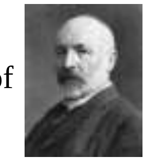
CHAPTER 2

SETS, FUNCTIONS AND GROUPS

Set



George Cantor * (1845-1918)

A **well-defined** collection of distinct objects is called a **set.** For example: set of planets, set of natural numbers etc.

There are **three** different ways of describing a **set.**

WAY OF DESCRIPTION	DEFINITION	EXAMPLE
The Descriptive Method	In this method the set can be described in words.	The set of vowels in English alphabets
The Tabular Method	In this method the set is described by listing its elements with in brackets.	$A=\{a,e,i,o,u\}$
Set Builder Method	Set Builder notation is a mathematical notation used to describe a set by enumerating its elements or demonstrating its properties that its members must satisfy.	$A=\{x \mid x \text{ is vowel}\}$

Equal Sets

Two sets A and B are **equal** i.e. A = B if and only if they have the same elements. For example the sets $\{1,2,3\}$ and $\{2,1,3\}$ are equal.

Equivalent Sets:

Two sets A and B are **equivalent** if there is a one-to-one correspondence between the sets. For example

$$\begin{array}{cccc} \{a, & b, & c, & d\} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \{1, & 2, & 3, & 4\} \end{array}$$

> **Q: Differentiate between Equal and Equivalent sets with example. (RWP 2023)**

Order of a Set

The number of elements in a set is called the **order** of the set. For example, If A = $\{a, b, c\}$, then order of set A is three i.e., $n(A)=3$

Singleton set

A set having only one element is called **singleton** set. For example $A=\{a\}$.

*The theory of sets is attributed to the German Mathematics George Cantor (1845-1918)

Empty set or Null set

A set with no elements is called the **empty set or null** set and it is denoted by the symbol ϕ or $\{\ \}$.

NAME OF THE SET	DEFINITION	EXAMPLE
Finite Set	If a set has definite number of elements present in it.	$\{1,2,3,\cdots,10\}$
Infinite Set	If a set has indefinite number of elements present in it.	$N=\{1,2,3,\cdots\}$ $Z=\{0,\pm1,\pm2,\cdots\}$
Subset	If every element of a set A is an element of set B, then A is said to be a **subset** of B	$A=\{a,b\}$ and $B=\{a,b,c,d\}$ then A is subset of B.
Singleton Set	A set having only one element.	$\{7\}$
Proper Subset	If A is a subset of B and B contains at least one element which is not an element of A, then A is said to be **a proper subset** of B.	$A=\{a,b\}$ and $B=\{a,b,c\}$ then A is a proper subset of B.
Improper subset	If A is subset of B and A = B, then we say that A is an **improper subset** of B.	$A=\{a,b\}$ and $B=\{a,b\}$ then A is improper subset of B.

Do you know?
- The empty set has no proper subsets.
- An empty set is a proper subset of every non empty set.
- Every set is an improper subset of itself.

Universal set

The set containing all the elements of under consideration sets is called **universal set**.

Power set

Let A be any set then the set containing all its subsets is called **power set** of A.

For example

If $A=\{a,b\}$, then $P(A)=\{\varphi,\{a\},\{b\},\{a,b\}\}$.

In general

if $n(A)=m$ then $n(P(A))=2^m$

Key Facts:
- If a set has n-elements then it has 2^n subsets.
- If a set has n-elements then there are 2^n elements in its power set.

www.ilmkidunya.com

EXERCISE 2.1

Q.1 Write the following sets in set builder form.

(i) $\{1,2,3,...,1000\}$

Solution:

$\{x \mid x \in \mathbb{N} \wedge x \leq 1000\}$ **Answer**

(ii) $\{0,1,2,3,...,100\}$

Solution:

$\{x \mid x \in W \wedge x \leq 100\}$ **Answer**

(iii) $\{0,\pm 1,\pm 2,...,\pm 1000\}$ **(SGD 2021)**

Solution:

$\{x \mid x \in \mathbb{Z} \wedge -1000 \leq x \leq 1000\}$ **Answer**

(iv) $\{0,-1,-2,-3,...,-500\}$

Solution:

$\{x \mid x \in \mathbb{Z} \wedge -500 \leq x \leq 0\}$ **Answer**

(v) $\{100,101,102,...,400\}$

Solution:

$\{x \mid x \in \mathbb{N} \wedge 100 \leq x \leq 400\}$ **Answer**

(vi) $\{-100,-101,-102,...,-500\}$

Solution:

$\{x \mid x \in \mathbb{Z} \wedge -500 \leq x \leq -100\}$ **Answer**

(vii) {Peshawar, Lahore, Karachi, Quetta}

Solution: { $x \mid x$ is a capital of a province of Pakistan} **Answer**

(viii) {January, June, July}

Solution: { $x \mid x$ is a month that starts with J} **Answer**

(ix) The set of all odd natural numbers

Solution:

$\{x \mid x \in O \wedge x > 0\}$ **Answer**

(x) The set of all rational numbers

Solution: $\{x \mid x \in \mathbb{Q}\}$ **Answer**

(xi) The set of all real numbers between 1 and 2.

Solution: $\{x \mid x \in \mathbb{R} \wedge 1 < x < 2\}$ **Ans.**

(xii) The set of all integers between -100 and 1000

Solution:

$\{x \mid x \in \mathbb{Z} \wedge -100 < x < 1000\}$ **Ans.**

Q.2 Write each of the following sets in the descriptive and tabular forms.

(i) $\{x \mid x \in \mathbb{N} \wedge x \leq 10\}$

(MTN 2021, RWP 2022, GRW 2022)

Descriptive: The set of first ten natural numbers

Tabular: $\{1,2,3,...,10\}$ **Answer**

(ii) $\{x \mid x \in N \wedge 4 < x < 12\}$

(RWP 2023)

Descriptive: The set of natural numbers between 4 and 12.

Tabular: $\{5,6,7,...,11\}$ **Answer**

(iii) $\{x \mid x \in \mathbb{Z} \wedge -5 < x < 5\}$

Descriptive: Set of integers between –5 and 5

Tabular: $\{0,\pm 1,\pm 2,\pm 3,\pm 4\}$ **Ans.**

(iv) $\{x \mid x \in E \wedge 2 < x \leq 4\}$

Descriptive: The set of even integers greater than 2 and less than or equal to 4

Tabular: $\{4\}$ **Answer**

(v) $\{x \mid x \in P \wedge x < 12\}$

(MTN 2022, LHR 2022, 23, GRW 2023)

Descriptive: The set of prime numbers less than 12

Tabular: $\{2,3,5,7,11\}$ **Answer**

(vi) $\{x \mid x \in O \wedge 3 < x < 12\}$

(FSD 2021, BWP 2021)

Descriptive: The set of odd integers between 3 and 12.

Tabular: $\{5,7,9,11\}$ **Answer**

(vii) $\{x \mid x \in E \wedge 4 \leq x \leq 10\}$

(BWP 2022)

Descriptive: The set of even integers from 4 up to 10.

Tabular: $\{4,6,8,10\}$ **Answer**

(viii) $\{x \mid x \in E \wedge 4 < x < 6\}$

Descriptive: The set of even integers between 4 and 6.

Tabular: ϕ **Answer**

(ix) $\{x \mid x \in O \wedge 5 \leq x \leq 7\}$

(DGK 2021)

Descriptive: The set of odd integers from 5 up to 7.

Tabular: $\{5,7\}$ **Answer**

(x) $\{x \mid x \in O \wedge 5 \leq x < 7\}$

Descriptive: The set of odd integers greater than or equal to 5 and less than 7.

Tabular: $\{5\}$ **Answer**

(xi) $\{x \mid x \in N \wedge x + 4 = 0\}$

(MTN 2023)

Descriptive: The set of natural numbers x satisfying $x + 4 = 0$

Tabular: ϕ **Answer**

(xii) $\{x \mid x \in \mathbb{Q} \wedge x^2 = 2\}$

(GRW 2021)

Descriptive: The set of rational numbers x satisfying $x^2 = 2$.

Tabular: ϕ **Answer**

(xiii) $\{x \mid x \in \mathbb{R} \wedge x = x\}$

Descriptive: The set of real numbers x satisfying $x = x$.

Tabular: $\mathbb{R}$ **Answer**

(xiv) $\{x \mid x \in \mathbb{Q} \wedge x = -x\}$

Descriptive: The set of rational numbers x satisfying $x = -x$.

Tabular: $\{0\}$ **Answer**

(xv) $\{x \mid x \in \mathbb{R} \wedge x \neq x\}$

(DGK 2023)

Descriptive: The set of real numbers x satisfying $x \neq x$.

Tabular: ϕ **Answer**

(xvi) $\{x \mid x \in \mathbb{R} \wedge x \notin \mathbb{Q}\}$

Descriptive: The set of real numbers x which are not rational.

Tabular: Q' **Answer**

Q.3 Which of the following sets are finite and which of these are infinite?

(i) The set of students of your class.

Solution: finite

(ii) The set of all schools in Pakistan.

Solution: finite

(iii) The set of natural numbers between 3 and 10.

Solution: finite

(iv) The set of rational numbers between 3 and 10.

Solution: infinite

(v) The set of real numbers between 0 and 1.

Solution: infinite

(vi) The set of rationales between 0 and 1.

Solution: infinite

(vii) The set of whole numbers between 0 and 1.

Solution: finite

(viii) The set of all leaves of trees in Pakistan.

