gamma\cot\beta$

$\Rightarrow \quad \cot\alpha\cot\beta+\cot\beta\cot\gamma+\cot\gamma\cot\alpha=1$

Q.14 Express the following in the form of $r\sin(\theta+\phi)$ or $r\sin(\theta-\phi)$, where the terminal sides of the angles of measure θ and ϕ are in the first quadrant.

(i) $12\sin\theta+5\cos\theta$

Solution:

Let $12=r\cos\phi$ (i)

$5=r\sin\phi$ (ii)

Squaring (i) & (ii) and then adding, we get,

$r^2(\cos^2\phi+\sin^2\phi)=144+25$

$r^2=169$

$\Rightarrow \quad r=13$

Dividing (ii) by (i) we get,

$\dfrac{r\sin\phi}{r\cos\phi}=\dfrac{5}{12}$

$$\tan\phi = \frac{5}{12}$$

$$\phi = \tan^{-1}\left(\frac{5}{12}\right)$$

Now given expression can be written as,

$$12\sin\theta + 5\cos\theta = r\cos\phi\sin\theta + r\sin\phi\cos\theta$$

$$= r(\sin\theta\cos\phi + \cos\theta\sin\phi)$$

$$= r\sin(\theta + \phi)$$

Where $r = 13$ and $\phi = \tan^{-1}\frac{5}{12}$

(ii) $3\sin\theta - 4\cos\theta$

Solution:

Let $3 = r\cos\phi$ (i)

$4 = r\sin\phi$ (ii)

Squaring (i) and (ii) and then adding, we get,

$$r^2(\cos^2\phi + \sin^2\phi) = 9 + 16$$

$$r^2 = 25$$

$$r = 5$$

Dividing (ii) by (i) we get,

$$\frac{r\sin\phi}{r\cos\phi} = \frac{4}{3}$$

$$\tan\phi = \frac{4}{3}$$

$$\phi = \tan^{-1}\frac{4}{3}$$

Given expression can be written as,

$$3\sin\theta - 4\cos\theta = r\cos\phi\sin\theta - r\sin\phi\cos\theta$$

$$= r(\sin\theta\cos\phi - \cos\theta\sin\phi)$$

$$= r\sin(\theta - \phi)$$

Where $r = 5$ and $\phi = \tan^{-1}\frac{4}{13}$

(iii) $\sin\theta - \cos\theta$

Solution:

Let $r\cos\phi = 1$ (i)

$r\sin\phi = 1$ (ii)

Squaring (i) and (ii) and then adding, we get,

$$r^2\left(\cos^2\phi + \sin^2\phi\right) = 1^2 + 1^2$$

$$r^2 = 2$$

$\Rightarrow$ $r = \sqrt{2}$

Dividing (ii) by (i) we get .

$$\frac{r\sin\phi}{r\cos\phi} = \frac{1}{1}$$

$$\tan\phi = 1$$

$\Rightarrow$ $\phi = \tan^{-1}1$

Given expression can be written as,

$$\sin\theta - \cos\theta = r\cos\phi\sin\theta - r\sin\phi\cos\theta$$

$$= r\left(\sin\theta\cos\phi - \cos\theta\sin\phi\right)$$

$$= r\sin\left(\theta - \phi\right)$$

Where $r = \sqrt{2}$ and $\phi = \tan^{-1}1$

(iv) $5\sin\theta - 4\cos\theta$

Solution:

Let $r\cos\phi = 5$ (i)

$r\sin\phi = 4$ (ii)

Squaring (i) & (ii) and then adding, we get,

$$r^2\left(\cos^2\phi + \sin^2\phi\right) = 25 + 16$$

$$r^2 = 41$$

$\Rightarrow$ $r = \sqrt{41}$

Dividing (ii) by (i) we get,

$$\frac{r\sin\phi}{r\cos\phi} = \frac{4}{5}$$

$$\tan\phi = \frac{4}{5}$$

$\Rightarrow \quad \phi = \tan^{-1}\frac{4}{5}$

Given expression can be written as,

$5\sin\theta - 4\cos\theta = r\cos\phi\sin\theta - r\sin\phi\cos\theta$

$= r(\sin\theta\cos\phi - \cos\theta\sin\phi)$

$= r\sin(\theta - \phi)$

Where $r = \sqrt{41}$ and $\phi = \tan^{-1}\frac{4}{5}$

(v) sinθ + cosθ

Solution:

Let $r\cos\phi = 1$ (i)

$r\sin\phi = 1$ (ii)

Squaring (i) and (ii) and then adding, we get,

$r^2(\cos^2\phi + \sin^2\phi) = 1^2 + 1^2$

$r^2 = 2$

$\Rightarrow \quad r = \sqrt{2}$

Dividing (ii) by (i) we get,

$\frac{r\ \sin\phi}{r\ \cos\phi} = \frac{1}{1}$

$\tan\phi = 1$

$\Rightarrow \quad \phi = \tan^{-1}(1)$

Given expression can be written as,

$\sin\theta + \cos\theta = r\cos\phi\sin\theta + r\sin\phi\cos\theta$

$= r(\sin\theta\cos\phi + \cos\theta\sin\phi)$

$= r\sin(\theta + \phi)$

Where $r = \sqrt{2}$ and $\phi = \tan^{-1}1$

(vi) 3sinθ – 5cosθ

Solution:

Let $r\cos\phi = 3$ (i)

$r\sin\phi = 5$ (ii)

Squaring (i) and (ii) and then adding we get,

$r^2(\cos^2\phi + \sin^2\phi) = 3^2 + 5^2$

$r^2 = 9 + 25$

$\Rightarrow \quad r = \sqrt{34}$

Dividing (ii) by (i) we get,

$$\frac{r\sin\phi}{r\cos\phi}=\frac{5}{3}$$

$$\tan\phi=\frac{5}{3}$$

$$\Rightarrow \quad \phi=\tan^{-1}\frac{5}{3}$$

Given expression can be written as,

$$3\sin\theta-5\cos\theta=r\cos\phi\sin\theta-r\sin\phi\cos\theta$$

$$=r\left(\sin\theta\cos\phi-\cos\theta\sin\phi\right)$$

$$=r\sin\left(\theta-\phi\right)$$

Where $r=\sqrt{34}$ and $\phi=\tan^{-1}\frac{5}{3}$

Double Angle Identities:

(i) $\mathbf{\sin 2\alpha = 2\sin\alpha\cos\alpha}$

(ii) $\mathbf{\cos 2\alpha = \cos^2\alpha - \sin^2\alpha}$

$$\mathbf{= 2\cos^2\alpha - 1}$$

$$\mathbf{= 1 - 2\sin^2\alpha}$$

(iii) $\mathbf{\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}}$

Proof:

(i)

We know that

$$\sin\left(\alpha+\beta\right)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

Replace 'β' by 'α' we get

$$\sin\left(\alpha+\alpha\right)=\sin\alpha\cos\alpha+\cos\alpha\sin\alpha$$

Hence $\boxed{\sin 2\alpha=2\sin\alpha\cos\alpha}$

(ii)

We know that

$$\cos\left(\alpha+\beta\right)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

Replace β by α

$$\cos\left(\alpha+\alpha\right)=\cos\alpha\cos\alpha-\sin\alpha\sin\alpha$$

Hence $\boxed{\cos 2\alpha=\cos^2\alpha-\sin^2\alpha}$

Now

$$\cos 2\alpha=\cos^2\alpha-\sin^2\alpha$$

$$=\cos^2\alpha-\left(1-\cos^2\alpha\right)$$

$$=\cos^2\alpha-1+\cos^2\alpha$$

Hence $\boxed{\cos 2\alpha=2\cos^2\alpha-1}$

Again $\cos 2\alpha=\cos^2\alpha-\sin^2\alpha$

$$=1-\sin^2\alpha-\sin^2\alpha$$

Hence $\boxed{\cos 2\alpha=1-2\sin^2\alpha}$

(iii)

We know that

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}.$$

Replace 'β' by 'α' we get.

$$\tan(\alpha+\alpha)=\frac{\tan\alpha+\tan\alpha}{1-\tan\alpha\tan\alpha}$$

Hence $\boxed{\tan 2\alpha=\dfrac{2\tan\alpha}{1-\tan^2\alpha}}$

Half Angle Identities:

(i) $\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$

(ii) $\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$

(iii) $\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$

Proof:

(i)

We know that

$\cos 2\theta = 2\cos^2\theta - 1$

Replace "θ" by "$\frac{\alpha}{2}$" we get.

$\cos 2\left(\frac{\alpha}{2}\right) = 2\cos^2\left(\frac{\alpha}{2}\right) - 1$

$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1$

$1+\cos\alpha = 2\cos^2\frac{\alpha}{2}$

$\frac{1+\cos\alpha}{2} = \cos^2\frac{\alpha}{2}$

$\pm\sqrt{\frac{1+\cos\alpha}{2}} = \cos\frac{\alpha}{2}$

Hence $\boxed{\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}}$

(ii)

We know that

$\cos 2\theta = 1 - 2\sin^2\theta$

Replace "θ" by "$\frac{\alpha}{2}$" we get.

$\cos 2\left(\frac{\alpha}{2}\right) = 1 - 2\sin^2\frac{\alpha}{2}$

$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}$

$2\sin^2\frac{\alpha}{2} = 1 - \cos\alpha$

$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}$

Hence $\boxed{\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}}$

(iii)

We know that

$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$

$= \pm\frac{\sqrt{\frac{1-\cos\alpha}{2}}}{\sqrt{\frac{1+\cos\alpha}{2}}}$

$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{\frac{1-\cos\alpha}{2}}{\frac{1+\cos\alpha}{2}}}$

Hence $\boxed{\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}}$

Triple angle identities:

(i) $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$

(ii) $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

(iii) $\tan 3\alpha = \dfrac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$

Proof:

(i)

L.H.S $\sin 3\alpha = \sin(2\alpha + \alpha)$

$$= \sin 2\alpha \cos\alpha + \cos 2\alpha \sin\alpha$$
$$= (2\sin\alpha\cos\alpha)\cos\alpha + (1 - 2\sin^2\alpha)\sin\alpha$$
$$= 2\sin\alpha\cos^2\alpha + \sin\alpha - 2\sin^3\alpha$$
$$= 2\sin\alpha(1 - \sin^2\alpha) + \sin\alpha - 2\sin^3\alpha$$
$$= 2\sin\alpha - 2\sin^3\alpha + \sin\alpha - 2\sin^3\alpha$$

Hence $\boxed{\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha}$

(ii)

L.H.S $\cos 3\alpha = \cos(2\alpha + \alpha)$

$$= \cos 2\alpha \cos\alpha - \sin 2\alpha \sin\alpha$$
$$= (2\cos^2\alpha - 1)\cos\alpha - (2\sin\alpha\cos\alpha)\sin\alpha$$
$$= 2\cos^3\alpha - \cos\alpha - 2\sin^2\alpha\cos\alpha$$
$$= 2\cos^3\alpha - \cos\alpha - 2(1 - \cos^2\alpha)\cos\alpha$$
$$= 2\cos^3\alpha - \cos\alpha - 2\cos\alpha + 2\cos^3\alpha$$

Hence $\boxed{\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha}$

(iii)

L.H.S $\tan 3\alpha = \tan(2\alpha + \alpha)$

$$= \frac{\tan 2\alpha + \tan\alpha}{1 - \tan 2\alpha \tan\alpha}$$
$$= \frac{\dfrac{2\tan\alpha}{1 - \tan^2\alpha} + \tan\alpha}{1 - \dfrac{2\tan\alpha}{1 - \tan^2\alpha}\,.\tan\alpha}$$
$$= \frac{\dfrac{2\tan\alpha + \tan\alpha(1 - \tan^2\alpha)}{1 - \tan^2\alpha}}{\dfrac{(1 - \tan^2\alpha) - 2\tan\alpha\,.\tan\alpha}{1 - \tan^2\alpha}}$$
$$= \frac{2\tan\alpha + \tan\alpha - \tan^3\alpha}{1 - \tan^2\alpha - 2\tan^2\alpha}$$

Hence $\boxed{\tan 3\alpha = \dfrac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}}$

Exercise 10.3

Q.1 Find the values of sin 2α, cos 2α and tan 2α , when:

(i) $\sin\alpha = \frac{12}{13}$

Where $0 < \alpha < \frac{\pi}{2}$

Solution:

Given that: $\sin\alpha = \frac{12}{13}$

$$\Rightarrow \cos\alpha = \sqrt{1-\sin^2\alpha}$$
$$= \sqrt{1-\left(\frac{12}{13}\right)^2}$$
$$= \sqrt{1-\frac{144}{169}}$$
$$= \sqrt{\frac{169-144}{169}}$$
$$= \sqrt{\frac{25}{169}}$$
$$\cos\alpha = \frac{5}{13}$$

