CHAPTER 13

INVERSE TRIGONOMETRIC FUNCTIONS

Introduction:

We have been finding the values of trigonometric functions for given measures of the angles. But in the application of trigonometry, the problem has also been the other way round and we are required to find the measure of the angle when the value of its trigonometric function is given. For this purpose, we need to have the knowledge of inverse trigonometric functions.

Note:

We know that only a one-to-one function will have an inverse. If a function is not one-to-one, it may be possible to restrict its domain to make it one-to-one so that its inverse can be found.

Sine Function:

The sine function is defined as $y = \sin x$ where its domain is $-\infty < x < \infty$ and its range is $-1 \le y \le 1$.

Note:

The graph of $y = \sin x$, $-\infty < x < \infty$, is shown in the figure (1)

$y - axis$

$\uparrow$

Note:

We observe that every horizontal line between the lines $y = -1$ and $y = 1$ intersects the graph infinitely many times. It follows that the sine function is not one-to-one.

Principal Sine Function:

If we restrict the domain of $y=\sin x$ to the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, then the restricted function $y=\sin x$, $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$ is called the principal sine function. Its graph is shown in figure (2)

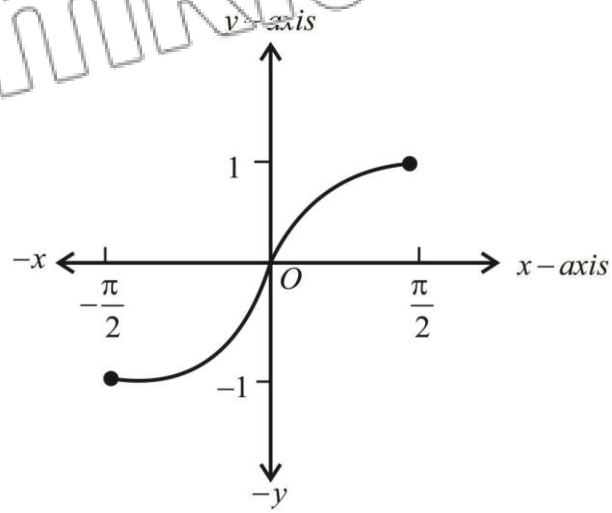


Fig (2)
Graph of principal sine function

Note:

The principle sine function is now one to one and hence will have an inverse figure (2)

The Inverse Sine Function:

The inverse sine function is defined by $y=\sin^{-1}x$ if and only if $x=\sin y$

where $-1\le x\le 1$ and $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$

Note:

The graph of $y=\sin^{-1}x$ is obtained by reflecting the restricted portion of the graph of principal sine function about the line $y=x$ as shown in figure (3).

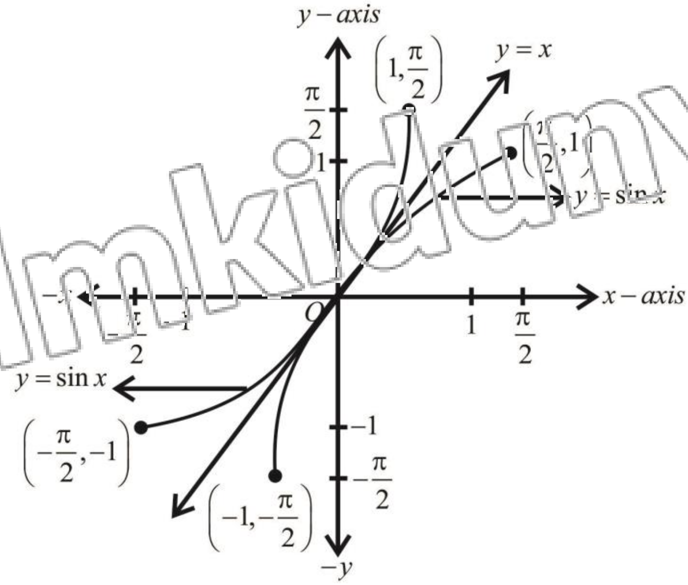


Fig (3)

Graph of principal sine of function $y=\sin x$ Vs $y=\sin^{-1}x$

Note:

We notice that the graph of $y=\sin x$ is along the $x-axis$ where as the graph of $y=\sin^{-1}x$ is along the $y-axis$ and graph of $y=\sin x$ and $y=\sin^{-1}x$ are symmetry about $y=x$

Cosine Function:

The cosine function is defined as $y=\cos x$ where its domain is $-\infty<x<\infty$ and its range is $-1\le y\le 1$. Its graph is shown in the figure (4).

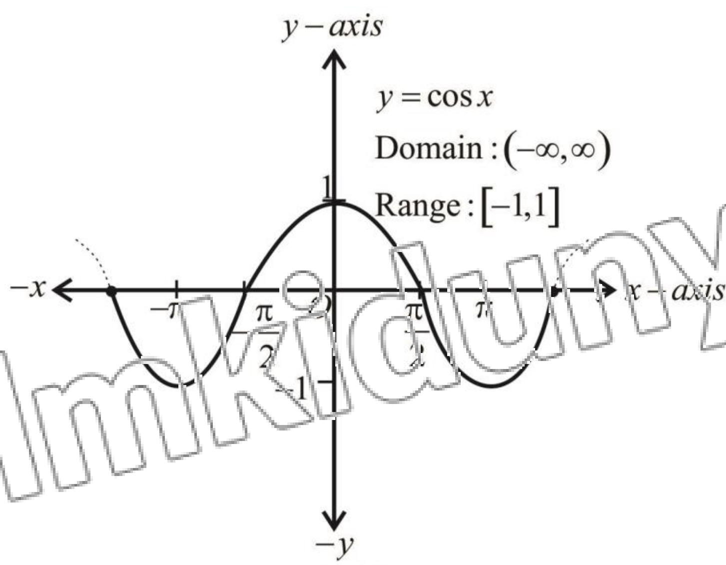


Fig (4)

Graph of $y=\cos x$

Note:

We observe that every horizontal line between the lines $y=-1$ and $y=1$ intersects the graph infinitely many times. It follows that cosine function is not one-to-one.

Principal Cosine Function:

If we restrict the domain of $y=\cos x$ to the interval $[0,\pi]$ then the restricted function $y=\cos x, 0\le x\le \pi$ is called the principal cosine function. Its graph is shown in figure (5)

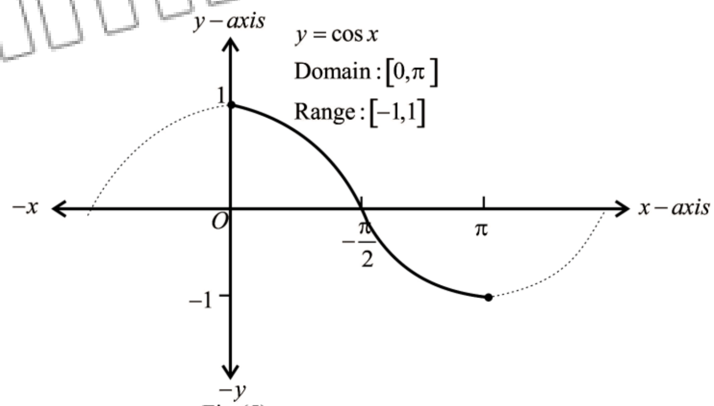


Fig (5)
Graph of the principal cosine function

Note:

The principal cosine function is now one-to-one and hence will have an inverse.

The Inverse Cosine Function:

The inverse cosine function is defined by $y=\cos^{-1}x$ if and only if $x=\cos y$ where $-1\le x\le 1$ and $0\le y\le \pi$. Its graph is shown in figure (6a).

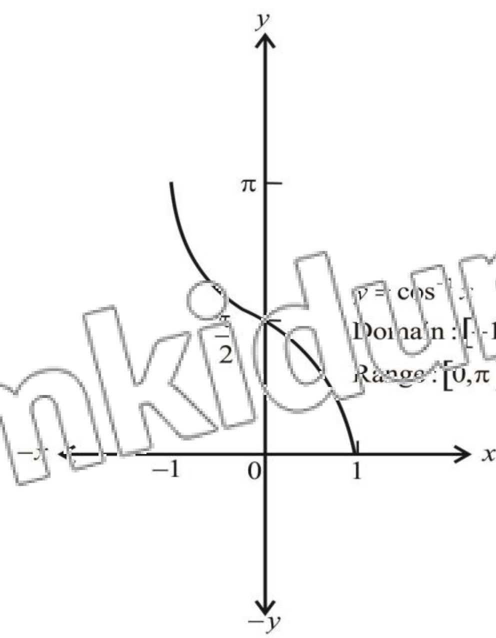


Fig (6a)
Graph of $y=\cos^{-1}x$

Note:

The graph of $y=\cos^{-1}x$ is obtained by reflecting the restricted portion of the graph of $y=\cos x$ about the line $y=x$ as shown in figure (6b).

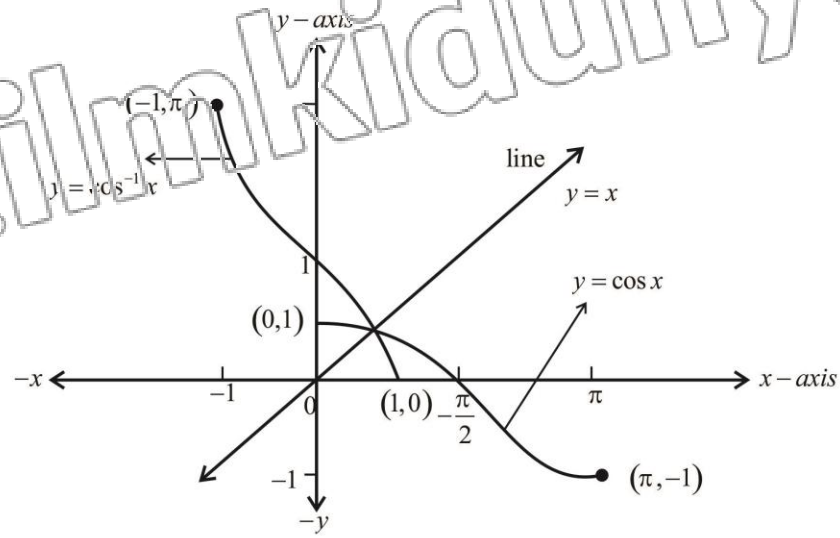


Fig (6b)

Graph of principal cosine function $y=\cos x$ Vs $y=\cos^{-1}x$

Note:

We notice that the graph of $y=\cos x$ is along the $x-axis$ whereas the graph of $y=\cos^{-1}x$ is along the $y=axis$.

Tangent Function:

The tangent function is defined by as $y=\tan x$ where its domain is $\left\{x : x \neq (2n+1)\dfrac{\pi}{2}, n \in Z\right\}$ and its range is $-\infty < y < \infty$.

Note:

The graph of $y=\tan x$ is shown in the figure (7)

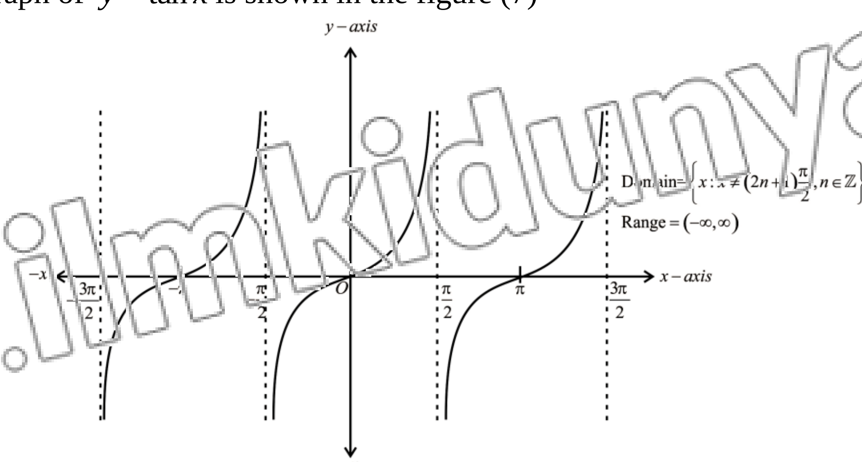


Fig (7)

Graph of $y=\tan x$

Note:

We observe that there are many horizontal lines between the lines $y=-1$ and $y=1$ intersects the graph infinitely many times. It follows that tangent function is not one-to-one.

Principal Tangent Function:

If we restrict the domain of $y=\tan x$ to the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ then the restricted function $y=\tan x$, $-\frac{\pi}{2}<x<\frac{\pi}{2}$ is called the principal tangent function. Its graph is shown in figure (8).