Solution: finite

(ix) $P(N)$

Solution: infinite

(x) $P\{a,b,c\}$

Solution: finite

(xi) $\{1,2,3,4,...\}$

Solution: infinite

(xii) $\{1,2,3,...,100000000\}$

Solution: finite

(xiii) $\{x \mid x \in \mathbb{R} \wedge x \neq x\}$

Solution: finite

(xiv) $\{x \mid x \in \mathbb{R} \wedge x^2 = -16\}$

Solution: finite

(xv) $\{x \mid x \in \mathbb{Q} \wedge x^2 = 5\}$

Solution: finite

(xvi) $\{x \mid x \in \mathbb{Q} \wedge 0 \leq x \leq 1\}$

Solution: infinite

Q.4 Write two proper subsets of each of the following sets.

(i) $\{a,b,c\}$

(MTN 2021, 22, DGK 2021, RWP 2021, GRW 2023)

Solution: $\{a,b\}$, $\{b,c\}$

(ii) $\{0,1\}$

Solution: $\{0\}$, $\{1\}$

(iii) □

Solution: $\{1\},\{2\}$

(iv) □

Solution: $\{0\},\{-1,0\}$

(v) □

Solution: $\{0\},\left\{\frac{1}{2},\frac{1}{4}\right\}$

(vi) $\mathbb{R}$

Solution: $\left\{\frac{1}{2}\right\},\left\{\frac{1}{\sqrt{2}}\right\}$

(vii) W

Solution: $\{0,1\},\{2,3\}$

(viii) $\{x \mid x \in \mathbb{Q} \wedge 0 < x \leq 2\}$

Solution: $\left\{1,\frac{1}{2}\right\},\left\{\frac{1}{3},\frac{2}{5}\right\}$

Q.5 Is there any set which has no proper subset? If so, name that set.

Solution: Yes. The empty set has no proper subset.

Q.6 What is the difference between $\{a,b\}$ and $\{\{a,b\}\}$? (SHW 2023)

Solution: The set $\{a,b\}$ has two elements a and b while the set $\{\{a,b\}\}$ has only one element $\{a,b\}$.

Q.7 Which of the following sentences are true and which of them are false?

(i) $\{1,2\} = \{2,1\}$ Ans : True

(ii) $\phi \subseteq \{\{a\}\}$ Ans : True

(iii) $\{a\} \subseteq \{\{a\}\}$ Ans : False

(iv) $\{a\} \in \{\{a\}\}$ Ans : True

(v) $a \in \{\{a\}\}$ Ans : False

(vi) $\phi \in \{\{a\}\}$ Ans : False

Q.8 What is the number of elements of power set of each of the following sets?

(i) $\{\ \}$ Ans : $2^0 = 1$

(ii) $\{0,1\}$ Ans : $2^2 = 4$

(iii) $\{1,2,3,4,5,6,7\}$

Ans: $2^7 = 128$

(iv) $\{0,1,2,3,4,5,6,7\}$

Ans: $2^8 = 256$

(v) $\{a,\{b,c\}\}$ Ans : $2^2 = 4$

(vi) $\{\{a,b\},\{b,c\},\{d,e\}\}$

Ans : $2^3 = 8$

Q.9 Write down the power set of each of the following sets.

(i) $\{9,11\}$

(GRW 2022, RWP 2023)

Solution: Let $A = \{9,11\}$

$P(A) = \{\phi,\{9\},\{11\},\{9,11\}\}$

(ii) $\{+,-,\times,\div\}$ **(SGD 2023)**

Solution: Let $A = \{+,-,\times,\div\}$

$P(A) = \{\phi,\{+\},\{-\},\{\times\},\{\div\},\{+,-\}\{+,\times\}$
$,\{+,\div\},\{-,\times\},\{-,\div\},\{\times,\div\},\{+,-,\times\},$
$\{+,-,\div\},\{-,\times,\div\},\{+,\times,\div\},\{+,-,\times,\div\}\}$

(iii) $\{\phi\}$

Solution: Let $A = \{\phi\}$

$P(A) = \{\phi,\{\phi\}\}$

(iv) $\{a,\{b,c\}\}$

(SGD 2021, 22, RWP 2022, GRW 2022)

Solution: Let $A = \{a,\{b,c\}\}$

$P(A) = \{\phi,\{a\},\{\{b,c\}\},\{a,\{b,c\}\}\}$

Q.10 Which pairs of sets are equivalent? Which of them are also equal?

(i) $\{a,b,c\},\{1,2,3\}$

Solution: Equivalent but not equal.

(ii) The set of first ten whole numbers, $\{0,1,2,3,...,9\}$

Solution: Equivalent and equal.

(iii) Set of angles of a quadrilateral *ABCD*, set of sides of same quadrilateral.

Solution: Equivalent but not equal.

(iv) Set of sides of hexagon *ABCDEF*, Set of angles of same hexagon.

Solution: Equivalent but not equal.

(v) $\{1,2,3,4,...\},\{2,4,6,8,...\}$

Solution: Equivalent but not equal.

(vi) $\{1,2,3,4,...\},\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},...\right\}$

Solution: Equivalent but not equal.

(vii) $\{5,10,15,...,55555\},\{5,10,15,20,...\}$

Solution: Neither Equivalent nor equal.

Operations on Sets

NAME OF OPERATION	DEFINITION
Union of Two Sets	$A\cup B=\{x \mid x\in A\vee x\in B\}$
Intersection of Two Sets	$A\cap B=\{x \mid x\in A\wedge x\in B\}$
Difference of Two Sets	$A-B=\{x \mid x\in A\wedge x\notin B\}$

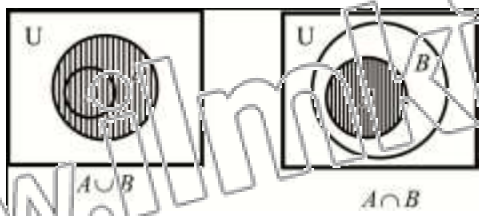
Sets, Based on the Operations

NAME OF THE SET	DEFINITION
Disjoint Sets	Two sets A & B are disjoint iff $A\cap B=\phi$.
Overlapping Sets	Two sets A & B are overlapping if $A\cap B\neq\phi$ but neither $A\subseteq B$ nor $B\subseteq A$
Complement of a Set	$A'=A^c=U-A=\{x / x\in U\wedge x\notin A\}$ where U is universal set and $A\subseteq U$

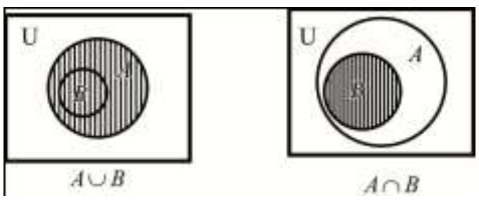
EXERCISE 2.2

Q.1 Exhibit $A\cup B$ and $A\cap B$ by Venn diagrams in the following cases.

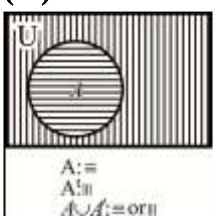
(i) $A\subseteq B$



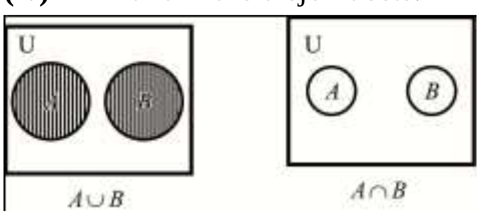
(ii) $B\subseteq A$



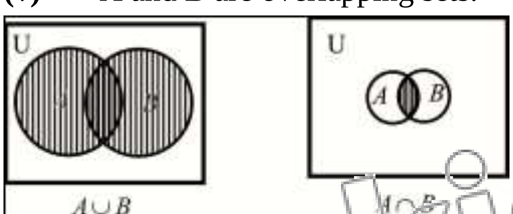
(iii) $A\cup A'$



(iv) A and B are disjoint sets.

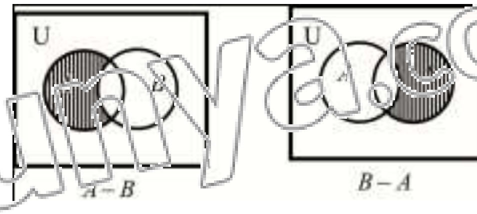


(v) A and B are overlapping sets.

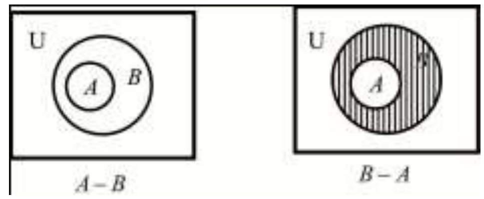


Q.2 Show $A-B$ and $B-A$ by Venn diagrams when:

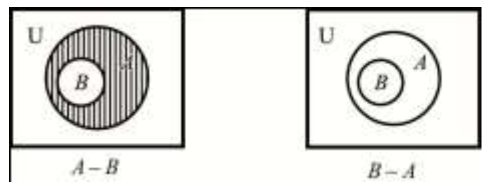
(i) A and B are overlapping sets. **(SHW 2021, BWP 2023)**



(ii) $A\subseteq B$ **(SHW 2022)**



(iii) $B\subseteq A$ **(DGK 2022, MTN 2022)**



Q.3 Under what conditions on A and B are the following statements true?