Using $\sin 2\alpha = 2\sin\alpha\cos\alpha$

$$= 2\left(\frac{12}{13}\right)\left(\frac{5}{13}\right)$$
$$\sin 2\alpha = \frac{120}{169}$$

Now $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$$\cos 2\alpha = \left(\frac{5}{13}\right)^2 - \left(\frac{12}{13}\right)^2$$
$$= \frac{25}{169} - \frac{144}{169}$$
$$= \frac{25-144}{169}$$
$$\cos 2\alpha = \frac{-119}{169}$$

Now $\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{\frac{120}{169}}{\frac{-119}{169}}$

$$\tan 2\alpha = \frac{-120}{119}$$

(ii) $\cos\alpha = \frac{3}{5}$

Solution:

Given that: $\cos\alpha = \frac{3}{5}$

$$\sin\alpha = \sqrt{1-\cos^2\alpha}$$
$$= \sqrt{1-\left(\frac{3}{5}\right)^2}$$
$$= \sqrt{1-\frac{9}{25}}$$
$$= \sqrt{\frac{25-9}{25}}$$
$$= \sqrt{\frac{16}{25}}$$
$$\sin\alpha = \frac{4}{5}$$

Using $\sin 2\alpha = 2\sin\alpha\cos\alpha$

$$= 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)$$
$$\sin 2\alpha = \frac{24}{25}$$

Now $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$$= \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2$$
$$= \frac{9}{25} - \frac{16}{25}$$
$$= \frac{9-16}{25}$$
$$\cos 2\alpha = \frac{-7}{25}$$

Now $\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{\frac{24}{25}}{\frac{-7}{25}}$

$$\tan 2\alpha = \frac{-24}{7}$$

Prove the following Identities:

Q.2 $\cot\alpha - \tan\alpha = 2\cot 2\alpha$

Solution:

$$\text{L.H.S.} = \cot\alpha - \tan\alpha$$
$$= \frac{\cos\alpha}{\sin\alpha} - \frac{\sin\alpha}{\cos\alpha}$$
$$= \frac{\cos^2\alpha - \sin^2\alpha}{\sin\alpha\cos\alpha}$$
$$= \frac{\cos 2\alpha}{\sin\alpha\cos\alpha}$$
$$= \frac{2\cos 2\alpha}{2\sin\alpha\cdot\cos\alpha}$$
$$= \frac{2\cos 2\alpha}{\sin 2\alpha}$$
$$= 2\cot 2\alpha = \text{R.H.S.}$$

Q.3 $\frac{\sin 2\alpha}{1+\cos 2\alpha} = \tan\alpha$

Solution:

$$\text{L.H.S.} = \frac{\sin 2\alpha}{1+\cos 2\alpha}$$
$$= \frac{2\sin\alpha\cos\alpha}{2\cos^2\alpha}$$
$$= \frac{\sin\alpha}{\cos\alpha}$$
$$= \tan\alpha = \text{R.H.S.}$$

Q.4 $\frac{1-\cos\alpha}{\sin\alpha} = \tan\frac{\alpha}{2}$

Solution:

$$\text{L.H.S.} = \frac{1-\cos\alpha}{\sin\alpha}$$
$$= \frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$$
$$= \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$$
$$= \tan\frac{\alpha}{2} = \text{R.H.S.}$$

Q.5 $\frac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha} = \sec 2\alpha - \tan 2\alpha$

Solution:

$$\text{R.H.S.} = \sec 2\alpha - \tan 2\alpha$$
$$= \frac{1}{\cos 2\alpha} - \frac{\sin 2\alpha}{\cos 2\alpha}$$
$$= \frac{1-\sin 2\alpha}{\cos 2\alpha}$$
$$= \frac{\sin^2\alpha + \cos^2\alpha - 2\sin\alpha\cos\alpha}{\cos^2\alpha - \sin^2\alpha}$$
$$= \frac{(\cos\alpha - \sin\alpha)^2}{(\cos\alpha - \sin\alpha)(\cos\alpha + \sin\alpha)}$$
$$= \frac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha} = \text{L.H.S.}$$

Alternate Solution:

$$\text{L.H.S.} = \frac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha}$$
$$= \frac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha} \times \frac{\cos\alpha - \sin\alpha}{\cos\alpha - \sin\alpha}$$
$$= \frac{(\cos\alpha - \sin\alpha)^2}{\cos^2\alpha - \sin^2\alpha}$$
$$= \frac{\cos^2\alpha + \sin^2\alpha - 2\sin\alpha\cos\alpha}{\cos 2\alpha}$$
$$= \frac{1-\sin 2\alpha}{\cos 2\alpha}$$
$$= \frac{1}{\cos 2\alpha} - \frac{\sin 2\alpha}{\cos 2\alpha} = \sec 2\alpha - \tan 2\alpha$$
$$= \text{R.H.S.}$$

Q.6 $\sqrt{\dfrac{1+\sin\alpha}{1-\sin\alpha}} = \dfrac{\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}-\cos\frac{\alpha}{2}}$

Solution:

$$\text{L.H.S} = \sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}}$$

$$= \sqrt{\frac{\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}-2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}}$$

$$= \sqrt{\frac{\left(\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}\right)^2}{\left(\sin\frac{\alpha}{2}-\cos\frac{\alpha}{2}\right)^2}}$$

$$= \frac{\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}-\cos\frac{\alpha}{2}} = \text{R.H.S.}$$

Q.7 $\dfrac{\operatorname{cosec}\theta+2\operatorname{cosec}2\theta}{\sec\theta} = \cot\dfrac{\theta}{2}$

Solution:

$$\text{L.H.S.} = \frac{\operatorname{cosec}\theta+2\operatorname{cosec}2\theta}{\sec\theta}$$

$$= \frac{\operatorname{cosec}\theta}{\sec\theta}+\frac{2\operatorname{cosec}2\theta}{\sec\theta}$$

$$= \frac{\cos\theta}{\sin\theta}+\frac{2\cos\theta}{\sin 2\theta}$$

$$= \frac{\cos\theta}{\sin\theta}+\frac{2\cos\theta}{2\sin\theta\cos\theta}$$

$$= \frac{\cos\theta}{\sin\theta}+\frac{1}{\sin\theta}$$

$$= \frac{1+\cos\theta}{\sin\theta}$$

$$= \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$= \cot\frac{\theta}{2} = \text{R.H.S}$$