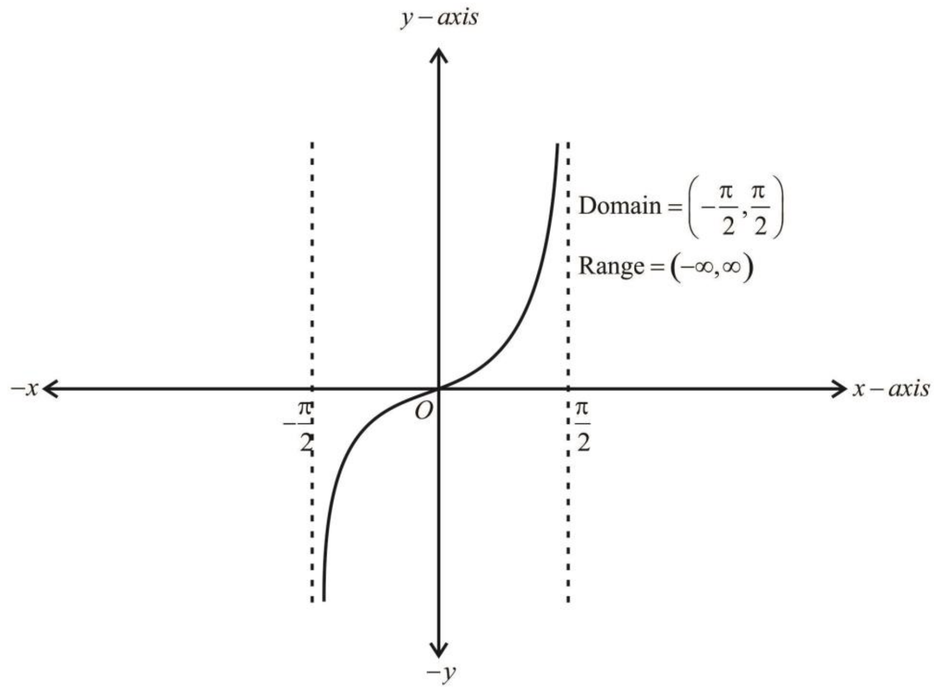


Fig (8)
Graph of the principal tangent function

Note:

The principal tangent function is now one-to-one and hence will have an inverse as shown in figure (9).

Inverse Tangent Function:

The inverse tangent function is defined by $y=\tan^{-1}x$, if and only if $x=\tan y$ where $-\infty<x<\infty$ and $-\frac{\pi}{2}<y<\frac{\pi}{2}$

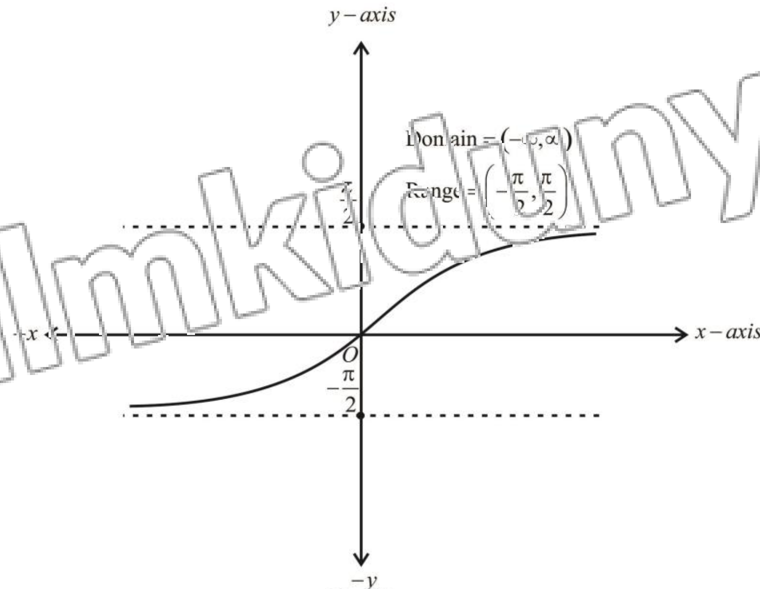


Fig (9)
Graph of $y=\tan^{-1}x$

Note:

The graph of $y=\tan^{-1}x$ is obtained by reflecting the restricted portion of the graph of $y=\tan x$ about the line $y=x$ as shown in figure 9 (b)

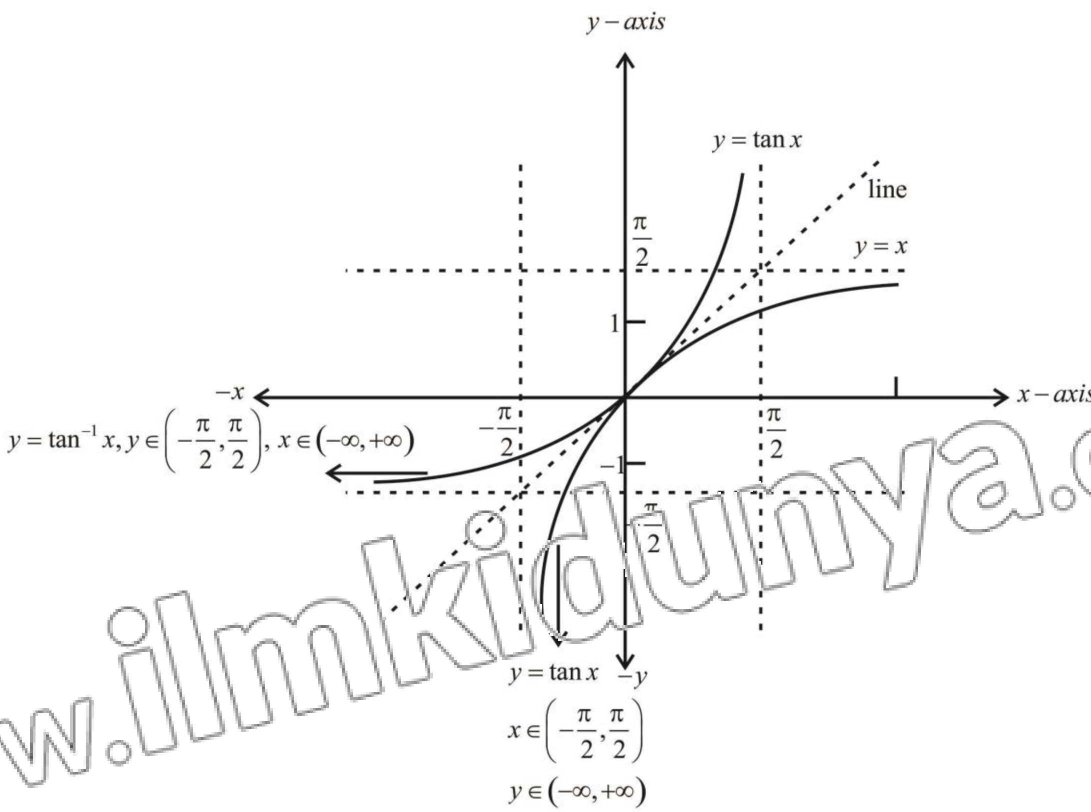


Fig 9 (b)

Graph of principal tangent function $y=\tan x$ Vs $y=\tan^{-1}x$

Note:

We notice that the graph of $y = \tan x$ is along the $x - axis$ where as the graph of $y = \tan^{-1} x$ is along the $y - axis$.

Cotangent Function:

The cotangent function is defined by as $y = \cot x$ where its domain is $\{x : x \neq n\pi, n \in \mathbb{Z}\}$ and its range is $-\infty < y < \infty$

Note:

The graph of $y = \cot x$, $-\infty < x < \infty$ $(x \neq n\pi, n \in \mathbb{Z})$ is shown in the figure (10)

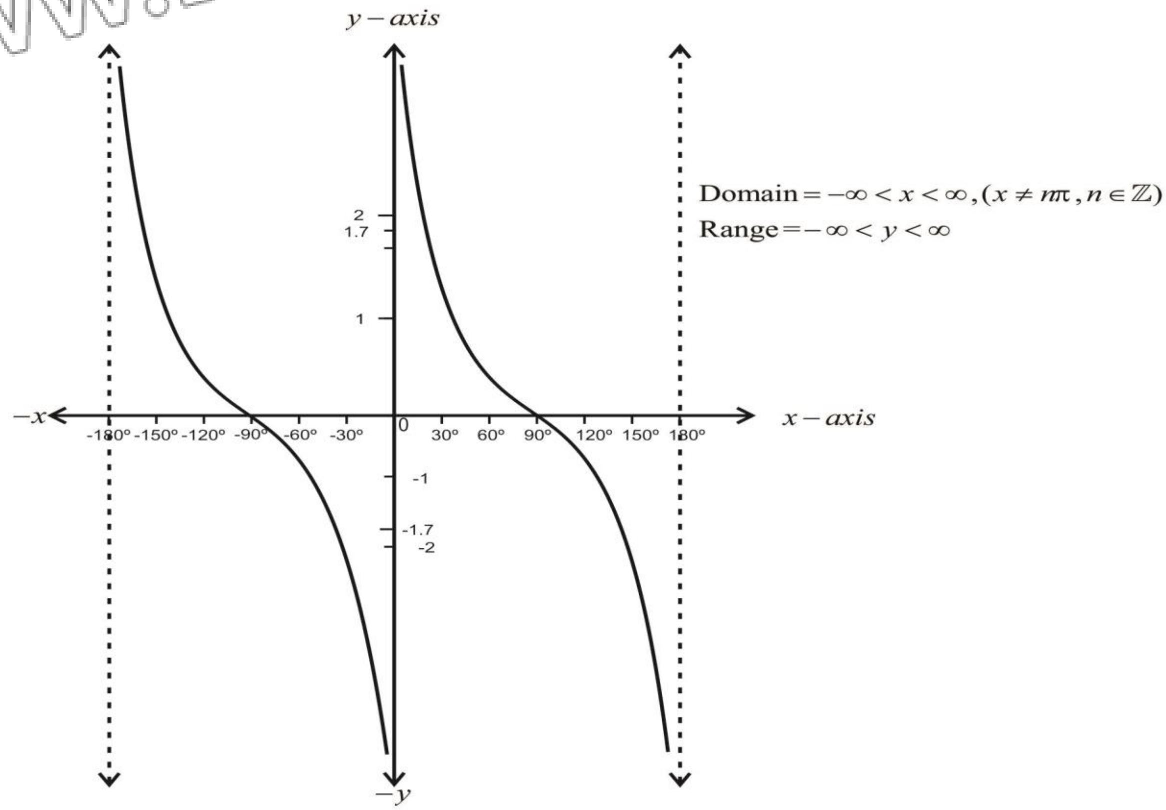


Fig 10

Graph of $y = \cot x$

Note:

We notice that every horizontal line intersects the graph infinitely many times. It follows that cotangent function is not one-to-one.

Principal Cotangent Function:

If we restrict the domain of $y = \cot x$ to the interval $(0, \pi)$.

Then the restricted function $y = \cot x$, $0 < x < \pi$ and $-\infty < y < \infty$ is called the principal cotangent function.

