CHAPTER 12

APPLICATION OF TRIGONOMETRY

Introduction:

A triangle has six important elements; three angles and three sides. In a triangle *ABC*, the measures of the three angles are usually denoted by α, β, γ and the three measures of the three sides opposite to them are denoted by *a, b, c* respectively**.**

If any three out of these six elements, out of which at least one side, are given, the remaining three elements can be determined. This process of finding the unknown elements is called the **solution of the triangle.**

EXERCISE 12.1

Q.1 Find the values of

(i)

Solution:

$\sin 53°40'$

In the first column on the left hand side headed by degrees (in the Natural Sine table) we read the number $53°$ till the minute column number $36'$ is reached, we get the number 0.8049, then we see the right hand column headed by mean differences. Running down the column under $4'$ till the row of $53°$ is reached. We find 7 as the difference for 4. Adding 7 to 8049, we get 8056.

Hence

$\sin\ 53°\ 40' = 0.8056$

Alternate Solution:

(ii) $\cos 36°\ 20'$

Solution:

$$\cos\ (36°\ 20') = \cos(90° - 53°40')$$
$$= \sin 53°40'$$
$$= 0.8056$$

(iii) $\tan 19°\ 30'$

Solution:

$\tan 19°\ 30'$

In the first column on the left hand side headed by degrees in Natural Tangents table we reached the number $19°$ till the minute column number $30'$ is reached, we get the number 0.3541.

Hence

$\tan 19°\ 30' = 0.3541.$

(iv) $\cot\ 33°\ 50'$

Solution:

$$\cot\ 33°\ 50' = \cot(90° - 56°10') = \tan 56°\ 10'$$

In the first column on the left hand side headed by degrees in the Natural Tangents Table we read the number $56°$ till the minute column number $6'$ is reached, we get the number 0.4882, then we see the right hand column headed by mean differences. Running down then column under $4'$ till the row of $56°$ is reached. We find 38 as the difference for $4'$. Adding 38 to 4882, we get 4920. The integral part of the figure just next to $56°$ in the horizontal line is 1. Hence

$cot\ 33°\ 50' = \tan 56°10' = 1.4920$

(v) $\cos 42°\ 38'$

Solution:

$$\cos 42°\ 38'$$
$$= \cos(90° - 47°22')$$
$$= \sin 47°22'$$

In the first column on the let hand side headed by degrees (in the Natural Sine Table) ,We read the number 47°. Looking along the row of 47° till the minute column number $18'$ is reached, we get the number 0.7349, then we see the right hand column headed by mean differences. Running down the column under $4'$ till the row of $47°$ is reached. We find 8 as the difference for $4'$. Adding 8 to 7.349, we get 7357. Hence

$\cos 42°38' = \sin\ 47°22' = 0.7357$

(vi) $\tan\ 25°\,34'$

Solution:

In the first column on the left hand side headed by degrees (in the Natural Tangents Table) we read the number 25°.Looking along the row, of 25° till the minute column number $30'$ is reached, we get the number 0.4770, then we see the right hand column headed by mean differences. Running down the column under $4'$ till the row of 25° is reached. We find 14 as the difference for $4'$. Adding 14 to 4770 we get 4784.Hence

$\tan\ 25°34' = 0.4784$

(vii) $\sin 18°31'$

Solution:

In the first column on the left hand side headed by degrees (in the Natural Sine Table) we read the number 18° till the minute column number $30'$ is reached, we get the number 0.3173 , then we see the right hand column headed by mean differences. Running down the column under 1` till the row of 18° is reached. We find 3 as the difference for $1'$. Adding 3 to 3173, we get 3176. Hence

$\sin\ 18°31' = 0.3176$

(viii) $\cos 52°13'$

Solution:

$$\cos 52°13' = \cos\left(90° - 37°47'\right)$$
$$= \sin 37°47'$$

In the first column on the left hand side headed by degrees 9 in the (Natural Sine Table) we read the number 37°. Looking along the row of 37° till the minute column number $42'$ is reached, we get the number the number 0.6115, then we see the right hand column headed by mean difference . Running down the column under $5'$ till the row of 37^0 is reached, we 12 as the difference for $5'$. Adding 12 to 6115, we get 6127. Hence

$\cos 52°13' = \sin 37°47' = 0.6127$

(ix) $\tan 9°51'$

Solution:

$\tan 9°51'$

In the first column on the left hand side headed by degrees in the (Natural Tangents Table) we read the number 9°. Looking along the row of 9° till the minute column number $48'$ is reached, we get the number 0.1727, then we see the right hand column headed by mean differences. Running down the column under $3'$ till the row of 9° is reached. We find 9 as the difference for $3'$. Adding 9 to 1727, we get 1736.

Hence $\tan\ 9°51' = 0.0149$

Q.2 Find θ , if :

(i) $\sin\theta = 0.579$

Solution:

In the table of Natural Sine, we get the number 5793 (nearest to 5790) which lies at the intersection of the row beginning with 35° and the column headed by $24'$. The difference between 5793 and 5790 is 3 which does not occur in the row of 35° , so we take 2 which occurs in the row of 35° under the mean difference column by $1'$, so we subtract $1'$ from $35°\,24'$ as get

$$\boxed{\theta = \sin^{-1}\left(0.579\right) = 35°23'}$$

(ii) $\cos\theta = 0.9316$

Solution:

$$\cos\theta = 0.9316$$

$$\Rightarrow \sin(90° - \theta) = 0.9316$$

In the table of natural sine , we get the number 9317 (nearest to 9316) which lies at the intersection of the row beginning with $68°$ and the column headed by $42'$. The difference between 9317 and 9316 is 1 which occurs in the row of $68°$ under the mean difference column by $1'$, so we subtract $1'$ from $68°42'$ and get

$$90° - \theta = \sin^{-1}(0.9316)$$

$$90° - \theta = 68°41'$$

$$\theta = 21°19'$$

(iii) $\cos\theta = 0.5272$

Solution:

$$\cos\theta = 0.5272$$

$$\sin(90° - \theta) = 0.5272$$

In the table of Natural sine, we get the number 5255 (nearest to 5257) which lies at the intersection of the rows beginning with $31°$ and the column headed by $42'$. The difference between 5257 and 5255 is 2 which occurs in the row of $31°$ under the mean difference column by $1'$ in $31°42'$ and get

$$90° - \theta = \sin^{-1}(0.5257) = 31°43'$$

$$\theta = 58°17'$$

(iv) $\tan\theta = 1.705$

Solution:

$$\tan\theta = 1.705$$

In the Table of Natural Tangents , we get the number 7045 (nearest to 7050) which lies at the intersection of the row beginning with $59°$ and the column headed by $36'$. The difference between 7050 and 7045 is 5 which does not occur in the row of $59°$, so we ignore it and get

$$\theta = \tan^{-1}(1.705)$$

$$\theta = 59°36'$$

(v) $\tan\theta = 21.943$

Solution:

In the Table of Natural Tangents, we get the number 21.20 (nearest to 21.943 which lies at the intersection of the row beginning with $87°$ and the column headed by $18'$. The difference does not occur in the Table of Tangents, So $5'$ are added in $87°18'$ (we cannot take the value of $87°\ 24'$ from the table).

$$\theta = \tan^{-1}(21.493)$$

$$\theta = 87°\ 23'$$

(vi) $\sin\theta = 0.5186$

Solution:

$$\sin\theta = 0.5186$$

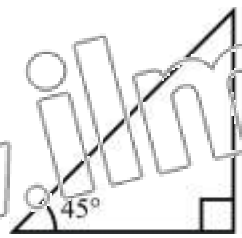
In the table of Natural sine, we get the number 5180 (nearest to 5186) which lies at the intersection of the row beginning with $31°$ and the column headed by $12'$. the difference between 5186 and 5180 is 6 which does not occur in the row of $31°$, so we take 5 which occurs in the row of $31°$ under the mean difference column by $2'$, so we add $2'$ in $31°\ 12'$ and get

$$\theta = \sin^{-1}(0.5186) = 31°14'$$

EXERCISE 12.2

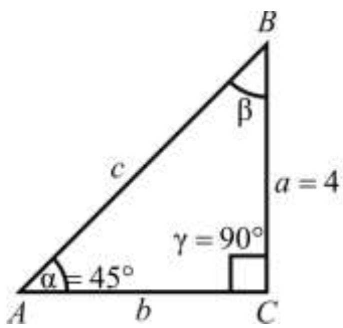
Q.1 Find the unknown angles and sides of the following triangles.

(i)



Solution:

Labeling the given triangle .



We have

$a=4,\quad \alpha=45°,\quad \gamma=90°$

We have to find b, c and β .

As we know that

$\alpha+\beta+\gamma=180°$

$\beta=180°-\alpha-\gamma$

$\beta=180°-90°-45°$

$\Rightarrow \quad \boxed{\beta=45°}$

From figure

$\tan\alpha = \frac{a}{b}$

$\tan 45° = \frac{4}{b}$

$1 = \frac{4}{b}$

$\Rightarrow \boxed{b = 4}$

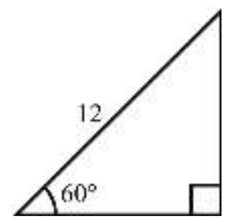
$\sin\alpha = \frac{a}{c}$

$\sin 45° = \frac{4}{c}$

$\frac{1}{\sqrt{2}} = \frac{4}{c}$

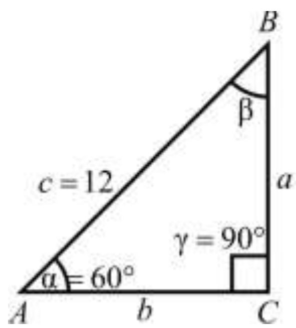
$\Rightarrow \boxed{c = 4\sqrt{2}}$

(ii)



Solution:

Labeling the given triangle



We have

$c=12\ ,\ \alpha=60^0\ ,\ \gamma=90^0$

We have to find a , b and β

As we know that

$\alpha+\beta+\gamma=180°$

$\beta=180°-\alpha-\gamma$

$\beta=180°-60°-90°$

$\Rightarrow \quad \boxed{\beta=30°}$

From figure

$\sin\alpha = \frac{a}{c}$

$$\sin 60° = \frac{a}{12}$$

$$\frac{\sqrt{3}}{2} = \frac{a}{12}$$

$$\frac{\sqrt{3}}{2} \times 12 = a$$

$$\boxed{6\sqrt{3} = a}$$

and

$$\cos\alpha = \frac{b}{c}$$

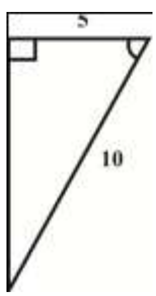
$$\cos 60° = \frac{b}{12}$$

$$\frac{1}{2} = \frac{b}{12}$$

$$\frac{1}{2} \times 12 = b$$

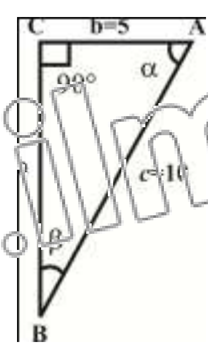
$$\boxed{6 = b}$$

(iii)



Solution:

Labeling the given triangle.



We have .

$b = 5$, $c = 10$, $\gamma = 90^0$

We have to find a , α and β

From figure

$$\cos\alpha = \frac{b}{c}$$

$$\cos\alpha = \frac{5}{10}$$

$$\cos\alpha = \frac{1}{2}$$

$$\alpha = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\Rightarrow \quad \boxed{\alpha = 60°}$$

We know that

$$\alpha + \beta + \gamma = 180°$$

$$\beta = 180° - \alpha - \gamma$$

$$\beta = 180° - 60° - 90°$$

$$\Rightarrow \quad \boxed{\beta = 30°}$$

Also from figure

$$\sin\alpha = \frac{a}{c}$$

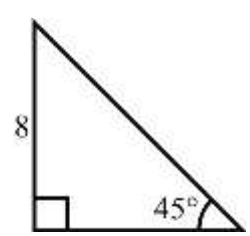
$$\sin 60° = \frac{a}{10}$$

$$\frac{\sqrt{3}}{2} = \frac{a}{10}$$

$$\frac{\sqrt{3}}{2} \times 10 = a$$

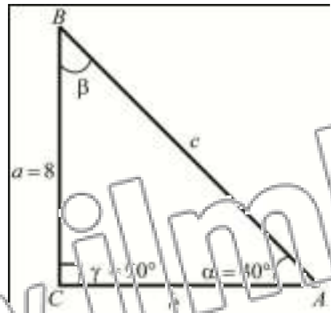
$$\boxed{5\sqrt{3} = a}$$

(iv)



Solution:

Labeling the given triangle



We have

$a = 8$, $\alpha = 40^0$, $\gamma = 90^0$

We have to find b, c and β

As we know that

$\alpha + \beta + \gamma = 180°$

$\beta = 180° - \alpha - \gamma$

$\beta = 180° - 90° - 40°$

$\Rightarrow \boxed{\beta = 50°}$

From figure

$\sin\alpha = \frac{a}{c}$

$\sin 40^o = \frac{8}{c}$

$c = \frac{8}{\sin 40^o}$

$c = \frac{8}{0.643}$

$\Rightarrow \boxed{c = 12.45}$

and

$\tan\alpha = \frac{a}{b}$

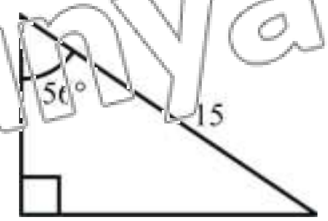
$\tan 40^o = \frac{8}{b}$

$b = \frac{8}{\tan 40^o}$

$= \frac{8}{0.839}$

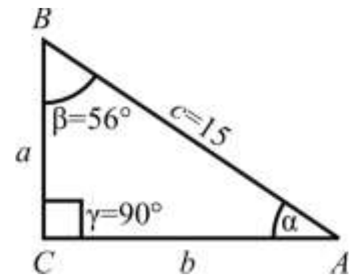
$\Rightarrow \boxed{b = 9.535}$

(v)



Solution:

Labeling the given triangle .



We have

$c = 15$, $\beta = 56^o$, $\gamma = 90^o$

We have to find a, b, *and* α

We know that

$\alpha + \beta + \gamma = 180°$

$\alpha = 180° - \beta - \gamma$

$\alpha = 180° - 56° - 90°$

$\Rightarrow \boxed{\alpha = 50°}$

From figure

$\sin\alpha = \frac{a}{c}$

$\sin 34^o = \frac{a}{15}$

$15 \times 0.559 = a$

$8.387 = a$

$\Rightarrow \boxed{a = 8.387}$

and

$\cos\alpha = \frac{b}{c}$

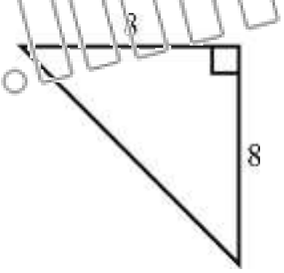
$$\cos 34^\circ = \frac{b}{15}$$

$$15 \times 0.829 = b$$

$$12.435 = b$$

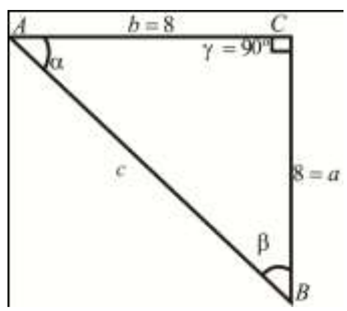
$$\Rightarrow \quad \boxed{b = 12.435}$$

(vi)



Solution:

Labeling the given triangle .



We have

$a = 8, \quad b = 8, \quad \gamma = 90^0$

We have to find α, β and c

From figure

$$\tan \alpha = \frac{a}{b}$$

$$\tan \alpha = \frac{8}{8}$$

$$\tan \alpha = 1$$

$$\alpha = \tan^{-1}(1)$$

$$\Rightarrow \quad \boxed{\alpha = 45^\circ}$$

We know that

$$\alpha + \beta + \gamma = 180^\circ$$

$$\beta = 180^\circ - \alpha - \gamma$$

$$\beta = 180^\circ - 45^\circ - 90^\circ$$

$$\Rightarrow \quad \boxed{\beta = 45^\circ}$$

and

$$\sin \alpha = \frac{a}{c}$$

$$\sin 45^\circ = \frac{8}{c}$$

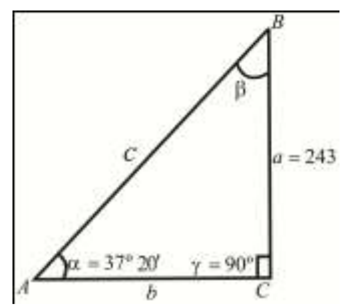
$$\frac{1}{\sqrt{2}} = \frac{8}{c}$$

$$\Rightarrow \quad \boxed{c = 8\sqrt{2}}$$

Solve the right triangle ABC, in which $\gamma = 90^0$

Q.2 $\alpha = 37^\circ\ 20'$, $a = 243$

Solution:



We have

$\gamma = 90^\circ$, $\alpha = 37^\circ\ 20'$, $a = 243$

We have to find β, b and c

We know that

$$\alpha + \beta + \gamma = 180^\circ$$

$$\beta = 180^\circ - \alpha - \gamma$$

$$\beta = 180^\circ - 37^\circ 20' - 90^\circ$$

$$\Rightarrow \quad \boxed{\beta = 52^\circ 40'}$$

From figure

$$\sin \alpha = \frac{a}{c}$$

$$\sin(37^\circ 20') = \frac{243}{c}$$

$$0.626 = \frac{243}{c}$$

$$c = \frac{243}{0.626}$$

$$\Rightarrow \boxed{c = 400.69}$$

and

$$\tan\alpha = \frac{a}{b}$$

$$\tan 37^\circ 20' = \frac{243}{b}$$

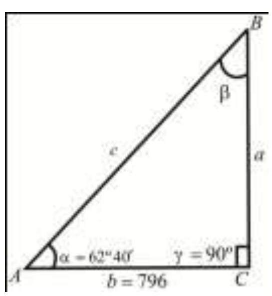
$$0.762 = \frac{243}{b}$$

$$b = \frac{243}{0.762}$$

$$\Rightarrow \boxed{b = 318.89}$$

Q.3 $\alpha = 62^\circ 40'$, $b = 796$, $\gamma = 90^\circ$

Solution:



We have

$\alpha = 62^\circ\ 40'$, $\gamma = 90^\circ$, $b = 796$

We have to find β, a and c.