(i) $A\cup B=A$

Solution: $B\subseteq A$

(ii) $A\cup B=B$

Solution: $A\subseteq B$

(iii) $A-B=A$

Solution: $A\cap B=\phi$

(iv) $A\cap B=B$

Solution: $B\subseteq A$

(v) $n(A\cup B)=n(A)+n(B)$

Solution: $A\cap B=\phi$

(vi) $n(A\cap B)=n(A)$

Solution: $A\subseteq B$

(vii) $A-B=A$

Solution: $A\cap B=\phi$

(viii) $n(A\cap B)=0$

Solution: $A\cap B=\phi$

(ix) $A\cup B=U$

Solution: $B=A'$

(x) $A\cup B=B\cup A$

Solution: It holds always

(xi) $n(A\cap B)=n(B)$

Solution: $B\subseteq A$

Q.4 Let $\boldsymbol{U}=\{\mathbf{1,2,3,4,5,6,7,8,9,10}\}$

$\boldsymbol{A}=\{\mathbf{2,4,6,8,10}\}$

$\boldsymbol{B}=\{\mathbf{1,2,3,4,5}\}$ **and** $\boldsymbol{C}=\{\mathbf{1,3,5,7,9}\}$

List the members of each of the following sets.

(i) A^c , $A^c=U-A=\{1,3,5,7,9\}$

(ii) B^c, $B^c=U-B=\{6,7,8,9,10\}$

(iii) $A\cup B$,

$A\cup B=\{1,2,3,4,5,6,8,10\}$

(iv) $A-B$, $A-B=\{6,8,10\}$

(v) $A\cap C$, $A\cap C=\phi$

(vi) $A^c\cup C^c$, $A^c=\{1,3,5,7,9\}$,

$C^c=\{2,4,6,8,10\}$ and

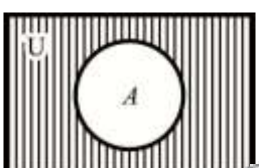
$A^c\cup C^c=\{1,2,3,...,10\}=U$

(vii) $A^c\cup C$, $A^c\cup C=\{1,3,5,7,9\}$

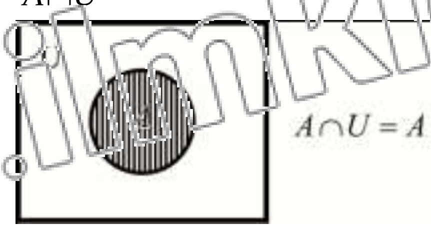
(viii) U^c , $U^c=\phi$

Q.5 **Using Venn diagrams, if necessary, find the single sets equal to the following.**

(i) $A^c=U-A$



(ii) $A\cap U$

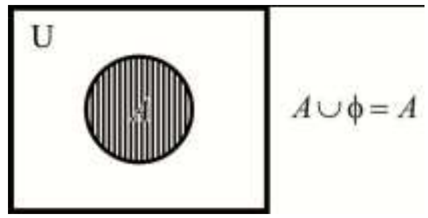


(iii) $A\cup U$

(xii) $U-A=\phi$

Solution: $U=A$

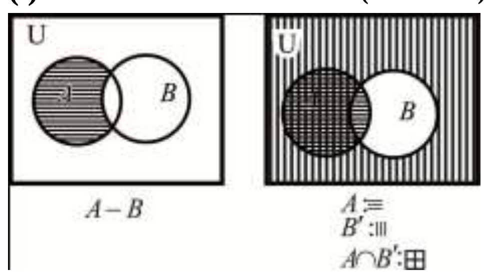
(iv) $A\cup\phi$



(v) $\phi\cap\phi=\phi$

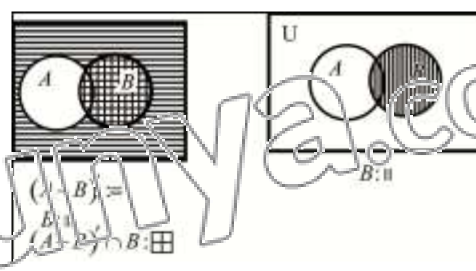
Q.6 **Use Venn diagrams to verify the following**

(i) $A-B=A\cap B^c$ **(BWP 2022)**



From Venn diagrams $A-B=A\cap B^c$

(ii) $(A-B)^c\cap B=B$



From Venn diagrams

$(A-B)^c\cap B=B$

<u>De Morgan's laws:</u>

(i) $(A\cup B)'=A'\cap B'$

(ii) $(A\cap B)'=A'\cup B'$

Proof:-

(i) $(A\cup B)' = A'\cap B'$

Let $x\in (A\cup B)'$

$\Rightarrow x\notin A\cup B$

$\Rightarrow x\notin A$ and $x\notin B$

$\Rightarrow x\in A'$ and $x\in B'$

$\Rightarrow x\in A'\cap B'$

But x is an arbitrary member of $(A\cup B)'$.

Therefore $(A\cup B)'\subseteq A'\cap B'$ (i)

Conversely, suppose that

$y\in A'\cap B'$

$\Rightarrow y\in A'$ and $y\in B'$

$\Rightarrow y\notin A$ and $y\notin B$

$\Rightarrow y\notin A\cup B$

$\Rightarrow y\in (A\cup B)'$

Therefore $A'\cap B'\subseteq (A\cup B)'$ (ii)

From (i) and (ii) we conclude that

$(A\cup B)' = A'\cap B'$

(ii) $(A\cap B)' = A'\cup B'$

Let $x\in (A\cap B)'$

$\Rightarrow x\notin A\cap B$

$\Rightarrow x\notin A$ or $x\notin B$

$\Rightarrow x\in A'$ or $x\in B'$

$\Rightarrow x\in A'\cup B'$

Therefore $(A\cap B)'\subseteq A'\cup B'$ (i)

Conversely, suppose that

$y\in A'\cup B'$

$\Rightarrow y\in A'$ or $y\in B'$

$\Rightarrow y\notin A$ or $y\notin B$

$\Rightarrow y\notin (A\cap B)$

$\Rightarrow y\in (A\cap B)'$

Therefore $A'\cup B'\subseteq (A\cap B)'$ (ii)

From (i) and (ii) we conclude that

$(A\cap B)' = A'\cup B'$

Distributive laws

(i) $A\cup (B\cap C) = (A\cup B)\cap (A\cup C)$

(ii) $A\cap (B\cup C) = (A\cap B)\cup (A\cap C)$

Proof:

(i) $A\cup (B\cap C) = (A\cup B)\cap (A\cup C)$

Let $x\in A\cup (B\cap C)$

$\Rightarrow x\in A$ or $x\in B\cap C$

$\Rightarrow x\in A$ or $x\in B$ and $x\in C$

$\Rightarrow x\in A$ or $x\in B$ and $x\in A$ or $x\in C$

$\Rightarrow x\in (A\cup B)$ and $x\in (A\cup C)$

$\Rightarrow x\in (A\cup B)\cap (A\cup C)$

Thus

$A\cup (B\cap C)\subseteq (A\cup B)\cap (A\cup C)$ (i)

Conversely suppose that

$y\in (A\cup B)\cap (A\cup C)$

$\Rightarrow y\in (A\cup B)$ and $y\in (A\cup C)$

$\Rightarrow y\in A$ or $y\in B$ and $y\in A$ or $y\in C$

$\Rightarrow y\in A$ or $y\in B$ and $y\in C$

$\Rightarrow y\in A$ or $y\in (B\cap C)$

$\Rightarrow y\in A\cup (B\cap C)$

Thus

$(A\cup B)\cap (A\cup C)\subseteq A\cup (B\cap C)$ (ii)

From (i) and (ii)

$A\cup (B\cap C) = (A\cup B)\cap (A\cup C)$

(ii) $A\cap (B\cup C) = (A\cap B)\cup (A\cap C)$

Let $x\in A\cap (B\cup C)$

$\Rightarrow x\in A$ and $x\in (B\cup C)$

$\Rightarrow x\in A$ and $x\in B$ or $x\in C$

$\Rightarrow x\in A$ and $x\in B$ or $x\in A$ and $x\in C$

$\Rightarrow x\in (A\cap B)$ or $x\in (A\cap C)$

$\Rightarrow x\in (A\cap B)\cup (A\cap C)$

Therefore

$A\cap (B\cup C)\subseteq (A\cap B)\cup (A\cap C)$ (i)

Conversely let

$y\in (A\cap B)\cup (A\cap C)$

$\Rightarrow y\in (A\cap B)$ or $y\in (A\cap C)$

$\Rightarrow y\in A$ and $y\in B$ or $y\in A$ and $y\in C$

$\Rightarrow y \in A$ and $y \in B$ or $y \in C$

$\Rightarrow y \in A$ and $y \in (B \cup C)$

$\Rightarrow y \in A \cap (B \cup C)$

Therefore

$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ (ii)

From (i) and (ii)

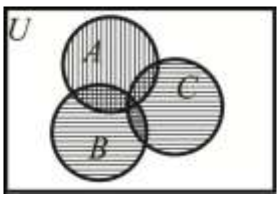
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Verification of the properties with the help of Venn diagrams