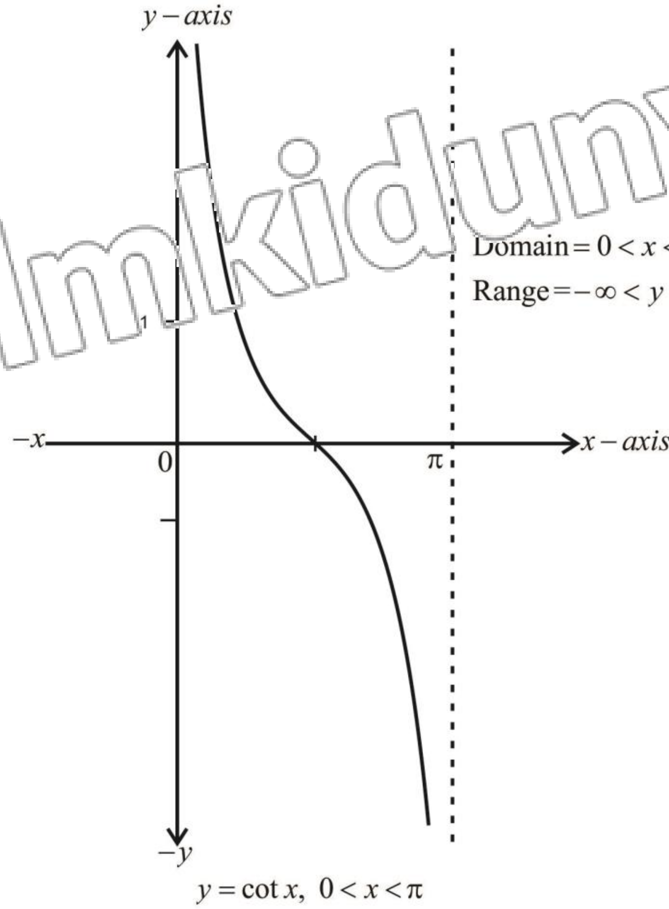


Fig 11
Graph of principal cotangent function

Note:

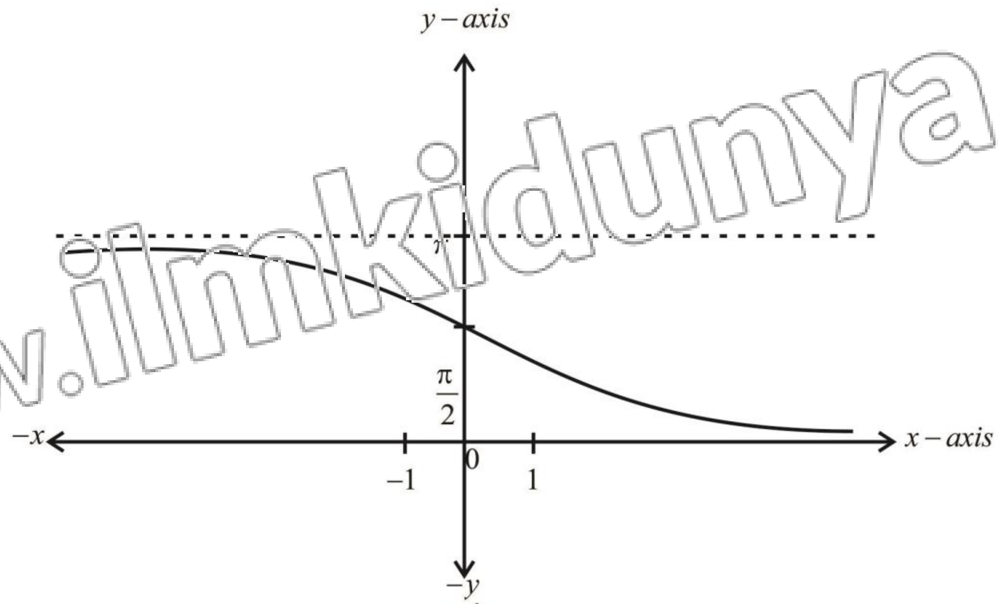
The principal cotangent function is now one-to-one and hence will have an inverse as shown in figure (11)

Inverse Cotangent Function:

The inverse cotangent function is defined by $y=\cot^{-1}x$ if and only if $x=\cot y$ where $-\infty < x < \infty$ and $0 < y < \pi$.

Note:

The graph of $y=\cot^{-1}x$ is obtained by reflecting the restricted portion of the graph of $y=\cot x$ about the line $y=x$ as shown in the figure (12)



$y = \cot^{-1} x$

Domain= $(-\infty, \infty)$

Range $= \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

Fig 12

Graph of $y = \cot^{-1} x$

Note:

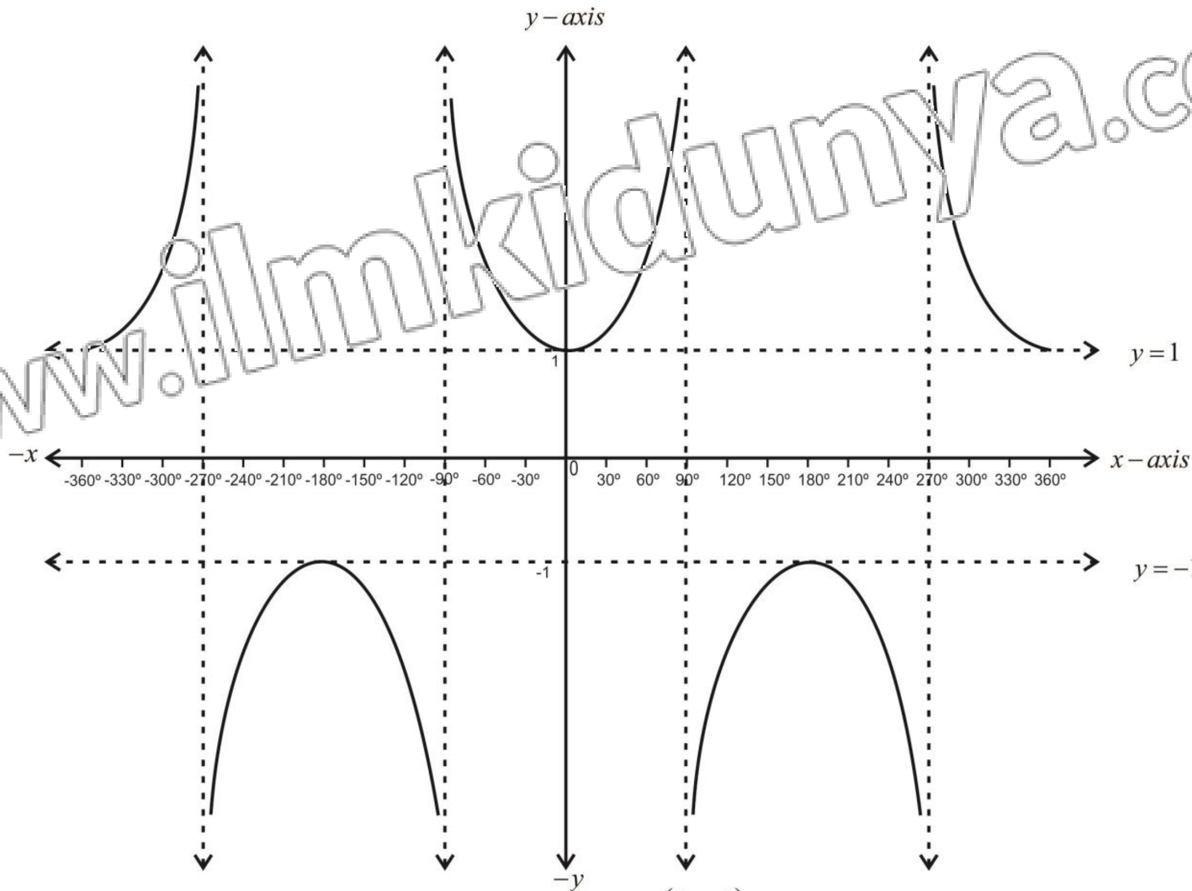
We notice that the graph of $y = \cot x$ is along $y - axis$ where as the graph of $y = \tan^{-1} x$ is along $x - axis$.

Secant Function:

The secant function is defined by as $y = \sec x$ where its domain is $-\infty < x < \infty$, $\left\{x \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}\right\}$ and its range is $y \leq -1$ or $y \geq 1$.

Note:

The graph of $y = \sec x$, $-\infty < x < \infty \left\{x \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}\right\}$ is shown in the figure (13)



Domain= $-\infty < x < +\infty, x \neq \frac{(2n+1)\pi}{2}, n \in Z$

Range= $y \geq 1$ or $y \leq -1$

Fig 13

Graph of $y = \sec x$

Note:

We notice that every horizontal line except, between the lines $y = 1$ and $y = -1$ intersects the graph infinitely many times. It follows that secant function is not one-to-one.

Principal Secant Function:

If we restrict the domain of $y = \sec x$ to the interval $[0, \pi]$, $x \neq \frac{\pi}{2}$ then the restricted function $y = \sec x$, $0 \leq x \leq \pi$ where $(x \neq \frac{\pi}{2})$ and $y \leq 1$ or $y \geq -1$ is called the principal secant function.

Note:

The principal secant function is now one-to-one and hence will have an inverse as shown is figure (14)

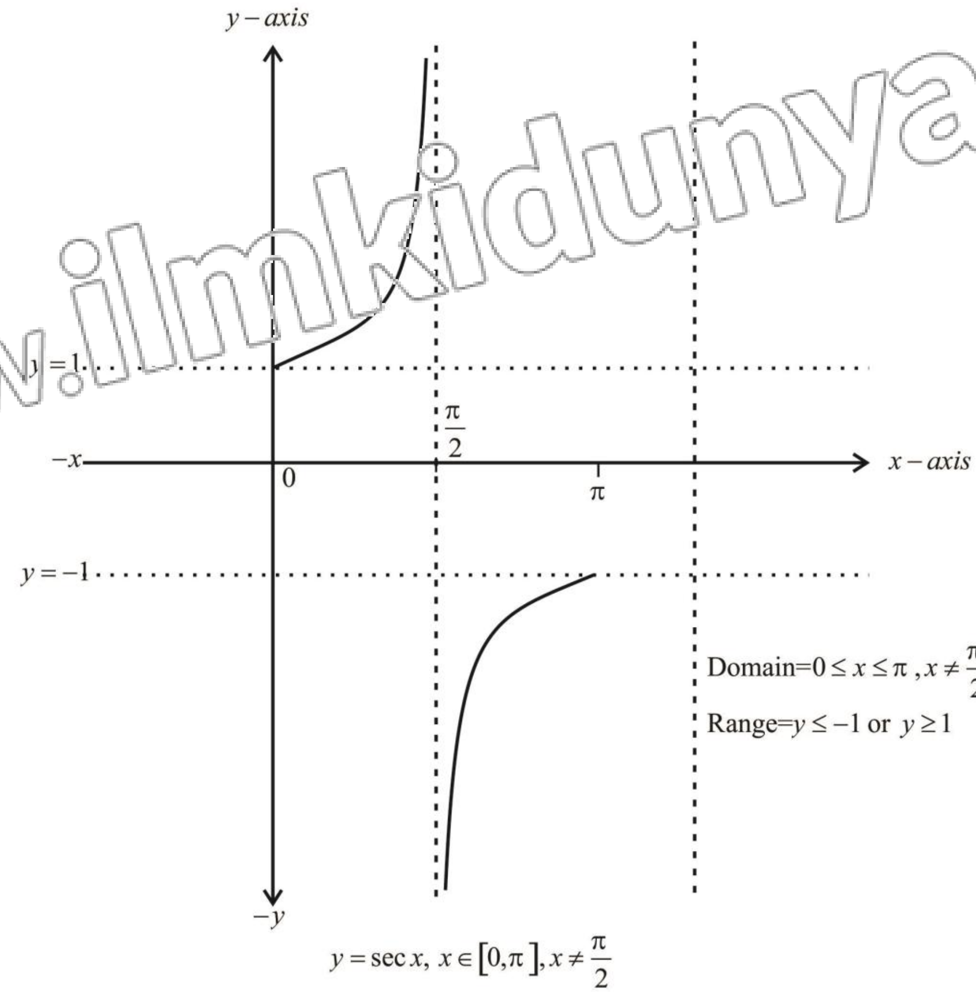


Fig 14

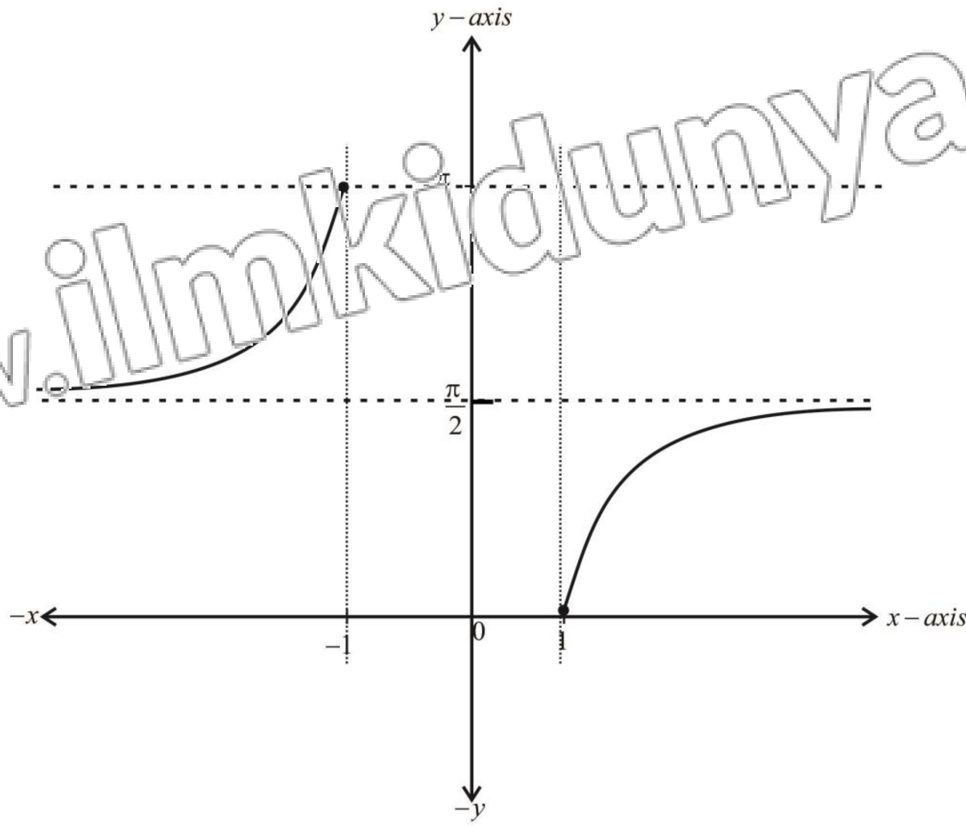
Graph of principal secant function

Inverse Secant Function:

The inverse secant function is defined by $y = \sec^{-1} x$ if and only if $x = \sec y$ where $x \le -1$ or $x \ge 1$ and $0 \le y \le \pi,\ y \ne \dfrac{\pi}{2}$.