We know that

$$\alpha + \beta + \gamma = 180^\circ$$

$$\beta = 180^\circ - \alpha - \gamma$$

$$\beta = 180^\circ - 90^\circ - 62^\circ 40'$$

$$\Rightarrow \boxed{\beta = 27^\circ 20'}$$

From figure

$$\tan\alpha = \frac{a}{b}$$

$$\tan 62^\circ 40' = \frac{a}{796}$$

$$796 \times 1.934 = a$$

$$\Rightarrow \boxed{1540.02 = a}$$

and

$$\cos\alpha = \frac{b}{c}$$

$$\cos 62^\circ 40' = \frac{796}{c}$$

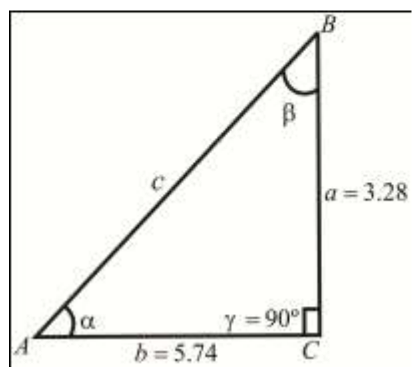
$$0.459 = \frac{796}{c}$$

$$c = \frac{796}{0.459}$$

$$\Rightarrow \boxed{c = 1733.57}$$

Q.4 $a = 3.28$, $b = 5.74$, $\gamma = 90^\circ$

Solution:



We have

$\gamma = 90^\circ$, $a = 3.28$, $b = 5.74$

We have to find α, β and c.

From figure

$$\tan\alpha = \frac{a}{b}$$

$$\tan\alpha = \frac{3.28}{5.74}$$

$$\tan\alpha = 0.571$$

$$\alpha = \tan^{-1}(0.571)$$

$\Rightarrow \alpha = 29^{o}44'$

We know that

$\alpha+\beta+\gamma=180°$

$\beta=180°-\alpha-\gamma$

$\beta=180°-90°-29°44'$

$\Rightarrow \boxed{\beta=59°16'}$

and

$\cos\alpha = \frac{b}{c}$

$\cos(29^{o}44') = \frac{5.74}{c}$

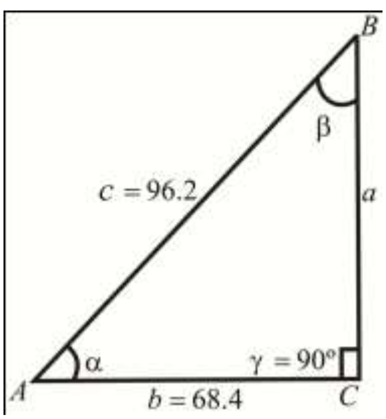
$0.868 = \frac{5.74}{c}$

$c = \frac{5.74}{0.868}$

$\Rightarrow c = 6.61$

Q.5 $b = 68.4,\ c = 96.2,\ \gamma = 90^{o}$

Solution:



We have

$b = 68.4,\ c = 96.2,\ \gamma = 90^{o}$

We have to find a, α and β

From figure

$\cos\alpha = \frac{b}{c}$

$\cos\alpha = \frac{68.4}{96.2}$

$\cos\alpha = 0.711$

$\alpha = \cos^{-1}(0.711)$

$\Rightarrow \boxed{\alpha = 44^{o}40'}$

We know that

$\alpha+\beta+\gamma=180°$

$\beta=180°-\alpha-\gamma$

$\beta=180°-44°44'-90°$

$\Rightarrow \boxed{\beta=45°20'}$

and

$\tan\alpha = \frac{a}{b}$

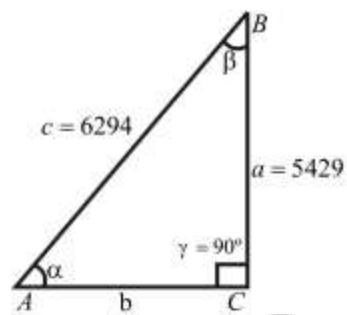
$\tan 44^{o}40' = \frac{a}{68.4}$

$1.01\times 68.4 = a$

$\Rightarrow \boxed{69.084 = a}$

Q.6 $a = 5429,\ c = 6294,\ \gamma = 90°$

Solution:



We have

$a = 5429,\ c = 6294,\ \gamma = 90°$

We have to find α, β and b

From figure

$\sin\alpha = \frac{a}{c}$

$\sin\alpha = \frac{5429}{6294}$

$\sin\alpha = 0.862$

$$\alpha = \sin^{-1}(0.862)$$

$\Rightarrow$ $\boxed{\alpha = 59^\circ 36'}$

We know that

$$\alpha + \beta + \gamma = 180^\circ$$

$$\beta = 180^\circ - \alpha - \gamma$$

$$\beta = 180^\circ - 59^\circ 36' - 90^\circ$$

$\Rightarrow$ $\boxed{\beta = 30^\circ 24'}$

Also from figure

$$\cos\alpha = \frac{b}{c}$$

$$\cos(59^\circ 36') = \frac{b}{6294}$$

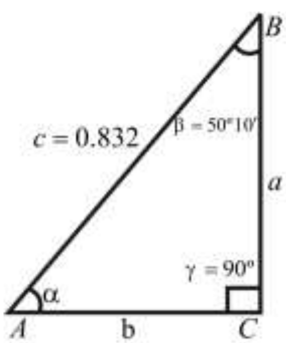
$\Rightarrow$ $\boxed{3184.97 = b}$

Q.7 $\beta = 50^\circ 10'$, $c = 0.832$, $\gamma = 90^\circ$

Solution:

We have $\beta = 50^\circ 10'$, $c = 0.832$, $\gamma = 90^\circ$

We have to find : α, a *and* b.



We know that

$$\alpha + \beta + \gamma = 180^\circ$$

$$\alpha = 180^\circ - \beta - \gamma$$

$$\alpha = 180^\circ - 50^\circ 10' - 90^\circ$$

$\Rightarrow$ $\boxed{\alpha = 39^\circ 50'}$

From figure

$$\sin\alpha = \frac{a}{c}$$

$$\sin(39^\circ 50') = \frac{a}{0.832}$$

$$0.64 \times 0.832 = a$$

$$0.532 = a$$

$\Rightarrow$ $\boxed{a = 0.532}$

$$\cos\alpha = \frac{b}{c}$$

$$\cos(39^\circ 50') = \frac{b}{0.832}$$

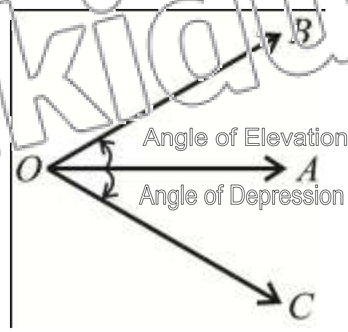
$$0.767 \times 0.832 = b$$

$$0.628 = b$$

$\Rightarrow$ $\boxed{b = 0.638}$

Angles of Elevation and Depression

Let $\overrightarrow{OA}$ be the horizontal line and O be the position of observer B and C are two points such that B is above the horizontal line and C is below the horizontal line.



From Figure

(i) For looking at B above the horizontal ray, we have to raise our eye, and $< AOB$ is called the **Angle of Elevation** and

(ii) for looking at C below the horizontal ray we have to lower our eye, and $< AOC$ is called the **Angle of Depression.**

EXERCISE 12.3

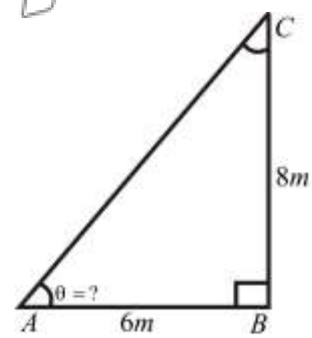
Q.1 A vertical pole is 8m high and the length of its shadow is 6m. what is the angle of elevation of the sun at that moment?

Solution:

Let height of pole is $\overline{BC} = 8\ m$

Length of its shadow is $\overline{AB} = 6\ m$

From figure.



$$\tan\theta = \frac{\text{Perp}}{\text{Base}} = \frac{8}{6} = \frac{4}{3}$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right)$$

$$\boxed{\theta = 53.1^\circ}$$

Or $\boxed{\theta = 53^\circ 8'}$

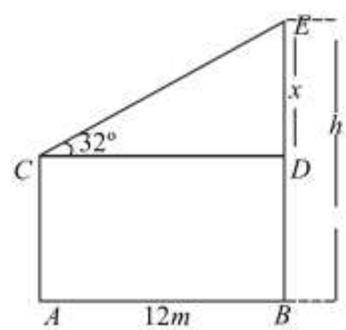
Q.2 A man 18dm tall observes that the angle of elevation of the top of a tree at a distance of 12m from him is 32°. What is the height of the tree?

Solution:

Let height of tree is $\overline{BE} = h$

Height of man is $\overline{AC} = 18\ dm = 1.8m$

Distance between Man and Tree is $\overline{AB} = 12\,m$



From figure

$$m\overline{AB} = m\ \overline{CD} = 12m$$

$$m\overline{AC} = m\ \overline{BD} = 1.8m$$

$$\tan 32^\circ = \frac{\overline{DE}}{\overline{CD}}$$

$$(0.6248)(12) = x$$

$$x = 7.498m$$

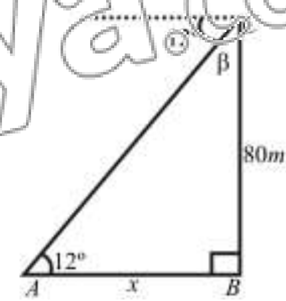
Height of tree is h $= m\ \overline{BD} + m\ \overline{DE}$

$= 1.8 + 7.49$

$$\boxed{h = 9.29\,m}$$

Q.3 At the top of a cliff 80m high, the angle of depression of a boat is $12°$. How far is the boat from the cliff?

Solution:



Let A be the position of boat and C be the top of cliff.

Distance between boat and cliff is $\overline{AB} = x$

Height of cliff is $m\overline{BC} = 80\ m$

From figure

$m\angle ACD = m\angle BAC$ (Alternate angles)

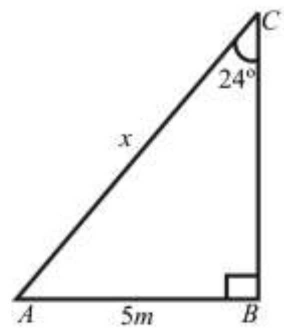
$$\tan 12° = \frac{m\overline{BC}}{m\overline{AB}}$$

$$x = \frac{80}{\tan 12^0}$$

$$\boxed{x = 376.3m}$$

Q.4 A ladder leaning against a vertical wall makes an angle of 24^0 with the well. Its foot is $5m$ from the wall. Find its length

Solution:



Let the length of ladder is $\overline{AC} = x$

Distance between ladder and wall is $m\overline{AB} = 5m$

From figure

$$\sin 24° = \frac{5}{x}$$

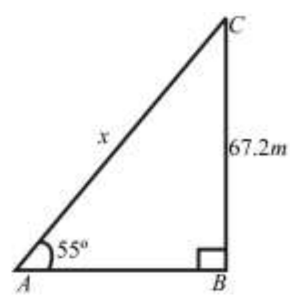
$$x = \frac{5}{\sin 24°}$$

$$\boxed{x = 12.29m}$$

Note: Answer of this question in book is wrong .

Q.5 A kite flying at a height of $67.2m$ is attached to a fully stretched string inclined at an angle of $55°$ to horizontal. Find the length of the string .

Solution:



Let C be the position of kite

Height of kite is $m\overline{BC} = 67.2\,m$

Length of string is $m\overline{AC} = x$

Form figure

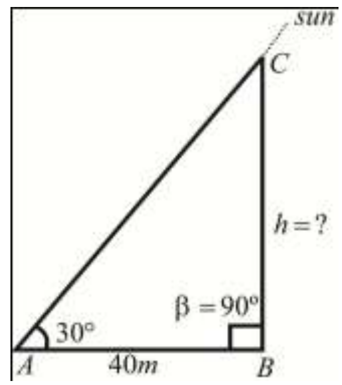
$$\sin 55° = \frac{67.2}{x}$$

$$x = \frac{67.2}{\sin 55°}$$

$$\boxed{x = 82.036m}$$

Q.6 When the angle between the ground and the sun is 30^0, flag pole casts a shadow of $40m$ long. Find the height of the top of the flag.

Solution



Let the height of flag pole is $m\overline{BC} = h$

Length of its shadow is $m\overline{AB} = 40\,m$

From figure

$$\tan 30° = \frac{h}{40}$$

$$h = 40 \times (0.5773)$$

$$\boxed{h = 23.09m}$$

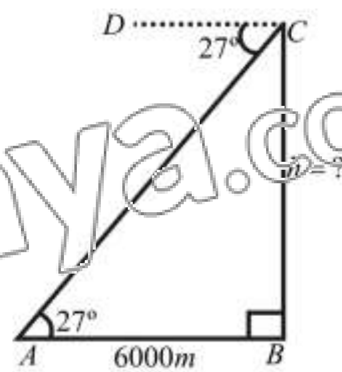
Q.7 A plane flying directly above a post 6000m away from an anti-aircraft gun observe the gun at an angle of depression of 27°. Find the height of the plane.

Solution:

Let A be the position of anti-aircraft gun, B be the position of check post and height of plane is $m\overline{BC} = h$

distance between place and anti-aircraft Gun is $m\overline{AB} = 6000\,m$

From figure



$$m\angle ACD = m\angle BAC \quad \text{(Alternative Angles)}$$

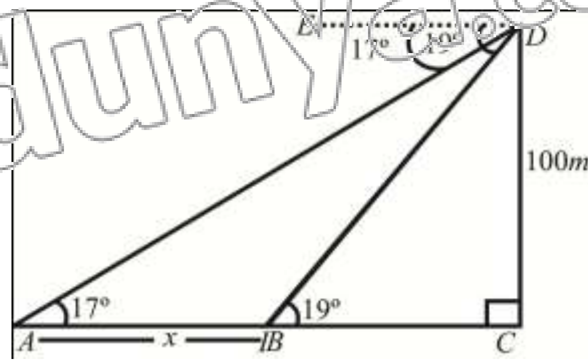
in ΔABC

$$\tan 27^0 = \frac{h}{6000}$$

$$h = 6000 \times 0.5095$$

$$\boxed{h = 3057.15m}$$

Q.8 A man on the top of $100m$ high light house is in the line with two ships on the same side of it, whose angles of depression from the man are 17° and 19° respectively. Find the distance between the ships.



Solution:

Let A and B be the position of ships,

and distance between ships is $m\overline{AB} = x$

height of light house is $m\overline{CD} = 100m$

From figure,

$$m\angle ADE = m\angle CAD$$

[∵ Alternate Angles]

$$m\angle BDE = m\angle CBD$$

In ΔBDC

$$\tan 19^\circ = \frac{100}{\overline{BC}}$$

$$m\overline{BC} = \frac{100}{\tan 19^0} = 290.42m$$

Now In ΔACD

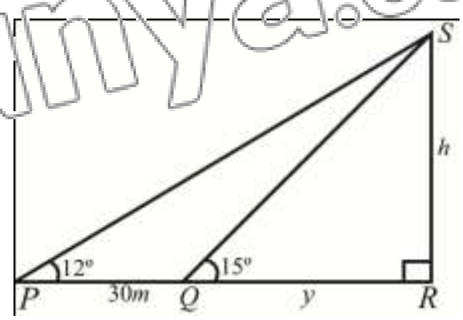
$$\tan 17^\circ = \frac{100}{x + 290.42}$$

$$x + 290.42 = \frac{100}{\tan 17^\circ}$$

$$x = 327.08 - 290.42$$

Distance between Ships $x = 36.66m$

Q.9 P and Q are two points in line with a tree. If the distance between 'P' and 'Q' be $30m$ and the angles of elevation of the top of the tree at P and Q be 12° and 15° respectively, find the height of the tree.



Solution:

Let height of tree is $m\overline{RS} = h$

Distance between P and Q is $m\overline{PQ} = 30m$

In ΔQRS

$$\tan 15^\circ = \frac{h}{y}$$

$$h = y \tan 15^\circ \qquad \text{(i)}$$

In ΔPRS

$$\tan 12° = \frac{h}{30+y}$$

$$\boxed{h = (30+y)\tan 12°} \quad \text{(ii)}$$

Comparing (i) & (ii)

$$y\tan 15° = (30+y)\tan 12°$$

$$y\tan 15° = 30\tan 12° + y\tan 12°$$

$$y\tan 15° - y\tan 12° = 6.3766$$

$$y(0.0553) = 6.3766$$

$$\boxed{y = 115.11m}$$

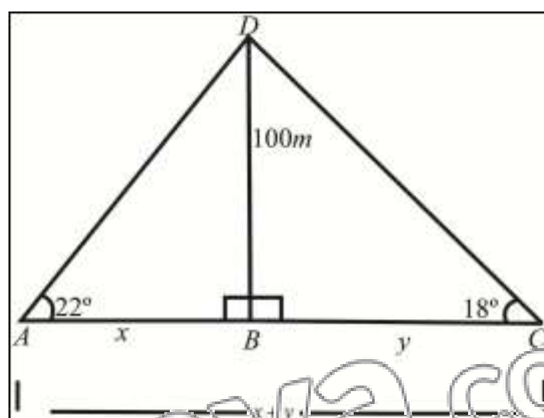
Put in (i)

$$h = (115.11)(0.267)$$

$$\boxed{h = 30.84m}$$

Q.10 Two men are on the opposite sides of a 100m high tower. If the measures of the angles of elevation of the top of the tower are $18°$ and $22°$ respectively. Find the distance between them.