Q.8 $1+\tan\alpha\tan2\alpha = \sec2\alpha$

Solution:

$$\text{L.H.S.} = 1+\tan\alpha\tan2\alpha$$

$$= 1+\frac{\sin\alpha}{\cos\alpha}.\frac{\sin2\alpha}{\cos2\alpha}$$

$$= \frac{\cos2\alpha\cos\alpha+\sin2\alpha.\sin\alpha}{\cos\alpha\cos2\alpha}$$

$$= \frac{\cos(2\alpha-\alpha)}{\cos\alpha\cos2\alpha}$$

$$= \frac{\cos\alpha}{\cos\alpha\ \cos2\alpha}$$

$$= \frac{1}{\cos2\alpha}$$

$$= \sec2\alpha = \text{R.H.S.}$$

Q.9 $\dfrac{2\sin\theta\sin2\theta}{\cos\theta+\cos3\theta} = \tan2\theta\tan\theta$

Solution:

$$\text{L.H.S.} = \frac{2\sin\theta\sin2\theta}{\cos\theta+\cos3\theta}$$

$$= \frac{2\sin\theta\sin2\theta}{\cos\theta+\left(4\cos^3\theta-3\cos\theta\right)}$$

$$= \frac{2\sin\theta\sin2\theta}{\cos\theta\left[1+4\cos^2\theta-3\right]}$$

$$= \frac{2\sin\theta\sin2\theta}{\cos\theta\left(4\cos^2\theta-2\right)}$$

$$= \frac{2\sin\theta\sin2\theta}{2\cos\theta\left(2\cos^2\theta-1\right)}$$

$$= \frac{2\sin\theta \sin 2\theta}{2\cos\theta . \cos 2\theta}$$

$$= \frac{\sin\theta}{\cos\theta} \cdot \frac{\sin 2\theta}{\cos 2\theta}$$

$$= \tan\theta . \tan 2\theta = \text{R.H.S.}$$

Q.10 $\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = 2$

Solution:

$$\text{L.H.S} = \frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta}$$

$$= \frac{\sin 3\theta\cos\theta - \cos 3\theta\sin\theta}{\sin\theta\cos\theta}$$

$$= \frac{\sin(3\theta - \theta)}{\sin\theta\cos\theta}$$

$$= \frac{\sin 2\theta}{\sin\theta\cos\theta}$$

$$= \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta}$$

$$= 2 = \text{R.H.S.}$$

Q.11 $\frac{\cos 3\theta}{\cos\theta} + \frac{\sin 3\theta}{\sin\theta} = 4\cos 2\theta$

Solution:

$$\text{L.H.S} = \frac{\cos 3\theta}{\cos\theta} + \frac{\sin 3\theta}{\sin\theta}$$

$$= \frac{\cos 3\theta\sin\theta + \sin 3\theta\cos\theta}{\cos\theta\sin\theta}$$

$$= \frac{\sin 3\theta\cos\theta + \cos 3\theta\sin\theta}{\sin\theta\cos\theta}$$

$$= \frac{\sin(3\theta + \theta)}{\cos\theta\sin\theta}$$

$$= \frac{\sin 4\theta}{\cos\theta\sin\theta}$$

$$= \frac{2\sin 2\theta\cos 2\theta}{\cos\theta\sin\theta}$$

$\because$ multiply and divide by 2

$$= \frac{2\times 2\sin 2\theta\cos 2\theta}{2\cos\theta\sin\theta}$$

$$= \frac{4\sin 2\theta\cos 2\theta}{\sin 2\theta}$$

$$= 4\cos 2\theta = \text{R.H.S}$$

Q.12 $\frac{\tan\frac{\theta}{2} + \cot\frac{\theta}{2}}{\cot\frac{\theta}{2} - \tan\frac{\theta}{2}} = \sec\theta$

Solution:

$$\text{L.H.S} = \frac{\tan\frac{\theta}{2} + \cot\frac{\theta}{2}}{\cot\frac{\theta}{2} - \tan\frac{\theta}{2}}$$

$$= \frac{\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} + \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}}{\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} - \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}$$

$$= \frac{\frac{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}}{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}}{\frac{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}}$$

$$= \frac{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}$$

$$= \frac{1}{\cos\theta} = \sec\theta = \text{R.H.S.}$$

Q.13 $\frac{\sin 3\theta}{\cos\theta}+\frac{\cos 3\theta}{\sin\theta} = 2\cot 2\theta$

Solution:

$$\text{L.H.S.} =\frac{\sin 3\theta}{\cos\theta}+\frac{\cos 3\theta}{\sin\theta}$$
$$=\frac{\sin 3\theta\sin\theta+\cos 3\theta.\cos\theta}{\cos\theta.\sin\theta}$$
$$=\frac{\cos(3\theta-\theta)}{\cos\theta.\sin\theta}$$
$$=\frac{\cos 2\theta}{\cos\theta\sin\theta}$$
$$=\frac{2\cos 2\theta}{2\sin\theta\cos\theta}$$
$$=\frac{2\cos 2\theta}{\sin 2\theta}$$
$$=2\cot 2\theta=\text{R.H.S.}$$

Q.14 Reduce $\sin^4\theta$ to an expression involving only function of multiples of θ raised to the first power.

Solution:

Here

$$\sin^4\theta=\left(\sin^2\theta\right)^2$$
$$=\left(\frac{1-\cos 2\theta}{2}\right)^2$$
$$=\frac{1}{4}\left(1-\cos 2\theta\right)^2$$
$$=\frac{1}{4}\left(1+\cos^2 2\theta-2\cos 2\theta\right)$$
$$=\frac{1}{4}\left(1+\frac{1+\cos 4\theta}{2}-2\cos 2\theta\right)$$
$$=\frac{1}{4}\left(1+\frac{1}{2}+\frac{\cos 4\theta}{2}-2\cos 2\theta\right)$$
$$=\frac{1}{4}\left(\frac{3}{2}+\frac{\cos 4\theta}{2}-2\cos 2\theta\right)$$
$$=\frac{3}{8}+\frac{\cos 4\theta}{8}-\frac{2\cos 2\theta}{4}$$
$$=\frac{3+\cos 4\theta-4\cos 2\theta}{8}$$
$$=\frac{3-4\cos 2\theta+\cos 4\theta}{8}$$

