CHAPTER 3

MATRICES AND DETERMINANTS

Matrix:

A rectangular array of numbers enclosed by a pair of brackets e.g $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 4 & 7 \end{bmatrix}$ is called matrix.

Points to Remember:

- Horizontal lines of numbers are called rows.
- Vertical lines of numbers are called columns.
- If there are m rows and n columns then order of matrix is defined as $m \times n$.

Row Matrix or Row Vector:

A matrix, which has only one row, i.e, a $1 \times n$ matrix of the form $\begin{bmatrix} a_{i1} & a_{i2} & a_{i3} & \dots & a_{in} \end{bmatrix}$ is said to be a row matrix or row vector. For example $\begin{bmatrix} 1 & -1 & 3 & 4 \end{bmatrix}$ is a row matrix having 4 columns.

Column Matrix or Column Vector:

A matrix which has only one column i.e $m \times 1$ matrix of the form $\begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{mj} \end{bmatrix}$ is said to be a column matrix or a column vector. For example $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ is a column matrix having 3 rows.

Rectangular Matrix:

If $m \neq n$, then the matrix is called a rectangular matrix of order $m \times n$, that is, the matrix in which number of rows are not equal to the number of columns, is said to be a rectangular matrix for example; $\begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 4 \end{bmatrix}$ is rectangular matrix of order 2×3.

Square Matrix:

If $m=n$, then the matrix of the order $m\times n$ is said to be a square matrix of order n or m. i.e., the matrix which has same number of rows and columns is called a square matrix. For example;

$\begin{bmatrix} 2 & 5 \\ -1 & 6 \end{bmatrix}$ is a square matrix of order 2.

Points to Remember:

- Let $A[a_{ij}]$ be a square matrix of order n, then the entries $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ form a **principal diagonal** for the matrix A and the entries $a_{1n}, a_{2\,n-1}, a_{3\,n-2}, ..., a_{n-1\,2}, a_{n\,1}$ form secondary diagonal for the matrix A.
- The principal diagonal is also named as **leading diagonal** or **main diagonal**.
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Diagonal Matrix:

Let $A=\left[a_{ij}\right]$ be a square matrix of order 'n'. If $a_{ij}=0\ \forall i\neq j$ and at least one $a_{ij}\neq 0\ for\ i=j$, that is, some elements of the principal diagonal of A may be zero but not all, then the matrix A is called a **diagonal matrix**. The matrices

$[7], \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ are diagonal matrices.

Scalar Matrix:

Let $A=\left[a_{ij}\right]$ be a square matrix of order 'n'. If $a_{ij}=0\ \forall i\neq j$ and $a_{ij}=k$ (some non-zero scalar) $\forall i=j$, then the matrix is called a **scalar matrix** of order 'n'. For example

$\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ are scalar matrices of order 2 and 4.

Unit Matrix or Identity Matrix:

Let $A=\left[a_{ij}\right]$ be a square matrix of order n. If $a_{ij}=0\ \forall i\neq j$ and $a_{ij}=1\ \ \forall\ i=j$, then the matrix A is called a **unit matrix** or **identity matrix** of order n denoted by I_n. The identity matrix of order is 3 is denoted by I_3, that is, $I_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Null Matrix or Zero Matrix:

A square or rectangular matrix whose each element is zero, is called a **null** or **zero** matrix. An $m\times n$ matrix with all its elements equal to zero is denoted by $O_{m\times n}$. Here are some example

$$[0],[0 \quad 0 \quad 0],\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix}$$

O may be used to denote null matrix of any order if there is no confusion.

Equal Matrices:

Two matrices of the same order are said to be **equal** if their corresponding entries are equal e.g. $A=\left[a_{ij}\right]_{m\times n}$ and $B=\left[b_{ij}\right]_{m\times n}$ are equal i.e., $A=B$ iff $a_{ij}=b_{ij}$ for $i=1,2,3,\ldots,m,\ j=1,2,3,\ldots,n$.

Points to Remember:

- Two matrices are confirmable for addition if they are of the same order.
- If $A=\left[a_{ij}\right]$ is $m\times n$ matrix and k is a scalar then $kA=\left[ka_{ij}\right]$.
- Two matrices *A* and *B* are said to be confirmable for the product *AB* if the number of columns of *A* is equal to the number of rows of *B*.

If $O(A)=2\times 3$ and $O(B)=3\times 2$

$\Rightarrow O(AB)=2\times 2$

EXERCISE 3.1

Q.1 If $A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 7 \\ 6 & 4 \end{bmatrix}$, then show that

(i) $4A - 3A = A$

Solution:

$4A - 3A = A$

$$\text{L.H.S} = 4A - 3A$$

$$= 4\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} - 3\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 12 \\ 4 & 20 \end{bmatrix} - \begin{bmatrix} 6 & 9 \\ 3 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 8-6 & 12-9 \\ 4-3 & 20-15 \end{bmatrix}$$

$$\text{L.H.S} = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} = A = \text{R.H.S}$$

L.H.S = R.H.S

Hence $4A - 3A = A$

(ii) $3B - 3A = 3(B - A)$

Solution: $\text{L.H.S} = 3B - 3A$

$$= 3\begin{bmatrix} 1 & 7 \\ 6 & 4 \end{bmatrix} - 3\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 21 \\ 18 & 12 \end{bmatrix} - \begin{bmatrix} 6 & 9 \\ 3 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 3-6 & 21-9 \\ 18-3 & 12-15 \end{bmatrix}$$

$$\text{L.H.S} = \begin{bmatrix} -3 & 12 \\ 15 & -3 \end{bmatrix} \quad \text{(i)}$$

$$\text{R.H.S} = 3(B - A)$$

$$= 3\left(\begin{bmatrix} 1 & 7 \\ 6 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}\right)$$

$$= 3\left(\begin{bmatrix} 1-2 & 7-3 \\ 6-1 & 4-5 \end{bmatrix}\right)$$

$$= 3\begin{bmatrix} -1 & 4 \\ 5 & -1 \end{bmatrix}$$

$$\text{R.H.S} = \begin{bmatrix} -3 & 12 \\ 15 & -3 \end{bmatrix} \quad \text{(ii)}$$

From (i) and (ii)

L.H.S = R.H.S

$3B - 3A = 3(B - A)$

Q.2 If $A = \begin{bmatrix} i & 0 \\ 1 & -i \end{bmatrix}$ show that $A^4 = I_2$

Solution: We know that

$$A^2 = A.A$$

$$= \begin{bmatrix} i & 0 \\ 1 & -i \end{bmatrix} . \begin{bmatrix} i & 0 \\ 1 & -i \end{bmatrix}$$

$$= \begin{bmatrix} i \times i + 0 \times 1 & i \times 0 + 0 \times (-i) \\ 1 \times i + (-i \times 1) & 1 \times 0 + (-i) \times (-i) \end{bmatrix}$$

$$= \begin{bmatrix} i^2 + 0 & 0 + 0 \\ i - i & 0 + i^2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^4 = A^2.A^2$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} . \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} (-1 \times (-1)) + (0 \times 0) & (-1 \times 0) + (0 \times (-1)) \\ (0 \times -1) + (-1 \times 0) & (0 \times 0) + (-1 \times (-1)) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I_2$$

Hence $A^4 = I_2$

Q.3 Find x and y if

(i) $\begin{bmatrix} \mathbf{x+3} & \mathbf{1} \\ \mathbf{-3} & \mathbf{3y-4} \end{bmatrix} = \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{-3} & \mathbf{2} \end{bmatrix}$

Solution:

$$\begin{bmatrix} x+3 & 1 \\ -3 & 3y-4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

By definition of equal matrices, if two matrices are equal, then their corresponding elements must be equal.