Solution:



Let A and C be the position of men

Height of tower is $m\overline{BD} = 100m$

Distance between men is $m\overline{AC} = x + y$

In ΔBCD

$$\tan 18° = \frac{100}{y}$$

$$y = \frac{100}{\tan 18°}$$

$$\boxed{y = 307.8m}$$

In ΔABD

$$\tan 22° = \frac{100}{x}$$

$$x = \frac{100}{\tan 22°}$$

$$\boxed{x = 247.5m}$$

So, Distance between points = $x + y = 307.8 + 247.5 = 555.3m$

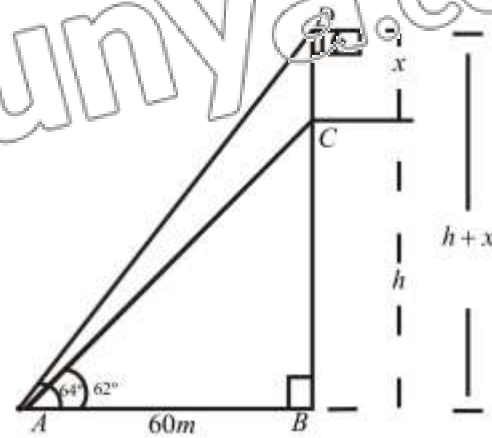
Q.11 A man standing $60m$ away from a tower notices that the angles of elevation of the top and bottom of a flag staff on the top of the tower are $64°$ and $62°$ respectively, Find length of flag staff .

Solution:

Let the height of tower is $m\overline{BC} = h$

Height of flag staff is $m\overline{CD} = x$

Distance between men and power is $m\overline{AB} = 60m$



In ΔABC

$$\tan 62° = \frac{h}{60}$$

$$h = 60(\tan 62°)$$

$$\boxed{h = 112.8435m}$$

In ΔABD

$$\tan 64° = \frac{h+x}{60}$$

$$h + x = 60(\tan 64°)$$

$$112.84 + x = 123.01$$

$$x = 123.01 - 112.84$$

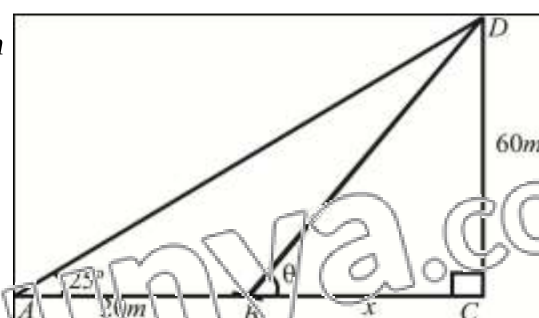
$$\boxed{x = 10.17m}$$

Q.12 The angle of elevation of the top of a $60m$ high tower from a point A, on the same level as the foot of the tower, is $25°$. Find the angle of elevation of the top of the tower from a point B, $20m$ nearer to A from the foot of the tower.

Solution:

Let the height of tower is $m\overline{CD} = 60m$

The distance between A and B is $m\overline{AB} = 20m$



In ΔACD

$$\tan 25° = \frac{60}{20+x}$$

$$(20+x) = \frac{60}{\tan 25°}$$

$$x = 128.6 - 20$$

$$\boxed{x = 108.67m}$$

In ΔBCD

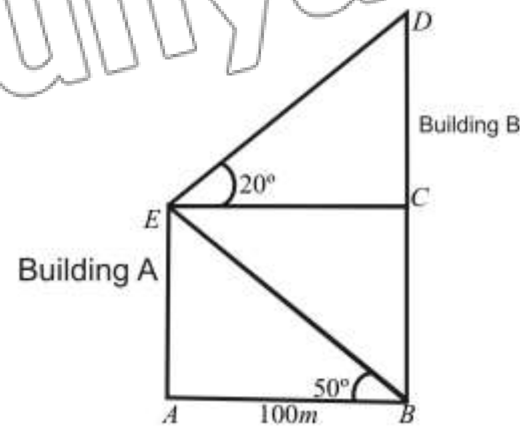
$$\tan \theta = \frac{60}{x}$$

$$\theta = \tan^{-1}\left(\frac{60}{108.67}\right)$$

$$\theta = 28.9°$$

Q.13 Two buildings A and B are $100m$ apart. The angle of elevation from the top of the building A to the top of the building B in $20°$. The angle of elevation from the base of building B to the top of the building A in $50°$. Find the height of the building B.

Solution:



Let distance between building A and Building B is $m\overline{AB} = 100m$.

Height of building A is $m\overline{AE} = x$

Height of building B is $m\overline{BD} = x + y$

In ΔAEB

$$\tan 50° = \frac{x}{100}$$

$$x = 100 \times 1.1917$$

$$x = 119.17 = m\overline{BC} \quad (\text{From figure})$$

In ΔDEC

$$\tan 20° = \frac{y}{m\overline{CE}}$$

$$\tan 20° = \frac{y}{m\overline{AB}}$$

$$y = 100 \times \tan 20°$$

$$\boxed{y = 36.39m}$$

Height of Building $B = m\overline{BC} + m\overline{DC}$

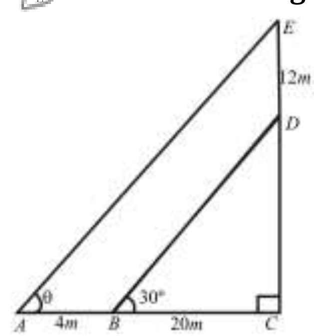
$= x + y$

$= 119.17 + 36.39$

$= 155.56m$

Q.14 A window washer is working in a hotel building. An observer at a distance of $20m$ from the building finds the angle of elevation of the worker to be of $30°$. The worker climbs up $12m$ and the observer moves $4m$ farther away from the building. Find the new angle of elevation of the worker.

Solution:



Let D be the position of window washer and B be the position of observer.

Angle between observer and the worker is $30°$

Distance between observer and building is $m\overline{BC} = 20m$

Height of building is $m\overline{CE} = x + 12$

In ΔBCD

$$\tan 30° = \frac{m\overline{CD}}{20}$$

$$x = 20 \times \tan 30°$$

$$\boxed{x = 11.54m}$$

In ΔACE

$$\tan\theta = \frac{m\overline{CD} + m\overline{DE}}{m\overline{AB} + m\overline{BC}}$$

$$= \frac{11.55 + 12}{4 + 20}$$

$$= \frac{23.54}{24}$$

$$\theta = \tan^{-1}(0.9811)$$

$$\boxed{\theta = 44.4°}$$

Q.15 A man standing on the bank of Canal observes that the measure of the angle of elevation of a tree on the other side of the canal, is $60°$. On retreating $40m$ from the bank, he finds the measure of the angle of elevation of the tree as $30°$. Find the height of the tree and width of the Canal

Solution:

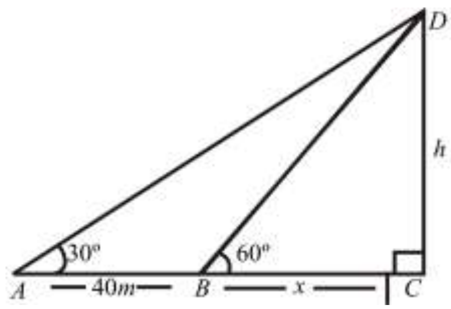
Let B be the position of man making angle $60°$ with tree.

After retreating

A be the position of man making angle $30°$ with tree.

Width of canal is $m\overline{BC} = x$

height of tree is $m\overline{CD} = h$



In ΔBCD

$$\tan 60° = \frac{h}{x}$$

$$\boxed{h = x\tan 60°} \quad \text{(i)}$$

In ΔACD

$$\tan 30° = \frac{h}{40 + x}$$

$$\boxed{(40 + x)\tan 30° = h} \quad \text{(ii)}$$

On Comparing (i) and (ii)

$$x\tan 60° = (40 + x)\tan 30°$$

$$x\tan 60° = 23.094 + x\tan 30°$$

$$x(\tan 60° - \tan 30°) = 23.094$$

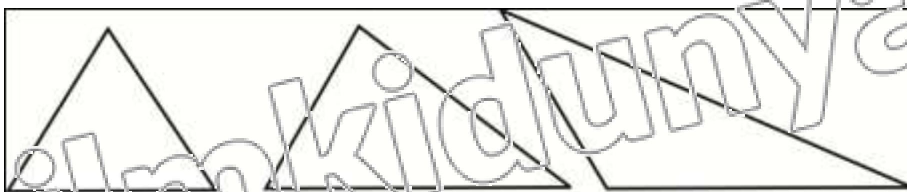
Width of Canal $\boxed{x = 20m}$

From (i) $h = 20\tan 60° = 20\sqrt{3}$

$$\boxed{h = 34.64m}$$

Oblique Triangles:

A triangle which is not right, is called an oblique triangle,



All the above triangles are oblique triangles.

The Law of Cosine:

In any triangle ABC, with usual notations, prove that:

(i) $a^2 = b^2 + c^2 - 2bc\cos\alpha$

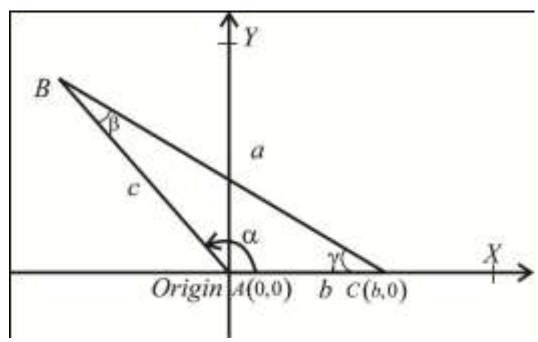
Proof:

Let side $\overline{AC}$ of triangle ABC along the positive direction of $x-axis$ with vertex A at origin, then $\angle BAC$ will be in the standard position.

Since $\overline{AB} = c$ and $m\angle BAC = \alpha$

$\therefore$ Coordinates of B are $(c\cos\alpha, c\sin\alpha)$

Since $\overline{AC} = b$ and point C is on the $x-axis$,



$\therefore$ Coordinates of C is $(b, 0)$

By Distance formula

$|BC| = \sqrt{(c\cos\alpha - b)^2 + (c\sin\alpha - o)^2}$

Squaring on both sides

$|BC|^2 = (c\cos\alpha - \mathrm{b})^2 + (c\sin\alpha - o)^2$

$|BC|^2 = c^2\cos^2\alpha + b^2 - 2bc\cos\alpha + c^2\sin^2\alpha$

$|BC|^2 = c^2(\cos^2\alpha + \sin^2\alpha) + b^2 - 2bc\cos\alpha$

$|BC|^2 = c^2 + b^2 - 2bc\cos\alpha \qquad \because \cos^2\alpha + \sin^2\alpha = 1$

$a^2 = b^2 + c^2 - 2bc\cos\alpha \qquad \because |BC| = a$

Hence proved

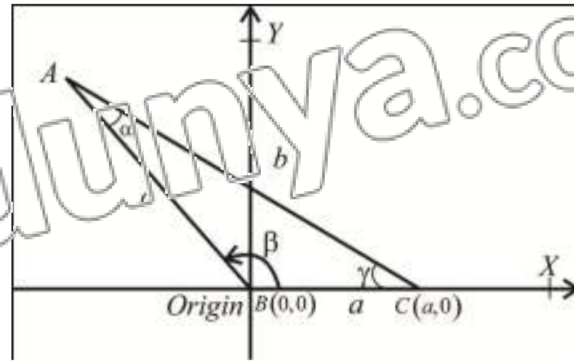
(ii) $b^2 = a^2 + c^2 - 2ac\cos\beta$

Proof:

Let side $\overline{BC}$ of triangle ABC be along the positive direction of the $x-axis$ with vertex B at origin than $\angle ABC$ will be in the standard position.

Since $\overline{BA} = c$ and $m\angle ABC = \beta$

$\therefore$ Coordinates of A are $(c\cos\beta,\ c\sin\beta)$



Also $\overline{BC} = b$ and point C is on the $x-axis$

$\therefore$ Coordinates of C is $(a,0)$

By distance formula

$$|AC| = \sqrt{(c\cos\beta - a)^2 + (c\sin\beta - 0)^2}$$

Squaring on both sides

$$|AC|^2 = (c\cos\beta - a)^2 + (c\sin\beta - 0)^2$$

$$|AC|^2 = c^2\cos^2\beta + a^2 - 2ac\cos\beta + c^2\sin^2\beta$$

$$|AC|^2 = c^2(\cos^2\beta + \sin^2\beta) + a^2 - 2ac\cos\beta$$

$$|AC|^2 = c^2 + a^2 - 2ac\cos\beta \qquad \because \cos^2\beta + \sin^2\beta = 1$$

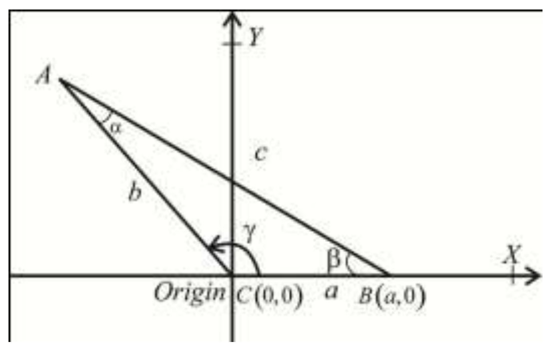
$$b^2 = a^2 + c^2 - 2ac\cos\beta \qquad \because |AC| = b$$

Hence proved

(iii) $c^2 = a^2 + b^2 - 2ab\cos\gamma$

Proof:

Let side $\overline{CB}$ of triangle ABC be along the positive
direction of the x-axis with vertex C at the origin then
$\angle ACB$ will be in the standard position.



Since $\overline{CA} = b$ and $m\angle ACB = \gamma$

$\therefore$ Coordinates of A are $(b\cos\gamma,\ b\sin\gamma)$

Also $\overline{CB} = a$ and point B is on the x-axis

$\therefore$ Coordinates of B are $(a,0)$

By distance formula

$$|AB| = \sqrt{(b\cos\gamma - a)^2 + (b\sin\gamma - 0)^2}$$

Squaring both sides

$$|AB|^2 = (b\cos\gamma - a)^2 + (b\sin\gamma - 0)^2$$

$$|AB|^2 = b^2\cos^2\gamma - a^2 - 2ab\cos\gamma + b^2\sin^2\gamma$$

$$|AB|^2 = b^2(\cos^2\gamma + \sin^2\gamma) + a^2 - 2ab\cos\gamma$$

$$|AB|^2 = b^2 + a^2 - 2ab\cos\gamma \qquad \because \cos^2\gamma + \sin^2\gamma = 1$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \qquad \because |AB| = c$$

Hence proved

Laws (i), (ii) and (iii) can also be written as

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos\beta = \frac{c^2 + a^2 - b^2}{2ca}$$

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

Note:

If ΔABC is right, then

Law of cosine reduces to Pythagoras Theorem

If $\alpha = 90^\circ$ then $a^2 = b^2 + c^2$

If $\beta = 90^\circ$ then $b^2 = a^2 + c^2$

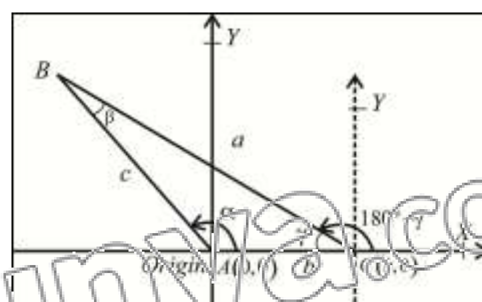
If $\gamma = 90^\circ$ then $c^2 = a^2 + b^2$

The Law of Sines

In any triangle *ABC*, with usual notations, prove that:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Proof:



Let side $\overline{AC}$ of triangle *ABC* be along the positive direction of the $x-axis$ with vertex A at origin, then $\angle BAC$ will be in the standard position.