Q.15 Find the value of $\sin\theta$ and $\cos\theta$ without using table or calculator, when θ is (i) 18° (ii) 36° (iii) 54° (iv) 72°

Hence prove that: $\cos 36°\cos 72°\cos 108°\cos 144° = \frac{1}{16}$

Solution:

(i)

Let $\theta=18°$

$\Rightarrow 5\theta=90°$

$\Rightarrow 3\theta+2\theta=90°$

$\Rightarrow 3\theta=90°-2\theta$

$\Rightarrow \sin 3\theta=\sin\left(90°-2\theta\right)$

$3\sin\theta-4\sin^3\theta=\cos 2\theta$

$$3\sin\theta - 4\sin^3\theta = 1 - 2\sin^2\theta$$

$$4\sin^3\theta - 2\sin^2\theta - 3\sin\theta + 1 = 0$$

Using synthetic division here

1	4	-2	-3	1
		4	2	-1
	4	2	-1	0

So one of the root of equation is

$\sin\theta = 1 \Rightarrow \theta = 90^\circ$ (neglect it because our $\theta = 18^\circ$)

Depressed equation is

$$4\sin^2\theta + 2\sin\theta - 1 = 0$$

Here $a = 4, b = 2, c = -1$

$$\sin\theta = \frac{-2 \pm \sqrt{4 - 4(4)(-1)}}{2(4)}$$

$$= \frac{-2 \pm \sqrt{4+16}}{8}$$

$$= \frac{-2 \pm \sqrt{20}}{8}$$

$$= \frac{-2 \pm 2\sqrt{5}}{8}$$

$$= \frac{-1 \pm \sqrt{5}}{4}$$

$$\sin\theta = \frac{\sqrt{5}-1}{4} > 0 \quad , \quad \sin\theta = \frac{-\sqrt{5}-1}{4} < 0$$

As here $\theta = 18^\circ$ lies in 1st quadrant, so $\sin\theta$ should be positive

So, $\boxed{\sin 18^\circ = \frac{\sqrt{5}-1}{4}}$

Now $\cos 72^\circ = \cos(90^\circ - 18^\circ)$

$= \sin 18^\circ$

$$\boxed{\cos 72^\circ = \sin 18^\circ = \frac{\sqrt{5}-1}{4}}$$

Next let $\theta = 36^\circ$

$\Rightarrow \quad 5\theta = 36^\circ \times 5$

$5\theta = 180^\circ$

$3\theta + 2\theta = 180^\circ$

$3\theta = 180^\circ - 2\theta$

$\sin 3\theta = \sin(180^\circ - 2\theta)$

$3\sin\theta - 4\sin^3\theta = \sin 2\theta$

$3\sin\theta - 4\sin^3\theta - 2\sin\theta\cos\theta = 0$

$\sin\theta\left(3 - 4\sin^2\theta - 2\cos\theta\right) = 0$

$\sin\theta \neq 0$ because here $\theta = 36^\circ$ so $\sin 36^\circ \neq 0$

and $\quad 3 - 4\sin^2\theta - 2\cos\theta = 0$

$4\sin^2\theta + 2\cos\theta - 3 = 0$

$4\left(1 - \cos^2\theta\right) + 2\cos\theta - 3 = 0$

$4 - 4\cos^2\theta + 2\cos\theta - 3 = 0$

$-4\cos^2\theta + 2\cos\theta + 1 = 0$

$4\cos^2\theta - 2\cos\theta - 1 = 0$

Here $a = 4, b = -2, c = -1$

Using $\cos\theta = \dfrac{2 \pm \sqrt{4 - 4(4)(-1)}}{2(4)}$

$= \dfrac{2 \pm \sqrt{4 + 16}}{8}$

$= \dfrac{2 \pm \sqrt{20}}{8}$

$= \dfrac{2 \pm 2\sqrt{5}}{8}$

$= \dfrac{1 \pm \sqrt{5}}{4}$

Here $\cos\theta = \dfrac{1 - \sqrt{5}}{4} < 0$ and $\cos\theta = \dfrac{1 + \sqrt{5}}{4} > 0$

So neglect $\cos\theta = \dfrac{1 - \sqrt{5}}{4} < 0$ because here $\theta = 36^\circ$ lies in 1^{st} quadrant so $\cos 36^\circ$ should be positive.

So $\boxed{\cos 36° = \dfrac{1+\sqrt{5}}{4}}$

Also $\cos 36° = \cos(90° - 54°)$

$= \sin 54°$

$\therefore \boxed{\cos 36° = \sin 54° = \dfrac{1+\sqrt{5}}{4}}$

Now $\cos^2 18° = 1 - \sin^2 18°$

$$= 1 - \left(\frac{\sqrt{5}-1}{4}\right)^2$$

$$= 1 - \left(\frac{5+1-2\sqrt{5}}{16}\right)$$

$$= \frac{16-\left(6-2\sqrt{5}\right)}{16}$$

$$\cos^2 18° = \frac{10+2\sqrt{5}}{16}$$

$$\Rightarrow \quad \cos 18° = \pm\sqrt{\frac{10+2\sqrt{5}}{16}}$$

$\therefore \theta = 18°$ lies in quad.I so $\cos 18° > 0$

$$\boxed{\cos 18° = \frac{\sqrt{10+2\sqrt{5}}}{4}}$$

Also $\cos 18° = \cos(90° - 72°)$

$= \sin 72°$

$$\boxed{\therefore \cos 18° = \sin 72° = \frac{\sqrt{10+2\sqrt{5}}}{4}}$$

Now $\cos^2 54° = 1 - \sin^2 54°$

$$= 1 - \left(\frac{1+\sqrt{5}}{4}\right)^2$$

$$= 1 - \left(\frac{1+5+2\sqrt{5}}{16}\right)$$

$$= \frac{16-\left(1+5+2\sqrt{5}\right)}{16}$$

$$= \frac{10-2\sqrt{5}}{16}$$

$$\cos 54^\circ = \pm\frac{\sqrt{10-2\sqrt{5}}}{4}$$

$\therefore \theta = 54^\circ$ lies in quad.I so $\cos 54^\circ > 0$

$$\boxed{\cos 54^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}}$$

Also $\cos 54^\circ = \cos\left(90^\circ - 36^\circ\right)$

$= \sin 36^\circ$

$$\boxed{\sin 36^\circ = \cos 54^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}}$$