(i) Associative property of union

$A \cup (B \cup C) = (A \cup B) \cup C$

L.H.S = $A \cup (B \cup C)$

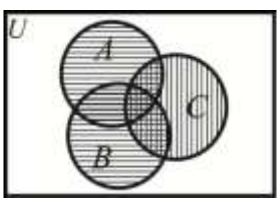


A: |||

$B \cup C$: ≡

$A \cup (B \cup C)$: |||, ≡ or ⊞

R.H.S= $(A \cup B) \cup C$



$A \cup B$: ≡

C: |||

$(A \cup B) \cup C$: |||, ≡ or ⊞

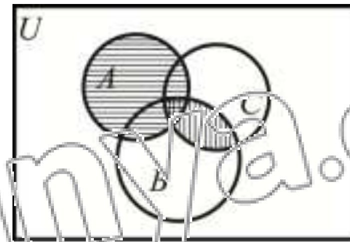
From two diagrams, we can see that

$A \cup (B \cup C) = (A \cup B) \cup C$

(ii) Associative property of intersection

$A \cap (B \cap C) = (A \cap B) \cap C$

L.H.S = $A \cap (B \cap C)$

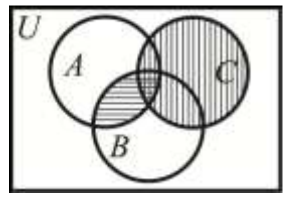


A: ≡

$B \cap C$: |||

$A \cap (B \cap C)$: ⊞

R.H.S = $(A \cap B) \cap C$



$A \cap B$: ≡

C: |||

$(A \cap B) \cap C$: ⊞

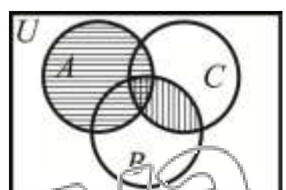
From two diagrams we can see that

$A \cap (B \cap C) = (A \cap B) \cap C$

(iii) Distributive laws

Proof: (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

L.H.S $= A \cup (B \cap C)$

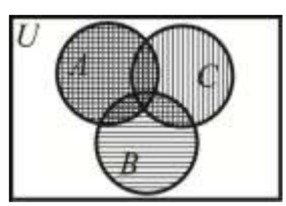


A: ≡

$B \cap C$: |||

$A \cup (B \cap C)$: ≡, ||| or ⊞

R.H.S $= (A \cup B) \cap (A \cup C)$



$A \cup B$:

$A \cup C$:

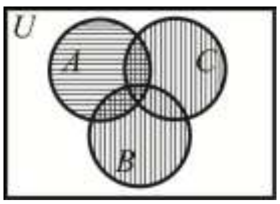
$(A \cup B) \cap (A \cup C)$:

From two diagrams, we can see that

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

L.H.S = $A \cap (B \cup C)$

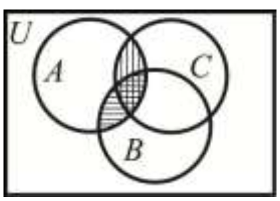


A:

$B \cup C$:

$A \cap (B \cup C)$:

R.H.S = $(A \cap B) \cup (A \cap C)$



$A \cap B$:

$A \cap C$:

$(A \cap B) \cup (A \cap C)$: , or

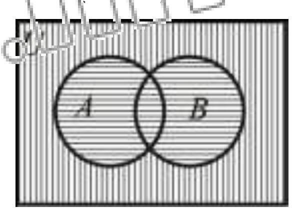
From two diagrams

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(iv) De Morgan's Laws

(a) $(A \cup B)' = A' \cap B'$

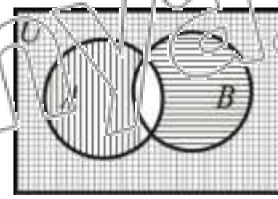
L.H.S = $(A \cup B)'$



$A \cup B$:

$(A \cup B)'$:

R.H.S = $A' \cap B'$



A' :

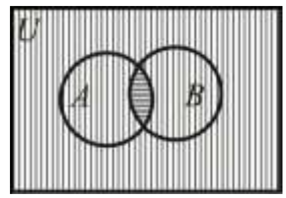
B' :

$A' \cap B'$:

From two diagrams $(A \cup B)' = A' \cap B'$

(b) $(A \cap B)' = A' \cup B'$

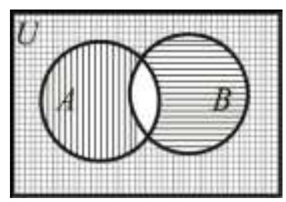
L.H.S = $(A \cap B)'$



$A \cap B$:

$(A \cap B)'$:

R.H.S = $A' \cup B'$



A' :

B' :

$A' \cup B'$: , or

From two diagrams $(A \cap B)' = A' \cup B'$

Exercise 2.3

Q.1 Verify the commutative properties of union & intersection for the following pairs of sets.
(DGK 2021, LHR 2022, BWP 2023, MTN 2023, GRW 2023)

(i) $A=\{1,2,3,4,5\}$, $B=\{4,6,8,10\}$

Solution:

$A\cup B=\{1,2,3,4,5,6,8,10\}$ (i)

$B\cup A=\{1,2,3,4,5,6,8,10\}$ (ii)

From (i) and (ii) commutative property of union is satisfied.

Now $A\cap B=\{4\}$ (iii)

$B\cap A=\{4\}$ (iv)

From (iii) and (iv) commutative property of intersection is satisfied.

(i) □ ,□

Solution: ¥ ∪¢ = ¢ = ¢ ∪¥

Which satisfies commutative property of union. Also,

¥ ∩¢ = ¥ = ¢ ∩¥

Which satisfies commutative property of intersection.

(ii) $A=\{x\,|\,x\in$ ¡ $\wedge x\geq 0\}$, $B=$ ¡

Solution:

$A\cup B=\{x\,|\,x\in$ ¡ $\wedge x\geq 0\}\cup$ ¡ $=$ ¡

and $B\cup A=A$ $\quad(\because A\subseteq B)$

So commutative property of union is satisfied.

Also $A\cap B=$

$\{x\,|\,x\in$ ¡ $\wedge x\geq 0\}\cap$ ¡ $=\{x\,|\,x\in$ ¡ $\wedge x\geq 0\}=A$

and $B\cap A=A$ $\quad(\because A\subseteq B)$

Which satisfies commutative property of intersection.

Q.2 Verify the properties for the sets A, B and C given below.

(a) $A=\{1,2,3,4\}$, $B=\{3,4,5,6,7,8\}$, $C=\{5,6,7,9,10\}$

(i) Associativity of union.

$A\cup B=\{1,2,3,....,8\}$

$(A\cup B)\cup C=\{1,2,3,...,10\}$

$B\cup C=\{3,4,5,...,10\}$

$A\cup(B\cup C)=\{1,2,3,...,10\}$

Hence proved that

$A\cup(B\cup C)=(A\cup B)\cup C$

(ii) Associativity of intersection.

$A\cap B=\{3,4\}$

$(A\cap B)\cap C=\phi$

$B\cap C=\{5,6,7\}$

$A\cap(B\cap C)=\phi$

Hence proved that

$(A\cap B)\cap C=A\cap(B\cap C)$

(iii) Distributivity of union over intersection.

$B\cap C=\{5,6,7\}$

$A\cup(B\cap C)=\{1,2,3,4,5,6,7\}$

(i)

$A\cup B=\{1,2,3,...,8\}$

$A\cup C=\{1,2,3,4,5,6,7,9,10\}$

Now

$(A\cup B)\cap(A\cup C)=\{1,2,3,4,5,6,7\}$

(ii)

From (i) and (ii) we conclude that

$A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$

Which satisfies the property

(iv) Distributivity of intersection over union.

$B\cup C=\{3,4,5,...,10\}$

$A\cap(B\cup C)=\{3,4\}$ (i)

$A\cap C=\phi,\quad A\cap B=\{3,4\}$

$(A\cap B)\cup(A\cap C)=\{3,4\}$ (ii)

From (i) and (ii) we conclude that

$A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

Which satisfies the property.

(b) $A=\phi, B=\{0\}, C=\{0,1,2\}$

(i) Associativity of union.

$A\cup B=\{0\}$

$(A\cup B)\cup C=\{0,1,2\}$ (i)

$B\cup C=\{0,1,2\}$

$A\cup(B\cup C)=\{0,1,2\}$ (ii)

From (i) and (ii) we get the proof.

(ii) Associativity of intersection.

$A\cap B=\phi$

$(A\cap B)\cap C=\phi$ (i)

$B\cap C=\{0\}$

$A\cap(B\cap C)=\phi$ (ii)

From (i) and (ii) we conclude the result.

(iii) Distributivity of union over intersection.

$B\cap C=\{0\}$

$A\cup(B\cap C)=\{0\}$ (i)

$A\cup B=\{0\}$

$A\cup C=\{0,1,2\}$

$(A\cup B)\cap(A\cup C)=\{0\}$ (ii)

From (i) and (ii) we get the proof.

(iv) Distributivity of intersection over union.

$B\cup C=\{0,1,2\}$

$A\cap(B\cup C)=\phi$ (i)

$A\cap B=\phi,\quad A\cap C=\phi$

$(A\cap B)\cup(A\cap C)=\phi$ (ii)

From (i) and (ii) we get the proof.

(c) □ ,□ ,□

(i) Associativity of union.

¥ ∪ ¢ = ¢ (∵□ ⊂□)

(¥ ∪ ¢)∪¤ = ¢ ∪¤ = ¤

(Q¢ ⊂¤) (i)

¢ ∪¤ = ¤ (∵□ ⊂□)

¥ ∪(¢ ∪¤)=¥ ∪¤ = ¤

(Q¥ ⊂¤) (ii)

(i) and (ii) verify the property.

(ii) Associativity of intersection.

¥ ∩¢ = ¥ (∵□ ⊂□)

(¥ ∩¢)∩¤ = ¥ ∩¤ = ¥

(Q¥ ⊂¤) (i)

¢ ∩¤ = ¢ (∵□ ⊂□)

¥ ∩(¢ ∩¤)=¥ ∩¢ = ¥

(Q¥ ⊂¢) (ii)

From (i) and (ii) we have proof.

(iii) Distributivity of union over intersection

¢ ∩¤ = ¢ (Q ¢ ⊂¤)

¥ ∪(¢ ∩¤)=¥ ∪¢ = ¢ (i)

(Q¥ ⊂¢)

(¥ ∪¢)∩(¥ ∪¤)=¢ ∩¤

= ¢ (Q¢ ⊂¤) (ii)

From (i) & (ii) we get the result.

(iv) Distributivity of intersection over union.