Since $\overline{AB} = c$ and $m\angle BAC = \alpha$

$\therefore$ coordinates of the point B are $(c\cos\alpha, c\sin\alpha)$

If the origin A is shifted to C, then $\angle BCX$ will be in the standard position Since $\overline{BC} = a$ and $m\angle BCX = 180^\circ - \gamma$

Therefore the coordinates of B are $\left(a\cos(180^\circ - \gamma),\ a\sin(180^\circ - \gamma)\right)$

In both the cases, the *y*-coordinate of *B* remains the same.

$$a\,sin(180^\circ - \gamma) = c\sin\alpha$$

$$a\sin\gamma = c\sin\alpha$$

$$\Rightarrow \frac{a}{\sin\alpha} = \frac{c}{\sin\gamma} \qquad \text{(i)}$$

In a similar way, with side $\overline{AB}$ along positive $x-axis$, we can prove that

$$\Rightarrow \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \qquad \text{(ii)}$$

From (i) and (ii), we have

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Hence proved

The Law of Tangents

In any triangle ABC, with usual notations, Prove that:

(i) $$\frac{a-b}{a+b} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)}$$

(ii) $$\frac{b-c}{b+c} = \frac{\tan\left(\frac{\beta-\gamma}{2}\right)}{\tan\left(\frac{\beta+\gamma}{2}\right)}$$

(iii) $$\frac{c-a}{c+a} = \frac{\tan\left(\frac{\gamma-\alpha}{2}\right)}{\tan\left(\frac{\gamma+\alpha}{2}\right)}$$

Proof: **(i)**

By law of sine, we have

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$$\Rightarrow \frac{a}{b} = \frac{\sin\alpha}{\sin\beta}$$

By componendo and dividendo property, we have

$$\frac{a-b}{a+b} = \frac{\sin\alpha-\sin\beta}{\sin\alpha+\sin\beta}$$

$$= \frac{2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}$$

Divide up and down by $\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ we have

$$=\frac{\dfrac{\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}}{\dfrac{\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}}$$

$$=\frac{\dfrac{\sin\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}}{\dfrac{\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}}=\frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)}$$

$$\frac{a-b}{a+b}=\frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)}$$

Hence proved

Similarly We can prove that:

$$\frac{b-c}{b+c}=\frac{\tan\left(\frac{\beta-\gamma}{2}\right)}{\tan\left(\frac{\beta+\gamma}{2}\right)} \quad \text{and} \quad \frac{c-a}{c+a}=\frac{\tan\left(\frac{\gamma-\alpha}{2}\right)}{\tan\left(\frac{\gamma+\alpha}{2}\right)}$$

Half Angle Formulas:-

(a) **The sine of half the angle in terms of the sides:**

In any triangle ABC, we have

(i) $\sin\frac{\alpha}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$

(ii) $\sin\frac{\beta}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}}$ where $2s=a+b+c$

(iii) $\sin\dfrac{\gamma}{2}=\sqrt{\dfrac{(s-a)(s-b)}{ab}}$

Proof: (i)

We know that

$$2\sin^2\frac{\alpha}{2}=1-\cos\alpha \qquad \text{(i)}$$

$$2\sin^2\frac{\alpha}{2}=1-\frac{b^2+c^2-a^2}{2bc} \qquad \because \cos\alpha=\frac{b^2+c^2-a^2}{2bc}$$

$$=\frac{2bc-(b^2+c^2-a^2)}{2bc}$$

$$=\frac{2bc-b^2-c^2+a^2}{2bc}$$

$$=\frac{a^2-b^2-c^2+2bc}{2bc}$$

$$=\frac{a^2-(b^2+c^2-2bc)}{2bc}$$

$$=\frac{a^2-(b-c)^2}{2bc}$$

$$=\frac{[a-(b-c)][a+(b-c)]}{2bc}$$

$$=\frac{(a-b+c)(a+b-c)}{2bc}$$

$$=\frac{(a+c-b)(a+b-c)}{2bc} \qquad \text{(ii)}$$

$\because 2s=a+b+c$

$\Rightarrow a+c=2s-b$

and $a+b=2s-c$

$$=\frac{(2s-b-b)(2s-c-c)}{2bc}$$

$$=\frac{(2s-2b)(2s-2c)}{2bc}$$

$$2\sin^2\frac{\alpha}{2}=\frac{(2s-2b)(2s-2c)}{2bc}$$

$$\sin^2\frac{\alpha}{2}=\frac{2(s-b)(s-c)}{2bc}$$

$$\sin^2\frac{\alpha}{2}=\frac{(s-b)(s-c)}{bc}$$

$\Rightarrow \sin\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ $\quad \because \alpha$ is the measure of an angle of ΔABC,

$\frac{\alpha}{2} < 90° \Rightarrow \sin\frac{\alpha}{2} = +ve$

Hence proved

In Similar way, we can prove that: $\sin\frac{\beta}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}$ and $\sin\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(b) <u>The Cosine of Half the Angle in Term of the Sides.</u>

In any triangle *ABC*, with usual notation, Prove that:

(i) $\cos\frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}$

(ii) $\cos\frac{\beta}{2} = \sqrt{\frac{s(s-b)}{ac}}$

(iii) $\cos\frac{\gamma}{2} = \sqrt{\frac{s(s-c)}{ab}}$ $\quad$ where $2s = a+b+c$

<u>Proof:</u> (i)

We know that

$$2\cos^2\frac{\alpha}{2} = 1+\cos\alpha \qquad \text{(i)}$$

$$= 1+\frac{b^2+c^2-a^2}{2bc} \qquad \because \cos\alpha = \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{2bc+b^2+c^2-a^2}{2bc}$$

$$= \frac{(b+c)^2-a^2}{2bc}$$

$$= \frac{(b+c-a)(b+c+a)}{2bc}$$

$$= \frac{(b+c-a)(a+b+c)}{2bc} \qquad \text{(ii)}$$

$$2\cos^2\frac{\alpha}{2} = \frac{(2s-a-a)(2s)}{2bc} \qquad \because 2s = a+b+c$$

$$\cos^2\frac{\alpha}{2} = \frac{(2s-2a).2s}{2.2bc} \qquad \Rightarrow b+c = 2s-a$$

$$= \frac{\cancel{2}.\cancel{2}.s(s-a)}{\cancel{2}.\cancel{2}bc}$$

$$\cos^2\frac{\alpha}{2} = \frac{s(s-a)}{bc}$$

$$\Rightarrow \cos\frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$\because \alpha$ is the measure of an angle of ΔABC

Hence proved

$$\therefore \frac{\alpha}{2} < 90° \Rightarrow \cos\frac{\alpha}{2} = +ve$$

In Similar way, we can prove that:

$$\cos\frac{\beta}{2} = \sqrt{\frac{s(s-b)}{ac}} \quad \text{and} \quad \cos\frac{\gamma}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

(c) The Tangent of Half the Angle in Terms of the Sides.

In any triangle ABC, with usual notation, Prove that:

(i) $\tan\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

(ii) $\tan\frac{\beta}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$ Where $2s = a+b+c$

(iii) $\tan\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

<u>Proof:</u> (i)

We know that

$$\sin\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} ----(i)$$

$$\cos\frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}} ----(ii)$$

Divide (i) by (ii)

$$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{s(s-a)}{bc}}}$$

$$= \sqrt{\frac{(s-b)(s-c)}{bc}} \times \sqrt{\frac{bc}{s(s-a)}}$$

$$= \sqrt{\frac{(s-b)(s-c)}{\cancel{bc}} \times \frac{\cancel{bc}}{s(s-a)}}$$

$$\Rightarrow \tan\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

Hence proved

In similar way. We can prove that:

$$\tan\frac{\beta}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} \text{ and } \tan\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

Solution of Oblique Triangles

We can solve an oblique triangle if

(i) One side and two angles are known, Or

(ii) Two sides and their included angle are known Or

(iii) Three sides are known.

Case	Given	Use
1.	One side and Two angles are given	Law of sines
2.	Two sides and their included angle are given	(i) First law of cosine and then law of sines, or (ii) First law of tangents and then law of sines.
3.	Three sides are given	(i) Law of cosine or (ii) the half angles formulas

EXERCISE 12.4

Solve the triangle ABC, if

Q.1 $\beta = 60°,\ \gamma = 15°,\ b = \sqrt{6}$

Solution:

$\because \quad \alpha + \beta + \gamma = 180°$

$\alpha = 180° - \beta - \gamma$

$\alpha = 180° - 60° - 15°$

$\boxed{\alpha = 105°}$

By law of sines, we have

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$$\Rightarrow a = b\frac{\sin\alpha}{\sin\beta}$$

$$a = \frac{\sqrt{6}\sin 105°}{\sin 60°}$$

$$= \frac{\sqrt{6}\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}{\left(\frac{\sqrt{3}}{2}\right)}$$

$$= \sqrt{6}\,\frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{2}{\sqrt{3}}$$

$\boxed{a = \sqrt{3}+1}$

Again

$$\frac{c}{\sin\gamma} = \frac{b}{\sin\beta}$$

$$c = b\frac{\sin\gamma}{\sin\beta}$$

$$c = \frac{\sqrt{6}\sin 15°}{\sin 60°}$$

$$= \frac{\sqrt{6}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)}{\left(\frac{\sqrt{3}}{2}\right)}$$

$$= \sqrt{6}\,\frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{2}{\sqrt{3}}$$

$\boxed{c = \sqrt{3}-1}$

Q.2 $\beta = 52°,\ \gamma = 89°35',\ a = 89.35$

Solution:

$\because \alpha + \beta + \gamma = 180°$

$\alpha = 180° - \beta - \gamma$

$\alpha = 180° - 52° - 89°35'$

$\boxed{\alpha = 38°25'}$

By Law of sines, we have

$$\frac{b}{\sin\beta} = \frac{a}{\sin\alpha}$$

$$b = a\frac{\sin\beta}{\sin\alpha}$$

$$b = \frac{89.35 \times \sin 52°}{\sin 38°25'}$$

$\boxed{b = 113.31}$

Again

$$\frac{c}{\sin\gamma} = \frac{a}{\sin\alpha}$$

$$c = a\frac{\sin\gamma}{\sin\alpha}$$

$$c = \frac{89.35 \times \sin 89°35'}{\sin 38°25'}$$

$\boxed{c = 143.79}$

Q.3 $b = 125,\ \gamma = 53°,\ \alpha = 47°$

Solution:

$\because \quad \alpha + \beta + \gamma = 180°$

$\beta = 180° - \alpha - \gamma$

$\beta = 180° - 53° - 47°$

$\boxed{\beta = 80°}$

By law of sines, We have

$$\frac{c}{\sin\gamma}=\frac{b}{\sin\beta}$$

$$c=b\frac{\sin\gamma}{\sin\beta}$$

$$c=\frac{125\times\sin 53^\circ}{\sin 80^\circ}$$

$$\boxed{c\ =\ 92.82}$$

Again

$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}$$

$$a=b\frac{\sin\alpha}{\sin\beta}$$

$$a=\frac{125\times\sin 47^\circ}{\sin 80^\circ}$$

$$\boxed{a=\ 101.36}$$

Q.4 $c=16.1,\ \alpha\ =\ 42^\circ45',\ \gamma\ =\ 74^\circ32'$

Solution:

$\because\ \alpha\ +\ \beta\ +\ \gamma\ =\ 180^\circ$

$$\beta=180^\circ-\alpha-\gamma$$

$$\beta=180^\circ-42^\circ45'-74^\circ32'$$

$$\boxed{\beta=62^\circ43'}$$

By law of sines, we have

$$\frac{a}{\sin\alpha}=\frac{c}{\sin\gamma}$$

$$a=c\frac{\sin\alpha}{\sin\gamma}$$

$$a=\frac{16.1\times\sin 42^\circ45'}{\sin 74^\circ32'}$$

$$\boxed{a\ =\ 11.33}$$

Again

$$\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}$$

$$b=c\frac{\sin\beta}{\sin\gamma}$$

$$b=\frac{16.1\times\sin 62^\circ43'}{\sin\ 74^\circ32'}$$

$$\boxed{b\ =\ 14.84}$$

Q.5 $a=53,\ \beta=\ 88^\circ36',\ \gamma\ =31^\circ54'$

Solution:

$\because\ \alpha+\beta+\gamma=180^\circ$

$$\alpha\ =\ 180^\circ-\beta-\gamma$$

$$\alpha=180^\circ-88^\circ36'-31^\circ54'$$

$$\boxed{\alpha\ =\ 59^\circ30'}$$

By law of sines, we have

$$\frac{b}{\sin\beta}=\frac{a}{\sin\alpha}$$

$$b=a\frac{\sin\beta}{\sin\alpha}$$

$$b=\frac{53\times\sin 88^\circ36'}{\sin\ 59^\circ30'}$$

$$\boxed{b\ =\ 61.49}$$

Again

$$\frac{c}{\sin\gamma}=\frac{a}{\sin\alpha}$$

$$c=a\frac{\sin\gamma}{\sin\alpha}$$

$$c=\frac{53\times\sin 31^\circ54'}{\sin 59^\circ30'}$$

$$\boxed{c\ =\ 32.50}$$

EXERCISE 12.5

Q.1 $b = 95, c = 34$ **and** $\alpha = 52°$

Solution:

By cosine law,

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$a^2 = 95^2 + 34^2 - 2(95)(34)\cos 52°$$

$$a^2 = 6203.8$$

$$\sqrt{a^2} = \sqrt{6203.8}$$

$$\boxed{a = 78.76}$$

By using sine law,

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$$\sin\beta = \frac{b\sin\alpha}{a}$$

$$\sin\beta = \frac{95\times\sin 52°}{78.76}$$

$$\sin\beta = 0.95$$

$$\beta = \sin^{-1}(0.95)$$

$$\boxed{\beta = 71°53'}$$

$$\because\ \alpha + \beta + \gamma = 180°$$

$$\gamma = 180° - \alpha - \beta$$

$$= 180° - 52° - 71°53'$$

$$\boxed{\gamma = 56°7'}$$

Q.2 $b = 12.5,\ c = 23,\ \alpha = 38°20'$

Solution:

By cosine law,

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$a^2 = (12.5)^2 + (23)^2 - 2(12.5)(23)\cos(38°20')$$

$$a^2 = 243.21$$

$$\sqrt{a^2} = \sqrt{243.21}$$

$$\boxed{a = 15.3}$$

By using sine law,

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$$\sin\beta = \frac{b\sin\alpha}{a}$$

$$\sin\beta = \frac{12.5\times\sin(38°20')}{15.3}$$

$$\sin\beta = 0.506$$

$$\beta = \sin^{-1}(0.506)$$

$$\boxed{\beta = 30°26'}$$

$$\because\ \alpha + \beta + \gamma = 180°$$

$$\gamma = 180° - \alpha - \beta$$

$$= 180° - 38°20' - 30°26'$$

$$\boxed{\gamma = 111°14'}$$

Q.3 $a = \sqrt{3} - 1,\ b = \sqrt{3} + 1,\ \gamma = 60°$

Solution:

By cosine law,

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

$$c^2 = \left(\sqrt{3}+1\right)^2 + \left(\sqrt{3}-1\right)^2 - 2\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)\cos 60°$$

$$\because\ (a+b)^2 + (a-b)^2 = 2\left(a^2 + b^2\right)$$

$$\Rightarrow c^2 = 2\left[\left(\sqrt{3}\right)^2 + (1)^2\right] - 2\left[\left(\sqrt{3}\right)^2 - (1)^2\right].\frac{1}{2}$$

$$c^2 = 2(3+1) - 2(3-1).\frac{1}{2}$$

$$c^2 = 2(4) - 2$$

$$c^2 = 6$$

$$\sqrt{c^2} = \sqrt{6}$$

$$\boxed{c = \sqrt{6}}$$

By using sine law,

$$\frac{a}{\sin\alpha}=\frac{c}{\sin\gamma}$$

$$\sin\alpha=\frac{a\sin\gamma}{c}$$

$$\sin\alpha=\frac{(\sqrt{3}-1)\sin 60^\circ}{\sqrt{6}}$$

$$=\frac{(\sqrt{3}-1)}{\sqrt{3}\sqrt{2}}\frac{\sqrt{3}}{2}$$

$$\sin\alpha=\frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\alpha=\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)$$

$$\boxed{\alpha=15^\circ}$$

$\because\ \alpha+\beta+\gamma=180^\circ$

$$\beta=180^\circ-\alpha-\gamma$$

$$=180^\circ-15^\circ-60^\circ$$

$$\boxed{\beta=105^\circ}$$

Q.4 $a=3,\quad c=6,\quad \beta=36^\circ 20'$

Solution:

By cosine law,

$$b^2=a^2+c^2-2ac\cos\beta$$

$$b^2=3^2+6^2-2(3)(6)\cos 36^\circ 20'$$

$$b^2=9+36-36(0.805)$$

$$b^2=15.99$$

$$\sqrt{b^2}=\sqrt{15.99}$$

$$\boxed{b=3.99}$$

By using sine law,

$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}$$

$$\sin\alpha=\frac{a\sin\beta}{b}$$

$$\sin\alpha=\frac{3\times\sin(36^\circ 20')}{3.99}$$

$$\sin\alpha=0.445$$

$$\alpha=\sin^{-1}(0.445)$$

$$\boxed{\alpha=26^\circ 27'}$$

$\because\ \alpha+\beta+\gamma=180^\circ$

$$\gamma=180^\circ-\alpha-\beta$$

$$\gamma=180^\circ-26^\circ 27'-36^\circ 20'$$

$$\boxed{\gamma=117^\circ 13'}$$

Q.5 $a=7\quad b=3\quad \gamma=38^\circ 13'$

Solution:

By cosine law,

$$c^2=a^2+b^2-2ab\cos\gamma$$

$$c^2=7^2+3^2-2(7)(3)\cos(38^\circ 13')$$

$$c^2=25$$

$$\sqrt{c^2}=\sqrt{25}$$

$$\boxed{c=5}$$

By using sine law,

$$\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}$$

$$\sin\beta=\frac{b\sin\gamma}{c}$$

$$\sin\beta=\frac{3\times\sin(38^\circ 13')}{5}$$

$$\sin\beta=0.371$$

$$\beta=\sin^{-1}(0.371)$$

$$\boxed{\beta=21^\circ 47'}$$

$\because\ \alpha+\beta+\gamma=180^\circ$

$$\alpha=180^\circ-\beta-\gamma$$

$$\alpha=180^\circ-21^\circ 47'-38^\circ 13'$$

$$\boxed{\alpha=120^\circ}$$

Q.6 Solve the following triangles, using first law of tangents and then law of sines:

$a = 36.21,\ b = 42.09,\ \gamma = 44°29'$

Solution:

Here $b > a \quad \therefore \beta > \alpha$

$\because \alpha + \beta + \gamma = 180°$

$\alpha + \beta = 180° - \gamma$

$\alpha + \beta = 180° - 44°29'$

$\alpha + \beta = 135°31' \qquad \text{(i)}$

$\Rightarrow \dfrac{\alpha + \beta}{2} = 67°45'$

By law of tangents

$$\frac{\tan\left(\frac{\beta - \alpha}{2}\right)}{\tan\left(\frac{\beta + \alpha}{2}\right)} = \frac{b - a}{b + a}$$

$$\Rightarrow \tan\left(\frac{\beta - \alpha}{2}\right) = \frac{b - a}{b + a}\tan\left(\frac{\beta + \alpha}{2}\right)$$

So

$$\tan\left(\frac{\beta - \alpha}{2}\right) = \frac{42{\cdot}09 - 36{\cdot}21}{42{\cdot}09 + 36{\cdot}21}\tan 67°45'$$

$$= \frac{5.88}{78{\cdot}3} \times 2{\cdot}444$$

$$\tan\left(\frac{\beta - \alpha}{2}\right) = 0.1836$$

$$\frac{\beta - \alpha}{2} = \tan^{-1}(0.1836)$$

$$\frac{\beta - \alpha}{2} = 10°23'$$

$\beta - \alpha = 20°46'$

$\alpha - \beta = -20°46' \qquad \text{(ii)}$

Adding equation (i) and equation (ii)

$\alpha + \beta = 135°31'$

$\alpha - \beta = -20°46'$

$2\alpha = 114°45'$

$\boxed{\alpha = 57°22'}$

Putting in equation (i)

$\alpha + \beta = 135°31'$

$57°22' + \beta = 135°31'$

$\beta = 135°31' - 57°22'$

$\boxed{\beta = 78°9'}$

To find c, We use law of sines

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma}$$

$$c = a\frac{\sin\gamma}{\sin\alpha}$$

$$c = \frac{36.21 \times \sin 44°29'}{\sin 57°22'}$$

$\boxed{c = 30.12}$

Q.7 $a = 93,\ c = 101,\ \beta = 80°$

Solution:

Here $c > a \quad \therefore \gamma > \alpha$

$\because \alpha + \beta + \gamma = 180°$

$\alpha + \gamma = 180° - \beta$

$\alpha + \gamma = 180° - 80°$

$\alpha + \gamma = 100° \qquad \text{(i)}$

$\Rightarrow \dfrac{\alpha + \gamma}{2} = 50°$

By law of tangents of

$$\frac{\tan\left(\frac{\gamma - \alpha}{2}\right)}{\tan\left(\frac{\gamma + \alpha}{2}\right)} = \frac{c - a}{c + a}$$

$$\Rightarrow \tan\left(\frac{\gamma - \alpha}{2}\right) = \frac{c - a}{c + a}\tan\left(\frac{\gamma + \alpha}{2}\right)$$

So, $\tan\left(\frac{\gamma-\alpha}{2}\right)=\frac{101-93}{101+93}\tan 50^\circ$

$=\frac{8}{194}\times 1.1915$

$\tan\left(\frac{\gamma-\alpha}{2}\right)=0.049$

$\frac{\gamma-\alpha}{2}=\tan^{-1}(0.049)$

$\frac{\gamma-\alpha}{2}=2^\circ 49'$

$\gamma-\alpha=5^\circ 37'$

$\alpha-\gamma=-5^\circ 37'$ (ii)

Adding equation (i) and equation (ii)

$\alpha+\gamma=100^\circ$

$\alpha-\gamma=-5^\circ 37'$

$2\alpha=94^\circ 23'$

$\boxed{\alpha=47^\circ 11'}$

Putting in equation (i)

$\alpha+\gamma=100^\circ$

$47^\circ 11'+\gamma=100^\circ$

$\gamma=100^\circ-47^\circ 11'$

$\boxed{\gamma=52^\circ 49'}$

To find

$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}$

$\frac{93}{\sin 47^\circ 11'}=\frac{b}{\sin 80^\circ}$

$\frac{93\times\sin 80^\circ}{\sin 47^\circ 11'}=b$

$124.8=b$

$\boxed{b=124.8}$

Q.8 $b=14.8,\ c=16.1,\ \alpha=42^\circ 45'$

Solution:

Here $c>b$ $\therefore \gamma>\beta$

$\because \alpha+\beta+\gamma=180^\circ$

$\beta+\gamma=180^\circ-\alpha$

$\beta+\gamma=180^\circ-42^\circ 45'$

$\beta+\gamma=137^\circ 15'$ (i)

$\Rightarrow \frac{\beta+\gamma}{2}=68^\circ 37'$

By law of tangent

$$\frac{\tan\left(\frac{\gamma-\beta}{2}\right)}{\tan\left(\frac{\gamma+\beta}{2}\right)}=\frac{c-b}{c+b}$$

$$\Rightarrow \tan\left(\frac{\gamma-\beta}{2}\right)=\frac{c-b}{c+b}\tan\left(\frac{\gamma+\beta}{2}\right)$$

So,

$\tan\left(\frac{\gamma-\beta}{2}\right)=\frac{16.1-14.8}{16.1+14.8}\tan 68^\circ 37'$

$=\frac{1.3}{30.9}\times 2.554$

$\tan\left(\frac{\gamma-\beta}{2}\right)=0.107$

$\frac{\gamma-\beta}{2}=\tan^{-1}(0.107)$

$\frac{\gamma-\beta}{2}=6^\circ 8'$

$\gamma-\beta=12^\circ 16'$

$\beta-\gamma=-12^\circ 16'$ (ii)

Adding equation (i) and equation (ii)

$\beta+\gamma=137^\circ 15'$

$\beta-\gamma=-12^\circ 16'$

$2\beta=124^\circ 59'$

$\boxed{\beta=62^\circ 29'}$

Putting in equation (i)

$$\beta+\gamma=137°25'$$

$$62°29'+\gamma=137°25'$$

$$\gamma=137°25'-62°29'$$

$$\boxed{\gamma=74°56'}$$

To find a, We use law of sines.

$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}$$

$$a=b\frac{\sin\alpha}{\sin\beta}$$

$$a=\frac{14.8\times\sin 42°45'}{\sin 62°29'}$$

$$\boxed{a=11.3}$$

Q.9 $a=319$, $b=168$, $\gamma=110°22'$

Solution:

Here $a>b$ $\quad\therefore\ \alpha>\beta$

Q $\alpha+\beta+\gamma=18°$

$$\alpha+\beta=180°-\gamma$$

$$\alpha+\beta=180°-110°22'$$

$$\alpha+\beta=69°38' \qquad \text{(i)}$$

$$\Rightarrow \frac{\alpha+\beta}{2}=34°49'$$

By law of tangent

$$\frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)}=\frac{a-b}{a+b}$$

$$\Rightarrow \tan\left(\frac{\alpha-\beta}{2}\right)=\frac{a-b}{a+b}\tan\left(\frac{\alpha+\beta}{2}\right)$$

So,

$$\tan\left(\frac{\alpha-\beta}{2}\right)=\frac{319-168}{319+168}\tan 34°49'$$

$$=\frac{151}{487}\times 0.695$$

$$\tan\left(\frac{\alpha-\beta}{2}\right)=0.2156$$

$$\frac{\alpha-\beta}{2}=\tan^{-1}(0.2156)$$

$$\frac{\alpha-\beta}{2}=12°10'$$

$$\alpha-\beta=24°20' \qquad \text{(ii)}$$

Adding equation (i) and equation (ii)

$$\alpha+\beta=69°38'$$

$$\alpha-\beta=24°20'$$

$$2\alpha=93°58'$$

$$\alpha=\frac{93°58'}{2}$$

$$\boxed{\alpha=46°59'}$$

Putting in equation (i)

$$\alpha+\beta=69°38'$$

$$46°59'+\beta=69°38'$$

$$\beta=69°38'-46°59'$$

$$\boxed{\beta=22°39'}$$

To find c, We use law of sines,

$$\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}$$

$$c=b\frac{\sin\gamma}{\sin\beta}$$

$$c=\frac{168\times\sin(110°22')}{\sin(22°39')}$$

$$\boxed{c=408.97}$$

Q.10 $b=61$, $c=32$, $\gamma=59°30'$

Solution:

Here $b>c$ $\quad\therefore\ \beta>\gamma$

Q $\alpha+\beta+\gamma=18°$

$$\beta+\gamma=180°-\gamma$$

$$\beta+\gamma=180°-59°30'$$

$$\beta+\gamma=120°30' \qquad \text{(i)}$$

$$\Rightarrow \quad \frac{\beta+\gamma}{2}=60°15'$$

By law of tangent

$$\frac{\tan\left(\frac{\beta-\gamma}{2}\right)}{\tan\left(\frac{\beta+\gamma}{2}\right)}=\frac{b-c}{b+c}$$

$$\Rightarrow \tan\left(\frac{\beta-\gamma}{2}\right)=\frac{b-c}{b+c}\tan\left(\frac{\beta+\gamma}{2}\right)$$

So,

$$\tan\left(\frac{\beta-\gamma}{2}\right)=\frac{61-32}{61+32}\tan 60°15'$$

$$=\frac{29}{93}\times 1.7496$$

$$\tan\left(\frac{\beta-\gamma}{2}\right)=0.5456$$

$$\frac{\beta-\gamma}{2}=\tan^{-1}(0.5456)$$

$$\frac{\beta-\gamma}{2}=28°36'$$

$$\beta-\gamma=57°13' \qquad \text{(ii)}$$

By adding equation (i) and equation (ii)

$$\beta+\gamma=120°30'$$

$$\beta-\gamma=57°13'$$

$$2\beta=177°43'$$

$$\boxed{\beta=88°51'}$$

Putting in equation (i)

$$\beta+\gamma=120°30'$$

$$88°51'+\gamma=120°30'$$

$$\gamma=120°30'-88°51'$$

$$\boxed{\gamma=31°39'}$$

To find a, use law of sines,

$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}$$

$$a=b\frac{\sin\alpha}{\sin\beta}$$

$$a=\frac{61\times\sin 59°30'}{\sin 88°51'}$$

$$\boxed{a=52.56}$$

Q.11 Measures of two sides of a triangle are in the ratio 3:2 and they include an angle of measure 57°. Find the remaining two angles.

Solution:

Consider a triangle ABC with usual notations. Such that

$$a:b=3:2 \text{ and } \gamma=57°$$

$$\because \alpha+\beta+\gamma=180°$$

$$\alpha+\beta=180°-\gamma$$

$$\alpha+\beta=180°-57°$$

$$\alpha+\beta=123° \qquad \text{(i)}$$

$$\Rightarrow \frac{\alpha+\beta}{2}=61°30'$$

Also given

$$a:b=3:2$$

$$\frac{a}{b}=\frac{3}{2}$$

By Componendo-Dividendo theorem

$$\frac{a-b}{a+b}=\frac{3-2}{3+2}$$

$$\frac{a-b}{a+b}=\frac{1}{5} \qquad \text{(ii)}$$

By law of tangent

$$\frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)}=\frac{a-b}{a+b}$$

$$\Rightarrow \tan\left(\frac{\alpha-\beta}{2}\right)=\frac{a-b}{a+b}\tan\left(\frac{\alpha+\beta}{2}\right)$$

So, $\tan\left(\frac{\alpha-\beta}{2}\right)=\frac{1}{5}\tan 61°30'$

$$=\frac{1}{5}\times 1.8418$$

$$\tan\left(\frac{\alpha-\beta}{2}\right)=0.3684$$

$$\Rightarrow \frac{\alpha-\beta}{2}=\tan^{-1}(0.3684)$$

$$\frac{\alpha-\beta}{2}=20°13'$$

$$\alpha-\beta=40°26' \qquad \text{(iii)}$$

Adding equation (i) and equation (iii)

$$\alpha+\beta=123°$$

$$\alpha-\beta=40°26'$$

$$2\alpha=163°26'$$

$$\boxed{\alpha=81°43'}$$

Putting in equation (i)

$$\alpha+\beta=123°$$

$$81°43'+\beta=123°$$

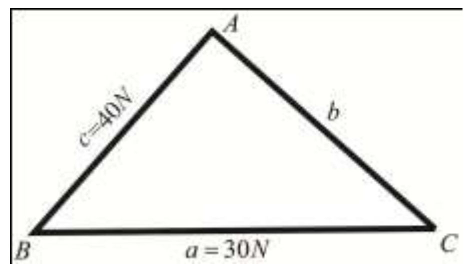
$$\beta=123°-81°43'$$

$$\boxed{\beta=41°17'}$$

Q.12 Two forces of $40N$ and $30N$ are represented by $\vec{AB}$ and $\vec{BC}$ which are inclined at an angle of $147°25'$. Find $\vec{AC}$, the resultant of $\vec{AB}$ and $\vec{BC}$.

Solution:

Consider a triangle ABC with usual notation



Here $a=30N$, $c=40N$,

$\beta=147°25'$

To find b

By cosine law

$$b^2=c^2+a^2-2ca\cos\beta$$

$$b^2=40^2+30^2-2(40)(30)\cos(147°25')$$

$$b^2=1600+900-2400\cos(147°25')$$

$$b^2=4522.26$$

$$\sqrt{b^2}=\sqrt{4522.26}$$

$$\boxed{b=67.24}$$

EXERCISE 12.6

Q.1 $a=7,\ b=7,\ c=9$

Solution:

We know that

$$\cos\alpha=\frac{b^2+c^2-a^2}{2bc}$$

$$\therefore \cos\alpha=\frac{7^2+9^2-7^2}{2(7)(9)}$$

$$\cos\alpha=\frac{81}{2\times7\times9}$$

$$\cos\alpha=\frac{9}{14}$$

$$\alpha=\cos^{-1}\left(\frac{9}{14}\right)$$

$$\boxed{\alpha=50^\circ}$$

Again by cosine law

$$\cos\beta=\frac{a^2+c^2-b^2}{2ac}$$

$$=\frac{7^2+9^2-7^2}{2\times7\times9}$$

$$\cos\beta=\frac{81}{2\times7\times9}$$

$$\beta=\cos^{-1}\left(\frac{9}{14}\right)$$

$$\boxed{\beta=50^\circ}$$

Q $\alpha+\beta+\gamma=180^\circ$

$$\gamma=180^\circ-50^\circ-50^\circ$$

$$\boxed{\gamma=80^\circ}$$

Q.2 $a=32,\ b=40,\ c=66$

Solution:

We know that

$$\cos\alpha=\frac{b^2+c^2-a^2}{2bc}$$

$$\therefore \cos\alpha=\frac{40^2+66^2-32^2}{2\times40\times66}$$

$$\cos\alpha=\frac{4932}{5280}$$

$$\cos\alpha=\frac{411}{440}$$

$$\alpha=\cos^{-1}\left(\frac{411}{440}\right)$$

$$\boxed{\alpha=20^\circ55'}$$

Again by cosine law

$$\cos\beta=\frac{a^2+c^2-b^2}{2ac}$$

$$\cos\beta=\frac{32^2+66^2-40^2}{2\times32\times66}$$

$$\cos\beta=\frac{3780}{4224}$$

$$\cos\beta=\frac{315}{352}$$

$$\beta=\cos^{-1}\left(\frac{315}{352}\right)$$

$$\boxed{\beta=26^\circ30'}$$

Q $\alpha+\beta+\gamma=180^\circ$

$$\gamma=180^\circ-20^\circ55'-26^\circ30'$$

$$\boxed{\gamma=132^\circ35'}$$

Q.3 $a=28.3,\ b=31.7,\ c=42.8$

Solution:

We know that

$$\cos\alpha=\frac{b^2+c^2-a^2}{2bc}$$

$$\therefore \cos\alpha=\frac{(31.7)^2+(42.8)^2-(28.3)^2}{2\times31.7\times42.8}$$

$$\cos\alpha=\frac{2035.84}{2713.52}$$

$\cos\alpha = 0.75$

$\alpha = \cos^{-1}(0.75)$

$\boxed{\alpha = 41°23'}$

Again by cosine law

$$\cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos\beta = \frac{(28.3)^2 + (42.8)^2 - (31.7)^2}{2\times 28.3\times 42.8}$$

$$\cos\beta = \frac{1627.84}{2422.48}$$

$\cos\beta = 0.672$

$\beta = \cos^{-1}(0.672)$

$\boxed{\beta = 47°46'}$

$\because \alpha + \beta + \gamma = 180°$

$\gamma = 180° - 41°23' - 47°46'$

$\boxed{\gamma = 90°51'}$

Q.4 $a = 31.9$, $b = 56.31$, $c = 40.27$

Solution:

We know that

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\therefore \cos\alpha = \frac{(56.31)^2 + (40.27)^2 - (31.9)^2}{2\times 56031\times 2074}$$

$\cos\alpha = 0.832$

$\alpha = \cos^{-1}(0.832)$

$\boxed{\alpha = 33°39'}$

Again by cosine law

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos\gamma = \frac{(31.9)^2 + (56.31)^2 - (40.27)^2}{2(31.9)(56.31)}$$

$$\cos\gamma = \frac{2566.7532}{3592.578}$$

$\cos\gamma = 0.714$

$\gamma = \cos^{-1}(0.714)$

$\boxed{\gamma = 44°24'}$

$\because \alpha + \beta + \gamma = 180°$

$\beta = 180° - 33°39' - 44°24'$

$\boxed{\beta = 101°57'}$

Q.5 $a = 4584$, $b = 5140$, $c = 3624$

Solution:

We know that

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\therefore \cos\alpha = \frac{(5140)^2 + (3624)^2 - (4584)^2}{2\times 5140\times 3624}$$

$$\cos\alpha = \frac{18539920}{37254720}$$

$\cos\alpha = 0.497$

$\alpha = \cos^{-1}(0.497)$

$\boxed{\alpha = 60°9'}$

Again by cosine law

$$\cos\beta = \frac{a^2 + c^2 + b^2}{2ac}$$

$$\cos\beta = \frac{(4584)^2 + (3624)^2 - (5140)^2}{2(4584)(3624)}$$

$$\cos\beta = \frac{7726832}{33224832}$$

$\cos\beta = 0.232$

$\beta = \cos^{-1}(0.232)$

$\boxed{\beta = 76°33'}$

$\because \alpha + \beta + \gamma = 180°$

$\gamma = 180° - 60°9' - 76°33'$

$\boxed{\gamma = 43°18'}$

Q.6 Find the smallest angle of the triangle ABC, When

$a = 37.37$, $b = 3.24$, $c = 35.06$

Solution:

As $b = 3.24$ is the smallest side of the triangle, then β the opposite angle of side b is the smallest angle of the triangle.