$x+3=2$ and $3y-4=2$

$\Rightarrow x=2-3$, $\Rightarrow 3y=6$

$\Rightarrow x=-1$ $\Rightarrow y=2$

Hence $x=-1$ and $y=2$ are the required values of x and y.

(ii) $\begin{bmatrix} \mathbf{x+3} & \mathbf{1} \\ \mathbf{-3} & \mathbf{3y-4} \end{bmatrix} = \begin{bmatrix} \mathbf{y} & \mathbf{1} \\ \mathbf{-3} & \mathbf{2x} \end{bmatrix}$

Solution:

By definition of equal matrices, if two matrices are equal, then their corresponding elements must be equal.

$x+3=y$ and $3y-4=2x$

$x-y=-3$ (i)

$2x-3y=-4$ (ii)

Multiply 2 to equation (i) and subtracting from (ii)

$$2x-3y=-4$$
$$\underline{-2x\mp 2y=\mp 6}$$
$$-y=2$$
$$\Rightarrow y=-2$$

Put the value of y in (i)

$$x-(-2)=-3$$
$$x+2=-3$$
$$\Rightarrow x=-5$$

Hence $x=-5$ and $y=-2$ are the required values

Q.4 If $\mathbf{A} = \begin{bmatrix} \mathbf{-1} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{0} & \mathbf{2} \end{bmatrix}$ **and** $\mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathbf{3} & \mathbf{2} \\ \mathbf{1} & \mathbf{-1} & \mathbf{2} \end{bmatrix}$

Find the following matrices,

(i) 4A – 3B

Solution:

$$4A-3B = 4\begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} - 3\begin{bmatrix} 0 & 3 & 2 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 8 & 12 \\ 4 & 0 & 8 \end{bmatrix} - \begin{bmatrix} 0 & 9 & 6 \\ 3 & -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -4-0 & 8-9 & 12-6 \\ 4-3 & 0+3 & 8-6 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -1 & 6 \\ 1 & 3 & 2 \end{bmatrix}$$

Hence

$$4A-3B=\begin{bmatrix}-4 & -1 & 6\\ 1 & 3 & 2\end{bmatrix}$$

(ii) $\boldsymbol{A+3(B-A)}$

Solution:

Consider $B-A=\begin{bmatrix}0 & 3 & 2\\ 1 & -1 & 2\end{bmatrix}-\begin{bmatrix}-1 & 2 & 3\\ 1 & 0 & 2\end{bmatrix}$

$$=\begin{bmatrix}0+1 & 3-2 & 2-3\\ 1-1 & -1-0 & 2-2\end{bmatrix}$$

$$B-A=\begin{bmatrix}1 & 1 & -1\\ 0 & -1 & 0\end{bmatrix}$$

Now $A+3(B-A)$

$$=\begin{bmatrix}-1 & 2 & 3\\ 1 & 0 & 2\end{bmatrix}+3\begin{bmatrix}1 & 1 & -1\\ 0 & -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}-1 & 2 & 3\\ 1 & 0 & 2\end{bmatrix}+\begin{bmatrix}3 & 3 & -3\\ 0 & -3 & 0\end{bmatrix}$$

$$=\begin{bmatrix}-1+3 & 2+3 & 3-3\\ 1+0 & 0-3 & 2+0\end{bmatrix}$$

$$=\begin{bmatrix}2 & 5 & 0\\ 1 & -3 & 2\end{bmatrix}$$

Hence

$$A+3(B-A)=\begin{bmatrix}2 & 5 & 0\\ 1 & -3 & 2\end{bmatrix}$$

Q.5 Find x and y if $\begin{bmatrix}\mathbf{2} & \mathbf{0} & x\\ \mathbf{1} & y & \mathbf{3}\end{bmatrix}+\mathbf{2}\begin{bmatrix}\mathbf{1} & x & y\\ \mathbf{0} & \mathbf{2} & \mathbf{-1}\end{bmatrix}=\begin{bmatrix}\mathbf{4} & \mathbf{-2} & \mathbf{3}\\ \mathbf{1} & \mathbf{6} & \mathbf{1}\end{bmatrix}$

Solution: $\begin{bmatrix}2 & 0 & x\\ 1 & y & 3\end{bmatrix}+2\begin{bmatrix}1 & x & y\\ 1 & 2 & -1\end{bmatrix}=\begin{bmatrix}4 & -2 & 3\\ 1 & 6 & 1\end{bmatrix}$

$$\begin{bmatrix}2 & 0 & x\\ 1 & y & 3\end{bmatrix}+\begin{bmatrix}2 & 2x & 2y\\ 0 & 4 & -2\end{bmatrix}=\begin{bmatrix}4 & -2 & 3\\ 1 & 6 & 1\end{bmatrix}$$

$$\begin{bmatrix}2+2 & 0+2x & x+2y\\ 1+0 & y+4 & 3-2\end{bmatrix}=\begin{bmatrix}4 & -2 & 3\\ 1 & 6 & 1\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}4 & 2x & x+2y\\ 1 & y+4 & 1\end{bmatrix}=\begin{bmatrix}4 & -2 & 3\\ 1 & 6 & 1\end{bmatrix}$$

By definition of equal matrices, if two matrices are equal, then their corresponding elements must be equal.

$2x=-2$ and $y+4=6$

$\Rightarrow x=-1$ and $y=2$

So,

$x=-1$ and $y=2$

Are the required values of *x* and *y*.

Q.6 If $\mathbf{A}=\left[a_{ij}\right]_{3\times3}$, show that

(i) $\lambda(\mu A)=(\lambda\mu)A$

Solution: We have

$$A=\left[a_{ij}\right]_{3\times3}$$

$$=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

If μ is a scalar then by scalar multiplication

$$\mu A=\mu\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\mu A=\begin{bmatrix} \mu a_{11} & \mu a_{12} & \mu a_{13} \\ \mu a_{21} & \mu a_{22} & \mu a_{23} \\ \mu a_{31} & \mu a_{32} & \mu a_{33} \end{bmatrix}$$

Now consider λ is another scalar then

$$\lambda(\mu A)=\lambda\begin{bmatrix} \mu a_{11} & \mu a_{12} & \mu a_{13} \\ \mu a_{21} & \mu a_{22} & \mu a_{23} \\ \mu a_{31} & \mu a_{32} & \mu a_{33} \end{bmatrix}$$

$$\lambda(\mu A)=\begin{bmatrix} \lambda\mu a_{11} & \lambda\mu a_{12} & \lambda\mu a_{13} \\ \lambda\mu a_{21} & \lambda\mu a_{22} & \lambda\mu a_{23} \\ \lambda\mu a_{31} & \lambda\mu a_{32} & \lambda\mu a_{33} \end{bmatrix}$$

$$=(\lambda\mu)\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \because \text{ by taking common } \lambda\mu$$

$$=(\lambda\mu)A$$

Hence $\lambda(\mu A)=(\lambda\mu)A$

(ii) $(\lambda+\mu)A=\lambda A+\mu A$

Solution: Consider R.H.S $=\lambda A+\mu A$

$$=\lambda\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}+\mu\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33} \end{bmatrix} + \begin{bmatrix} \mu a_{11} & \mu a_{12} & \mu a_{13} \\ \mu a_{21} & \mu a_{22} & \mu a_{23} \\ \mu a_{31} & \mu a_{32} & \mu a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda a_{11}+\mu a_{11} & \lambda a_{12}+\mu a_{12} & \lambda a_{13}+\mu a_{13} \\ \lambda a_{21}+\mu a_{21} & \lambda a_{22}+\mu a_{22} & \lambda a_{23}+\mu a_{23} \\ \lambda a_{31}+\mu a_{31} & \lambda a_{32}+\mu a_{32} & \lambda a_{33}+\mu a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} (\lambda+\mu) a_{11} & (\lambda+\mu) a_{12} & (\lambda+\mu) a_{13} \\ (\lambda+\mu) a_{21} & (\lambda+\mu) a_{22} & (\lambda+\mu) a_{23} \\ (\lambda+\mu) a_{31} & (\lambda+\mu) a_{32} & (\lambda+\mu) a_{33} \end{bmatrix}$$

$$= (\lambda+\mu) \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= (\lambda+\mu) A = \text{L.H.S}$$