¢ ∪¤ =¤ (∵ ⊂)

¥ ∩(¢ ∪¤)=¥ ∩¤ =¥

(Q¥ ⊂¤) (i)

¥ ∩¢ =¥ (Q¥ ⊂¢)

¥ ∩¤ =¥ (Q¥ ⊂¤)

(¥ ∩¢)∪(¥ ∩¤)=¥ ∪¥ =¥

(ii)

From (i) and (ii) we get the proof.

Q.3 Verify de-Morgan's Laws for the following sets.

$U=\{1,2,3,...,20\}$

$A=\{2,4,6,...,20\}$

$B=\{1,3,5,7,...,19\}$

Solution: $A\cup B=\{1,2,3,...,20\}$

$(A\cup B)'=U-(A\cup B)$

$=\phi$

$A'=U-A$

$=\{1,3,5,...,19\}$

$B'=U-B$

$=\{2,4,6,...,20\}$

$A'\cap B'=\phi$

Which shows that $(A\cup B)'=A'\cap B'$

Now $A\cap B=\phi$

$(A\cap B)'=\{1,2,3,...,20\}$

$A'\cup B'=\{1,2,3,...,20\}$

Which shows that $(A\cap B)'=A'\cup B'$

Hence the proof.

Q.4 Let *U*= The set of English alphabets. (FSD 2023)

$A=\{x\,|\,x \text{ is a vowel}\}$

$B=\{y\,|\,y \text{ is a consonant}\}$

Verify de-Morgan's Laws.

Now $U=\{a,b,c,d,e,...,z\}$

$A=\{a,e,i,o,u\}$

$B=\left\{\begin{matrix} b,c,d,f,g,h,j,k,l,m,n,p,q,r, \\ s,t,v,w,x,y,z \end{matrix}\right\}$

Now $A\cup B=\{a,b,c,...,z\}$

$(A\cup B)'=\phi$ (i)

$A'=U-A$

$=\left\{\begin{matrix} b,c,d,f,g,h,j,k,l,m,n,p,q,r, \\ s,t,v,w,x,y,z \end{matrix}\right\}$

$B'=U-B$

$=\{a,e,i,o,u\}$

$A'\cap B'=\phi$ (ii)

From (*i*) and (*ii*) $(A\cup B)'=A'\cap B'$

Also $A\cap B=\phi$

$(A\cap B)'=U-\phi=U$ (iii)

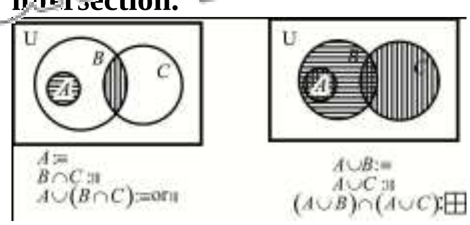
$A'\cup B'=\{a,b,c,...,z\}=U$ (iv)

Form (iii) and (iv) $(A\cap B)'=A'\cup B'$

Q.5 With the help of Venn diagrams, verify the two distributive properties in the following cases with respect to union and intersection.

(i) $A\subseteq B$, $A\cap C=\phi$ **and** B **&** C **are over lapping.**

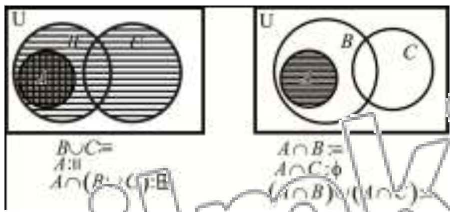
Distributivity of union over intersection.



From Venn diagrams

$A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$

Distributivity of intersection over union.

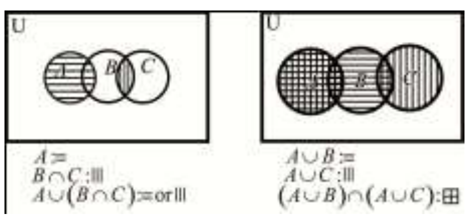


From Venn diagrams

$A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

(ii) **A & B are overlapping, B & C are overlapping, but A & C are disjoint.**

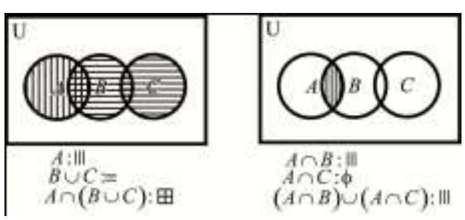
Distributivity of union over intersection.



From Venn diagrams

$A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$

Distributivity of intersection over union.



From Venn diagrams

$A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

Q.6 Taking any set, say $A=\{1,2,3,4,5\}$ **verify the following.**

(i) $A\cup\phi=A$

$A\cup\phi=\{1,2,3,4,5\}\cup\phi=\{1,2,3,4,5\}=A$

(ii) $A\cup A=A$

$A\cup A=\{1,2,3,4,5\}\cup\{1,2,3,4,5\}$

$=\{1,2,3,4,5\}$

$=A$

(iii) $A\cap A=A$

$A\cap A=\{1,2,3,4,5\}\cap\{1,2,3,4,5\}$

$=\{1,2,3,4,5\}$

$=A$

Q.7 If $U=\{1,2,3,...,20\}, A=\{1,3,5,...,19\}$ **verify the following.**

(i) $A\cup A'=U$ **(GRW 21, FSD 2021)**

$A'=U-A$

$=\{2,4,6,...,20\}$

Now

$A\cup A'=\{1,3,5,...19\}\cup\{2,4,6,...,20\}$

$\{1,2,3,...,20\}$

$=U$

(ii) $A\cap U=A$

$A\cap U=\{1,3,5,...,19\}\cap\{1,2,3,...,20\}$

$=\{1,3,5,...,19\}$

$=A$

(iii) $A\cap A'=\phi$

$A'=\{2,4,6,...,20\}$

$A\cap A'=\{1,3,5,...,19\}\cap\{2,4,6,...,20\}$

$=\phi$

Q.8 From suitable properties of union and intersection deduce the following results.

(i) $A\cap(A\cup B)=A\cup(A\cap B)$ **(RWP 2022)**

Solution: L.H.S. $=A\cap(A\cup B)$

$=(A\cap A)\cup(A\cap B)$

(Distributive Law)

$=A\cup(A\cap B)$ $(\because A\cap A=A)$

= R.H.S.

(ii) $A\cup(A\cap B)=A\cap(A\cup B)$

Solution: L.H.S. $=A\cup(A\cap B)$

$=(A\cup A)\cap(A\cup B)$

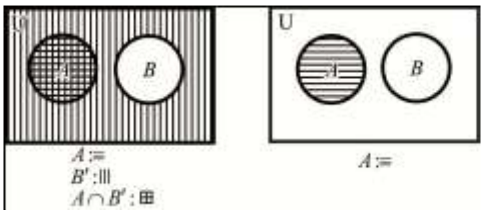
(Distributive Law)

$=A\cap(A\cup B)$ $(\because A\cup A=A)$

= R.H.S

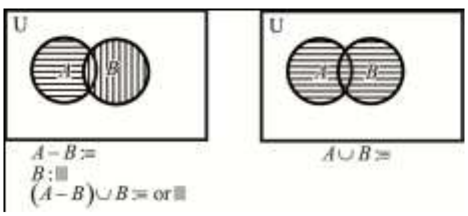
Using Venn diagrams, verify the following results.

(i) $A\cap B'=A$ iff $A\cap B=\phi$



From Venn diagrams $A\cap B'=A$

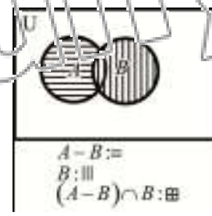
(ii) $(A-B)\cup B=A\cup B$



From Venn diagrams

$(A-B)\cup B=A\cup B$

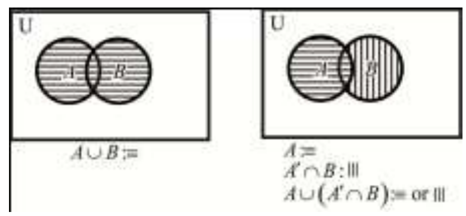
(iii) $(A-B)\cap B=\phi$



As such region does not exist so

$(A-B)\cap B=\phi$

(iv) $A\cup B=A\cup(A'\cap B)$



From Venn diagrams

$A\cup B=A\cup(A'\cap B)$

EXERCISE 2.4

Q.1 Write the converse, inverse and contrapositive of the following conditionals:-

(i) : $p\to q$

(GRW 2021, 23, RWP 2021, LHR 2022, DGK 2023, SGD 2023)

Converse: $q\to\sim p$

Inverse: $\sim p\to\sim q$

Contrapositive: $\sim q\to\sim p$

(ii) $q\to p$ **(SGD 2022)**

Converse: $p\to q$

Inverse: $\sim q\to\sim p$

Contrapositive: $\sim p\to\sim q$

(iii) $\sim p\to\sim q$ **(FSD 2021, SHW 2021)**

Converse: $\sim q\to\sim p$

Inverse: $p\to q$

Contrapositive: $q\to p$

(iv) $\sim q\to\sim p$

Converse: $\sim p\to\sim q$

Inverse: $q\to p$

Contrapositive: $p\to q$

Q.2 Construct truth tables for the following statements:

(i) $(p \to \sim p) \vee (p \to q)$ **(LHR 2022, RWP 2023, GRW 2023)**

p	q	$\sim p$	$p \to \sim p$	$p \to q$	$(p \to \sim p) \vee (p \to q)$
T	T	F	F	T	T
T	F	F	F	F	F
F	T	T	T	T	T
F	F	T	T	T	T

(ii) $(p \wedge \sim p) \to q$ **(MTN 2021, 22, 23, RWP 2022, 23)**

p	q	$\sim p$	$p \wedge \sim p$	$(p \wedge \sim p) \to q$
T	T	F	F	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

(iii) $\sim (p \to q) \leftrightarrow (p \wedge \sim q)$ **(LHR 2021, FSD 2023)**

p	q	$\sim q$	$p \to q$	$\sim (p \to q)$	$p \wedge \sim q$	$\sim (p \to q) \leftrightarrow (p \wedge \sim q)$
T	T	F	T	F	F	T
T	F	T	F	T	T	T
F	T	F	T	F	F	T
F	F	T	T	F	F	T

Q.3 Show that each of the following statements is a tautology.