Next we show that

$$\cos 36^\circ \cos 72^\circ \cos 108^\circ \cos 144^\circ = \frac{1}{16}$$

L.H.S. $= \cos 36^\circ \cos 72^\circ \cos 108^\circ \cos 144^\circ$

$= \cos 36^\circ \cos 72^\circ \cos\left(180^\circ - 72^\circ\right)\cos\left(180^\circ - 36^\circ\right)$

$= \cos 36^\circ \cos 72^\circ \left(-\cos 72^\circ\right)\left(-\cos 36^\circ\right)$

$= \cos^2 36.\cos^2 72$

$$= \left(\frac{1+\sqrt{5}}{4}\right)^2\left(\frac{\sqrt{5}-1}{4}\right)^2$$

$$= \left(\frac{\left(\sqrt{5}+1\right)\left(\sqrt{5}-1\right)}{4\times 4}\right)^2$$

$$= \left(\frac{\left(\sqrt{5}\right)^2-(1)^2}{16}\right)^2$$

$$= \left(\frac{5-1}{16}\right)^2$$

$$= \left(\frac{4}{16}\right)^2$$

$$= \left(\frac{1}{4}\right)^2$$

$$= \frac{1}{16} = \text{R.H.S.}$$

Sum, Difference and Product of Sines and Cosines:

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta)+\sin(\alpha-\beta)$$

$$2\cos\alpha\sin\beta = \sin(\alpha+\beta)-\sin(\alpha-\beta)$$

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta)+\cos(\alpha-\beta)$$

$$-2\sin\alpha\sin\beta = \cos(\alpha+\beta)-\cos(\alpha-\beta)$$

and

$$\sin P+\sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$$

$$\sin P-\sin Q = 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$$

$$\cos P+\cos Q = 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$$

$$\cos P-\cos Q = -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$$

Exercise 10.4

Q.1 Express the following products as sum or differences:

(i) $2\sin 3\theta\cos\theta$

Solution:

We know that

$2\sin\alpha\cos\beta$

$=\sin(\alpha+\beta)+\sin(\alpha-\beta)$

So

$2\sin 3\theta\cos\theta$

$=\sin(3\theta+\theta)+\sin(3\theta-\theta)$

$=\sin 4\theta+\sin 2\theta$

(ii) $2\cos 5\theta\sin 3\theta$

Solution:

We know that

$2\cos\alpha\sin\beta$

$=\sin(\alpha+\beta)-\sin(\alpha-\beta)$

So

$2\cos 5\theta\sin 3\theta$

$=\sin(5\theta+3\theta)-\sin(5\theta-3\theta)$

$=\sin 8\theta-\sin 2\theta$

(iii) $\sin 5\theta\cos 2\theta$

Solution:

$\sin 5\theta\cos 2\theta$

$=\dfrac{1}{2}(2\sin 5\theta\cos 2\theta)$

$=\dfrac{1}{2}\left[\sin(5\theta+2\theta)+\sin(5\theta-2\theta)\right]$

$=\dfrac{1}{2}\left[\sin 7\theta+\sin 3\theta\right]$

(iv) $2\sin 7\theta\sin 2\theta$

Solution:

$2\sin 7\theta\sin 2\theta$

$=-(-2\sin 7\theta\sin 2\theta)$

$=-\left[\cos(7\theta+2\theta)-\cos(7\theta-2\theta)\right]$

$=-\left[\cos 9\theta-\cos 5\theta\right]$

$=\cos 5\theta-\cos 9\theta$

(v) $\cos(x+y)\sin(x-y)$

Solution:

$\cos(x+y)\sin(x-y)$

$=\dfrac{1}{2}2\cos(x+y)\sin(x-y)$

$=\dfrac{1}{2}\left[\sin(x+y+x-y)-\sin(x+y-x+y)\right]$

$=\dfrac{1}{2}\left[\sin 2x-\sin 2y\right]$

(vi) $\cos(2x+30°)\cos(2x-30°)$

Solution:

$\cos(2x+30°)\cos(2x-30°)$

$=\dfrac{1}{2}\left[2\cos(2x+30°)\cos(2x-30°)\right]$

$=\dfrac{1}{2}\begin{bmatrix}\cos(2x+30°+2x-30°)\\+\cos(2x+30°-2x+30°)\end{bmatrix}$

$=\dfrac{1}{2}\left[\cos 4x+\cos 60°\right]$

(vii) $\sin 12°\sin 46°$

Solution:

$\sin 12°\sin 46°$

$=\dfrac{-1}{2}\left[-2\sin 46°\sin 12°\right]$

$=\dfrac{-1}{2}\left[\cos(46°+12°)-\cos(46°-12°)\right]$

$=\dfrac{-1}{2}\left[\cos 58°-\cos 34°\right]$

$=\dfrac{1}{2}\left[\cos 34°-\cos 58°\right]$

(viii) $\sin(x+45^\circ)\ \sin(x-45^\circ)$

Solution:

$\sin(x+45^\circ)\sin(x-45^\circ)$

$= \frac{-1}{2}\left[-2\sin(x+45^\circ)\sin(x-45^\circ)\right]$

$= \frac{-1}{2}\left[\begin{array}{l}\cos(x+45^\circ+x-45^\circ)\\ -\cos(x+45^\circ-x+45^\circ)\end{array}\right]$

$= \frac{-1}{2}\left[\cos 2x-\cos 90^\circ\right]$

$= \frac{1}{2}\left[\cos 90^\circ-\cos 2x\right]$

Q.2 Express the following sums or differences as products;

(i) $\sin 5\theta+\sin 3\theta$

Solution:

$\sin 5\theta+\sin\theta$

$= 2\sin\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right)$

$= 2\sin\frac{8\theta}{2}\cos\frac{2\theta}{2}$

$= 2\sin 4\theta\cos\theta$

(ii) $\sin 8\theta-\sin 4\theta$

Solution:

$\sin 8\theta-\sin 4\theta$

$= 2\cos\left(\frac{8\theta+4\theta}{2}\right)\sin\left(\frac{8\theta-4\theta}{2}\right)$

$= 2\cos\frac{12\theta}{2}\sin\frac{4\theta}{2}$

$= 2\cos 6\theta\sin 2\theta$

(iii) $\cos 6\theta+\cos 3\theta$

Solution:

$\cos 6\theta+\cos 3\theta$

$= 2\cos\frac{6\theta+3\theta}{2}\cos\frac{6\theta-3\theta}{2}$

$= 2\cos\frac{9\theta}{2}\cos\frac{3\theta}{2}$

(iv) $\cos 7\theta-\cos\theta$

Solution:

$\cos 7\theta-\cos\theta$

$= -2\sin\frac{7\theta+\theta}{2}\sin\frac{7\theta-\theta}{2}$

$= -2\sin\frac{8\theta}{2}\sin\frac{6\theta}{2}$

$= -2\sin 4\theta\sin 3\theta$

(v) $\cos 12^\circ+\cos 48^\circ$

Solution:

$\cos 12^\circ+\cos 48^\circ$

$= 2\cos\frac{12^\circ+48^\circ}{2}\cos\frac{12^\circ-48^\circ}{2}$

$= 2\cos\frac{60^\circ}{2}\cos\frac{-36^\circ}{2}$

$= 2\cos 30^\circ\cos(-18^\circ)$

$= 2\cos 30^\circ\cos 18^\circ$

(vi) $\sin(x+30^\circ)+\sin(x-30^\circ)$

Solution:

$\sin(x+30^\circ)+\sin(x-30^\circ)$

$= 2\sin\left(\frac{x+30^\circ+x-30^\circ}{2}\right)\cos\left(\frac{x+30^\circ-x+30^\circ}{2}\right)$

$= 2\sin\frac{2x}{2}\cos\frac{60^\circ}{2}$

$= 2\sin x\cos 30^\circ$

Q.3 Prove the following identities :

(i) $\frac{\sin 3x-\sin x}{\cos x-\cos 3x}=\cot 2x$

Solution:

L.H.S. $= \frac{\sin 3x-\sin x}{\cos x-\cos 3x}$

$= \frac{2\cos\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right)}{-2\sin\left(\frac{x+3x}{2}\right)\sin\left(\frac{x-3x}{2}\right)}$

$$= \frac{2\cos\frac{4x}{2}\sin\frac{2x}{2}}{-2\sin\frac{4x}{2}\sin\frac{-2x}{2}}$$

$$= \frac{2\cos 2x\sin x}{-2\sin 2x\sin(-x)}$$

$$= \frac{2\cos 2x\sin x}{-2\sin 2x\sin(-x)}$$

$$= \frac{2\cos 2x\sin x}{+2\sin 2x\sin x}$$

$$= \cot 2x = \text{R.H.S.}$$

(ii) $\mathbf{\frac{\sin 8x + \sin 2x}{\cos 8x + \cos 2x} = \tan 5x}$

Solution:

L.H.S. $= \frac{\sin 8x + \sin 2x}{\cos 8x + \cos 2x}$

$$= \frac{2\sin\frac{8x+2x}{2}\cos\frac{8x-2x}{2}}{2\cos\frac{8x+2x}{2}\cos\frac{8x-2x}{2}}$$

$$= \frac{2\sin\frac{10x}{2}\cos\frac{6x}{2}}{2\cos\frac{10x}{2}\cos\frac{6x}{2}}$$

$$= \frac{2\sin 5x\cos 3x}{2\cos 5x\cos 3x}$$

$$= \tan 5x = \text{R.H.S.}$$

(iii) $\mathbf{\frac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta} = \tan\left(\frac{\alpha-\beta}{2}\right)\cot\left(\frac{\alpha+\beta}{2}\right)}$

Solution:

L.H.S. $= \frac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta}$

$$= \frac{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}$$

$$= \tan\frac{\alpha-\beta}{2}\cot\frac{\alpha+\beta}{2} = \text{R.H.S.}$$

Q.4 Prove that:

(i) $\mathbf{\cos 20° + \cos 100° + \cos 140° = 0}$

Solution:

L.H.S. $= \cos 20° + \cos 100° + \cos 140°$

$$= 2\cos\frac{20°+100°}{2}\cos\frac{20°-100°}{2} + \cos 140°$$

$$= 2\cos\left(\frac{120°}{2}\right)\cos\left(\frac{-80°}{2}\right) + \cos 140°$$

$$= 2\cos 60°\cos(-40°) + \cos 140°$$

$$= 2\times\frac{1}{2}\cos 40° + \cos 140°$$

$$= \cos 40° + \cos 140°$$

$$= 2\cos\left(\frac{40°+140°}{2}\right)\cos\left(\frac{40°-140°}{2}\right)$$

$$= 2\cos\left(\frac{180°}{2}\right)\cos\left(\frac{-100°}{2}\right)$$

$$= 2\cos 90°\cos(-50°)$$

$$= 2(0)\cos(50°)$$

$$= 0 = \text{R.H.S}$$

(ii) $\mathbf{\sin\left(\frac{\pi}{4}-\theta\right)\sin\left(\frac{\pi}{4}+\theta\right) = \frac{1}{2}\cos 2\theta}$

Solution:

L.H.S. $= \sin\left(\frac{\pi}{4}-\theta\right)\sin\left(\frac{\pi}{4}+\theta\right)$

$$= -\frac{1}{2}\left[-2\sin\left(\frac{\pi}{4}-\theta\right)\sin\left(\frac{\pi}{4}+\theta\right)\right]$$

$$= -\frac{1}{2}\left[\cos\left(\frac{\pi}{4}-\theta+\frac{\pi}{4}+\theta\right) - \cos\left(\frac{\pi}{4}-\theta-\frac{\pi}{4}-\theta\right)\right]$$

$$= -\frac{1}{2}\left[\cos\left(\frac{\pi}{2}\right) - \cos(-2\theta)\right]$$

$$= -\frac{1}{2}[0 - \cos 2\theta]$$

$$= \frac{1}{2}\cos 2\theta = \text{R.H.S}$$

(iii) $\dfrac{\sin\theta+\sin3\theta+\sin5\theta+\sin7\theta}{\cos\theta+\cos3\theta+\cos5\theta+\cos7\theta}=\tan4\theta$

Solution:

L.H.S. $=\dfrac{\sin\theta+\sin3\theta+\sin5\theta+\sin7\theta}{\cos\theta+\cos3\theta+\cos5\theta+\cos7\theta}$

$=\dfrac{(\sin7\theta+\sin\theta)+(\sin5\theta+\sin3\theta)}{(\cos7\theta+\cos\theta)+(\cos5\theta+\cos3\theta)}$

$=\dfrac{2\sin\left(\frac{7\theta+\theta}{2}\right)\cos\left(\frac{7\theta-\theta}{2}\right)+2\sin\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right)}{2\cos\left(\frac{7\theta+\theta}{2}\right)\cos\left(\frac{7\theta-\theta}{2}\right)+2\cos\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right)}$