Hence $(\lambda+\mu)A = \lambda A + \mu A$

(iii) $\boldsymbol{\lambda A - A = (\lambda - 1)A}$

Solution: Consider L.H.S = $\lambda A - A$

$$= \lambda \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda a_{11}-a_{11} & \lambda a_{12}-a_{12} & \lambda a_{13}-a_{13} \\ \lambda a_{21}-a_{21} & \lambda a_{22}-a_{22} & \lambda a_{23}-a_{23} \\ \lambda a_{31}-a_{31} & \lambda a_{32}-a_{32} & \lambda a_{33}-a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} (\lambda-1) a_{11} & (\lambda-1) a_{12} & (\lambda-1) a_{13} \\ (\lambda-1) a_{21} & (\lambda-1) a_{22} & (\lambda-1) a_{23} \\ (\lambda-1) a_{31} & (\lambda-1) a_{32} & (\lambda-1) a_{33} \end{bmatrix}$$

$$= (\lambda-1) \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= (\lambda-1) A$$

= R.H.S

L.H.S = R.H.S

So $\lambda A - A = (\lambda - 1)A$

Q.7 If $\mathbf{A} = [\mathbf{a}_{ij}]_{2\times3}$ and $\mathbf{B} = [\mathbf{b}_{ij}]_{2\times3}$, show that $\lambda(\mathbf{A}+\mathbf{B}) = \lambda\mathbf{A} + \lambda\mathbf{B}$

Solution: Let $A = [a_{ij}]_{2\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and $B = [b_{ij}]_{2\times3} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

Consider R.H.S = $\lambda A + \lambda B$

$$= \lambda \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \lambda \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \end{bmatrix} + \begin{bmatrix} \lambda b_{11} & \lambda b_{12} & \lambda b_{13} \\ \lambda b_{21} & \lambda b_{22} & \lambda b_{23} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda a_{11} + \lambda b_{11} & \lambda a_{12} + \lambda b_{12} & \lambda a_{13} + \lambda b_{13} \\ \lambda a_{21} + \lambda b_{21} & \lambda a_{22} + \lambda b_{22} & \lambda a_{23} + \lambda b_{23} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda(a_{11} + b_{11}) & \lambda(a_{12} + b_{12}) & \lambda(a_{13} + b_{13}) \\ \lambda(a_{21} + b_{21}) & \lambda(a_{22} + b_{22}) & \lambda(a_{23} + b_{23}) \end{bmatrix}$$

$$= \lambda \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{bmatrix} \quad \text{(i)}$$

Now consider

$$A + B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{bmatrix} \quad \text{(ii)}$$

So by using (ii), (i) becomes

$\lambda(A+B) =$ L.H.S

So L.H.S = R.H.S

Hence $\lambda(A+B) = \lambda A + \lambda B$

Q.8 If $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{a} & \mathbf{b} \end{bmatrix}$ and $\mathbf{A}^2 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, find the values of a and b.

Solution: We know that

$A^2 = A.A$

$$= \begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix} . \begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix}$$

$$=\begin{bmatrix}1\times1+2\times a & 1\times2+2\times b\\ a\times1+b\times a & a\times2+b\times b\end{bmatrix}$$

$$\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}1+2a & 2+2b\\ a+ab & 2a+b^2\end{bmatrix}$$

Now by definition of equal matrices

$1+2a=0$ and $2+2b=0$

$\Rightarrow 2a=-1$

$\Rightarrow a=-\dfrac{1}{2}$ and $2b=-2$

$\Rightarrow b=-1$

So, $a=-\dfrac{1}{2}$ and $b=-1$ are required values of a and b.

Q.9 If A = $\begin{bmatrix}\mathbf{1} & \mathbf{-1}\\ \mathbf{a} & \mathbf{b}\end{bmatrix}$ and A^2 = $\begin{bmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}\end{bmatrix}$, Find the value of *a* and *b*.

Solution: We know that

$A^2 = A.A$

$$=\begin{bmatrix}1 & -1\\ a & b\end{bmatrix}.\begin{bmatrix}1 & -1\\ a & b\end{bmatrix}$$

$$=\begin{bmatrix}1\times1+(-1)\times a & 1\times(-1)+(-1)\times b\\ a\times1+b\times a & a\times(-1)+b\times b\end{bmatrix}$$

$$=\begin{bmatrix}1-a & -1-b\\ a+ab & -a+b^2\end{bmatrix}$$

$$=\begin{bmatrix}1-a & -1-b\\ a+ab & -a+b^2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Now by definition of equal matrices

$1-a=1$ and $-1-b=0$

$\Rightarrow -a=0$ $\Rightarrow b=-1$

$\Rightarrow a=0$

Hence $a=0$ and $b=-1$ are required values of a and b.

Q.10 If A = $\begin{bmatrix}\mathbf{1} & \mathbf{-1} & \mathbf{2}\\ \mathbf{0} & \mathbf{3} & \mathbf{1}\end{bmatrix}$ and B = $\begin{bmatrix}\mathbf{2} & \mathbf{3} & \mathbf{0}\\ \mathbf{1} & \mathbf{2} & \mathbf{-1}\end{bmatrix}$, then show that $(\mathbf{A}+\mathbf{B})^t = \mathbf{A}^t+\mathbf{B}^t$

Solution: As $A=\begin{bmatrix}1 & -1 & 2\\ 0 & 3 & 1\end{bmatrix}$, then $A^t=\begin{bmatrix}1 & 0\\ -1 & 3\\ 2 & 1\end{bmatrix}$

$B=\begin{bmatrix}2 & 3 & 0\\ 1 & 2 & -1\end{bmatrix}$, then $B^t=\begin{bmatrix}2 & 1\\ 3 & 2\\ 0 & -1\end{bmatrix}$

Now

$$A^t + B^t = \begin{bmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2 & 0+1 \\ -1+3 & 3+2 \\ 2+0 & 1+(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ 2 & 5 \\ 2 & 0 \end{bmatrix} \qquad \text{(i)}$$

Now consider

$$A+B = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2 & -1+3 & 2+0 \\ 0+1 & 3+2 & 1-1 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 5 & 0 \end{bmatrix}$$

By taking transpose

$$(A+B)^t = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 5 & 0 \end{bmatrix}^t$$

$$= \begin{bmatrix} 3 & 1 \\ 2 & 5 \\ 2 & 0 \end{bmatrix} \text{———} (\text{ii})$$

From (i) and (ii)

$$(A+B)^t = A^t + B^t$$

Q.11 Find A^3 if A = $\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{3} \\ \mathbf{5} & \mathbf{2} & \mathbf{6} \\ \mathbf{-2} & \mathbf{-1} & \mathbf{-3} \end{bmatrix}$