By cosine law

$$\cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos\beta = \frac{(37.34)^2 + (35.06)^2 - (3.24)^2}{2 \times 37.34 \times 35.06}$$

$$\cos\beta = \frac{2612.9816}{2618.2808}$$

$$\cos\beta = 0.997$$

$$\beta = \cos^{-1}(0.997)$$

$$\boxed{\beta = 3°38'}$$

Q.7 Find the measure of the greatest angle, if sides of the triangle are 16,20,33.

Solution:

As $c = 33$ is the longest side, then angle γ opposite to side c is the greatest angle.

By cosine law

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{(16)^2 + (20)^2 - (33)^2}{2(16)(20)}$$

$$= \frac{256 + 400 - 1089}{640}$$

$$= \frac{-433}{640}$$

$$\cos\gamma = -0.6766$$

$$\gamma = \cos^{-1}(-0.6766)$$

$$\boxed{\gamma = 132°34'}$$

Q.8 The sides of a triangle are $x^2 + x + 1$, $2x + 1$ and $x^2 - 1$. Prove that the greatest angle of the triangle is $120°$.

Solution:

Suppose: $a = x^2 + x + 1$, $b = 2x + 1$, $c = x^2 - 1$

Clearly a is the greatest side, then the angle 'α' opposite to the side 'a' is greatest.

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(2x+1)^2 + (x^2-1)^2 - (x^2+x+1)^2}{2(2x+1)(x^2-1)}$$

$$= \frac{4x^2 + 1 + 4x + x^4 + 1 - 2x^2 - (x^4 + x^2 + 1 + 2x^3 + 2x^2 + 2x)}{2(2x^3 - 2x + x^2 - 1)}$$

$$= \frac{x^4 + 2x^2 + 4x + 2 - (x^4 + 2x^3 + 3x^2 + 2x + 1)}{2(2x^3 - 2x + x^2 - 1)}$$

$$= \frac{-2x^3 - x^2 + 2x + 1}{2\left(2x^3 + x^2 - 2x - 1\right)}$$

$$= \frac{-\left(2x^3 + x^2 - 2x - 1\right)}{2\left(2x^3 + x^2 - 2x - 1\right)}$$

$$\cos\alpha = -\frac{1}{2}$$

$$\alpha = \cos^{-1}\left(-\frac{1}{2}\right)$$

$$\alpha = 120°$$

$$\boxed{\alpha = 120°}$$

Q.9 The measures of side of a triangular plot are 413,214 and 375 meters. Find the measures of the corner angles of the plot.

Solution:

Suppose: $a = 413$, $b = 214$, $c = 375$

By cosine law

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos\alpha = \frac{(214)^2 + (375)^2 - (413)^2}{2(214)(375)}$$

$$\cos\alpha = \frac{15852}{160500}$$

$$\cos\alpha = 0.098$$

$$\alpha = \cos^{-1}(0.098)$$

$$\boxed{\alpha = 84°19'}$$

Again by cosine law

$$\cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos\beta = \frac{(413)^2 + (375)^2 - (214)^2}{2(413)(375)}$$

$$\cos\beta = \frac{265398}{309750}$$

$$\cos\beta = 0.856$$

$$\beta = \cos^{-1}(0.856)$$

$$\boxed{\beta = 31°2'}$$

Q $\alpha + \beta + \gamma = 180°$

$$84°19' + 31°2' + \gamma = 180°$$

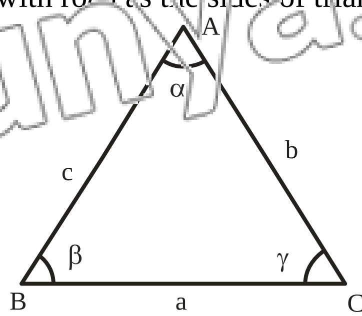
$$\gamma = 180° - 84°19' - 31°21'$$

$$\boxed{\gamma = 64°39'}$$

Q.10 Three villages A,B and C are connected by straight roads 6km, 9km and 13km. What angle these roads make with each other?

Solution:

Consider the three villages A,B and C are the vertices of a triangle ABC with road as the sides of triangle.



Suppose $a = 6km$, $b = 9km$, $c = 13km$

By cosine law

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{9^2+13^2-6^2}{2\times 9\times 13}$$

$$= \frac{81+169-36}{2\times 9\times 13}$$

$$= \frac{214}{234}$$

$$\cos\alpha = \frac{107}{117}$$

$$\alpha = \cos^{-1}\left(\frac{107}{117}\right)$$

$$\boxed{\alpha = 23°51'}$$

Again by cosine law

$$\cos\beta = \frac{a^2+c^2-b^2}{2ac}$$

$$\cos\beta = \frac{6^2+13^2-9^2}{2\times 6\times 13}$$

$$= \frac{124}{156}$$

$$\cos\beta = \frac{31}{39}$$

$$\beta = \cos^{-1}\left(\frac{31}{39}\right)$$

$$\boxed{\beta = 37°21'}$$

As $\alpha+\beta+\gamma = 180°$

$$\gamma = 180° - \alpha - \beta$$

$$\gamma = 180° - 23°51' - 37°21'$$

$$\boxed{\gamma = 118°48'}$$

Area of Triangle

Case-I:

Area of Triangle in Terms of the Measures of Two Sides and Their Included Angle.

With usual notations, Prove that:

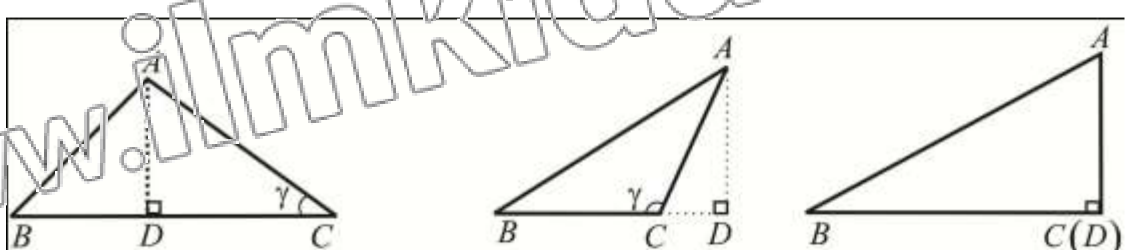
Area of triangle $ABC = \frac{1}{2}bc\sin\alpha = \frac{1}{2}ca\sin\beta = \frac{1}{2}ab\sin\gamma$

Proof :

Consider three different kinds of triangles ABC with $m\angle C = \gamma$ as

(i) Acute (ii) Obtuse (iii) Right

From A, draw $\overline{AD} \perp \overline{BC}$ or $\overline{BC}$ produced.



fig(i) fig(ii) fig(iii)

In figure (i), $\frac{AD}{AC} = \sin\gamma$

In figure (ii), $\frac{AD}{AC} = \sin(180^o - \gamma) = \sin\gamma$

In figure (iii), $\frac{AD}{AC} = 1 = \sin 90^o = \sin\gamma$

In all the three cases, we have

$AD = AC\sin\gamma = b\sin\gamma$ $\qquad \because AC = b$

Let Δ denote the area of triangle ABC

By elementary geometry, we know that

$$\Delta = \frac{1}{2}\ \text{(base)(altitude)}$$

$$\Delta = \frac{1}{2}\ (BC)(AD)$$

$$\Delta = \frac{1}{2}\ ab\sin\gamma$$

Similarly We can prove that:

$$\Delta = \frac{1}{2} bc\sin\alpha = \frac{1}{2} ca\sin\beta$$

Case-II:

Area of Triangle in Terms of the Measures of One Side and Two Angles.

In a triangle *ABC*, with usual notations, we have

$$\text{Area of triangle } = \frac{a^2\sin\beta\sin\gamma}{2\sin\alpha} = \frac{b^2\sin\alpha\sin\gamma}{2\sin\beta} = \frac{c^2\sin\alpha\sin\beta}{2\sin\gamma}$$

Proof:

By the law of sines, we know that:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

$$\Rightarrow \frac{b}{\sin\beta} = \frac{a}{\sin\alpha} \Rightarrow b = \frac{a\sin\beta}{\sin\alpha} \qquad \text{(i)}$$

$$\text{and } \frac{c}{\sin\gamma} = \frac{a}{\sin\alpha} \Rightarrow c = \frac{a\sin\gamma}{\sin\alpha} \qquad \text{(ii)}$$

Also we know that area of triangle is

$$\Delta = \frac{1}{2} bc\sin\alpha \qquad \text{(iii)}$$

Putting values from (i) and (ii) in (iii), we get

$$\Delta = \frac{1}{2}\left(\frac{a\sin\beta}{\sin\alpha}\right)\left(\frac{a\sin\gamma}{\cancel{\sin\alpha}}\right)\cancel{\sin\alpha}$$

$$\Delta = \frac{a^2\sin\beta\sin\gamma}{2\sin\alpha}$$

Hence proved

Case-III:

Area of Triangle in Terms of the Measures of its Sides.

In a triangle ABC, with usual notation, prove that:

Area of triangle $=\sqrt{s(s-a)(s-b)(s-c)}$

Proof:

We know that area of triangle ABC is

$$\Delta = \frac{1}{2}bc\sin\alpha \qquad \text{(i)}$$

$$\Delta = \frac{1}{\cancel{2}}bc\,.\,\cancel{2}\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \qquad \because\ \sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$

$$\Delta = bc\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \qquad \text{(ii)}$$

$$\Delta = bc\sqrt{\frac{(s-b)(s-c)}{bc}}\ \sqrt{\frac{s(s-a)}{bc}} \qquad \text{(by half angle formulas)}$$

$$\Delta = bc\sqrt{\frac{(s-b)(s-c)s(s-a)}{(bc)^2}}$$

$$\Delta = bc\frac{\sqrt{s(s-a)(s-b)(s-c)}}{\sqrt{(bc)^{\cancel{2}}}}$$

$$\Delta = \cancel{bc}\,\frac{\sqrt{s(s-a)(s-b)(s-c)}}{\cancel{bc}}$$

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

Which is also called **Hero's formula**

Case No.	Given	Area of Triangle
1	Two sides and one angle is given	$\Delta = \frac{1}{2}bc\sin\alpha = \frac{1}{2}ca\sin\beta = \frac{1}{2}ab\sin\gamma$
2	One side and two angles are given	$\Delta = \frac{a^2\sin\beta\sin\gamma}{2\sin\alpha} = \frac{b^2\sin\alpha\sin\gamma}{2\sin\beta} = \frac{c^2\sin\alpha\sin\beta}{2\sin\gamma}$
3	Three sides are given	$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

EXERCISE 12.7

Q.1 Find the area of triangle *ABC*, given two sides and their included angle:

(i) $a=200$, $b=120$, $\gamma=150°$

Solution: We know that

$$\Delta\ ABC=\frac{1}{2}ab\sin\gamma$$

$$=\frac{1}{2}\times 200\times 120\sin 150°$$

$$=100\times 120\times\frac{1}{2}$$

$$\boxed{=6000\ \text{sq.units}}$$

(ii) $b=37$, $c=45$, $\alpha=30°50'$

Solution: We know that

$$\Delta\ ABC=\frac{1}{2}bc\sin\alpha$$

$$=\frac{1}{2}\times 37\times 45\times\sin 30°50'$$

$$\boxed{=426.69\ \text{sq.units}}$$

(iii) $a=4.33$, $b=9.25$, $\gamma=56°44'$

Solution: We know that

$$\Delta\ ABC=\frac{1}{2}ab\sin\gamma$$

$$=\frac{1}{2}\times 4.33\times 9.25\sin 56°44'$$

$$\boxed{=16.74\ \text{sq.units}}$$

Q.2 Find the area of the triangle *ABC*, given one side and two angles.

(i) $b=25.4$, $\gamma=36°41'$, $\alpha=45°17'$

Solution: $\because\ \alpha+\beta+\gamma=180°$

$$\beta=180°-45°17-36°41'=98°2'$$

We know that area of triangle *ABC* is

$$\Delta=\frac{b^2\sin\alpha\sin\gamma}{2\sin\beta}$$

$$=\frac{(25.4)^2\sin(45°17')\sin(36°41')}{2\sin(98°2')}$$

$$\boxed{=138.29\ \text{sq.units}}$$

(ii) $c=32$, $\alpha=47°24'$, $\beta=70°16'$

Solution: $\because\ \alpha+\beta+\gamma=180°$

$$\gamma=180°-\alpha-\beta$$

$$\gamma=180°-47°24'-70°16'=62°20'$$

We know that area of triangle *ABC* is

$$\Delta=\frac{c^2\sin\alpha\sin\beta}{2\sin\gamma}$$

$$=\frac{(32)^2\sin(47°24')\sin(70°16')}{2\sin(62°20')}$$

$$\boxed{=400.54\ \text{sq.units}}$$

(iii) $a=4.8$, $\alpha=83°42'$, $\gamma=37°12'$

Solution: $\because\ \alpha+\beta+\gamma=180°$

$$83°42'+\beta+37°12'=180°$$

$$\beta=180°-83°42'-37°12'$$

$$\beta=59°6'$$

$$\Delta=\frac{a^2\sin\beta\ \sin\gamma}{2\sin\alpha}$$

$$=\frac{(4.8)^2\sin(59°6')\sin(37°12')}{2\sin(83°42')}$$

$$\boxed{=6.01\ \text{sq.units}}$$

Q.3 Find the area of the triangle *ABC*, given three sides:

(i) $a=18$, $b=24$, $c=30$

Solution: $s=\dfrac{a+b+c}{2}$

$$s=\frac{18+24+30}{2}$$

$$s=\frac{72}{2}$$

$$s=36$$

Now $s-a=36-18=18$

$s-b=36-24=12$

$s-c=36-30=6$

Area of triangle:

$$\Delta=\sqrt{s(s-a)(s-b)(s-c)}$$

$$=\sqrt{36\times18\times12\times6}$$

$$=\sqrt{46656}$$

$$\boxed{=216\text{ sq.units}}$$

(ii) $a=524$, $b=276$, $c=315$

Solution: $s=\dfrac{a+b+c}{2}$

$$s=\frac{524+276+315}{2}$$

$$s=\frac{1115}{2}$$

$$s=557.5$$

Now $s-a=557.5-524=33.5$

$s-b=557.5-276=281.5$

$s-c=557.5-315=242.5$

Area of triangle:

$$\Delta=\sqrt{s(s-a)(s-b)(s-c)}$$

$$=\sqrt{557.5\times33.5\times281.5\times242.5}$$

$$\boxed{=35705.89\text{ sq.units}}$$

(iii) $a=32.65$, $b=42.81$, $c=64.92$

Solution: $s=\dfrac{a+b+c}{2}$

$$=\frac{32.65+42.81+64.92}{2}$$

$$=\frac{140.38}{2}$$

$$s=70.19$$

Now $s-a=70.19-32.65=37.54$

$s-b=70.19-42.81=27.38$

$s-c=70.19-64.92=5.27$

Area of triangle:

$$\Delta=\sqrt{s(s-a)(s-b)(s-c)}$$

$$=\sqrt{70.19\times37.54\times27.38\times5.27}$$

$$\boxed{=616.604\text{ sq.units}}$$

Q.4 The area of triangle is 2437. if $a=79$, and $c=97$, **then find angle** β.

Solution: Area of triangle: $\Delta=2437$

$a=79$, $c=97$, $\beta=?$

Area of triangle: $\Delta=\dfrac{1}{2}ac\sin\beta$

$$2437=\frac{1}{2}\times79\times97\sin\beta$$

$$\frac{2437\times2}{79\times97}=\sin\beta$$

$$0.636=\sin\beta$$

$$\sin^{-1}(0.636)=\beta$$

$$39^\circ29'=\beta$$

$$\boxed{\beta=39^\circ29'}$$

Q.5 The area of triangle is 121.34. if $\alpha=32^\circ15'$, $\beta=65^\circ37'$, **the find c and angle** γ.