Note:

The graph of $y = \sec^{-1} x$ is obtained by reflecting the restricted portion of the graph of $y = \sec x$ about the line $y = x$ as shown in the figure (15)



$y = \sec^{-1} x$

Domain: $x \in (-\infty, -1] \cup [1, \infty)$

Range: $y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, $y \neq \frac{\pi}{2}$

Fig 15

Graph of $y = \sec^{-1} x$

Note:

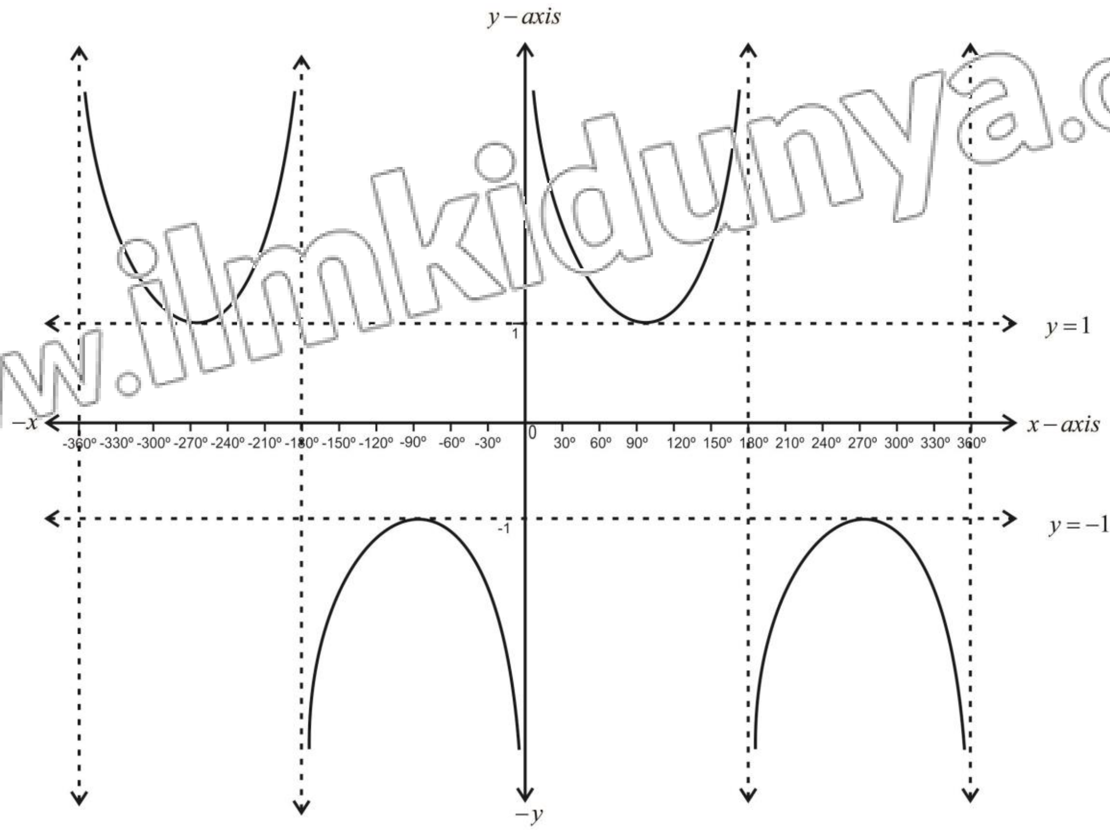
We notice that the graph of $y = \sec x$ is along $y - axis$ where as the graph of $y = \sec^{-1} x$ is along $x - axis$.

Cosecant Function:

The cosecant function is defined by as $y = \csc x$ where its domain is $-\infty < x < \infty$ $(x \neq n\pi, n \in \mathbb{Z})$ and its range is $y \leq -1$ or $y \geq 1$.

Note:

The graph of $y = \csc x$, $-\infty < x < \infty$, $(x \neq n\pi,\ n \in \mathbb{Z})$ is shown in the figure (16)



Domain= $-\infty < x < +\infty, x \neq n\pi, n \in Z$

Range= $y \geq 1$ or $y \leq -1$

Fig 16

Graph of $y = \text{cosec}\, x$

Note:

We notice that every horizontal line except, between the lines $y = 1$ and $y = -1$ intersects the graph infinitely many times. It follows that cosecant function is not one-to-one.

Principal Cosecant Function:

If we restrict the domain of $y = \csc x$ to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right], x \neq 0$ then the restricted function $y = \sec x,\ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\ (x \neq 0)$ and $y \leq 1$ or $y \geq -1$ is called the principal cosecant function. Its graph is shown in Figure 17.

Domain: $-\frac{\pi}{2} \le x \le \frac{\pi}{2}, x \neq 0$

Range: $y \le -1$ or $y \ge 1$

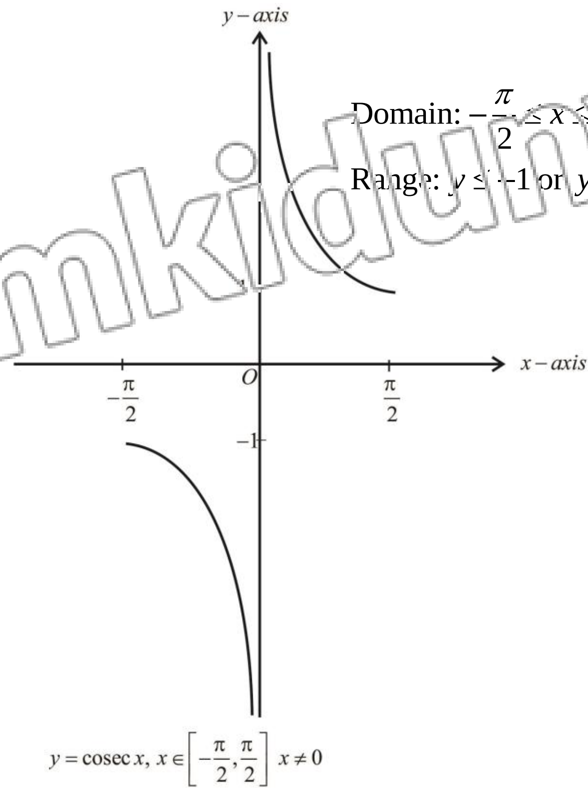


Fig 17

Graph of principal cosecant function

Note:

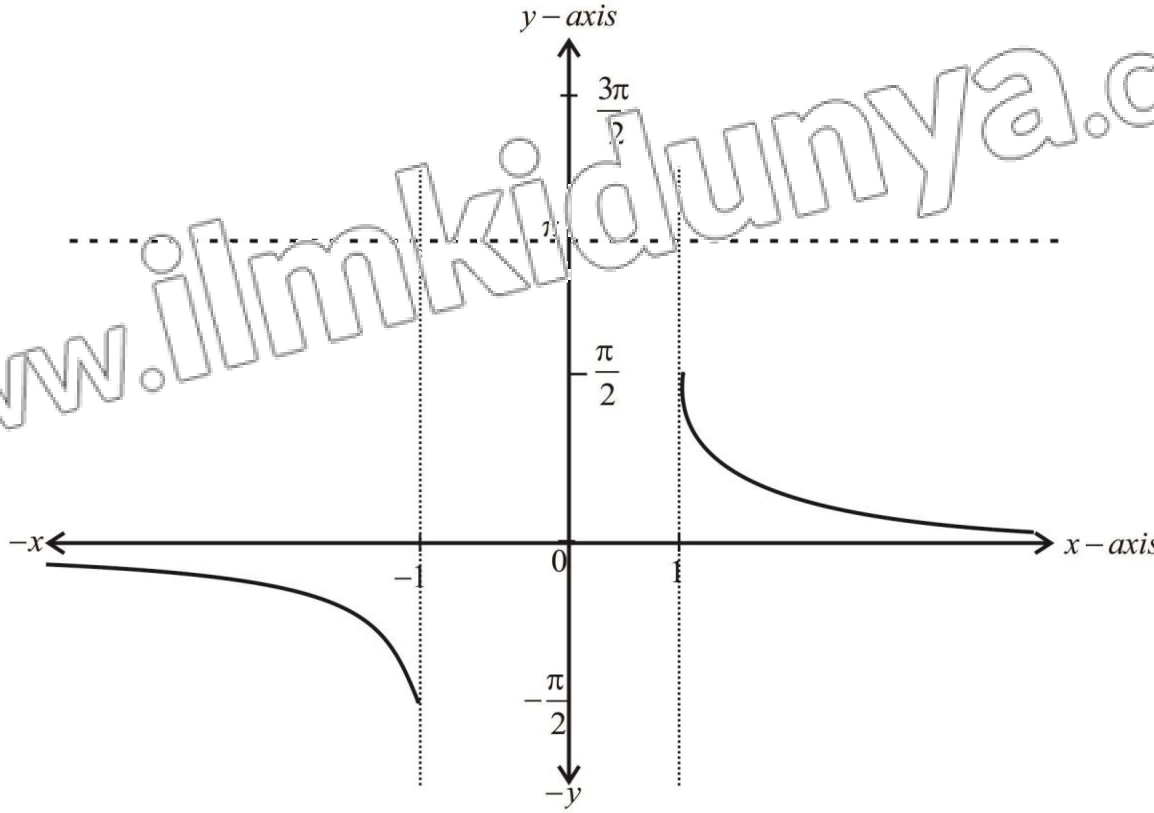
The principal secant function is now one-to-one and hence will have an inverse as shown in figure (17)

Inverse Cosecant Function:

The inverse cosecant function is defined by $y = \csc^{-1} x$ if and only if $x = \csc y$ where $x \le -1$ or $x \ge 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}, \quad y \neq 0$

Note:

The graph of $y = \csc^{-1} x$ is obtained by reflecting the restricted portion of the graph of $y = \csc x$ about the line $y = x$ as shown in the figure (13)



Domain: $x \in (-\infty, -1] \cup [1, \infty)$

Range: $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

Fig 18

Graph of $y = \text{cosec}^{-1} x$

Note:

We notice that the graph of $y = \csc x$ is along $y-axis$ where as the graph of $y = \csc^{-1} x$ is along $x-axis$.

Note:

It must be remembered that

$\sin^{-1} x \neq (\sin x)^{-1}$

$\cos^{-1} x \neq (\cos x)^{-1}$

$\tan^{-1} x \neq (\tan x)^{-1}$

$\cot^{-1} x \neq (\cot x)^{-1}$

$\sec^{-1} x \neq (\sec x)^{-1}$

$\csc^{-1} x \neq (\csc x)^{-1}$

Note:

While discussing the inverse trigonometric functions, we have seen in general, inverses of trigonometric functions does not exist, but restricting their domains to principal functions, we have made them as one-to-one functions and hence they will have inverse functions.