Solution: We know that

$$A^2 = A.A$$

$$= \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} . \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times1+1\times5+3\times(-2) & 1\times1+1\times2+3\times(-1) & 1\times3+1\times6+3\times(-3) \\ 5\times1+2\times5+6\times(-2) & 5\times1+2\times2+6\times(-1) & 5\times3+2\times6+6\times(-3) \\ -2\times1+(-1)\times5+(-3)\times(-2) & -2\times1+(-1)\times2+(-3)\times(-1) & -2\times3+(-1)\times6+(-3)\times(-3) \end{bmatrix}$$

$$=\begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix}$$

$$A^2=\begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

Now $A^3 = A^2.A$

$$=\begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}.\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$=\begin{bmatrix} 0\times1+0\times5+0\times(-2) & 0\times1+0\times2+0\times(-1) & 0\times3+0\times6+0\times(-3) \\ 3\times1+3\times5+9\times(-2) & 3\times1+3\times2+9\times(-1) & 3\times3+3\times6+9\times(-3) \\ -1\times1+(-1)\times5+(-3)\times(-2) & (-1)\times1+(-1)\times2+(-3)\times(-1) & -1\times3+(-1)\times6+(-3)\times(-3) \end{bmatrix}$$

$$=\begin{bmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 3+15-18 & 3+6-9 & 9+18-27 \\ -1-5+6 & -1-2+3 & -3-6+9 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence

$$A^3=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}=O_3$$

Q.12 Find the matrix *X* if

(i) $X\begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix}=\begin{bmatrix} -1 & 5 \\ 12 & 3 \end{bmatrix}$

Let

$$X=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix}=\begin{bmatrix} -1 & 5 \\ 12 & 3 \end{bmatrix}$$

$$\Rightarrow\begin{bmatrix} 5a-2b & 2a+b \\ 5c-2d & 2c+d \end{bmatrix}=\begin{bmatrix} -1 & 5 \\ 12 & 3 \end{bmatrix}$$

Now by definition of equal matrices

$5a-2b=-1$ (i)

$2a+b=5$ (ii)

$5c-2d=12$ (iii)

$2c+d=3$ (iv)

Multiply 2 to equation (ii) and adding in (i)

$5a-2b=-1$

$\underline{4a+2b=10}$

$9a=9$

$\Rightarrow a=1$

Substituting $a=1$ in (ii) we get

$2(1)+b=5$

$\Rightarrow b=3$

multiply 2 to equation (iv) and adding in (iii)

$5c-2d=12$

$\underline{4c+2d=6}$

$9c=18$

$\Rightarrow c=2$

Substituting $c=2$ in (iv) we get

$2(2)+d=3$

$\Rightarrow d=-1$

Hence $X=\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ is the enquired matrix

(ii) $\begin{bmatrix} \mathbf{5} & \mathbf{2} \\ \mathbf{-2} & \mathbf{1} \end{bmatrix}\mathbf{X}=\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{5} & \mathbf{10} \end{bmatrix}$

Let

$X=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then $\begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 2 & 1 \\ 5 & 10 \end{bmatrix}$

$\Rightarrow\begin{bmatrix} 5a+2c & 5b+2d \\ -2a+c & -2b+d \end{bmatrix}=\begin{bmatrix} 2 & 1 \\ 5 & 10 \end{bmatrix}$

By definition of equal matrices

$5a+2c=2$ (i)

$-2a+c=5$ (ii)

$5b+2d=1$ (iii)

$-2b+d=10$ (iv)

Multiply 2 to equation (ii) and subtracting from (i) we get

$5a+2c=2$

$\underline{\mp 4a\pm 2c=-10}$

$9a=-8$

multiply 2 to equation (iv) and subtracting from (iii) we get

$5b+2d=1$

$\underline{\mp 4b\pm 2d=-20}$

$9b=-19$

$\Rightarrow a = -\dfrac{8}{9}$

Substituting the value of a in (ii) we get

$-2\left(-\dfrac{8}{9}\right) + c = 5$

$\dfrac{16}{9} + c = 5$

$c = 5 - \dfrac{16}{9}$

$= \dfrac{45-16}{9}$

$\Rightarrow c = \dfrac{29}{9}$

$\Rightarrow b = \dfrac{-19}{9}$

Substituting the value of b in (iv) we get

$-2\left(-\dfrac{19}{9}\right) + d = 10$

$\dfrac{38}{9} + d = 10$

$d = 10 - \dfrac{38}{9}$

$= \dfrac{90-38}{9}$

$\Rightarrow d = \dfrac{52}{9}$

Hence $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -\dfrac{8}{9} & -\dfrac{19}{9} \\ \dfrac{29}{9} & \dfrac{52}{9} \end{bmatrix}$

Q.13 Find the matrix A if,

(i) $\begin{bmatrix} \mathbf{5} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{3} & \mathbf{1} \end{bmatrix} \mathbf{A} = \begin{bmatrix} \mathbf{3} & \mathbf{-7} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{7} & \mathbf{2} \end{bmatrix}$

Solution:

Since $\begin{bmatrix} 5 & -1 \\ 0 & 0 \\ 3 & 1 \end{bmatrix}_{3\times2} A_{2\times2} = \begin{bmatrix} 3 & -7 \\ 0 & 0 \\ 7 & 2 \end{bmatrix}_{3\times2}$

So let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$\begin{bmatrix} 5 & -1 \\ 0 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & -7 \\ 0 & 0 \\ 7 & 2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 5a-c & 5b-d \\ 0+0 & 0+0 \\ 3a+c & 3b+d \end{bmatrix} = \begin{bmatrix} 3 & -7 \\ 0 & 0 \\ 7 & 2 \end{bmatrix}$

By definition of equal matrices, we get

$5a - c = 3$ (i)

$3a + c = 7$ (ii)

$5b - d = -7$ (iii)

$3b + d = 2$ (iv)

Adding (i) and (ii)

$8a = 10$

$\Rightarrow a = \frac{5}{4}$

Substituting the value of a in (ii) we get

$3\left(\frac{5}{4}\right) + c = 7$

$\Rightarrow c = 7 - \frac{15}{4}$

$c = \frac{13}{4}$

adding (iii) and (iv)

$8b = -5$

$\Rightarrow b = -\frac{5}{8}$

substituting the value of b in (iv) we get

$3\left(-\frac{5}{8}\right) + d = 2$

$d = 2 + \frac{15}{8}$

$d = \frac{31}{8}$

Hence

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & \frac{-5}{8} \\ \frac{13}{4} & \frac{31}{8} \end{bmatrix}$ is the required matrix.

(ii) $\begin{bmatrix} \mathbf{2} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{2} \end{bmatrix} \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{-3} & \mathbf{8} \\ \mathbf{3} & \mathbf{3} & \mathbf{-7} \end{bmatrix}$

Solution:

Since $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}_{2\times2} \quad A_{2\times3} = \begin{bmatrix} 0 & -3 & 8 \\ 3 & 3 & -7 \end{bmatrix}_{2\times3}$

So let $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, then

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 0 & -3 & 8 \\ 3 & 3 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 2a - d & 2b - e & 2c - f \\ -a + 2d & -b + 2e & -c + 2f \end{bmatrix} = \begin{bmatrix} 0 & -3 & 8 \\ 3 & 3 & -7 \end{bmatrix}$$

By definition of equal matrices, we get

$2a - d = 0$ (i)

$-a + 2d = 3$ (ii)

$2b-e=-3$ (iii)

$-b+2e=3$ (iv)

$2c-f=8$ (v)

$-c+2f=-7$ (vi)

Multiply 2 to equation (i) and adding in (ii)

$4a-2d=0$

$\underline{-a+2d=3}$

$3a=3$

$\Rightarrow a=1$

Substituting $a=1$ in (i) we get

$2(1)-d=0$

$\Rightarrow d=2$

Multiply 2 to equation (iii) and adding in (iv)

$4b-2e=-6$

$\underline{-b+2e=3}$

$3b=-3$

$\Rightarrow b=-1$

Substituting $b=-1$, in (iv) we get

$-(-1)+2e=3$

$\Rightarrow 2e=2$

$\Rightarrow e=1$

Multiply 2 to equation (v) and adding in (vi) we get

$4c-2f=16$

$\underline{-c+2f=-7}$

$3c=9$

$\Rightarrow c=3$

Substituting $c=3$ in (vi) we get.