$=\dfrac{2\sin\left(\frac{8\theta}{2}\right)\cos\left(\frac{6\theta}{2}\right)+2\sin\left(\frac{8\theta}{2}\right)\cos\left(\frac{2\theta}{2}\right)}{2\cos\left(\frac{8\theta}{2}\right)\cos\left(\frac{6\theta}{2}\right)+2\cos\left(\frac{8\theta}{2}\right)\cos\left(\frac{2\theta}{2}\right)}$

$=\dfrac{2\sin4\theta\cos3\theta+2\sin4\theta\cos\theta}{2\cos4\theta\cos3\theta+2\cos4\theta\cos\theta}$

$=\dfrac{2\sin4\theta[\cos3\theta+\cos\theta]}{2\cos4\theta[\cos3\theta+\cos\theta]}$

$=\dfrac{\sin4\theta}{\cos4\theta}$

$=\tan4\theta$ = R.H.S.

Q.5 Prove that:

(i) $\cos20^\circ\cos40^\circ\cos60^\circ\cos80^\circ=\dfrac{1}{16}$

Solution:

L.H.S. $=\cos20^\circ\cos40^\circ\cos60^\circ\cos80^\circ$

$=\dfrac{1}{2}\cos80^\circ\cos40^\circ\cos20^\circ$

$=\dfrac{1}{4}\{2\cos80^\circ\cos40^\circ\}\cos20^\circ$

$=\dfrac{1}{4}\{\cos(80^\circ+40^\circ)+\cos(80^\circ-40^\circ)\}\cos20^\circ$

$=\dfrac{1}{4}\{\cos120^\circ+\cos40^\circ\}\cos20^\circ$

$$=\frac{1}{4}\left\{\frac{-1}{2}+\cos 40^\circ\right\}\cos 20^\circ$$

$$=\frac{-1}{8}\cos 20^\circ+\frac{1}{4}\cos 40^\circ\cos 20^\circ$$

$$=\frac{-1}{8}\cos 20^\circ+\frac{1}{8}\left(2\cos 40^\circ\cos 20^\circ\right)$$

$$=\frac{-1}{8}\cos 20^\circ+\frac{1}{8}\left[\cos\left(40^\circ+20^\circ\right)+\cos\left(40^\circ-20^\circ\right)\right]$$

$$=\frac{-1}{8}\cos 20^\circ+\frac{1}{8}\left(\cos 60^\circ+\cos 20^\circ\right)$$

$$=\frac{-1}{8}\cos 20^\circ+\frac{1}{8}\left(\frac{1}{2}+\cos 20^\circ\right)$$

$$=\frac{-1}{8}\cos 20^\circ+\frac{1}{16}+\frac{1}{8}\cos 20^\circ$$

$$=\frac{1}{16}=\text{R.H.S.}$$

(ii) $\sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{\pi}{3}\sin\frac{4\pi}{9}=\frac{3}{16}$

Solution:

L.H.S. $=\sin 20^\circ\sin 40^\circ\sin 60^\circ\sin 80^\circ$

$$=\frac{\sqrt{3}}{2}\left[\sin 80^\circ\sin 40^\circ\right]\sin 20^\circ$$

$$=-\frac{\sqrt{3}}{4}\left[-2\sin 80^\circ\sin 40^\circ\right]\sin 20^\circ$$

$$=\frac{-\sqrt{3}}{4}\left[\cos\left(80^\circ+40^\circ\right)-\cos\left(80^\circ-40^\circ\right)\right]\sin 20^\circ$$

$$=\frac{-\sqrt{3}}{4}\left[\cos 120^\circ-\cos 40^\circ\right]\sin 20^\circ$$

$$=\frac{-\sqrt{3}}{4}\left[\frac{-1}{2}-\cos 40^\circ\right]\sin 20^\circ$$

$$=\frac{\sqrt{3}}{8}\sin 20^\circ+\frac{\sqrt{3}}{4}\cos 40^\circ\sin 20^\circ$$

$$=\frac{\sqrt{3}}{8}\sin 20^\circ+\frac{\sqrt{3}}{8}\left(2\cos 40^\circ\sin 20^\circ\right)$$

$$=\frac{\sqrt{3}}{8}\sin 20° + \frac{\sqrt{3}}{8}\left[\sin(40°+20°) - \sin(40°-20°)\right]$$

$$=\frac{\sqrt{3}}{8}\sin 20° + \frac{\sqrt{3}}{8}(\sin 60° - \sin 20°)$$

$$=\frac{\sqrt{3}}{8}\sin 20° + \frac{\sqrt{3}}{8}\left(\frac{\sqrt{3}}{2} - \sin 20°\right)$$

$$=\frac{\sqrt{3}}{8}\sin 20° + \frac{3}{16} - \frac{\sqrt{3}}{8}\sin 20°$$

$$=\frac{3}{16} = \text{R.H.S.}$$

(iii) $\mathbf{\sin 10° \sin 30° \sin 50° \sin 70° = \frac{1}{16}}$

Solution:

L.H.S. $=\sin 10° \sin 30° \sin 50° \sin 70°$

$$=\frac{1}{2}(\sin 70° \sin 50°)\sin 10°$$

$$=\frac{1}{-4}(-2\sin 70° \sin 50°)\sin 10°$$

$$=\frac{1}{-4}\left[\cos(70°+50°) - \cos(70°-50°)\right]\sin 10°$$

$$=\frac{-1}{4}\left[\cos 120° - \cos 20°\right]\sin 10°$$

$$=\frac{-1}{4}\left[\frac{-1}{2} - \cos 20°\right]\sin 10°$$

$$=\frac{1}{8}\sin 10° + \frac{1}{4}\cos 20° \sin 10°$$

$$=\frac{1}{8}\sin 10° + \frac{1}{8}(2\cos 20° \sin 10°)$$

$$=\frac{1}{8}\sin 10° + \frac{1}{8}\left[\sin(20°+10°) - \sin(20°-10°)\right]$$

$$=\frac{1}{8}\sin 10° + \frac{1}{8}(\sin 30° - \sin 10°)$$

$$=\frac{1}{8}\sin 10° + \frac{1}{8}\left(\frac{1}{2} - \sin 10°\right)$$

$$=\frac{1}{8}\sin 10° + \frac{1}{16} - \frac{1}{8}\sin 10°$$

$$=\frac{1}{16} = \text{R.H.S.}$$