Domains and Ranges of Principal Trigonometric Functions:

Principal Trigonometric Function	Domain	Range
$y=\sin x$	$-\frac{\pi}{2}\le x\le\frac{\pi}{2}$	$-1\le y\le 1$
$y=\cos x$	$0\le x\le\pi$	$-1\le y\le 1$
$y=\tan x$	$-\frac{\pi}{2}< x<\frac{\pi}{2}$	$-\infty< y<\infty$
$y=\cot x$	$0< x<\pi$	$-\infty< y<\infty$
$y=\sec x$	$0\le x\le\pi,\ x\ne\frac{\pi}{2}$	$y\le -1\ or\ \ y\ge 1$
$y=\csc x$	$-\frac{\pi}{2}\le x\le\frac{\pi}{2},\ x\ne 0$	$y\le -1\ or\ \ y\ge 1$

Domains and Ranges of Inverse Trigonometric Functions:

Inverse Trigonometric Function	Domain	Range
$y=\sin^{-1}x$	$-1\le x\le 1$	$-\frac{\pi}{2}\le y\le\frac{\pi}{2}$
$y=\cos^{-1}x$	$-1\le x\le 1$	$0\le y\le\pi$
$y=\tan^{-1}x$	$-\infty< x<\infty$	$-\frac{\pi}{2}< y<\frac{\pi}{2}$
$y=\cot^{-1}x$	$-\infty< x<\infty$	$0< y<\pi$
$y=\sec^{-1}x$	$x\le -1\ or\ \ x\ge 1$	$0\le y\le\pi,\ y\ne\frac{\pi}{2}$
$y=\csc^{-1}x$	$x\le -1\ or\ \ x\ge 1$	$-\frac{\pi}{2}\le y\le\frac{\pi}{2},\ y\ne 0$

EXERCISE 13.1

Q.1 Evaluate without using tables/calculator.

(i) $\sin^{-1}(1)$

Solution:

Let $\sin^{-1}(1) = y$

$\Rightarrow \quad \sin y = 1 \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\Rightarrow \quad y = \frac{\pi}{2}$

$$\boxed{\sin^{-1}(1) = \frac{\pi}{2}}$$

(ii) $\sin^{-1}(-1)$

Solution:

Let $\sin^{-1}(-1) = y$

$\Rightarrow \quad \sin y = -1 \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\Rightarrow \quad y = -\frac{\pi}{2}$

$$\boxed{\sin^{-1}(-1) = -\frac{\pi}{2}}$$

(iii) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Solution:

Let $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = y$

$\Rightarrow \quad \cos y = \frac{\sqrt{3}}{2} \quad 0 \le y \le \pi$

$\Rightarrow \quad y = \frac{\pi}{6}$

$$\boxed{\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}}$$

(iv) $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

Solution:

Let $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = y$

$\Rightarrow \quad \tan y = \frac{-1}{\sqrt{3}}$

$-\frac{\pi}{2} < y < \frac{\pi}{2}$

$\Rightarrow \quad y = -\frac{\pi}{6}$

$$\boxed{\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6}}$$

(v) $\cos^{-1}\left(\frac{1}{2}\right)$

Solution:

Let $\cos^{-1}\left(\frac{1}{2}\right) = y$

$\Rightarrow \quad \cos y = \frac{1}{2} \quad 0 \le y \le \pi$

$\Rightarrow \quad y = \frac{\pi}{3}$

$$\boxed{\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}}$$

(Note: Answer of this part is wrong in the book)

(vi) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Solution:

Let $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = y$

$\Rightarrow \quad \tan y = \frac{1}{\sqrt{3}} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$

$\Rightarrow y = \frac{\pi}{6}$

$$\boxed{\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}}$$

(vii) $\cot^{-1}(-1)$

Solution:

Let $\cot^{-1}(-1) = y$

$\Rightarrow \cot y = -1 \quad 0 < y < \pi$

$\Rightarrow y = \frac{3\pi}{4}$

$$\boxed{\cot^{-1}(-1) = \frac{3\pi}{4}}$$

(viii) $\operatorname{cosec}^{-1}\left(\frac{-2}{\sqrt{3}}\right)$

Solution:

Let $\operatorname{cosec}^{-1}\left(\frac{-2}{\sqrt{3}}\right) = y$

$\Rightarrow \operatorname{cosec} y = \frac{-2}{\sqrt{3}}$

$-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$

$\Rightarrow y = \frac{-\pi}{3}$

$$\boxed{\operatorname{cosec}^{-1}\left(\frac{-2}{\sqrt{3}}\right) = \frac{-\pi}{3}}$$

(ix) $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

Solution:

Let $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right) = y$

$\Rightarrow \sin y = \frac{-1}{\sqrt{2}} \quad \frac{-\pi}{2} \le y \le \frac{\pi}{2}$

$\Rightarrow y = \frac{-\pi}{4}$

$$\boxed{\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right) = -\frac{\pi}{4}}$$

Q.2 Without using table/calculator show that:

(i) $\tan^{-1}\frac{5}{12} = \sin^{-1}\frac{5}{13}$

Solution:

Let $\alpha = \tan^{-1}\frac{5}{12} \Rightarrow \tan\alpha = \frac{5}{12}$

Using Pythagoras theorem

$(x)^2 = (12)^2 + (5)^2$

$x^2 = 144 + 25$

$x^2 = 169$

$x = 13$

So, $\sin\alpha = \frac{5}{13} \Rightarrow \alpha = \sin^{-1}\frac{5}{13}$

$\tan^{-1}\frac{5}{12} = \sin^{-1}\frac{5}{13}$

$$\boxed{\text{Hence : } \tan^{-1}\frac{5}{12} = \sin^{-1}\frac{5}{13}}$$

(ii) $2\cos^{-1}\frac{4}{5} = \sin^{-1}\frac{24}{25}$

Solution:

Let $\alpha = \cos^{-1}\frac{4}{5} \Rightarrow \cos\alpha = \frac{4}{5}$

Using Pythagoras theorem

$(5)^2 = (x)^2 + (4)^2$

$x^2 = 25 - 16$

$x^2 = 9$

$x = 3$

So, $\sin\alpha = \frac{3}{5}$

Using double angle identity.

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$$\sin 2\alpha = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right)$$

$$\sin 2\alpha = \frac{24}{25}$$

$$2\alpha = \sin^{-1}\frac{24}{25}$$

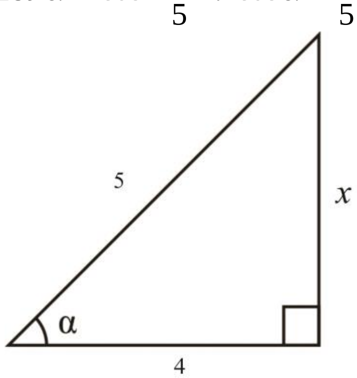
$$2\cos^{-1}\frac{4}{5} = \sin^{-1}\frac{24}{25}$$

$$\boxed{\text{Hence : } 2\cos^{-1}\frac{4}{5} = \sin^{-1}\frac{24}{25}}$$

(iii) $\mathbf{\cos^{-1}\frac{4}{5} = \cot^{-1}\frac{4}{3}}$

Solution:

Let $\alpha = \cos^{-1}\frac{4}{5} \Rightarrow \cos\alpha = \frac{4}{5}$



Using Pythagoras theorem

$$(5)^2 = x^2 + (4)^2$$
$$25 = x^2 + 16$$
$$x^2 = 25 - 16$$
$$x^2 = 9$$
$$x = 3$$

So, $\cot\alpha = \frac{4}{3} \quad \Rightarrow \quad \alpha = \cot^{-1}\frac{4}{3}$

$$\cos^{-1}\frac{4}{5} = \cot^{-1}\frac{4}{3}$$

$$\boxed{\text{Hence : } \cos^{-1}\frac{4}{5} = \cot^{-1}\frac{4}{3}}$$

Q.3 Find the value of each expression:

(i) $\mathbf{\cos\left(\sin^{-1}\frac{1}{\sqrt{2}}\right)}$

Solution:

Let $\sin^{-1}\frac{1}{\sqrt{2}} = y$

$\Rightarrow \sin y = \frac{1}{\sqrt{2}} \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\Rightarrow \qquad y = \frac{\pi}{4}$

So, $\sin^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$

$\therefore \cos\left(\sin^{-1}\frac{1}{\sqrt{2}}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$\boxed{\cos\left(\sin^{-1}\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}}$$

(ii) $\mathbf{\sec\left(\cos^{-1}\frac{1}{2}\right)}$

Solution:

Let $\cos^{-1}\frac{1}{2} = y$

$\Rightarrow \cos y = \frac{1}{2} \qquad 0 \le y \le \pi$

$\Rightarrow \qquad y = \frac{\pi}{3}$

So, $\cos^{-1}\frac{1}{2} = \frac{\pi}{3}$

$\therefore \sec\left(\cos^{-1}\frac{1}{2}\right) = \sec\frac{\pi}{3} = 2$

$$\boxed{\sec\left(\cos^{-1}\frac{1}{2}\right) = 2}$$

(iii) $\mathbf{\tan\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)}$

Solution:

Let $\cos^{-1}\frac{\sqrt{3}}{2} = y$

$\Rightarrow \cos y = \frac{\sqrt{3}}{2} \quad 0 \le y \le \pi$

$\Rightarrow y = \frac{\pi}{6}$

So, $\cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$

$\therefore \tan\left(\cos^{-1}\frac{\sqrt{3}}{2}\right) = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$\boxed{\tan\left(\cos^{-1}\frac{\sqrt{3}}{2}\right) = \frac{1}{\sqrt{3}}}$

(iv) $\mathbf{cosec\left(\tan^{-1}(-1)\right)}$

Solution:

Let $\tan^{-1}(-1) = y$

$\Rightarrow \tan y = -1 \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$

$\Rightarrow \quad y = \frac{-\pi}{4}$

So, $\tan^{-1}(-1) = \frac{-\pi}{4}$

$\therefore \operatorname{cosec}\left(\tan^{-1}(-1)\right)$

$= \operatorname{cosec}\left(\frac{-\pi}{4}\right) = -\sqrt{2}$

$\boxed{\operatorname{cosec}\left(\tan^{-1}(-1)\right) = -\sqrt{2}}$

(Note: Answer of this part is wrong in the Book)

(v) $\mathbf{\sec\left[\sin^{-1}\left(\frac{-1}{2}\right)\right]}$

Solution:

Let $\sin^{-1}\left(\frac{-1}{2}\right) = y$

$\Rightarrow \sin y = \frac{-1}{2} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\Rightarrow \quad y = \frac{-\pi}{6}$

So, $\sin^{-1}\left(\frac{-1}{2}\right) = \frac{-\pi}{6}$

$\therefore \sec\left[\sin^{-1}\left(\frac{-1}{2}\right)\right] = \sec\left(\frac{-\pi}{6}\right)$

$\sec\left[\sin^{-1}\left(\frac{-1}{2}\right)\right] = \sec\left(\frac{\pi}{6}\right)$

$\boxed{\sec\left[\sin^{-1}\left(\frac{-1}{2}\right)\right] = \frac{2}{\sqrt{3}}}$

(vi) $\mathbf{\tan\left(\tan^{-1}(-1)\right)}$

Solution:

Let $\tan^{-1}(-1) = y$

$\Rightarrow \tan y = -1 \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$

$\Rightarrow \quad y = \frac{-\pi}{4}$

So, $\tan^{-1}(-1) = \frac{-\pi}{4}$

$\therefore \tan\left(\tan^{-1}(-1)\right)$

$= \tan\left(\frac{-\pi}{4}\right) = -1$

$\boxed{\tan\left(\tan^{-1}(-1)\right) = -1}$

(vii) $\mathbf{\sin\left(\sin^{-1}\left(\frac{1}{2}\right)\right)}$

Solution:

Let $\sin^{-1}\frac{1}{2} = y$

$\Rightarrow \sin y = \frac{1}{2} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\Rightarrow y = \frac{\pi}{6}$

So, $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$$\therefore \sin\left(\sin^{-1}\left(\frac{1}{2}\right)\right) = \sin\frac{\pi}{6} = \frac{1}{2}$$

$$\boxed{\sin\left(\sin^{-1}\left(\frac{1}{2}\right)\right) = \frac{1}{2}}$$

(viii) $\tan\left[\sin^{-1}\left(\frac{-1}{2}\right)\right]$

Solution:

Let $\sin^{-1}\left(\frac{-1}{2}\right) = y$

$\Rightarrow \sin y = \frac{-1}{2} \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\Rightarrow \quad y = \frac{-\pi}{6}$

So, $\sin^{-1}\left(\frac{-1}{2}\right) = \frac{-\pi}{6}$

$$\therefore \tan\left[\sin^{-1}\left(\frac{-1}{2}\right)\right] = \tan\left[\frac{-\pi}{6}\right] - \frac{1}{\sqrt{3}}$$

$$\boxed{\tan\left[\sin^{-1}\left(\frac{-1}{2}\right)\right] = \frac{-1}{\sqrt{3}}}$$

(ix) $\sin\left[\tan^{-1}(-1)\right]$

Solution:

Let $\tan^{-1}(-1) = y$

$\Rightarrow \tan y = -1 \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$

$\Rightarrow y = \frac{-\pi}{4}$

So, $\tan^{-1}(-1) = \frac{-\pi}{4}$

$$\therefore \sin\left[\tan^{-1}(-1)\right] = \sin\left[\frac{-\pi}{4}\right] = -\frac{1}{\sqrt{2}}$$

$$\boxed{\sin\left[\tan^{-1}(-1)\right] = \frac{-1}{\sqrt{2}}}$$

Addition and Subtraction Formulas:

1. Prove that: $\mathbf{\sin^{-1}A + \sin^{-1}B = \sin^{-1}\left(A\sqrt{1-B^2} + B\sqrt{1-A^2}\right)}$

Proof:

Let $x = \sin^{-1}A \Rightarrow \sin x = A$

and $y = \sin^{-1}B \Rightarrow \sin y = B$

Now $\cos x = \pm\sqrt{1-\sin^2 x} = \pm\sqrt{1-A^2}$

$\cos x = \sqrt{1-A^2} \because \left\{\text{For } \sin x = A\text{, domain} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\text{, in which cosine is +ve}\right.$

Similarly $\cos y = \sqrt{1-B^2}$

Now $\sin(x+y) = \sin x\cos y + \cos x\sin y$

$= A\sqrt{1-B^2} + \sqrt{1-A^2}.B$

$\Rightarrow x+y = \sin^{-1}\left(A\sqrt{1-B^2} + B\sqrt{1-A^2}\right)$

$$\therefore \boxed{\sin^{-1}A + \sin^{-1}B = \sin^{-1}\left(A\sqrt{1-B^2} + B\sqrt{1-A^2}\right)}$$

2. Prove that: $\mathbf{\sin^{-1}A - \sin^{-1}B = \sin^{-1}\left(A\sqrt{1-B^2} - B\sqrt{1-A^2}\right)}$

Proof:

Let $x = \sin^{-1} A \;\Rightarrow\; \sin x = A$

and $y = \sin^{-1} B \;\Rightarrow\; \sin y = B$

Now $\cos x = \pm\sqrt{1-\sin^2 x} = \pm\sqrt{1-A^2}$

$\cos x = \sqrt{1-A^2}$ $\because \left\{ \text{For } \sin x = A \text{, domain} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{, in which cosine is } +ve \right.$

Similarly $\cos y = \sqrt{1-B^2}$

Now $\sin(x-y) = \sin x \cos y - \cos x \sin y$

$= A\sqrt{1-B^2} - \sqrt{1-A^2}.B$

$\Rightarrow\; x - y = \sin^{-1}\left(A\sqrt{1-B^2} - B\sqrt{1-A^2}\right)$

$\therefore$ $\boxed{\sin^{-1} A - \sin^{-1} B = \sin^{-1}\left(A\sqrt{1-B^2} - B\sqrt{1-A^2}\right)}$

3. Prove that: $\mathbf{\cos^{-1}A + \cos^{-1}B = \cos^{-1}\left(AB - \sqrt{(1-A^2)(1-B^2)}\right)}$

Proof:

Let $x = \cos^{-1} A \;\Rightarrow\; \cos x = A$

and $y = \cos^{-1} B \;\Rightarrow\; \cos y = B$

Now $\sin x = \pm\sqrt{1-\cos^2 x} = \pm\sqrt{1-A^2}$

$\sin x = \sqrt{1-A^2}$ $\because \left\{ \text{For } \cos x = A \text{, domain} = [0, \pi] \text{, in which sine is } +ve \right.$

Similarly, $\sin y = \sqrt{1-B^2}$

Now $\cos(x+y) = \cos x \cos y - \sin x \sin y$

$= A.B - \sqrt{1-A^2}\,.\sqrt{1-B^2}$

$\Rightarrow x + y = \cos^{-1}\left(AB - \sqrt{(1-A^2)(1-B^2)}\right)$

$\therefore$ $\boxed{\cos^{-1} A + \cos^{-1} B = \cos^{-1}\left(AB - \sqrt{(1-A^2)(1-B^2)}\right)}$

4. Prove that: $\mathbf{\cos^{-1}A - \cos^{-1}B = \cos^{-1}\left(AB + \sqrt{(1-A^2)(1-B^2)}\right)}$

Proof:

Let $x = \cos^{-1} A \Rightarrow \cos x = A$

and $y = \cos^{-1} B \Rightarrow \cos y = B$

Now $\sin x = \pm\sqrt{1-\cos^2 x} = \pm\sqrt{1-A^2}$

$\sin x = \sqrt{1-A^2}$ $\because \{$For $\cos x = A$, domain $= [0, \pi]$, in which sine is $+ve$

Similarly, $\sin y = \sqrt{1-B^2}$

Now $\cos(x-y) = \cos x \cos y + \sin x \sin y$

$= A.B + \sqrt{1-A^2}\,.\sqrt{1-B^2}$

$\Rightarrow x - y = \cos^{-1}\left(AB + \sqrt{(1-A^2)(1-B^2)}\right)$

$\therefore$ $\boxed{\cos^{-1} A - \cos^{-1} B = \cos^{-1}\left(AB + \sqrt{(1-A^2)(1-B^2)}\right)}$

5. Prove that: $\mathbf{\tan^{-1}A + \tan^{-1}B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)}$

Proof:

Let $x = \tan^{-1} A \Rightarrow \tan x = A$

and $y = \tan^{-1} B \Rightarrow \tan y = B$

Now $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{A+B}{1-A.B}$

$\Rightarrow x + y = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$

$\therefore$ $\boxed{\tan^{-1} A + \tan^{-1} B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)}$

6. Prove that: $\mathbf{\tan^{-1}A - \tan^{-1}B = \tan^{-1}\left(\frac{A-B}{1+AB}\right)}$

Proof:

Let $x = \tan^{-1} A \Rightarrow \tan x = A$

and $y = \tan^{-1} B \Rightarrow \tan y = B$

Now $\tan(x-y) = \dfrac{\tan x - \tan y}{1+\tan x \tan y} = \dfrac{A-B}{1+A.B}$

$$\Rightarrow x - y = \tan^{-1}\left(\frac{A-B}{1+AB}\right)$$

$\therefore$ $\boxed{\tan^{-1} A - \tan^{-1} B = \tan^{-1}\left(\frac{A-B}{1+AB}\right)}$

7. Prove that: $\mathbf{2\tan^{-1}A = \tan^{-1}\left(\frac{2A}{1-A^2}\right)}$

Proof:

$$\because \tan^{-1} A + \tan^{-1} B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$$

Put $B = A$

$$\tan^{-1} A + \tan^{-1} A = \tan^{-1}\left(\frac{A+A}{1-A.A}\right)$$

$$\boxed{2\tan^{-1} A = \tan^{-1}\left(\frac{2A}{1-A^2}\right)}$$

EXERCISE 13.2

Prove the following:

Q.1 $\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{7}{25}=\cos^{-1}\frac{253}{325}$

Solution:

$\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{7}{25}$

$\because \sin^{-1}A+\sin^{-1}B$

$=\sin^{-1}\left[A\sqrt{1-B^2}+B\sqrt{1-A^2}\right]$

$$\Rightarrow \quad =\sin^{-1}\left[\frac{5}{13}\sqrt{1-\left(\frac{7}{25}\right)^2}+\frac{7}{25}\sqrt{1-\left(\frac{5}{13}\right)^2}\right]$$

$$=\sin^{-1}\left[\frac{5}{13}\sqrt{1-\frac{49}{625}}+\frac{7}{25}\sqrt{1-\frac{25}{169}}\right]$$

$$=\sin^{-1}\left[\frac{5}{13}\sqrt{\frac{625-49}{625}}+\frac{7}{25}\sqrt{\frac{169-25}{169}}\right]$$

$$=\sin^{-1}\left[\frac{5}{13}\sqrt{\frac{576}{625}}+\frac{7}{25}\sqrt{\frac{144}{169}}\right]$$

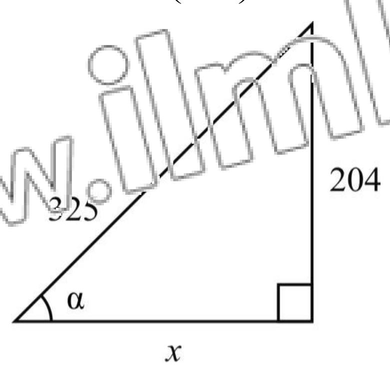
$$=\sin^{-1}\left[\frac{5}{13}\times\frac{24}{25}+\frac{7}{25}\times\frac{12}{13}\right]$$

$$=\sin^{-1}\left[\frac{120}{325}+\frac{84}{325}\right]$$

$$=\sin^{-1}\left[\frac{120+84}{325}\right]$$

$$=\sin^{-1}\left[\frac{204}{325}\right]$$

Let $\alpha=\sin^{-1}\left(\frac{204}{325}\right) \Rightarrow \sin\alpha=\frac{204}{325}$



Using According to Pythagoras theorem

$(325)^2=(x)^2+(204)^2$

$105,625=x^2+41,616$

$x^2=105,625-41,616$

$x^2=64,009$

$x=253$

So, $\cos\alpha=\frac{253}{325} \Rightarrow \alpha=\cos^{-1}\left(\frac{253}{325}\right)$

$\sin^{-1}\left(\frac{204}{325}\right)=\cos^{-1}\left(\frac{253}{325}\right)$

Hence : $\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{7}{25}$
$=\sin^{-1}\frac{204}{325}=\cos^{-1}\frac{253}{325}$

Q.2 $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{9}{19}$

Solution:

$$\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{5}=\tan^{-1}\left[\frac{\frac{1}{4}+\frac{1}{5}}{1-\left(\frac{1}{4}\right)\left(\frac{1}{5}\right)}\right]$$

$\because \tan^{-1}A+\tan^{-1}B=\tan^{-1}\left(\frac{A+B}{1-AB}\right)$

$$=\tan^{-1}\left[\frac{\frac{5+4}{20}}{\frac{20-1}{20}}\right]$$

$$=\tan^{-1}\left(\frac{9}{19}\right)$$

Hence : $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{9}{19}$

Q.3 $\mathbf{2\tan^{-1}\frac{2}{3} = \sin^{-1}\frac{12}{13}}$

Solution:

$$2\tan^{-1}\frac{2}{3} = \tan^{-1}\frac{2\left(\frac{2}{3}\right)}{1-\left(\frac{2}{3}\right)^2}$$

$$\because 2\tan^{-1}A = \tan^{-1}\left(\frac{2A}{1-A^2}\right)$$

$$= \tan^{-1}\frac{\frac{4}{3}}{1-\frac{4}{9}}$$

$$= \tan^{-1}\frac{\frac{4}{3}}{\frac{9-4}{9}}$$

$$= \tan^{-1}\frac{\frac{4}{3}}{\frac{5}{9}}$$

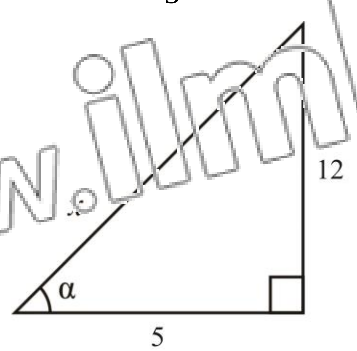
$$= \tan^{-1}\frac{4}{3}\times\frac{9}{5}$$

$$2\tan^{-1}\frac{2}{3} = \tan^{-1}\frac{12}{5}$$

Now we are to prove that

$$\tan^{-1}\frac{12}{5} = \sin^{-1}\frac{12}{13}$$

Let $\alpha = \tan^{-1}\frac{12}{5} \Rightarrow \tan\alpha = \frac{12}{5}$



Using Pythagoras theorem

$$x^2 = (12)^2 + (5)^2$$

$$x^2 = 144 + 25$$

$$x^2 = 169$$

$$x = 13$$

So, $\sin\alpha = \frac{12}{13} \Rightarrow \alpha = \sin^{-1}\frac{12}{13}$

$$\tan^{-1}\frac{12}{5} = \sin^{-1}\frac{12}{13}$$

$$\boxed{\text{Hence}: 2\tan^{-1}\frac{2}{3} = \tan^{-1}\frac{12}{5} = \sin^{-1}\frac{12}{13}}$$

Q.4 $\mathbf{\tan^{-1}\frac{120}{119} = 2\cos^{-1}\frac{12}{13}}$

Solution:

Let $\alpha = \cos^{-1}\frac{12}{13} \Rightarrow \cos\alpha = \frac{12}{13}$

Using Pythagoras theorem

$$(13)^2 = (12)^2 + x^2$$

$$x^2 = 169 - 144$$

$$x^2 = 25$$

$$x = 5$$

So, $\tan\alpha = \frac{5}{12}$

Using double angle identity of tangent

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$= \frac{2\left(\frac{5}{12}\right)}{1-\left(\frac{5}{12}\right)^2} = \frac{\frac{5}{6}}{1-\frac{25}{144}}$$

$$= \frac{\frac{5}{6}}{\frac{144-25}{144}} = \frac{\frac{5}{6}}{\frac{119}{144}}$$

$$= \frac{5}{6}\times\frac{144}{119}$$

$$\tan 2\alpha = \frac{120}{119} \Rightarrow 2\alpha = \tan^{-1}\frac{120}{119}$$

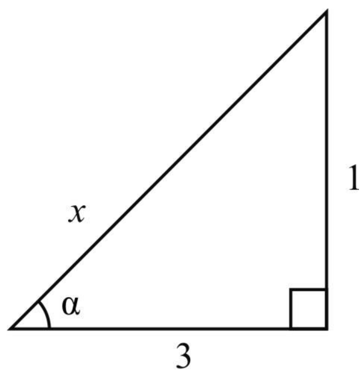
$$\text{Hence : } 2\cos^{-1}\frac{12}{13} = \tan^{-1}\frac{120}{119}$$

Q.5 $\sin^{-1}\frac{1}{\sqrt{5}} + \cot^{-1}3 = \frac{\pi}{4}$

Solution:

L.H.S

Let $\alpha = \cot^{-1}3 \Rightarrow \cot\alpha = 3$



Using Pythagoras theorem

$$x^2 = (1)^2 + (3)^2$$

$$x^2 = 10$$

$$x = \sqrt{10}$$

$$\sin\alpha = \frac{1}{\sqrt{10}} \quad \Rightarrow \alpha = \sin^{-1}\frac{1}{\sqrt{10}}$$

$$\therefore \cot^{-1}3 = \sin^{-1}\frac{1}{\sqrt{10}}$$

Now L.H.S becomes

$$\sin^{-1}\frac{1}{\sqrt{5}} + \cot^{-1}3$$

$$= \sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{10}}$$

$$\because \sin^{-1}A + \sin^{-1}B = \sin^{-1}\left[A\sqrt{1-B^2} + B\sqrt{1-A^2}\right]$$

$$= \sin^{-1}\left[\frac{1}{\sqrt{5}}\sqrt{1-\left(\frac{1}{\sqrt{10}}\right)^2} + \frac{1}{\sqrt{10}}\sqrt{1-\left(\frac{1}{\sqrt{5}}\right)^2}\right]$$

$$= \sin^{-1}\left[\frac{1}{\sqrt{5}}\sqrt{1-\frac{1}{10}} + \frac{1}{\sqrt{10}}\sqrt{1-\frac{1}{5}}\right]$$

$$= \sin^{-1}\left[\frac{1}{\sqrt{5}}\sqrt{\frac{10-1}{10}} + \frac{1}{\sqrt{10}}\sqrt{\frac{5-1}{5}}\right]$$

$$= \sin^{-1}\left[\frac{1}{\sqrt{5}} \times \frac{3}{\sqrt{10}} + \frac{1}{\sqrt{10}} \times \frac{2}{\sqrt{5}}\right]$$

$$= \sin^{-1}\left[\frac{3+2}{\sqrt{50}}\right]$$

$$= \sin^{-1}\frac{5}{\sqrt{25\times 2}}$$

$$= \sin^{-1}\frac{5}{5\sqrt{2}} = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\pi}{4}$$

$$\text{Hence : } \sin^{-1}\frac{1}{\sqrt{5}} + \cot^{-1}3 = \frac{\pi}{4}$$

Q.6 $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17} = \sin^{-1}\frac{77}{85}$

Solution:

L.H.S

$$\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17}$$

$$\because \sin^{-1}A + \sin^{-1}B = \sin^{-1}\left[A\sqrt{1-B^2} + B\sqrt{1-A^2}\right]$$

$$\Rightarrow \quad = \sin^{-1}\left[\frac{3}{5}\sqrt{1-\left(\frac{8}{17}\right)^2} + \frac{8}{17}\sqrt{1-\left(\frac{3}{5}\right)^2}\right]$$

$$= \sin^{-1}\left[\frac{3}{5}\sqrt{1-\frac{64}{289}} + \frac{8}{17}\sqrt{1-\frac{9}{25}}\right]$$

$$=\sin^{-1}\left[\frac{3}{5}\sqrt{\frac{289-64}{289}}+\frac{8}{17}\sqrt{\frac{25-9}{25}}\right]$$

$$=\sin^{-1}\left[\frac{3}{5}\times\sqrt{\frac{225}{289}}+\frac{8}{17}\sqrt{\frac{16}{25}}\right]$$

$$=\sin^{-1}\left[\frac{3}{5}\times\frac{15}{17}+\frac{8}{17}\times\frac{4}{5}\right]$$

$$=\sin^{-1}\left[\frac{45+32}{85}\right]$$

$$=\sin^{-1}\frac{77}{85}$$

$$\text{Hence}:\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}=\sin^{-1}\frac{77}{85}$$

Q.7 $\sin^{-1}\frac{77}{85}-\sin^{-1}\frac{3}{5}=\cos^{-1}\frac{15}{17}$

Solution:

L.H.S

$$\sin^{-1}\frac{77}{85}-\sin^{-1}\frac{3}{5}$$

$$\because \sin^{-1}A-\sin^{-1}B=\sin^{-1}\left[A\sqrt{1-B^2}-B\sqrt{1-A^2}\right]$$

$$\Rightarrow \quad =\sin^{-1}\left[\frac{77}{85}\sqrt{1-\left(\frac{3}{5}\right)^2}-\frac{3}{5}\sqrt{1-\left(\frac{77}{85}\right)^2}\right]$$

$$=\sin^{-1}\left[\frac{77}{85}\sqrt{1-\frac{9}{25}}-\frac{3}{5}\sqrt{1-\frac{5929}{7225}}\right]$$

$$=\sin^{-1}\left[\frac{77}{85}\sqrt{\frac{25-9}{25}}-\frac{3}{5}\sqrt{\frac{7225-5929}{7225}}\right]$$

$$=\sin^{-1}\left[\frac{77}{85}\times\frac{4}{5}-\frac{3}{5}\times\frac{36}{85}\right]$$

$$=\sin^{-1}\left[\frac{308-108}{425}\right]$$

$$=\sin^{-1}\frac{200}{425}$$

$$=\sin^{-1}\frac{8}{17} \qquad \text{(i)}$$

Let $\alpha=\sin^{-1}\frac{8}{17} \Rightarrow \sin\alpha=\frac{8}{17}$

Using Pythagoras theorem

$$(17)^2=(x)^2+(8)^2$$

$$289=x^2+64$$

$$x^2=289-64$$

$$x^2=225$$

$$x=15$$

So, $\cos\alpha=\frac{15}{17} \Rightarrow \alpha=\cos^{-1}\frac{15}{17}$

$$\therefore \sin^{-1}\frac{8}{17}=\cos^{-1}\frac{15}{17}$$

$$\text{Hence}:\sin^{-1}\frac{77}{85}-\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{8}{17}=\cos^{-1}\frac{15}{17}$$

Q.8 $\cos^{-1}\frac{63}{65}+2\tan^{-1}\frac{1}{5}=\sin^{-1}\frac{3}{5}$

Solution:

L.H.S

$$2\tan^{-1}\left(\frac{1}{5}\right)$$

$$\because 2\tan^{-1}A=\tan^{-1}\frac{2A}{1-A^2}$$

$$\Rightarrow \quad =\tan^{-1}\frac{2\left(\frac{1}{5}\right)}{1-\left(\frac{1}{5}\right)^2}$$

$$=\tan^{-1}\frac{\frac{2}{5}}{1-\frac{1}{25}}=\tan^{-1}\frac{\frac{2}{5}}{\frac{24}{25}}$$

$$=\tan^{-1}\left(\frac{2}{5}\times\frac{25}{24}\right)$$

$$2\tan^{-1}\left(\frac{1}{5}\right)=\tan^{-1}\frac{5}{12}$$

Let $\alpha = \tan^{-1}\frac{5}{12} \Rightarrow \tan\alpha = \frac{5}{12}$

Using Pythagoras theorem

$(x)^2 = (12)^2 + (5)^2$

$= 144 + 25$

$x^2 = 169$

$x = 13$

So, $\cos\alpha = \frac{12}{13} \Rightarrow \alpha = \cos^{-1}\frac{12}{13}$

$\tan^{-1}\frac{5}{12} = \cos^{-1}\frac{12}{13}$

Now the L.H.S becomes

$\cos^{-1}\frac{63}{65} + \cos^{-1}\frac{12}{13}$

$\because \cos^{-1}A + \cos^{-1}B = \cos^{-1}\left[AB - \sqrt{(1-A^2)(1-B^2)}\right]$

$\Rightarrow = \cos^{-1}\left[\frac{63}{65}\times\frac{12}{13} - \sqrt{\left(1-\left(\frac{63}{65}\right)^2\right)\left(1-\left(\frac{12}{13}\right)^2\right)}\right]$

$= \cos^{-1}\left[\frac{756}{845} - \sqrt{\left(1-\frac{3969}{4225}\right)\left(1-\frac{144}{169}\right)}\right]$

$= \cos^{-1}\left[\frac{756}{845} - \sqrt{\left(\frac{4225-3969}{4225}\right)\left(\frac{169-144}{169}\right)}\right]$

$= \cos^{-1}\left[\frac{756}{845} - \sqrt{\left(\frac{256}{4225}\times\frac{25}{169}\right)}\right]$

$= \cos^{-1}\left[\frac{756}{845} - \frac{16\times5}{65\times13}\right]$

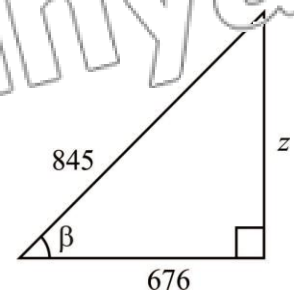
$= \cos^{-1}\left[\frac{756-80}{845}\right]$

$= \cos^{-1}\left[\frac{676}{845}\right]$

$\therefore \cos^{-1}\frac{63}{65} + \cos^{-1}\frac{12}{13} = \cos^{-1}\left[\frac{676}{845}\right]$

Let

$\beta = \cos^{-1}\frac{676}{845} \Rightarrow \cos\beta = \frac{676}{845}$



Using Pythagoras theorem

$(845)^2 = (676)^2 + z^2$

$z^2 = 714,025 - 456,976$

$z^2 = 275,049$

$z = 507$

$\sin\beta = \frac{507}{845} = \frac{3}{5}$

$\sin\beta = \frac{3}{5} \Rightarrow \beta = \sin^{-1}\frac{3}{5}$

$\cos^{-1}\frac{676}{845} = \sin^{-1}\frac{3}{5}$

$$\text{Hence}: \cos^{-1}\frac{63}{65} + 2\tan^{-1}\frac{1}{5} = \sin^{-1}\frac{3}{5}$$

Q.9 $\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} - \tan^{-1}\frac{8}{19} = \frac{\pi}{4}$

Solution:

L.H.S

$\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} - \tan^{-1}\frac{8}{19}$

$\because \tan^{-1}A + \tan^{-1}B = \tan^{-1}\frac{A+B}{1-AB}$

$\Rightarrow = \tan^{-1}\frac{\frac{3}{4}+\frac{3}{5}}{1-\left(\frac{3}{4}\right)\left(\frac{3}{5}\right)} - \tan^{-1}\frac{8}{19}$

$$= \tan^{-1}\frac{\frac{15+12}{20}}{\frac{20-9}{20}} - \tan^{-1}\frac{8}{19}$$

$$= \tan^{-1}\frac{27}{11} - \tan^{-1}\frac{8}{19}$$

$\because \tan^{-1}A - \tan^{-1}B = \tan^{-1}\left(\frac{A-B}{1+AB}\right)$

$$\Rightarrow = \tan^{-1}\left[\frac{\frac{27}{11}-\frac{8}{19}}{1+\left(\frac{27}{11}\right)\left(\frac{8}{19}\right)}\right]$$

$$= \tan^{-1}\frac{\frac{513-88}{209}}{\frac{209+216}{209}}$$

$$= \tan^{-1}\frac{425}{425}$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

$$\text{Hence: } \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} - \tan^{-1}\frac{8}{19} = \frac{\pi}{4}$$