$-(3)+2f=-7$

$2f=-4$

$\Rightarrow f=-2$

Hence $A=\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}=\begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & -2 \end{bmatrix}$

Q.14 Show that

$$\begin{bmatrix} r\cos\phi & 0 & -\sin\phi \\ 0 & r & 0 \\ r\sin\phi & 0 & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -r\sin\phi & 0 & r\cos\phi \end{bmatrix} = rI_3$$

Proof: Consider L.H.S $= \begin{bmatrix} r\cos\phi & 0 & -\sin\phi \\ 0 & r & 0 \\ r\sin\phi & 0 & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -r\sin\phi & 0 & r\cos\phi \end{bmatrix}$

$$= \begin{bmatrix} r\cos^2\phi + 0 + r\sin^2\phi & 0+0+0 & r\cos\phi\sin\phi + 0 - r\cos\phi\sin\phi \\ 0+0+0 & 0+r+0 & 0+0+0 \\ r\sin\phi\cos\phi + 0 + (-r\sin\phi)\cos\phi & 0+0+0 & r\sin^2\phi + 0 + r\cos^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} r\left(\cos^2\phi + \sin^2\phi\right) & 0 & r\sin\phi\cos\phi - r\sin\phi\cos\phi \\ 0 & r & 0 \\ r\sin\phi - r\sin\phi & 0 & r\left(\sin^2\phi + \cos^2\phi\right) \end{bmatrix}$$

$$= \begin{bmatrix} r(1) & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r(1) \end{bmatrix}$$

$$= r\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= rI_3$ = R.H.S

Since L.H.S = R.H.S hence proved.

Properties of Matrix Addition, Scalar Multiplication and Matrix Multiplication:

If A, B and C are $n \times n$ matrices and c, d are scalars, the following properties are true:

1. Commutative property w.r.t. addition: $A+B=B+A$

2. Associative property w.r.t addition: $(A+B)+C=A+(B+C)$

3. Associative property of scalar multiplication: $(cd)A=c(dA)$.

4. Existence of additive identity: $A+O=O+A=A$ (O is null matrix)

5. Existence of multiplicative identity: $IA=AI=A$ (I is unit/identity matrix)

6. Distributive property w.r.t scalar multiplication:

(a) $c(A+B)=cA+cB$ (b) $(c+d)A=cA+dA$

7. Associative property w.r.t multiplication: $A(BC)=(AB)C$

8. Left distributive property: $A(B+C)=AB+AC$

9. Right distributive property: $(A+B)C=AC+BC$

10. $c(AB)=(cA)B=A(cB)$.

EXERCISE 3.2

Q.1 If $A=\left[a_{ij}\right]_{3\times 4}$ then show that

i) $I_3A=A$ **ii)** $AI_4=A$

We have

$A=\left[a_{ij}\right]_{3\times 4}$

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$$I_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(i) $I_3A=A$

Consider L.H.S = I_3A

$$=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$$=\begin{bmatrix} 1\times a_{11}+0\times a_{21}+0\times a_{31} & 1\times a_{12}+0\times a_{22}+0\times a_{32} & 1\times a_{13}+0\times a_{23}+0\times a_{33} & 1\times a_{14}+0\times a_{24}+0\times a_{34} \\ 0\times a_{11}+1\times a_{21}+0\times a_{31} & 0\times a_{12}+1\times a_{22}+0\times a_{32} & 0\times a_{13}+1\times a_{23}+0\times a_{33} & 0\times a_{14}+1\times a_{24}+0\times a_{34} \\ 0\times a_{11}+0\times a_{21}+1\times a_{31} & 0\times a_{12}+0\times a_{22}+1\times a_{32} & 0\times a_{13}+0\times a_{23}+1\times a_{33} & 0\times a_{14}+0\times a_{24}+1\times a_{34} \end{bmatrix}$$

$$=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$=A=\text{R.H.S}$

L.H.S = R.H.S

Hence

$I_3A=A$

(ii) $AI_4=A$

Consider L.H.S = AI_4

$$=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} a_{11}\times1+a_{12}\times0+a_{13}\times0+a_{14}\times0 & a_{11}\times0+a_{12}\times1+a_{13}\times0+a_{14}\times0 & a_{11}\times0+a_{12}\times0+a_{13}\times1+a_{14}\times0 & a_{11}\times0+a_{12}\times0+a_{13}\times0+a_{14}\times1 \\ a_{21}\times1+a_{22}\times0+a_{23}\times0+a_{24}\times0 & a_{21}\times0+a_{22}\times1+a_{23}\times0+a_{24}\times0 & a_{21}\times0+a_{22}\times0+a_{23}\times1+a_{24}\times0 & a_{21}\times0+a_{22}\times0+a_{23}\times0+a_{24}\times1 \\ a_{31}\times1+a_{32}\times0+a_{33}\times0+a_{34}\times0 & a_{31}\times0+a_{32}\times1+a_{33}\times0+a_{34}\times0 & a_{31}\times0+a_{32}\times0+a_{33}\times1+a_{34}\times0 & a_{31}\times0+a_{32}\times0+a_{33}\times0+a_{34}\times1 \end{bmatrix}$$

$$=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$= A =$ R.H.S

L.H.S = R.H.S

Hence $AI_4 = A$

Q.2 Find the inverse of the following matrices.

(i) $\begin{bmatrix} \mathbf{3} & \mathbf{-1} \\ \mathbf{2} & \mathbf{1} \end{bmatrix}$

Solution: Let $A=\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$, then inverse of A will be A^{-1} given by

$$A^{-1}=\frac{AdjA}{|A|},\ |A|\neq 0$$

Now $AdjA=\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$

And $|A|=\begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix}=3\times1-(2)(-1)$

$=3+2$

$|A|=5\neq 0$

Now $A^{-1}=\dfrac{AdjA}{|A|}=\dfrac{\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}}{5}$

$$A^{-1}=\begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{-2}{5} & \frac{3}{5} \end{bmatrix}$$

Hence inverse of $\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$ is $\begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{-2}{5} & \frac{3}{5} \end{bmatrix}$

(ii) $\begin{bmatrix} \mathbf{-2} & \mathbf{3} \\ \mathbf{-4} & \mathbf{5} \end{bmatrix}$

Solution: Let $B=\begin{bmatrix} -2 & 3 \\ -4 & 5 \end{bmatrix}$, then inverse of B is given by

$$B^{-1}=\frac{AdjB}{|B|}, |B|\neq 0$$

Now $|B|=\begin{vmatrix} -2 & 3 \\ -4 & 5 \end{vmatrix}=(-2\times 5)-(-4)\times 3$

$=-10+12$

$|B|=2\neq 0$

Now $AdjB=\begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix}$

$$B^{-1}=\frac{AdjB}{|B|}=\frac{\begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix}}{2}$$

$$=\begin{bmatrix} \frac{5}{2} & \frac{-3}{2} \\ 2 & -1 \end{bmatrix}$$

Hence inverse of $\begin{bmatrix} -2 & 3 \\ -4 & 5 \end{bmatrix}$ is $\begin{bmatrix} \frac{5}{2} & \frac{-3}{2} \\ 2 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} \mathbf{2i} & \mathbf{i} \\ \mathbf{i} & \mathbf{-i} \end{bmatrix}$