(i) $(p \wedge q) \to p$

(DGK 2021, 23, SGD 2021, LHR 2021, RWP 2022, GRW 2022)

p	q	$p \wedge q$	$(p \wedge q) \to p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Last column shows that given statement is a tautology.

(ii) $p \to (p \vee q)$

(SGD 2021, MTN 2021)

p	q	$p \vee q$	$p \to (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Last column shows that given statement is a tautology.

(iii) $\sim(p \to q) \to p$ **(MTN 2023)**

p	q	$p \to q$	$\sim(p \to q)$	$\sim(p \to q) \to p$
T	T	T	F	T
T	F	F	T	T
F	T	T	F	T
F	F	T	F	T

Last column shows that given statement is a tautology.

(iv) $\sim q \wedge (p \to q) \to \sim p$ **(SHW 2022, 23)**

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \wedge (p \to q)$	$\sim q \wedge (p \to q) \to \sim p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

Last column shows that given statement is a tautology.

Q.4 Determine whether each of the following is a tautology, a contingency or an absurdity.

(i) $p \wedge \sim p$

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

Have $p \wedge \sim p$ is an absurdity.

(ii) $p \to (q \to p)$

p	q	$q \to p$	$p \to (q \to p)$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	T	T

Hence $p \to (q \to p)$ is a tautology.

(iii) $q \vee (\sim q \vee p)$ **(DGK 2022)**

p	q	$\sim q$	$\sim q \vee p$	$q \vee (\sim q \vee p)$
T	T	F	T	T
T	F	T	T	T
F	T	F	F	T
F	F	T	T	T

Hence $q \vee (\sim q \vee p)$ is a tautology.

Q.5 Prove that $p\vee(\sim p\wedge\sim q)\vee(p\wedge q)=p\vee(\sim p\wedge\sim q)$ **(SGD 2023)**

p	Q	$\sim p$	$\sim q$	$p\wedge q$	$\sim p\wedge\sim q$	$p\vee(\sim p\wedge\sim q)$	$p\vee(\sim p\wedge\sim q)\vee(p\wedge q)$
T	T	F	F	T	F	T	T
T	F	F	T	F	F	T	T
F	T	T	F	F	F	F	F
F	F	T	T	F	T	T	T

The last two columns show that

$p\vee(\sim p\wedge\sim q)\vee(p\wedge q)=p\vee(\sim p\wedge\sim q)$

Exercise 2.5

Convert the following theorems to logical form and prove them by constructing truth tables:

Q.1 $(A\cap B)'=A'\cup B'$ **(BWP 2021, FSD 2022)**

Solution: The corresponding logical form is $\sim(p\wedge q)=\sim p\vee\sim q$

p	q	$\sim p$	$\sim q$	$p\wedge q$	$\sim p\vee\sim q$	$\sim(p\wedge q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Last two columns show that $\sim(p\wedge q)=\sim p\vee\sim q$ and hence $(A\cap B)'=A'\cup B'$

Q.2 $(A\cup B)\cup C=A\cup(B\cup C)$

Solution: The corresponding logical form is $(p\vee q)\vee r=p\vee(q\vee r)$

p	q	r	$p\vee q$	$q\vee r$	$(p\vee q)\vee r$	$p\vee(q\vee r)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	F	T	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

Last two columns show that $(p\vee q)\vee r=p\vee(q\vee r)$ and hence

$(A\cup B)\cup C=A\cup(B\cup C)$

Q.3 $(A\cap B)\cap C = A\cap(B\cap C)$

Solution: The corresponding logical form is $(p\wedge q)\wedge r = p\wedge(q\wedge r)$

p	q	r	$p\wedge q$	$q\wedge r$	$(p\wedge q)\wedge r$	$p\wedge(q\wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	T	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

Last two columns show that $(p\wedge q)\wedge r = p\wedge(q\wedge r)$ and hence

$(A\cap B)\cap C = A\cap(B\cap C)$

Q.4 $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$ **(MTN 2022)**

Solution: The corresponding logical form is

$p\vee(q\wedge r) = (p\vee q)\wedge(p\vee r)$

p	q	r	$p\vee q$	$p\vee r$	$q\wedge r$	$p\vee(q\wedge r)$	$(p\vee q)\wedge(p\vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	T	F	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F
F	F	T	F	T	F	F	F
F	F	F	F	F	F	F	F

Last two columns show that $p\vee(q\wedge r) = (p\vee q)\wedge(p\vee r)$ and hence

$A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$

Exercise 2.6

Q.1 For $A=\{1,2,3,4\}$, find the following relations in A. State the domain and range of each relation. Also draw the graph of each.

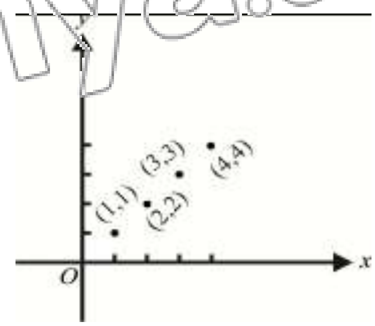
(i) $\{(x,y)\mid y=x\}$

Solution:

Let $R_1=\{(x,y)\mid y=x\}=\{(1,1),(2,2),(3,3),(4,4)\}$

The domain of R_1 is $\{1,2,3,4\}$

Range of R_1 is $\{1,2,3,4\}$



(ii) $\{(x,y)\mid y+x=5\}$ **(LHR 2021, DGK 2022)**

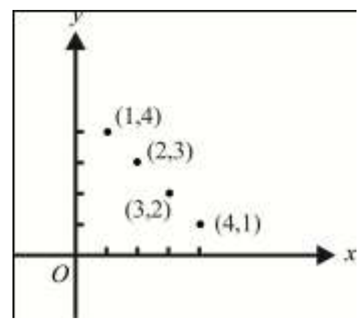
Solution:

Let $R_2=\{(x,y)\mid y+x=5\}$

$=\{(1,4),(2,3),(3,2),(4,1)\}$

Domain of $R_2=\{1,2,3,4\}$

Range of $R_2=\{1,2,3,4\}$



(iii) $\{(x,y)\mid x+y<5\}$ **(FSD 2022)**

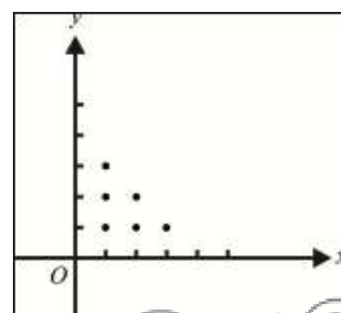
Solution:

Let $R_3=\{(x,y)\mid x+y<5\}$

$=\{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1)\}$

Domain of $R_3=\{1,2,3\}$

Range of $R_3=\{1,2,3\}$



(iv) $\{(x,y)\mid x+y>5\}$ **(MTN 2021, GRW 2021)**

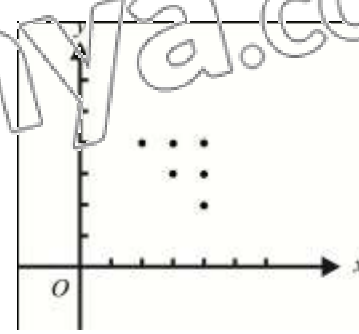
Solution:

Let $R_4=\{(x,y)\mid x+y>5\}$

$=\{(2,4),(3,3),(3,4),(4,2),(4,3),(4,4)\}$

Domain of $R_4=\{2,3,4\}$

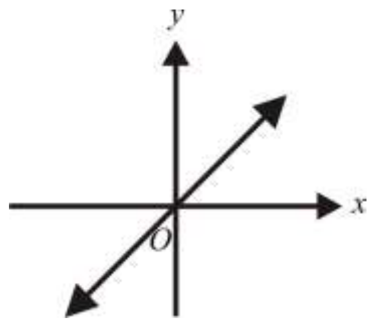
Range of $R_4=\{2,3,4\}$



Q.2 Repeat Q:1 when $A = \mathbb{R}$, the set of real numbers. Which of the real lines are functions?

(i) $\{(x, y) \mid y = x\}$

The domain of above relation is $\mathbb{R}$ and range is also $\mathbb{R}$. The graph gives straight line passing through origin. Given relation is a function since each value of x gives unique value of y.



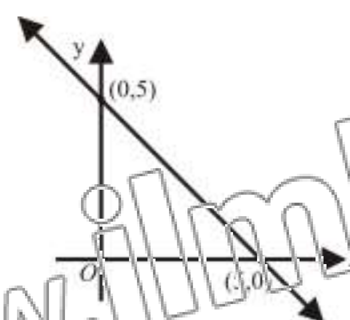
(ii) $\{(x, y) \mid y + x = 5\}$

Using

$y + x = 5,$ When $y = 0,$ $x = 5$

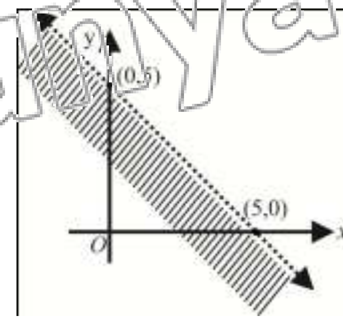
And when $x = 0,$ $y = 5$, so (5,0) and (0,5) lie on the graph. The domain and range is $\mathbb{R}$.