Q.10 $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} = \frac{\pi}{2}$

Solution:

L.H.S

$$= \sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65}$$

$\because \sin^{-1}A + \sin^{-1}B = \sin^{-1}\left[A\sqrt{1-B^2} + B\sqrt{1-A^2}\right]$

$$= \sin^{-1}\left[\frac{4}{5}\sqrt{1-\left(\frac{5}{13}\right)^2} + \frac{5}{13}\sqrt{1-\left(\frac{4}{5}\right)^2}\right] + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left[\frac{4}{5}\sqrt{\frac{169-25}{169}} + \frac{5}{13}\sqrt{\frac{25-16}{25}}\right] + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left[\frac{4}{5}\times\frac{12}{13} + \frac{5}{13}\times\frac{3}{5}\right] + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left[\frac{48+15}{65}\right] + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left[\frac{63}{65}\right] + \sin^{-1}\frac{16}{65}$$

$\because \sin^{-1}A + \sin^{-1}B = \sin^{-1}\left[A\sqrt{1-B^2} + B\sqrt{1-A^2}\right]$

$$\Rightarrow = \sin^{-1}\left[\frac{63}{65}\sqrt{1-\left(\frac{16}{65}\right)^2} + \frac{16}{65}\sqrt{1-\left(\frac{63}{65}\right)^2}\right]$$

$$= \sin^{-1}\left[\frac{63}{65}\sqrt{\frac{4225-256}{4225}} + \frac{16}{65}\sqrt{1-\left(\frac{63}{65}\right)^2}\right]$$

$$= \sin^{-1}\left[\frac{63}{65}\times\sqrt{\frac{3969}{4225}} + \frac{16}{65}\sqrt{\frac{256}{4225}}\right]$$

$$= \sin^{-1}\left[\frac{63}{65}\times\frac{63}{65} + \frac{16\times16}{65\times65}\right]$$

$$= \sin^{-1}\left[\frac{3969+256}{4225}\right]$$

$$= \sin^{-1}\left[\frac{4225}{4225}\right]$$

$$= \sin^{-1}(1)$$

$$= \frac{\pi}{2}$$

$$\text{Hence: } \sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} = \frac{\pi}{2}$$

Q.11 $\tan^{-1}\dfrac{1}{11}+\tan^{-1}\dfrac{5}{6}=\tan^{-1}\dfrac{1}{3}+\tan^{-1}\dfrac{1}{2}$

Solution:

L.H.S

$$=\tan^{-1}\frac{1}{11}+\tan^{-1}\frac{5}{6}$$

$$\because \tan^{-1}A+\tan^{-1}B=\tan^{-1}\frac{A+B}{1-AB}$$

$$\tan^{-1}\frac{1}{11}+\tan^{-1}\frac{5}{6}=\tan^{-1}\frac{\frac{1}{11}+\frac{5}{6}}{1-\left(\frac{1}{11}\right)\left(\frac{5}{6}\right)}$$

$$=\tan^{-1}\frac{\frac{6+55}{66}}{\frac{66-5}{66}}$$

$$=\tan^{-1}\frac{61}{61}$$

$$\tan^{-1}(1)$$

$$\tan^{-1}\frac{1}{11}+\tan^{-1}\frac{5}{6}=\frac{\pi}{4} \qquad \text{(i)}$$

R.H.S

$$=\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}$$

$$\because \tan^{-1}A+\tan^{-1}B=\tan^{-1}\left(\frac{A+B}{1-AB}\right)$$

$$\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}=\tan^{-1}\left[\frac{\frac{1}{3}+\frac{1}{2}}{1-\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)}\right]$$

$$=\tan^{-1}\left(\frac{\frac{2+3}{6}}{\frac{6-1}{6}}\right)$$

$$=\tan^{-1}\frac{5}{5}$$

$$=\tan^{-1}(1)$$

$$\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}=\frac{\pi}{4} \qquad \text{(ii)}$$

By comparing (i) and (ii)

$$\boxed{\text{Hence : }\tan^{-1}\frac{1}{11}+\tan^{-1}\frac{5}{6}=\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}}$$

Q.12 $2\tan^{-1}\dfrac{1}{3}+\tan^{-1}\dfrac{1}{7}=\dfrac{\pi}{4}$

Solution:

L.H.S

$$=2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}$$

$$\because 2\tan^{-1}A=\tan^{-1}\frac{2A}{1-A^2}$$

$$=\tan^{-1}\frac{2\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)^2}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{\frac{2}{3}}{1-\frac{1}{9}}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{\frac{2}{3}}{\frac{8}{9}}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}$$

$$2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}$$

$$\because \tan^{-1}A+\tan^{-1}B=\tan^{-1}\left(\frac{A+B}{1-AB}\right)$$

$$\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}=\tan^{-1}\left[\frac{\left(\frac{3}{4}\right)+\left(\frac{1}{7}\right)}{1-\left(\frac{3}{4}\right)\left(\frac{1}{7}\right)}\right]$$

$$=\tan^{-1}\left(\frac{\frac{21+4}{28}}{\frac{28-3}{28}}\right)$$

$= \tan^{-1}\dfrac{25}{25}$

$= \tan^{-1}(1)$

$= \dfrac{\pi}{4}$

$$\boxed{\text{Hence : } 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}}$$

Q.13 Show that $\cos(\sin^{-1}x) = \sqrt{1-x^2}$

Solution:

$\cos(\sin^{-1}x) = \sqrt{1-x^2}$

Put $\sin^{-1}x = \alpha$ (i)

$\sin\alpha = x$

$\because$ $\sin^2\alpha + \cos^2\alpha = 1$

$\cos^2\alpha = 1 - \sin^2\alpha$

$\cos\alpha = \sqrt{1-\sin^2\alpha}$ $\left(\because \dfrac{-\pi}{2} \le \alpha \le \dfrac{\pi}{2}\right)$

$\cos\alpha = \sqrt{1-x^2}$

From (i) $\sin^{-1}x = \alpha$

$\cos(\sin^{-1}x) = \sqrt{1-x^2}$

$$\boxed{\text{Hence : } \cos(\sin^{-1}x) = \sqrt{1-x^2}}$$

Q.14 Show that
$\sin(2\cos^{-1}x) = 2x\sqrt{1-x^2}$

Solution:

$\sin(2\cos^{-1}x) = 2x\sqrt{1-x^2}$

Put $\cos^{-1}x = \alpha$

$\cos\alpha = x$

$\sin^2\alpha + \cos^2\alpha = 1$

$\sin\alpha = \sqrt{1-\cos^2\alpha}$

$(0 \le \alpha \le \pi)$

$\sin\alpha = \sqrt{1-x^2}$

Using double angle identity of sine.

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

Putting values of α, $\cos\alpha$ and $\sin\alpha$.

$\sin(2\cos^{-1}x) = 2x\sqrt{1-x^2}$

$$\boxed{\text{Hence : } \sin(2\cos^{-1}x) = 2x\sqrt{1-x^2}}$$

Q.15 Show that $\cos(2\sin^{-1}x) = 1-2x^2$

Solution:

$\cos(2\sin^{-1}x) = 1-2x^2$

Put $\sin^{-1}x = \alpha$

$\sin\alpha = x \left(-\dfrac{\pi}{2} \le \alpha \le \dfrac{\pi}{2}\right)$

Using double angle identity of cosine

$\cos 2\alpha = 1 - 2\sin^2\alpha$

Putting value of α and $\sin\alpha$

$\cos(2\sin^{-1}x) = 1-2x^2$

$$\boxed{\text{Hence: } \cos(2\sin^{-1}x) = 1-2x^2}$$

Q.16 Show that $\tan^{-1}(-x) = -\tan^{-1}x$

Solution:

$\tan^{-1}(-x) = -\tan^{-1}x$

$\tan^{-1}(-x) + \tan^{-1}x$

$= \tan^{-1}\left(\dfrac{-x+x}{1-(-x)(x)}\right)$

$= \tan^{-1}\left(\dfrac{0}{1+x^2}\right)$

$= \tan^{-1}(0)$

$\tan^{-1}(-x) + \tan^{-1}x = 0$

$\tan^{-1}(-x) = -\tan^{-1}x$

$$\boxed{\text{Hence : } \tan^{-1}(-x) = -\tan^{-1}x}$$

Q.17 $\sin^{-1}(-x) = -\sin^{-1}x$

Solution:

$\sin^{-1}(-x) = -\sin^{-1}x$

$\sin^{-1}(-x) + \sin^{-1}x$

$= \sin^{-1}\left[(-x)\sqrt{1-x^2} + x\sqrt{1-(-x)^2}\right]$

$= \sin^{-1}\left[-x\sqrt{1-x^2} + \sqrt{1-x^2}\right]$

$= \sin^{-1}(0)$

$\sin^{-1}(-x) + \sin^{-1}x = 0$

$\sin^{-1}(-x) = -\sin^{-1}x$

$$\boxed{\text{Hence : } \sin^{-1}(-x) = -\sin^{-1}x}$$

Q.18 Show that $\cos^{-1}(-x)=\pi-\cos^{-1}x$

Solution:

$$\cos^{-1}(-x)=\pi-\cos^{-1}x$$

$$\cos^{-1}(-x)+\cos^{-1}x$$
$$=\cos^{-1}\left[(-x)(x)-\sqrt{\left(1-(-x)^2\right)\left(1-(x)^2\right)}\right]$$
$$=\cos^{-1}\left[-x^2-\sqrt{(1-x^2)(1-x^2)}\right]$$
$$=\cos^{-1}\left[-x^2-\sqrt{(1-x^2)^2}\right]$$
$$=\cos^{-1}\left[-x^2-(1-x^2)\right]$$
$$=\cos^{-1}\left[-x^2-1+x^2\right]$$
$$=\cos^{-1}(-1)$$
$$\cos^{-1}(-x)+\cos^{-1}x=\pi$$
$$\cos^{-1}(-x)=\pi-\cos^{-1}x$$

$$\boxed{\text{Hence : }\cos^{-1}(-x)=\pi-\cos^{-1}x}$$

Q.19 $\tan(\sin^{-1}x)=\frac{x}{\sqrt{1-x^2}}$

Solution:

$$\tan(\sin^{-1}x)=\frac{x}{\sqrt{1-x^2}}$$

Put $\sin^{-1}x=\alpha$ (i)

$\sin\alpha=x$

As $\sin^2\alpha+\cos^2\alpha=1$

$\cos^2\alpha=1-\sin^2\alpha$

$\cos\alpha=\sqrt{1-\sin^2\alpha}$ $\left(\because -\frac{\pi}{2}\le\alpha\le\frac{\pi}{2}\right)$

$\cos\alpha=\sqrt{1-x^2}$

$\tan\alpha=\frac{\sin\alpha}{\cos\alpha}$

From (i) $\alpha=\sin^{-1}x$

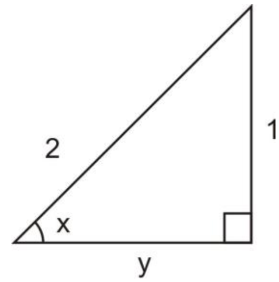
$\tan(\sin^{-1}x)=\frac{x}{\sqrt{1-x^2}}$

$$\boxed{\text{Hence : }\tan(\sin^{-1}x)=\frac{x}{\sqrt{1-x^2}}}$$

Q.20 Given that $x=\sin^{-1}\frac{1}{2}$ find the value of following trigonometric functions: $\sin x,\cos x,\tan x,\cot x,\sec x,\operatorname{cosec} x$

Solution:

Given that $x=\sin^{-1}\frac{1}{2}\Rightarrow\sin x=\frac{1}{2}$



Using Pythagoras theorem

$(\text{Hyp})^2=(\text{perp})^2+(\text{Base})^2$

$(2)^2=(1)^2+y^2$

$y^2=3$

$y=\sqrt{3}$

$\sin x=\frac{\text{perp}}{\text{Hyp}}=\frac{1}{2}$

$\cos x=\frac{\text{Base}}{\text{Hyp}}=\frac{\sqrt{3}}{2}$

$\tan x=\frac{\text{perp}}{\text{Base}}=\frac{1}{\sqrt{3}}$

$\cot x=\frac{\text{Base}}{\text{Perp}}=\sqrt{3}$

$\sec x=\frac{\text{Hyp}}{\text{Base}}=\frac{2}{\sqrt{3}}$

$\operatorname{cosec} x=\frac{\text{Hyp}}{\text{Perp}}=2$