Solution: Let $C=\begin{bmatrix} 2i & i \\ i & -i \end{bmatrix}$, then inverse of C is given by

$$C^{-1}=\frac{AdjC}{|C|}, |C|\neq 0$$

Now $|C|=\begin{vmatrix} 2i & i \\ i & -i \end{vmatrix}=2i\times(-i)-(i\times i)$

$=-2i^2-i^2$

$=-2(-1)-(-1)$

$=2+1$

$=3$

Now $AdjC = \begin{bmatrix} -i & -i \\ -i & 2i \end{bmatrix}$

$$C^{-1} = \frac{AdjC}{|C|}$$

$$= \frac{\begin{bmatrix} -i & -i \\ -i & 2i \end{bmatrix}}{3}$$

$$C^{-1} = \begin{bmatrix} \frac{-i}{3} & \frac{-i}{3} \\ \frac{-i}{3} & \frac{2i}{3} \end{bmatrix}$$

Hence inverse of $\begin{bmatrix} 2i & i \\ i & -i \end{bmatrix}$ is $\begin{bmatrix} \frac{-i}{3} & \frac{-i}{3} \\ \frac{-i}{3} & \frac{2i}{3} \end{bmatrix}$

(iv) $\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{6} & \mathbf{3} \end{bmatrix}$

Solution: Let $D = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$, then inverse of D is given by

$$D^{-1} = \frac{AdjD}{|D|}, |C| \neq 0$$

Now $|D| = \begin{vmatrix} 2 & 1 \\ 6 & 3 \end{vmatrix}$

$= 2\times3 - 6\times1$

$= 6 - 6$

Since $|D| = 0$ so inverse of $D = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$, is not possible

Q.3 Solve the following system of linear equations.
Hence equations

(i) $\left.\begin{matrix} \mathbf{2x_1 - 3x_2 = 5} \\ \mathbf{5x_1 + x_2 = 4} \end{matrix}\right\}$

Solution: The matrix form of the system $\left.\begin{matrix} 2x_1 - 3x_2 = 5 \\ 5x_1 + x_2 = 4 \end{matrix}\right\}$ is

$$\begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$AX = B$

$\Rightarrow X = A^{-1}B \ldots$(i)

Where $A=\begin{bmatrix}2 & -3\\ 5 & 1\end{bmatrix}, X=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ and $B=\begin{bmatrix}5\\ 4\end{bmatrix}$

Now

$$|A|=\begin{vmatrix}2 & -3\\ 5 & 1\end{vmatrix}=(2\times1).(-5\times(-3))$$

$$=2+15$$

$$|A|=17\neq0$$

And $AdjA=\begin{bmatrix}1 & 3\\ -5 & 2\end{bmatrix}$, therefore

$$A^{-1}=\frac{AdjA}{|A|}=\frac{\begin{bmatrix}1 & 3\\ -5 & 2\end{bmatrix}}{17}$$

Put the values of X, A^{-1} and B in (i)

$$X=A^{-1}B$$

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\frac{1}{17}\begin{bmatrix}1 & 3\\ -5 & 2\end{bmatrix}\begin{bmatrix}5\\ 4\end{bmatrix}$$

$$=\frac{1}{17}\begin{bmatrix}1\times5+3\times4\\ -5\times5+2\times4\end{bmatrix}$$

$$=\frac{1}{17}\begin{bmatrix}5+12\\ -25+8\end{bmatrix}$$

$$=\frac{1}{17}\begin{bmatrix}17\\ -17\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$\Rightarrow x_1=1$ and $x_2=-1$

(ii) $\left.\begin{aligned}\mathbf{4x_1+3x_2}&\mathbf{=5}\\ \mathbf{3x_1-x_2}&\mathbf{=7}\end{aligned}\right\}$

Solution: The matrix form of the system $\left.\begin{aligned}4x_1+3x_2&=5\\ 3x_1-x_2&=7\end{aligned}\right\}$ is

$$\begin{bmatrix}4 & 3\\ 3 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}5\\ 7\end{bmatrix}$$

$$AX=B$$

$$\Rightarrow X=A^{-1}B\ \ldots(i)$$

Where

$A=\begin{bmatrix}4 & 3\\ 3 & -1\end{bmatrix}, X=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ and $B=\begin{bmatrix}5\\ 7\end{bmatrix}$

$$|A| = \begin{vmatrix} 4 & 3 \\ 3 & -1 \end{vmatrix} = (-1\times 4) - (3\times 3)$$

$$= -4 - 9$$

$$|A| = -13 \neq 0$$

And $Adj A = \begin{bmatrix} -1 & -3 \\ -3 & 4 \end{bmatrix}$, therefore

$$A^{-1} = \frac{Adj A}{|A|} = \frac{\begin{bmatrix} -1 & -3 \\ -3 & 4 \end{bmatrix}}{-13}$$

Put the values of $X,\ A^{-1}$ and B in (i)

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\frac{1}{13}\begin{bmatrix} -1 & -3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\frac{1}{13}\begin{bmatrix} -1\times 5 + (-3)\times 7 \\ -3\times 5 + 4\times 7 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\frac{1}{13}\begin{bmatrix} -5 - 21 \\ -15 + 28 \end{bmatrix}$$

$$= \frac{-1}{13}\begin{bmatrix} -26 \\ 13 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow x_1 = 2, \qquad x_2 = -1$$

(iii) $\left.\begin{aligned} \mathbf{3x - 5y = 1} \\ \mathbf{-2x + y = -3} \end{aligned}\right\}$

Solution: The matrix form of the system

$$\left.\begin{aligned} 3x - 5y = 1 \\ -2x + y = -3 \end{aligned}\right\} \text{is}$$

$$\begin{bmatrix} 3 & -5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$AX = B$$

$$\Rightarrow X = A^{-1}B \ldots \text{(i)}$$

Where $A = \begin{bmatrix} 3 & -5 \\ -2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & -5 \\ -2 & 1 \end{vmatrix} = 3\times1-(-2)(-5)$$
$$= 3-10$$
$$|A| = -7 \neq 0$$

And $AdjA = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$

$$A^{-1} = \frac{AdjA}{|A|} = \frac{\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}}{-7}$$

Put the values of X, A^{-1} and B in (i)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-7}\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$
$$= \frac{-1}{7}\begin{bmatrix} 1\times1+5\times(-3) \\ 2\times1+3\times(-3) \end{bmatrix}$$
$$= \frac{-1}{7}\begin{bmatrix} 1-15 \\ 2-9 \end{bmatrix}$$
$$= \frac{-1}{7}\begin{bmatrix} -14 \\ -7 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$\Rightarrow x = 2 \quad \text{and} \quad y = 1$$

Q.4 If $\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix}$ **and** $\mathbf{C} = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 0 \\ 3 & 4 & -1 \end{bmatrix}$**, then find**

i) $A-B$ ii) $B-A$ iii) $(A-B)-C$ iv) $A-(B-C)$

(i) A – B

Solution:

$$A-B = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1-2 & -1-1 & 2+1 \\ 3-1 & 2-3 & 5-4 \\ -1+1 & 0-2 & 4-1 \end{bmatrix}$$
$$A-B = \begin{bmatrix} -1 & -2 & 3 \\ 2 & -1 & 1 \\ 0 & -2 & 3 \end{bmatrix}$$