Given relation is a function since each value of x gives unique value of y



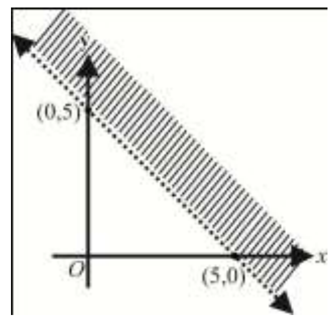
(iii) $\{(x, y) \mid x + y < 5\}$

Using $x + y = 5$, when $x = 0, y = 5$ and when $y = 0, x = 5$. The graph is shown in figure. The domain and range is $\mathbb{R}$. Clearly given relation is not a function.



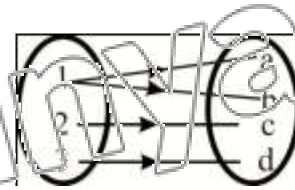
(iv) $\{(x, y) \mid x + y > 5\}$

Using $x + y = 5$, we get (5,0) and (0,5) on graph as shown in fig. The domain & range is $\mathbb{R}$. Clearly given relation is not a function.



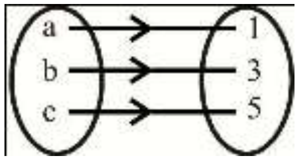
Q.3 Which of the following diagrams represent functions and of which type?

(i)



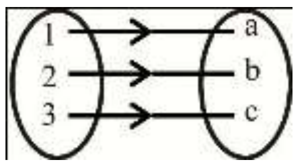
The above figure does not represent a function since element 1 has two images a and b, while for function each element in domain must have a unique image.

(ii)



The figure represents a function since every element in domain has a unique image. Also distinct elements have distinct images, therefore it is a one-to-one function. It is also an on to function. Hence given figure represents a bijective function.

(iii)



The figure represents a function since every element in domain has a unique image. Also distinct elements have distinct images, therefore it is a one-to-one function. It is also an on to function. Hence given figure represents a bijective function.

(iv)

Each element in domain has unique image, so this represents a function. But distinct elements do not have distinct images, so this is not a 1-1 function. As range $\neq \{x, y, z\}$, so given figure represents an into function.

Q.4 Find inverse of each of the following relations. Tell whether each relation and its inverse is a function or not.

(i) $\{(2,1),(3,2),(4,3),(5,4),(6,5)\}$

Solution:

Let $R=\{(2,1),(3,2),(4,3),(5,4),(6,5)\}$ then its inverse is

$R^{-1}=\{(1,2),(2,3),(3,4),(4,5),(5,6)\}$

Both R and R^{-1} are functions.

(ii) $\{(1,3),(2,5),(3,7),(4,9),(5,11)\}$ **(LGK 2021, 22)**

Solution:

Let $R=\{(1,3),(2,5),(3,7),(4,9),(5,11)\}$

Then $R^{-1}=\{(3,1),(5,2),(7,3),(9,4),(11,5)\}$

Both R and R^{-1} are functions.

(iii) $\{(x,y) \mid y = 2x+3, x \in \mathbb{R}\}$

Solution:

Let $R = \{(x,y) \mid y = 2x+3, x \in \mathbb{R}\}$

$y = 2x+3 \Rightarrow 2x = y-3 \Rightarrow x = \dfrac{y-3}{2}$

replace x by y

$y = \dfrac{x-3}{2}$

Then $R^{-1} = \left\{(x,y) \mid y = \dfrac{x-3}{2}, x \in \mathbb{R}\right\}$

Both R and R^{-1} are functions.

(iv) $\{(x,y) \mid y^2 = 4ax, x \geq 0\}$ **(SGD 2021)**

Solution:

Let $R = \{(x,y) \mid y^2 = 4ax, x \geq 0\}$

$y^2 = 4ax \Rightarrow \quad y = \pm 2\sqrt{ax}$

Which shows that we get two values of y for one value of x so the above relation is not a function.

Now $y^2 = 4ax \Rightarrow \quad x = \dfrac{y^2}{4a}$

Interchanging x and y, we get $y = \dfrac{x^2}{4a}$

Hence $R^{-1} = \left\{(x,y) \mid y = \dfrac{x^2}{4a}, y \geq 0\right\}$. Clearly R^{-1} is a function.

(v) $\{(x,y) \mid x^2 + y^2 = 9, |x| \leq 3, |y| \leq 3\}$

Solution:

Let $R = \{(x,y) \mid x^2 + y^2 = 9, |x| \leq 3, |y| \leq 3\}$

Using $x^2 + y^2 = 9$ we get $y = \pm\sqrt{9 - x^2}$

This shows that there are two values of y for one value of x. Hence R is not a function.

Interchanging x and y we get $y^2 + x^2 = 9$. Hence

$R^{-1} = \{(x,y) \mid y^2 + x^2 = 9, |x| \leq 3, |y| \leq 3\}$

Clearly R^{-1} is not a function.

www.ilmkidunya.com

EXERCISE 2.7

Q.1 Complete the table, indicating by a tick mark those properties which are satisfied by the specified set of numbers.

Property ↓ \ Set of Numbers →		Natural	Whole	Integers	Rational	Real
Closure	$\oplus$	✓	✓	✓	✓	✓
	$\otimes$	✓	✓	✓	✓	✓
Associative	$\oplus$	✓	✓	✓	✓	✓
	$\otimes$	✓	✓	✓	✓	✓
Identity	$\oplus$		✓	✓	✓	✓
	$\otimes$	✓	✓	✓	✓	✓
Inverse	$\oplus$			✓	✓	✓
	$\otimes$					
Commutative	$\oplus$	✓	✓	✓	✓	✓
	$\otimes$	✓	✓	✓	✓	✓

Q.2 What are field axioms? In what respect does the field of real numbers differ from that of complex numbers?

Solution: **Field:** A non empty set F is said to be a field if for all $x, y, z \in F$, the following axioms are satisfied.

1) $x + y \in F$

2) $x + (y + z) = (x + y) + z$

3) There exists $0 \in F$ such that $x + 0 = 0 + x = x$

4) There exists $-x \in F$ such that $x + (-x) = 0 = -x + x$

5) $x + y = y + x$

6) $xy \in F$

7) $x(yz) = (xy)z$

8) There exists $1 \in F$ such that $x.1 = 1.x = x$

9) There exists $\frac{1}{x} \in F$ such that $\frac{1}{x}.x = x.\frac{1}{x} = 1$ $\quad (x \neq 0)$

10) $xy = yx$

11) $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$

The field of real numbers differ from the field of complex numbers in a way that real field holds order axioms where as field of complex numbers does not hold order axioms.

Q.3 Show that the adjoining table is that of multiplication of the elements of the set of residue classes of modulo 5.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

The zeroes in the second row are produced by the 0 in the first column which shows that this is a product table. It is also noted that whenever a number equal to or greater than 5 is obtained, we divide it by 5 and write the reminder. Hence given table is that of multiplication of the elements of the set of residue classes modulo 5.

Q.4 Prepare a table of addition of the elements of the set of residue classes modulo 4.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Q.5 Which of the following binary operations shown in tables (a) and (b) is commutative?

(a)

$*$	a	b	c	d
a	a	c	b	d
b	b	c	b	a
c	c	d	b	c
d	a	a	b	b

(b)

$*$	a	b	c	d
a	a	c	b	d
b	c	d	b	a
c	b	b	a	c
d	d	a	c	d

In table (a) we have $a*c=b, c*a=c \Rightarrow a*c \neq c*a$ so operation $*$ is not commutative.

In table (b) elements across the diagonal are same, so operation $*$ is commutative e.g.

$a*b=b*a$

$c=c$

Q.6 Supply the missing elements of the third row of the given table so that the operation $*$ may be associative.

$*$	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c				
d	d	c	c	d

Solution:

$*$	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	p	q	r	s
d	d	c	c	d

Let missing elements be p, q, r and s

$p = c * a$ $= (d*b)*a$ $= d*(b*a)$ $= d*b$ $= c$	$q = c*b$ $= (d*b)*b$ $= d*(b*b)$ $= d*a$ $= d$	$r = c*c$ $= (d*b)*c$ $= d*(b*c)$ $= d*c$ $= c$	$s = c*d$ $= (d*b)*d$ $= d*(b*d)$ $= d*d$ $= d$

Q.7 Which operation is represented by the adjoining table? Name the identity element of the relevant set, if it exists. Is the operation associative? Find the inverses of 0,1,2,3, if they exist.

※	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

The second row of the given table is obtained by adding 0 in 0,1,2,3. This shows that the operation is addition. It is also noted that whenever a number equal to or greater than 4 is obtained, we divide it by 4 and write the remainder. So the binary operation is addition modulo 4.

0 is identity.

Clearly the binary operation is associative. e.g. $(1*2)*3 = 1*(2*3)$

$$3*3 = 1*1$$
$$2 = 2$$

The inverse of 0 is 0.
The inverse of 1 is 3
The inverse of 2 is 2
The inverse of 3 is 1

EXERCISE 2.8

Q.1 Operation $\oplus$ is performed on the two-member set G={0,1} is shown in the adjoining table. Answer the questions.

(i) Name the identity element if it exists.

$\oplus$	0	1
0	0	1
1	1	0

Solution: From the given table, $0+0=0,\ 0+1=1$ which shows that 0 is the identity element.

(ii) What is the inverse of 1?

Solution: Since $1+1=0$ (identity) so inverse of 1 is 1.

(iii) Is the set G, under the given operation a group? Abelian or non-Abelian?