Solution: $\alpha=32^\circ15'$, $\beta=65^\circ37'$, $c=?$

$\gamma=?$

Area of triangle: $\Delta=121.34$

$\because \alpha+\beta+\gamma=180^\circ$

$\gamma=180^\circ-32^\circ15'-65^\circ37'=82^\circ8'$

Area of triangle:

$$\Delta=\frac{c^2\sin\alpha\,\sin\beta}{2\sin\gamma}$$

$$121.34=c^2\frac{\sin(32^\circ15')\sin(65^\circ37')}{2\sin(82^\circ8')}$$

$$\frac{121.34\times 2\sin(82^\circ8')}{\sin(32^\circ15')\sin(65^\circ37')}=c^2$$

$$494.62=c^2$$

$$\sqrt{c^2}=\sqrt{494.62}$$

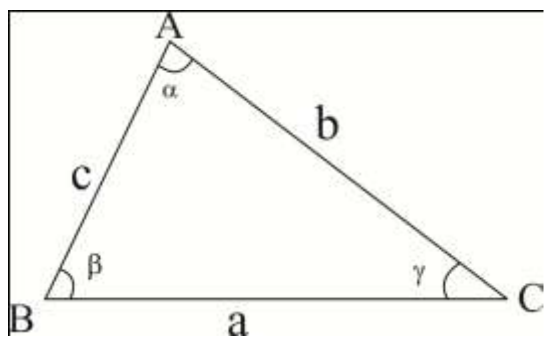
$$\boxed{c=22.24}$$

Q.6 One side of a triangular garden is 30m. if its two corner angles are $22^\circ\frac{1}{2}$ and $112^\circ\frac{1}{2}$, find the cost of planting the grass at the rate of Rs.5 per square meter.

Solution:

Consider a triangular garden ABC with usual notation.

Suppose: $a=30m$, $\beta=112^\circ\frac{1}{2}$, $\gamma=22^\circ\frac{1}{2}$



$\because \alpha+\beta+\gamma=180^\circ$

$\alpha=180^\circ-\beta-\gamma$

$\alpha=180^\circ-112.5^\circ-22.5^\circ$

$\alpha=45^\circ$

Area of Garden: $\Delta=\frac{a^2\sin\beta\,\sin\gamma}{2\sin\alpha}$

$$=\frac{30^2\sin(112.5^\circ)\sin(22.5^\circ)}{2\sin(45^\circ)}$$

Area of Garden $=225$ sq.meter

Cost of planting the grass at 1 sq.meter $=$ Rs.5

Cost of planting the grass at 225 sq.meter $=$ Rs. $225\times5=$ Rs.1125

Circles Connected with Triangle

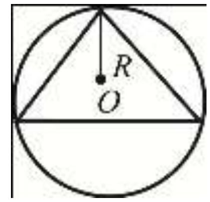
Following three kinds of circles related to a triangle

(i) Circum-circle (ii) In-circle (iii) Ex-circle

(i) Circum-Circle:

The circle passing through the three vertices of a triangle is called a circum-circle.

Circum-centre:



The centre of the circum-circle is called circum- centre,which is the point of intersection of the right bisectors of the sides of the triangle. In the figure, point O is called circum-centre.

Circum-radius:

The radius of the circum-circle is called the circum-radius and it is denoted by R

Theorem:

With usual notations, prove that

$$R=\frac{a}{2\sin\alpha}=\frac{b}{2\sin\beta}=\frac{c}{2\sin\gamma}$$

Proof:

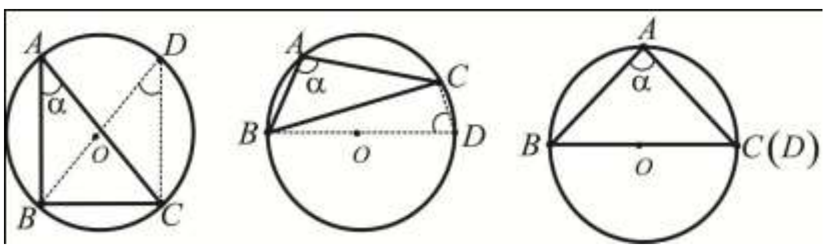


Fig (i) ($\angle BAC$ *is acute*) Fig(ii) ($\angle BAC$ *is obtuse*) Fig(iii) ($\angle BAC$ *is right*)

Consider three different kinds of triangle ABC with $m\angle A=\alpha$

(i) acute (ii) obtuse (iii) right

Let O be the circum-centre of ΔABC. Join B to O and produce $\overline{BO}$ to meet the circle again at D. Join C to D.

From figure $m\overline{BD}=2R$ (diameter of the circle)

and $m\overline{BC}=a$

Case-I:

In fig. (i) As angles in the same segment are equal, so

$m\angle BDC=m\angle A=\alpha$

In right triangle BCD,

$$\frac{m\overline{BC}}{m\overline{BD}}=\sin m\angle BDC=\sin\alpha$$

Case-II:

In fig. (ii) As sum of opposite angles of a cyclic quadrilateral is 180°, so

$m\angle BDC + m\angle A = 180^{o}$

$m\ \angle BDC + \alpha = 180^{o}$

$\Rightarrow m\angle BDC = 180^{o} - \alpha$

In right triangle BCD

$$\frac{m\overline{BC}}{m\overline{BD}} = \sin m\angle BDC \ = \sin(180^{o} - \alpha) = \sin\alpha$$

Case-III:

In fig. (iii), As angle inscribed in a semi-circle is always a right angle, so

$m\angle A = \alpha = 90^{o}$

$$\therefore\ \frac{m\overline{BC}}{m\overline{BD}} = 1 = \ \sin 90^{o} = \ \sin\alpha$$

In all the above three cases, we have proved that

$$\frac{m\overline{BC}}{m\overline{BD}} = \sin\alpha$$

$$\frac{a}{2R} = \sin\alpha \qquad \left(\because m\overline{BC} = a, m\overline{BD} = 2R\right)$$

$$\Rightarrow a = 2R\sin\alpha$$

$$\Rightarrow R = \frac{a}{2\sin\alpha}$$

Similarly, we can prove that $R = \dfrac{b}{2\sin\beta}$ and $R = \dfrac{c}{2\sin\gamma}$

Hence $R = \dfrac{a}{2\sin\alpha} = \dfrac{b}{2\sin\beta} = \dfrac{c}{2\sin\gamma}$

Deductions of Law of Sines:

We know that $R = \dfrac{a}{2\sin\alpha} = \dfrac{b}{2\sin\beta} = \dfrac{c}{2\sin\gamma}$

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R$$

Prove that $R=\dfrac{abc}{4\Delta}$

Proof:

We know that $R=\dfrac{a}{2\sin\alpha}$

$$R=\frac{a}{2.2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} \qquad \because \sin\alpha=2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$

$$=\frac{a}{4\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}} \qquad \text{By half angle formulas}$$

$$=\frac{a}{4\sqrt{\frac{(s-b)(s-c)\,.\,s(s-a)}{(bc)^2}}}$$

$$=\frac{a}{4\frac{\sqrt{s(s-a)(s-b)(s-c)}}{bc}}$$

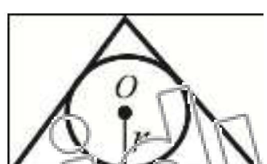
$$\Rightarrow \qquad R=\frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

$$\boxed{R=\frac{abc}{4\Delta}} \qquad \left(\because \Delta=\sqrt{s(s-a)(s-b)(s-c)}\right)$$

Hence proved

(ii) **In-Circle:**

The circle drawn inside a triangle touching its three sides is called in-circle or inscribed circle



In-Centre:

The centre of an inscribed circle is called in-centre, which is the point of intersection of the bisectors of angles of the triangle. In the above figure, point O is called in-centre.

In-Radius:

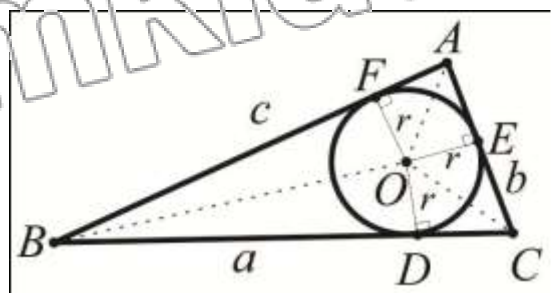
The radius of the inscribed circle is called in-radius and it is denoted by "*r*"

Theorem:

In any triangle ABC, with usual notations $r = \dfrac{\Delta}{S}$

Proof:

Let the internal bisectors of angles of triangle ABC meet at O, the in-centre.



Draw $\overline{OD} \perp \overline{BC}, \overline{OE} \perp \overline{AC}$ and $\overline{OF} \perp \overline{AB}$

Let $m\overline{OD} = m\overline{OE} = m\overline{OF} = r$

From the figure,

Area of $\Delta ABC =$ Area of $\Delta OBC +$ Area of $\Delta OCA +$ Area of ΔOAB

$$\Rightarrow \quad \Delta = \frac{1}{2}BC \times OD + \frac{1}{2}CA \times OE + \frac{1}{2}AB \times DF$$

$$= \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$$

$$= \frac{1}{2}r(a+b+c)$$

$$= \frac{1}{\cancel{2}}r(\cancel{2}s) \quad \because 2s = (a+b+c)$$

$$\Delta = rs$$

$$\frac{\Delta}{s} = r$$

$$\boxed{r = \frac{\Delta}{s}}$$

(iii) Escribed Circle:

A circle, which touches one side of the triangle externally and the other two produced sides internally is called an **escribed circle**, or **ex-circle** or **e-circle**.

Ex-Centre:

The centre of the escribed circle are called **ex-centre** which are the points where the internal bisector of one and the external bisectors of the other two angles of the triangle meet.

Note:

In ΔABC

(i) Centre of the ex-circle opposite to the vertex A is usually taken as I_1 and its radius is denoted by r_1

(ii) Centre of the ex-circle opposite to the vertex B is usually taken as I_2 and its radius is denoted by r_2

(iii) Centre of the ex-circle opposite to the vertex C is usually taken as I_3 and its radius is denoted by r_3

Theorem:

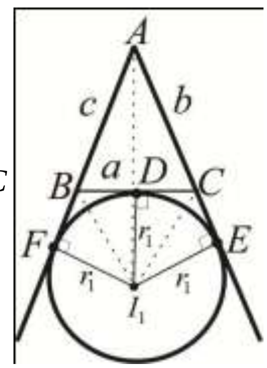
With usual notations, prove that

(i) $r_1 = \dfrac{\Delta}{s-a}$ (ii) $r_2 = \dfrac{\Delta}{s-b}$ (iii) $r_3 = \dfrac{\Delta}{s-c}$

Proof:

Let I_1 be the centre of the escribed circle opposite to the vertex A of ΔABC.

From I_1, draw $\overline{I_1D} \perp \overline{BC}$, $\overline{I_1E} \perp \overline{AC}$ produced and $\overline{I_1F} \perp \overline{AB}$ produced. Join I_1 to A, B and C.



Let $m\overline{I_1D} = m\overline{I_1E} = m\overline{I_1F} = r_1$

From the figure

Area of $\Delta\ ABC$ = Area of $\Delta\ I_1AB$ + Area of $\Delta\ I_1AC$ − Area of $\Delta\ I_1BC$

$$\Delta = \frac{1}{2}AB \times I_1F + \frac{1}{2}AC \times I_1E - \frac{1}{2}BC \times I_1D$$

$$= \frac{1}{2}cr_1 + \frac{1}{2}br_1 - \frac{1}{2}ar_1$$

$$\Delta = \frac{1}{2}r_1(c + b - a)$$

$$\Delta = \frac{1}{2}r_1(2s - a - a) \qquad \because 2s = a + b + c$$

$$2s - a = b + c$$

$$\Delta = \frac{1}{2}r_1(2s - 2a) = \frac{1}{\not{2}}.r_1 . \not{2}(s-a)$$

$$\Rightarrow \boxed{r_1 = \frac{\Delta}{s-a}}$$

Hence proved

Note:

In order to prove

(i) $r_2 = \dfrac{\Delta}{s-b}$, we take vertex B opposite to e-center I_2

(ii) $r_3 = \dfrac{\Delta}{s-c}$ we take vertex C opposite to e-center I_3

EXERCISE 12.8

Q.1 Show that

(i) $r = 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$

$$R.H.S = 4R\sin\frac{\alpha}{2}.\sin\frac{\beta}{2}.\sin\frac{\gamma}{2}$$

$$= 4\frac{abc}{4\Delta}\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$= \frac{abc}{\Delta}\sqrt{\frac{(s-a)^2(s-b)^2(s-c)^2}{a^2b^2c^2}}$$

$$= \frac{abc}{\Delta}.\frac{(s-a)(s-b)(s-c)}{abc}$$

$$= \frac{s(s-a)(s-b)(s-c)}{s\Delta}$$

$$= \frac{\Delta^2}{\Delta s}$$

$$= \frac{\Delta}{s}$$

$$= r$$

$$= L.\text{H.S}$$

(ii) $s = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

$$R.H.S = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

$$= 4\frac{abc}{4\Delta}\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{s(s-c)}{ab}}$$

$$= \frac{abc}{\Delta}\sqrt{\frac{s^2s(s-a)(s-b)(s-c)}{a^2b^2c^2}}$$

$$= \frac{abc}{\Delta}.\frac{s\Delta}{abc}$$

$$= s$$

$$= L.H.S$$

Q.2 Show that:

$$r = a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sec\frac{\alpha}{2} = b\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}\sec\frac{\beta}{2} = c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sec\frac{\gamma}{2}$$

First we show that $r = a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sec\frac{\alpha}{2}$

$$\text{R.H.S} = a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sec\frac{\alpha}{2}$$

$$= a\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}\sqrt{\frac{bc}{s(s-a)}}$$

$$= a\sqrt{\frac{bc(s-a)^2(s-b)(s-c)}{a^2bcs(s-a)}}$$

$$= a\sqrt{\frac{(s-a)(s-b)(s-c)}{a^2.s}}$$

$$= a\sqrt{\frac{s(s-a)(s-b)(s-c)}{a^2s^2}}$$

$$= a\frac{\Delta}{as}$$

$$= \frac{\Delta}{s}$$

$$= r$$

$$= L.H.S$$

Now, we show that

$$r = b\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}\sec\frac{\beta}{2}$$

$$\text{R.H.S} = b\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}\sec\frac{\beta}{2}$$

$$= b\sqrt{\frac{(s-a)(s-b)}{ab}}\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(ac)}{s(s-b)}}$$

$$= b\sqrt{\frac{ac(s-a)(s-b)^2(s-c)}{ab^2cs(s-b)}}$$

$$= b\sqrt{\frac{(s-a)(s-b)(s-c)}{b^2s}}$$

$$= b\sqrt{\frac{s(s-a)(s-b)(s-c)}{b^2 s^2}}$$

$$= b\frac{\Delta}{b.\text{s}}$$

$$= \frac{\Delta}{s}$$

$$= r$$

$$= R.H.S$$

Now, we prove that

$$r = c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sec\frac{\gamma}{2}$$

$$\text{R.H.S} = c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sec\frac{\gamma}{2}$$

$$= c\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{ab}{s(s-c)}}$$

$$= c\sqrt{\frac{ab(s-a)(s-b)(s-c)^2}{abc^2 s(s-c)}}$$

$$= c\sqrt{\frac{(s-a)(s-b)(s-c)}{c^2.s}}$$

$$= c\sqrt{\frac{s(s-a)(s-b)(s-c)}{c^2 s^2}}$$

$$= c\frac{\Delta}{c.s}$$

$$= \frac{\Delta}{s}$$

$$= r$$

$$= L.H.S$$

Q.3 Show that

(i) $r_1 = 4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

$$\text{R.H.S} = 4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

$$= 4\frac{abc}{4\Delta}\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{s(s-c)}{ab}}$$

$$= \frac{abc}{\Delta}\sqrt{\frac{s^2(s-b)^2(s-c)^2}{a^2b^2c^2}}$$

$$= \frac{abc}{\Delta}\cdot\frac{s(s-b)(s-c)}{abc}$$

$$= \frac{s(s-a)(s-b)(s-c)}{\Delta(s-a)}$$

$$= \frac{\Delta^2}{\Delta(s-a)}$$

$$= \frac{\Delta}{s-a}$$

$$= r_1$$

$$= L.H.S$$

(ii) $r_2 = 4R\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}$

$$\text{R.H.S} = 4R\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}$$

$$= 4\frac{abc}{4\Delta}\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{s(s-c)}{ab}}$$

$$= \frac{abc}{\Delta}\sqrt{\frac{s^2(s-a)^2(s-c)^2}{a^2b^2c^2}}$$

$$= \frac{abc}{\Delta}\ \frac{s(s-a)(s-c)}{abc}$$

$$= \frac{s(s-a)(s-b)(s-c)}{\Delta(s-b)}$$

$$= \frac{\Delta^2}{\Delta(s-b)}$$

$$= \frac{\Delta}{s-b}$$

$$= r_2 = L.H.S$$

(iii) $r_3 = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}$

$$R.H.S = 4R\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)$$

$$= 4\frac{abc}{4\Delta}\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$= \frac{abc}{\Delta}\sqrt{\frac{s^2(s-a)^2(s-b)^2}{a^2b^2c^2}}$$

$$= \frac{abc}{\Delta}\frac{s(s-a)(s-b)}{abc}$$

$$= \frac{s(s-a)(s-b)(s-c)}{\Delta(s-c)}$$

$$= \frac{\Delta^2}{\Delta(s-c)}$$

$$= \frac{\Delta}{s-c}$$

$$= r_3$$

$$= L.H.S$$

Q.4 Show that

(i) $r_1 = s\tan\frac{\alpha}{2}$

$$\text{R.H.S} = s\tan\frac{\alpha}{2}$$

$$= s\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$= \sqrt{\frac{s^2(s-b)(s-c)}{s(s-a)}}$$

$$= \sqrt{\frac{s(s-a)(s-b)(s-c)}{(s-a)^2}}$$

$$= \frac{\Delta}{s-a}$$

$$= r_1$$

$$= L.H.S$$

(ii) $r_2 = s\tan\frac{\beta}{2}$

$$\text{R.H.S} = s\tan\left(\frac{\beta}{2}\right)$$

$$= s\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$

$$= \sqrt{\frac{s^2(s-a)(s-c)}{s(s-b)}}$$

$$= \sqrt{\frac{s(s-a)(s-b)(s-c)}{(s-b)^2}}$$

$$= \frac{\Delta}{s-b}$$

$$= r_2$$

$$= L.H.S$$

(iii) $r_3 = s\tan\frac{\gamma}{2}$

$$\text{R.H.S} = s\tan\frac{\gamma}{2}$$

$$= s\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$= \sqrt{\frac{s^2(s-a)(s-b)}{s(s-c)}}$$

$$= \sqrt{\frac{s(s-a)(s-b)(s-c)}{(s-c)^2}}$$

$$= \frac{\Delta}{s-c}$$

$$= r_3$$

$$= L.H.S$$

Q.5 Prove that:

(i) $r_1r_2 + r_2r_3 + r_3r_1 = s^2$

$$L.H.S = r_1r_2 + r_2r_3 + r_3r_1$$

$$= \frac{\Delta}{s-a}.\frac{\Delta}{s-b} + \frac{\Delta}{s-b}.\frac{\Delta}{s-c} + \frac{\Delta}{s-c}.\frac{\Delta}{s-a}$$

$$= \frac{\Delta^2}{(s-a)(s-b)} + \frac{\Delta^2}{(s-b)(s-c)} + \frac{\Delta^2}{(s-c)(s-a)}$$

$$= \Delta^2\left[\frac{1}{(s-a)(s-b)} + \frac{1}{(s-b)(s-c)} + \frac{1}{(s-c)(s-a)}\right]$$

$$= \Delta^2\left[\frac{s-c+s-a+s-b}{(s-a)(s-b)(s-c)}\right]$$

$$= \Delta^2\left[\frac{3s-a-b-c}{(s-a)(s-b)(s-c)}\right]$$

$$= \Delta^2\left[\frac{3s-(a+b+c)}{(s-a)(s-b)(s-c)}\right]$$

$$= \Delta^2\frac{3s-2s}{(s-a)(s-b)(s-c)}$$

$$= \Delta^2\frac{s}{(s-a)(s-b)(s-c)}$$

$$= \frac{\Delta^2 s^2}{s(s-a)(s-b)(s-c)}$$

$$= \Delta^2\frac{s^2}{\Delta^2}$$

$$= s^2$$

$$= R.H.S$$

(ii) $rr_1r_2r_3=\Delta^2$

L.H.S $=rr_1r_2r_3$

$$=\frac{\Delta}{s}\cdot\frac{\Delta}{s-a}\cdot\frac{\Delta}{s-b}\cdot\frac{\Delta}{s-c}$$

$$=\frac{\Delta^4}{s(s-a)(s-b)(s-c)}$$

$$=\frac{\Delta^4}{\Delta^2}$$

$$=\Delta^2$$

$$=R.H.S$$

(iii) $r_1+r_2+r_3-r=4R$

L.H.S $=r_1+r_2+r_3-r$

$$=\frac{\Delta}{s-a}+\frac{\Delta}{s-b}+\frac{\Delta}{s-c}-\frac{\Delta}{s}$$

$$=\Delta\left[\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}\right]$$

$$=\Delta\left[\frac{s-b+s-a}{(s-a)(s-b)}+\frac{s-(s-c)}{(s-c)s}\right]$$

$$=\Delta\left[\frac{2s-a-b}{(s-a)(s-b)}+\frac{s-s+c}{s(s-c)}\right]$$

$$=\Delta\left[\frac{a+b+c-a-b}{(s-a)(s-b)}+\frac{c}{s(s-c)}\right]$$

$$=\Delta\left[\frac{c}{(s-a)(s-b)}+\frac{c}{s(s-c)}\right]$$

$$=\Delta c\left[\frac{1}{(s-a)(s-b)}+\frac{1}{s(s-c)}\right]$$

$$=\Delta c\left[\frac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)}\right]$$

$$=\Delta c\left[\frac{s^2-cs+s^2-as-bs+ab}{\Delta^2}\right]$$

$$=\frac{c}{\Delta}\left[2s^2-s(a+b+c)+ab\right]$$

$$=\frac{c}{\Delta}\left[2s^2-s(2s)+ab\right]$$

$$=\frac{c}{\Delta}\left[2s^2-2s^2+ab\right]$$

$$=\frac{c}{\Delta}[0+ab]$$

$$=\frac{abc}{\Delta}$$

$$=4\frac{abc}{4\Delta}$$

$$=4R$$

$$=R.H.S$$

(iv) $r_1r_2r_3=rs^2$

L.H.S $=r_1r_2r_3$

$$=\frac{\Delta}{s-a}\cdot\frac{\Delta}{s-b}\cdot\frac{\Delta}{s-c}$$

$$=\frac{\Delta^3}{(s-a)(s-b)(s-c)}$$

$$=\frac{s\Delta^3}{s(s-a)(s-b)(s-c)}$$

$$=\frac{s\Delta^3}{\Delta^2}$$

$$=s\Delta$$

$$=\frac{s^2\Delta}{s}$$

$$=s^2r$$

$$=\text{R.H.S}$$

Q.6 Find R, r, r_1, r_2 and r_3, if measure of the sides of triangle *ABC* are

(i) $a=13,\ b=14,\ c=15$

Solution:

Given: $a=13,\ b=14,\ c=15$

$\because \quad s=\dfrac{a+b+c}{2}$

$=\dfrac{13+14+15}{2}=\dfrac{42}{2}=21$

$s-a=21-13=8$

$s-b=21-14=7$

$s-c=21-15=6$

$\Delta=\sqrt{s(s-a)(s-b)(s-c)}$

$\Delta=\sqrt{21\times 8\times 7\times 6}$

$\Delta=\sqrt{7056}$

$\Delta=84$

$R=\dfrac{abc}{4\Delta}=\dfrac{13\times 14\times 15}{4\times 84}$

$=\dfrac{2730}{336}=8.125$

$r=\dfrac{\Delta}{s}=\dfrac{84}{21}=4$

$r_1=\dfrac{\Delta}{s-a}=\dfrac{84}{8}=10.5$

$r_2=\dfrac{\Delta}{s-b}=\dfrac{84}{7}=12$

$r_3=\dfrac{\Delta}{s-c}=\dfrac{84}{6}=14$

(ii) $a=34$, $b=20$, $c=42$

Solution:

Given: $a=34$, $b=20$, $c=42$

$\because \quad s=\dfrac{a+b+c}{2}=\dfrac{34+20+42}{2}$

$=\dfrac{96}{2}=48$

$s-a=48-34=14$

$s-b=48-20=28$

$s-c=48-42=6$

$\Delta=\sqrt{s(s-a)(s-b)(s-c)}$

$=\sqrt{48\times 14\times 28\times 6}$

$=\sqrt{112896}$

$\Delta=336$

$R=\dfrac{abc}{4\Delta}=\dfrac{34\times 20\times 42}{4\times 336}$

$=\dfrac{28560}{1344}=21.25$

$r=\dfrac{\Delta}{s}=\dfrac{336}{48}=7$

$r_1=\dfrac{\Delta}{s-a}=\dfrac{336}{14}=24$

$r_2=\dfrac{\Delta}{s-b}=\dfrac{336}{28}=12$

$r_3=\dfrac{\Delta}{s-c}=\dfrac{336}{6}=56$

Q.7 Prove that in an equilateral triangle

(i) $r:R:r_1=1:2:3$ **(ii)** $r:R:r_1:r_2:r_3=1:2:3:3:3$

Solution:

Let the length of each side of equilateral triangle is "a".

$s=\dfrac{a+b+c}{2}=\dfrac{a+a+a}{2}=\dfrac{3a}{2}$

$s-a=\dfrac{3a}{2}-a=\dfrac{3a-2a}{2}=\dfrac{a}{2}$

$$s-b=\frac{3a}{2}-a=\frac{a}{2}$$

$$s-c=\frac{3a}{2}-a=\frac{a}{2}$$

$$\Delta=\sqrt{s(s-a)(s-b)(s-c)}$$

$$=\sqrt{\frac{3a}{2}.\frac{a}{2}.\frac{a}{2}.\frac{a}{2}}$$

$$=\sqrt{\frac{3}{4}}a^2=\frac{\sqrt{3}}{2}a^2$$

$$R=\frac{abc}{4\Delta}=\frac{a.a.a}{4\frac{\sqrt{3}}{4}a^2}=\frac{a}{\sqrt{3}}$$

$$r=\frac{\Delta}{s}=\frac{\sqrt{3}a^2}{4}\div\frac{3a}{2}=\frac{\sqrt{3}a^2}{4}\times\frac{2}{3a}=\frac{a}{2\sqrt{3}}$$

$$r_1=\frac{\Delta}{s-a}=\frac{\sqrt{3}a^2}{4}\div\frac{a}{2}=\frac{\sqrt{3}a^2}{4}\times\frac{2}{a}=\frac{\sqrt{3}a}{2}$$

$$r_2=\frac{\Delta}{s-b}=\frac{\sqrt{3}a}{2}$$

$$r_3=\frac{\Delta}{s-c}=\frac{\sqrt{3}a}{2}$$

(i) $r:R:r_1=\frac{a}{2\sqrt{3}}:\frac{a}{\sqrt{3}}:\frac{\sqrt{3}a}{2}$

Multiplying by $\frac{2\sqrt{3}}{a}$

$$=\frac{a}{2\sqrt{3}}\times\frac{2\sqrt{3}}{a}:\frac{a}{\sqrt{3}}\times\frac{2\sqrt{3}}{a}:\frac{\sqrt{3}a}{2}\times\frac{2\sqrt{3}}{a}$$

$$=1:2:3=\text{R.H.S}$$

(ii) $r:R:r_1:r_2:r_3=1:2:3:3:3$

$\text{L.H.S}=r:R:r_1:r_2:r_3$

$$=\frac{a}{2\sqrt{3}}:\frac{a}{\sqrt{3}}:\frac{\sqrt{3}a}{2}:\frac{\sqrt{3}a}{2}:\frac{\sqrt{3}a}{2}$$

Multiplying by $\frac{2\sqrt{3}}{a}$

$$=\frac{a}{2\sqrt{3}}\times\frac{2\sqrt{3}}{a}:\frac{a}{\sqrt{3}}\times\frac{2\sqrt{3}}{a}:\frac{\sqrt{3}a}{2}\times\frac{2\sqrt{3}}{a}:\frac{\sqrt{3}a}{2}\times\frac{2\sqrt{3}}{a}:\frac{\sqrt{3}a}{2}\times\frac{2\sqrt{3}}{a}$$

$$=1:2:3:3:3=\text{R.H.S}$$

Q.8 Prove that:

(i) $\Delta = r^2 \cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$

$$\text{R.H.S} = r^2 \cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$$

$$= \frac{\Delta^2}{s^2}\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}\sqrt{\frac{s(s-b)}{(s-a)(s-c)}}\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \frac{\Delta^2}{s^2}\sqrt{\frac{s^3}{s(s-a)(s-b)(s-c)}}$$

$$= \frac{\Delta^2}{s^2}\sqrt{\frac{s^4}{s(s-a)(s-b)(s-c)}}$$

$$= \frac{\Delta^2}{s^2}\frac{s^2}{\Delta}$$

$$= \Delta$$

$$= L.\text{H.S}$$

(ii) $r = s\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

$$\text{R.H.S} = s\tan\frac{\alpha}{2}.\tan\frac{\beta}{2}.\tan\frac{\gamma}{2}$$

$$= s\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$= s\sqrt{\frac{(s-a)(s-b)(s-c)}{s^3}}$$

$$= s\sqrt{\frac{s(s-a)(s-b)(s-c)}{s^4}}$$

$$= s.\frac{\Delta}{s^2}$$

$$= \frac{\Delta}{s}$$

$$= r$$

$$= \text{L.H.S}$$

(iii) $\Delta = 4Rr\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

$$\text{R.H.S} = 4Rr\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

$$= 4\cdot\frac{abc}{4\Delta}\cdot\frac{\Delta}{s}\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{s(s-c)}{ab}}$$

$$= \frac{abc}{s}\cdot\sqrt{\frac{s^2\, s(s-a)(s-b)(s-c)}{a^2b^2c^2}}$$

$$= \frac{abc}{s}\cdot\frac{s.\Delta}{abc}$$

$$= \Delta$$

$$= \text{L.H.S}$$

Q.9 Show that

(i) $\frac{1}{2rR} = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$

$$\text{L.H.S} = \frac{1}{2rR}$$

$$= \frac{1}{2.\frac{\Delta}{s}.\frac{abc}{4\Delta}}$$

$$= \frac{1}{\frac{abc}{2s}} = \frac{2s}{abc}$$

$$= \frac{a+b+c}{abc}$$

$$= \frac{a}{abc} + \frac{b}{abc} + \frac{c}{abc}$$

$$= \frac{1}{bc} + \frac{1}{ac} + \frac{1}{ab}$$

$$= \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$$

$$= \text{R.H.S}$$

(ii) $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$

$$\text{R.H.S} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

$$= \frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta}$$

$$= \frac{s-a+s-b+s-c}{\Delta}$$

$$= \frac{3s-(a+b+c)}{\Delta}$$

$$= \frac{3s-2s}{\Delta}$$

$$= \frac{s}{\Delta}$$

$$= \frac{1}{r}$$

$$= \text{L.H.S}$$

Q.10 Prove that:

$$r = \frac{a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}} = \frac{b\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}}{\cos\frac{\beta}{2}} = \frac{c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}}$$

Solution: First we prove.

$$r = \frac{a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}}$$

$$R.H.S = \frac{a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}}$$

$$= a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sec\frac{\alpha}{2}$$

$$= a\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}\sqrt{\frac{bc}{s(s-a)}}$$

$$= a\sqrt{\frac{(s-a)(s-b)(s-c)}{s\,a^2}}$$

$$= a\sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2\,a^2}}$$

$$= a\frac{\Delta}{s.a}$$
$$= \frac{\Delta}{s}$$
$$= r$$
$$= L.H.S$$

Now, we prove

$$r = \frac{b\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}}{\cos\frac{\beta}{2}}$$

$$R.H.S = \frac{b\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}}{\cos\frac{\beta}{2}}$$
$$= b\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}\sec\frac{\beta}{2}$$
$$= b\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-a)(s-b)}{ab}}\sqrt{\frac{ac}{s(s-b)}}$$
$$= b\sqrt{\frac{(s-a)(s-b)(s-c)}{sb^2}}$$
$$= b\sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2b^2}}$$
$$= b.\frac{\Delta}{sb}$$
$$= \frac{\Delta}{s}$$
$$= r$$
$$= L.H.S$$

Now we prove

$$r = \frac{c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}}$$

$$R.H.S = \frac{c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}}$$
$$= c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sec\frac{\gamma}{2}$$

$$= c\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{ab}{s(s-c)}}$$

$$= c\sqrt{\frac{(s-a)(s-b)(s-c)}{sc^2}}$$

$$= c\sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2c^2}}$$

$$= c\frac{\Delta}{s.c}$$

$$= \frac{\Delta}{s}$$

$$= r$$

$$= \text{L.H.S}$$

Q.11 Prove that:

$$abc(\sin\alpha + \sin\beta + \sin\gamma) = 4\Delta s$$

$$\text{L.H.S} = abc(\sin\alpha + \sin\beta + \sin\gamma) \qquad \text{(i)}$$

$$\because\ R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma}$$

$$\sin\alpha = \frac{a}{2R},\ \sin\beta = \frac{b}{2R},\ \sin\gamma = \frac{c}{2R}$$

Equation (i) becomes

$$\text{L.H.S} = abc\left(\frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R}\right)$$

$$= abc\left(\frac{a+b+c}{2R}\right)$$

$$= abc\left(\frac{2S}{2R}\right)$$

$$= abc.\frac{s}{\frac{abc}{4\Delta}}$$

$$= \frac{s}{\frac{1}{4\Delta}}$$

$$= 4\Delta S$$

$$= \text{R.H.S}$$

Q.12 Prove that

(i) $(r_1+r_2)\tan\frac{\gamma}{2}=c$

Solution:

$$\text{L.H.S}=(r_1+r_2)\tan\frac{\gamma}{2}$$
$$=\left[\frac{\Delta}{s-a}+\frac{\Delta}{s-b}\right]\tan\frac{\gamma}{2}$$
$$=\Delta\left[\frac{1}{s-a}+\frac{1}{s-b}\right]\tan\frac{\gamma}{2}$$
$$=\Delta\left[\frac{s-b+s-a}{(s-a)(s-b)}\right]\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$
$$=\Delta[2s-a-b]\sqrt{\frac{(s-a)(s-b)}{s(s-a)^2(s-b)^2(s-c)}}$$
$$=\Delta[a+b+c-a-b]\sqrt{\frac{1}{s(s-a)(s-b)(s-c)}}$$
$$=\Delta.c.\frac{1}{\Delta}$$
$$=c$$
$$=\text{R.H.S}$$

(ii) $(r_3-r)\cot\frac{\gamma}{2}=c$

$$L.H.S=(r_3-r)\cot\frac{\gamma}{2}$$
$$=\left[\frac{\Delta}{s-c}-\frac{\Delta}{s}\right]\cot\frac{\gamma}{2}$$
$$=\Delta\left[\frac{1}{s-c}-\frac{1}{s}\right].\cot\frac{\gamma}{2}$$
$$=\Delta\left[\frac{s-(s-c)}{s(s-c)}\right]\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$
$$=\Delta[s-s+c]\sqrt{\frac{s(s-c)}{s^2(s-a)(s-b)(s-c)^2}}$$
$$=\Delta c\sqrt{\frac{1}{s(s-a)(s-b)(s-c)}}$$
$$=\Delta.\ c.\ \frac{1}{\Delta}$$
$$=c$$
$$=\text{R.H.S}$$