(ii) B – A

Solution:

$$B-A=\begin{bmatrix}2 & 1 & -1\\1 & 3 & 4\\-1 & 2 & 1\end{bmatrix}-\begin{bmatrix}1 & -1 & 2\\3 & 2 & 5\\-1 & 0 & 4\end{bmatrix}$$

$$=\begin{bmatrix}2-1 & 1+1 & -1-2\\1-3 & 3-2 & 4-5\\-1+1 & 2-0 & 1-4\end{bmatrix}$$

$$B-A=\begin{bmatrix}1 & 2 & -3\\-2 & 1 & -1\\0 & 2 & -3\end{bmatrix}$$

(iii) (A - B) - C

Consider

$$A-B=\begin{bmatrix}1 & -1 & 2\\3 & 2 & 5\\-1 & 0 & 4\end{bmatrix}-\begin{bmatrix}2 & 1 & -1\\1 & 3 & 4\\-1 & 2 & 1\end{bmatrix}$$

$$A-B=\begin{bmatrix}-1 & -2 & 3\\2 & -1 & 1\\0 & -2 & 3\end{bmatrix}$$

Now $(A-B)-C=\begin{bmatrix}-1 & -2 & 3\\2 & -1 & 1\\0 & -2 & 3\end{bmatrix}-\begin{bmatrix}1 & 3 & -2\\-1 & 2 & 0\\3 & 4 & -1\end{bmatrix}$

$$=\begin{bmatrix}-1-1 & -2-3 & 3+2\\2+1 & -1-2 & 1-0\\0-3 & -2-4 & 3+1\end{bmatrix}$$

$$(A-B)-C=\begin{bmatrix}-2 & -5 & 5\\3 & -3 & 1\\-3 & -6 & 4\end{bmatrix}$$

Q.5 **If** $A=\begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}, B=\begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix}$ **and** $C=\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$ **then show that**

(i) $(AB)C = A(BC)$ **(ii)** $(A+B)C = AC+BC$

(i) $(AB)C = A(BC)$

Consider $AB=\begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}\begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix}$

$$=\begin{bmatrix} i\times(-i)+2i\times 2i & i\times 1+2i\times i \\ 1\times(-i)+(-i)\times(2i) & 1\times 1+(-i)\times i \end{bmatrix}$$

$$=\begin{bmatrix} -i^2+4i^2 & i+2i^2 \\ -i-2i^2 & 1-i^2 \end{bmatrix}$$

$$=\begin{bmatrix} -1+4(-1) & i+2(-1) \\ -i-2(-1) & 1-(-1) \end{bmatrix}$$

$$=\begin{bmatrix} 1-4 & i-2 \\ -i+2 & 1+1 \end{bmatrix}$$

$$AB=\begin{bmatrix} -3 & i-2 \\ -i+2 & 2 \end{bmatrix}$$

Now

L.H.S $=(AB)C$

$$=\begin{bmatrix} -3 & i-2 \\ -i+2 & 2 \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$=\begin{bmatrix} (-3\times 2i)+((i-2)\times(-i)) & (-3)\times(-1)+(i-2)\times i \\ (-i+2)\times 2i+2\times(-i) & (-i+2)\times(-1)+2\times i \end{bmatrix}$$

$$=\begin{bmatrix} -6i-i^2+2i & 3+i^2-2i \\ -2i^2+2i & i-2+2i \end{bmatrix}$$

$$(AB)C=\begin{bmatrix} 1-4i & 2-2i \\ 2+2i & 3i-2 \end{bmatrix} \qquad \text{(i)}$$

Now Consider $BC=\begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$

$$=\begin{bmatrix} -i\times 2i+1\times(-i) & (-i)\times(-1)+1\times i \\ 2i\times 2i+i\times(-i) & 2i\times(-1)+i\times i \end{bmatrix}$$

$$=\begin{bmatrix} -2i^2-i & i+i \\ 4i^2-i^2 & -2i+i^2 \end{bmatrix}$$

$$BC=\begin{bmatrix} 2-i & 2i \\ -3 & -2i-1 \end{bmatrix}$$

Now R.H.S = $A(BC)$

$$=\begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}\begin{bmatrix} 2-i & 2i \\ -3 & -2i-1 \end{bmatrix}$$

$$=\begin{bmatrix} i\times(2-i)+2i\times(-3) & i\times 2i+2i(-2i-1) \\ 1\times(2-i)+(-i)(-3) & 1\times 2i+(-i)(-2i-1) \end{bmatrix}$$

$$=\begin{bmatrix} 2i-i^2-6i & 2i^2-4i^2-2i \\ 2-i+3i & 2i+2i^2+i \end{bmatrix}$$

$$A(BC)=\begin{bmatrix} -4i+1 & 2-2i \\ 2+2i & 3i-2 \end{bmatrix} \qquad \text{(ii)}$$

From (i) and (ii)

L.H.S = R.H.S

So $(AB)C=A(BC)$

(ii) $(A+B)C=AC+BC$

$$A+B=\begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}+\begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix}$$

$$=\begin{bmatrix} i-i & 2i+1 \\ 1+2i & -i+i \end{bmatrix}$$

$$A+B=\begin{bmatrix} 0 & 2i+1 \\ 1+2i & 0 \end{bmatrix}$$

Now L.H.S

$=(A+B)C$

$$=\begin{bmatrix} 0 & 1+2i \\ 1+2i & 0 \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$=\begin{bmatrix} 0\times 2i+(1+2i)(-i) & 0\times(-1)+(1+2i)\times i \\ (1+2i)\times 2i+0\times(-i) & (1+2i)\times(-1)+0\times i \end{bmatrix}$$

$$=\begin{bmatrix} 0-i-2i^2 & 0+i+2i^2 \\ 2i+4i^2+0 & -1-2i \end{bmatrix}$$

$$\text{L.H.S}=\begin{bmatrix} 2-i & i-2 \\ 2i-4 & -1-2i \end{bmatrix} \qquad \text{(i)}$$

Now consider R.H.S = $AC+BC$

$$=\begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}+\begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$=\begin{bmatrix} i\times 2i+2i\times(-i) & i\times(-1)+2i\times i \\ 1\times 2i+(-i)\times(-i) & 1\times(-1)+(-i)\times(i) \end{bmatrix}$$

$$+\begin{bmatrix} -i\times 2i+1\times(-i) & -i\times(-1)+1\times i \\ 2i\times 2i+i\times(-i) & 2i\times(-1)+i\times i \end{bmatrix}$$

$$=\begin{bmatrix} 2i^2-2i^2 & -i+2i^2 \\ 2i+i^2 & -1-i^2 \end{bmatrix}+\begin{bmatrix} -2i^2-i & i+i \\ 4i^2-i^2 & -2i+i^2 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & -i-2 \\ 2i-1 & -1+1 \end{bmatrix}+\begin{bmatrix} 2-i & 2i \\ -3 & -2i-1 \end{bmatrix}$$

$$=\begin{bmatrix} 0+2-i & -i-2+2i \\ 2i-1-3 & 0-2i-1 \end{bmatrix}$$

R.H.S = $\begin{bmatrix} 2-i & i-2 \\ 2i-4 & -2i-1 \end{bmatrix}$ (ii)