Solution: The numbers in table satisfy all the properties of abelian group so G is an abelian group under addition.

Q.2 The operation $\oplus$ as performed on the set $\{0,1,2,3\}$ is shown in the adjoining table, show that the set is an abelian group.

Solution:

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

(i) Each element of the table is an element of the given set $\{0,1,2,3\}$, so closure law holds.

(ii) Associative law holds.

e.g. (1 + 2) + 3 = 1 + (2 + 3)

3 + 3 = 1 + 1

2 = 2

(iii) 0 is the identity element.

(iv) The inverse of 0 is 0.

The inverse of 1 is 3.

The inverse of 2 is 2.

The inverse of 3 is 1

so inverse of every element exists.

(v) Commutative law also holds. e.g. 1 + 2 = 2 + 1

3 = 3

So the set $\{0,1,2,3\}$ is an abelian group under addition modulo 4.

Q.3 For each of the following sets, determine whether or not the set forms a group with respect to the indicated operation.

Set	**Operations**
(i) The set of rational numbers	$\times$
(ii) The set of rational numbers	$+$
(iii) The set of positive rational numbers	$\times$
(iv) The set of integers	$+$
(v) The set of integers	$\times$

Solution

(i) Let $Q =$ The set of rational numbers

(i) As product of any two rational numbers is also a rational number so Q is closed w.r.t. multiplication

(ii) Multiplication of rational numbers is always associative
i.e. $\forall a,b,c \in Q \Rightarrow (ab)c = a(bc)$

(iii) Here identity element is $1 \in Q$

(iv) Multiplicative inverse of $0 \in Q$ does not exist, so Q is not a group under multiplication

(ii) Let $Q =$ The set of rational numbers

(i) As sum of any two rational numbers is also a rational number so Q is closed w.r.t. addition.

(ii) Addition of rational numbers is always associative.
i.e. $\forall a,b,c \in Q \Rightarrow (a+b)+c = a+(b+c)$

(iii) Here identity element is $0 \in Q$

(iv) $\forall a \in Q$, the additive inverse is $-a \in Q$.
Hence Q is a group under addition

(iii) Let $Q^+ =$ The set of positive rational numbers

(i) As product of any two positive rational numbers is also a positive rational number so Q^+ is closed w.r.t. multiplication.

(ii) Multiplication of positive rational numbers is always associative.
i.e. $\forall a,b,c \in Q^+ \Rightarrow (ab)c = a(bc)$

(iii) Here identity element is $1 \in Q^+$

(iv) $\forall a \in Q^+$, the multiplicative inverse is $\frac{1}{a} \in Q^+$
Hence Q^+ is a group under multiplication

(iv) Let $Z =$ The set of integers

(i) As sum of any two integers is also an integer so Z is closed w.r.t. addition.

(ii) Addition of integers is always associative

i.e. $\forall a,b,c \in Z \Rightarrow (a+b)+c = a+(b+c)$

(iii) Here identity element is $0 \in Z$

(iv) $\forall a \in Z$, the additive inverse is $-a \in Z$.

Hence Z is a group under addition

(v) Let $Z =$ The set of integers

(i) As product of any two integers is also an integer so Z is closed w.r.t. multiplication

(ii) Multiplication of integers is always associative

i.e. $\forall a,b,c \in Z \Rightarrow (ab)c = a(bc)$

(iii) Here identity element is $1 \in Z$

(iv) Multiplicative inverse of any element of Z does not exist in Z except $\pm 1 \in Z$.

Hence Z is not a group under multiplication.

Q.4 Show that the adjoining table represents the sums of the elements of the set {E,O}.What is the identity element of this set? Show that this set is an abelian group.

$\oplus$	E	O
E	E	O
O	O	E

Solution: Since the sum of two even integers is also an even integer, so $E+E=E$

The sum of an even and an odd integers is odd, i.e. $E+O=O$

The sum of two odd integers is also even, i.e. $O+O=E$

Hence given table represents the sums of the elements of the set $\{E,O\}$.

Now since $E+E=E$ and $E+O=O$ so E is the identity element.

Now we show that this is an abelian group.

(i) Given set is closed under addition.

(ii) Associative law of addition holds in given set. e.g. $(E+O)+E = E+(O+E)$

$$O+E = E+O$$
$$O = O$$

(iii) Already proved that identity is E, so identity exists.

(iv) The inverse of E is E.

The inverse of O is O.

so inverse of each element exists.

(v) $O+E=O=E+O$ so commutative law holds.

Hence given set is an abelian group under addition.

Q.5 Show that the set $\{1,\omega,\omega^2\}$, when $\omega^3=1$, is an Abelian group w. r. t. ordinary multiplication.

Solution: Let $G=\{1,\omega,\omega^2\}$

$\times$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

(i) From the table it is clear that G is closed w.r.t. $\times$.

(ii) Multiplication of complex numbers is associative and $G\subset C$, so associative law of multiplication holds in G.

(iii) Identity element of G is 1

(iv) Inverse of each element exists.
Inverse of 1 is 1
Inverse of ω is ω^2
Inverse of ω^2 is ω

(v) Multiplication of complex numbers is commutative and $G\subset C$, so commutative law of multiplication holds in G
Hence G is an abelian group w.r.t. multiplication.

Q.6 If G is a group under operation $*$ and $a,b\in G$, find the solutions of the equations

(i) $a*x=b$

(ii) $x*a=b$

Solution:

(i) Since $a\in G$ and G is a group so $a^{-1}\in G$
Given $a*x=b$

$\Rightarrow a^{-1}*(a*x)=a^{-1}*b$

$\Rightarrow (a^{-1}*a)*x=a^{-1}*b$ (Associative Law)

$\Rightarrow e*x=a^{-1}*b$ $(a^{-1}*a=e)$

$\Rightarrow x=a^{-1}*b$ $(e*x=x)$

(ii) Since $a\in G$ and G is a group so $a^{-1}\in G$
Given $x*a=b$

$\Rightarrow (x*a)*a^{-1}=b*a^{-1}$

$\Rightarrow x*(a*a^{-1})=b*a^{-1}$ (Associative Law)

$\Rightarrow x*e=b*a^{-1}$ $(a*a^{-1}=e)$

$\Rightarrow x=b*a^{-1}$ $(x*e=x)$

Q.7 Show that the set consisting of elements of the form $a+\sqrt{3}b$, (a,b being rational) is an abelian group w .r .t addition.

Solution: Let $G=\{a+\sqrt{3}b \mid a,b\in Q\}$

Let x,y,z be any three elements of G and

$x=a+\sqrt{3}b,\quad y=c+\sqrt{3}d,\quad z=e+\sqrt{3}f$ where a, b, c, d, e, f are rational numbers.

(i) $x+y=\left(a+\sqrt{3}b\right)+\left(c+\sqrt{3}d\right)$

$=(a+c)+\sqrt{3}(b+d)\in G$ as $a+c, b+d\in Q$

So G is closed under addition.

(ii) Addition of real numbers is associative and $G\subset R$ so associative law of addition holds in G

(iii) $0=0+\sqrt{3}(0)$ is the identity element in G.

(iv) For all $x=a+\sqrt{3}b\in G,$ we have $-x=-a-\sqrt{3}b\in G$ such that

$x+(-x)=\left(a+\sqrt{3}b\right)+\left(-a-\sqrt{3}b\right)=0$. This shows that inverse of each element of G exists in G.

(v) Addition of real number is commutative and $G\subset R$ so commutative law of addition holds in G.

Hence G is an abelian group under addition

Q.8 Determine whether $(P(S),*)$ where $*$ stands for intersection is a semi-group, a monoid or neither. If it is a monoid, specify its identity.

(i) Since the intersection of two subsets of S is also its subset and will be contained by $P(S)$, so $P(S)$ is closed.

(ii) Intersection of sets is always associative.

i.e. $\forall A,B,C\in P(S)\Rightarrow (A\cap B)\cap C=A\cap(B\cap C)$

(iii) For all $A\in P(S)$, $A\cap S=A$ ($\because A$ is a subset of S). This shows that the identity element is $S\in P(S)$

This shows that $(P(S),*)$ is a monoid having identity S.

Q.9 Complete the following table to obtain a semi-group under $*$.

$*$	a	b	c
a	c	a	b
b	a	b	c
c			a

Solution:

Let missing elements be p and q

$*$	a	b	c
a	c	a	b
b	a	b	c
c	p	q	a

$$\begin{aligned} p &= c*a \\ &= (a*a)*a \quad (c = a*a) \\ &= a*(a*a) \quad (\text{Associative Law}) \\ &= a*c \\ &= b \end{aligned}$$

$$\begin{aligned} q &= c*b \\ &= (a*a)*b \quad (c = a*a) \\ &= a*(a*b) \quad (\text{Associative Law}) \\ &= a*a \\ &= c \end{aligned}$$

Q.10 Prove that all 2×2 non-singular matrices over the real field form a non-abelian group under multiplication.

Solution: Let G be the set of all 2×2 non-singular matrices over the real field.

(i) As product of any two 2 × 2 matrices is again a matrix of order 2 × 2, so G is closed under multiplication.

(ii) Associative law of multiplication holds in matrices confirmable for multiplication. i.e. $\forall\ A, B, C \in G \Rightarrow (AB)C = A(BC)$.

(iii) Since identity matrix of order 2×2 is also a non – singular matrix, so $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in G$ is identity element in G.

(iv) The inverse of every 2×2 non-singular matrix exists and is given by $A^{-1} = \frac{Adj A}{|A|} \in G$, so inverse of every matrix of G exists.

(v) Commutative law of multiplication does not hold in matrices i.e. generally, $AB \neq BA$.

So G is a non-abelian group under multiplication.