Since from (i) and (ii) L.H.S = R.H.S

Hence $(A+B)C = AC+BC$

Q.6 If A and B are square matrices of the same order, then explain why in general

i) $\mathbf{(A+B)^2 \neq A^2+2AB+B^2}$

ii) $\mathbf{(A-B)^2 \neq A^2-2AB+B^2}$

iii) $\mathbf{(A+B)(A-B) \neq A^2-B^2}$

(i) $\mathbf{(A+B)^2 \neq A^2+2AB+B^2}$

L.H.S $=(A+B)^2$

$=(A+B)(A+B)$

$=A.A+A.B+B.A+B.B$

$=A^2+AB+BA+B^2$

$=A^2+(AB+BA)+B^2$

In matrices $AB \neq BA$ in general

Therefore $AB+BA \neq 2AB$

$(A+B)^2 = A^2+(AB+BA)+B^2$

$\neq A^2+2AB+B^2$

Hence

$(A+B)^2 \neq A^2+2AB+B^2$

(ii) $\mathbf{(A-B)^2 \neq A^2 - 2AB + B^2}$

Consider

$$(A-B)^2 = (A-B)(A-B)$$
$$= A.A - AB - BA + B.B$$
$$= A^2 - (AB + BA) + B^2$$

In matrices $AB \neq BA$ in general

Therefore $AB + BA \neq 2AB$

So $(A-B)^2 \neq A^2 - 2AB + B^2$

Hence

$$(A-B)^2 \neq A^2 - 2AB + B^2$$

(iii) $(A+B)(A-B) \neq A^2 - B^2$

Consider $(A+B)(A-B)$

$$= A.A - AB + BA - B.B$$
$$= A^2 - A.B + BA - B^2$$

In matrices $AB \neq BA$ in general

Therefore $-AB + BA \neq 0$

So, $(A+B)(A-B) = A^2 - A.B + BA - B^2$

$$\neq A^2 - B^2$$

Hence $(A+B)(A-B) \neq A^2 - B^2$

Q.7 If $\mathbf{A = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ -3 & 5 & 2 & -1 \end{bmatrix}}$, then find $\mathbf{AA^t}$ and $\mathbf{A^tA}$

We have $A = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ -3 & 5 & 2 & -1 \end{bmatrix}$, then

$$A^t = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ -3 & 5 & 2 & -1 \end{bmatrix}^t$$

$$A^t = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix}$$

Now $AA^t = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ -3 & 5 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix}$

$$=\begin{bmatrix} 2\times2+(-1)\times(-1)+3\times3+0\times0 & 2\times1+(-1)\times0+3\times4+0\times(-2) & 2\times(-3)+(-1)\times5+3\times2+0\times(-1) \\ 1\times2+0\times(-1)+4\times3+(-2)\times0 & 1\times1+0\times0+4\times4+(-2)\times(-2) & 1\times(-3)+0\times5+4\times2+(-2)\times(-1) \\ -3\times2+5\times(-1)+2\times3+(-1)\times0 & -3\times1+5\times(0)+2\times4+(-1)\times(-2) & -3\times(-3)+5\times5+2\times2+(-1)\times(-1) \end{bmatrix}$$

$$=\begin{bmatrix} 4+1+9+0 & 2+0+12+0 & -6-5+6+0 \\ 2+0+12+0 & 1+0+16+4 & -3+0+8+2 \\ -6-5+6+0 & -3+0+8+2 & 9+25+4+1 \end{bmatrix}$$

$$AA^t=\begin{bmatrix} 14 & 14 & -5 \\ 14 & 21 & 7 \\ -5 & 7 & 39 \end{bmatrix}$$

Now

$$A^tA=\begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ -3 & 5 & 2 & -1 \end{bmatrix}$$

$$=\begin{bmatrix} 2\times2+1\times1+(-3)\times(-3) & 2\times(-1)+1\times0+(-3)\times(5) & 2\times3+1\times4+(-3)\times(2) & 2\times0+1\times(-2)+(-3)\times(-1) \\ -1\times2+0\times1+5\times(-3) & -1\times(-1)+0\times0+(5)\times(5) & -1\times3+0\times4+5\times(2) & -1\times0+0\times(-2)+5\times(-1) \\ 3\times2+4\times1+2\times(-3) & 3\times(-1)+4\times0+(2)\times(5) & 3\times3+4\times4+2\times2 & 3\times0+4\times(-2)+2\times(-1) \\ 0\times2+(-2)\times1+(-1)(-3) & 0\times(-1)+(-2)\times0+(-1)\times5 & 0\times3+(-2)\times4+(-1)\times2 & 0\times0+(-2)\times(-2)+(-1)\times(-1) \end{bmatrix}$$

$$=\begin{bmatrix} 4+1+9 & -2+0-15 & 6+4-6 & 0-2+3 \\ -2+0-15 & 1+0+25 & -3+0+10 & 0+0-5 \\ 6+4-6 & -3+0+10 & 9+16+4 & 0-8-2 \\ 0-2+3 & 0+0-5 & 0-8-2 & 0+4+1 \end{bmatrix}$$

$$A^tA=\begin{bmatrix} 14 & -17 & 4 & 1 \\ -17 & 26 & 7 & -5 \\ 4 & 7 & 29 & -10 \\ 1 & -5 & -10 & 5 \end{bmatrix}$$

Q.8 Solve the following equations for X.

(i) $3X - 2A = B$ if $A = \begin{bmatrix} 2 & 3 & -2 \\ -1 & 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix}$

Solution:

$3X - 2A = B$

Put the values of A and B

$$\Rightarrow 3X - 2\begin{bmatrix} 2 & 3 & -2 \\ -1 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix}$$

$$\Rightarrow 3X - \begin{bmatrix} 4 & 6 & -4 \\ -2 & 2 & 10 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix}$$

$$\Rightarrow 3X = \begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 6 & -4 \\ -2 & 2 & 10 \end{bmatrix}$$

$$\Rightarrow 3X = \begin{bmatrix} 2+4 & -3+6 & 1-4 \\ 5-2 & 4+2 & -1+10 \end{bmatrix}$$

$$3X = \begin{bmatrix} 6 & 3 & -3 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\Rightarrow X = \frac{1}{3}\begin{bmatrix} 6 & 3 & -3 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$

(ii) $2X - 3A = B$ if $A\begin{bmatrix} 1 & -1 & 2 \\ -2 & 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$

Solution: $2X - 3A = B$

Put the values of A and B

$$2X - 3\begin{bmatrix} 1 & -1 & 2 \\ -2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$$

$$\Rightarrow 2X - \begin{bmatrix} 3 & -3 & 6 \\ -6 & 12 & 15 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$$

$$\Rightarrow 2X = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -3 & 6 \\ -6 & 12 & 15 \end{bmatrix}$$

$$\Rightarrow 2X = \begin{bmatrix} 3+3 & -1-3 & 0+6 \\ 4-6 & 2+12 & 1+15 \end{bmatrix}$$

$$\Rightarrow 2X = \begin{bmatrix} 6 & -4 & 6 \\ -2 & 14 & 16 \end{bmatrix}$$

$$\Rightarrow X = \frac{1}{2}\begin{bmatrix} 6 & -4 & 6 \\ -2 & 14 & 16 \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} 3 & -2 & 3 \\ -1 & 7 & 8 \end{bmatrix}$$

Q.9 Solve the following matrix equations for A

(i) $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} A - \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 3 & 6 \end{bmatrix}$

Solution: $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} \mathrm{A} - \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 3 & 6 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} A = \begin{bmatrix} -1 & -4 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} A = \begin{bmatrix} -1+2 & -4+3 \\ 3-1 & 6-2 \end{bmatrix}$

$\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$ (i)

Now let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then (i) becomes

$\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 4a+3c & 4b+3d \\ 2a+2c & 2b+2d \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$

Now by definition of equal matrices

$4a + 3c = 1$ (ii)

$2a + 2c = 2$ (iii)

$4b + 3d = -1$ (iv)

$2b + 2d = 4$ (v)

Multiply 2 to equation (iii) and Subtracting from (ii)

$4a + 3c = 1$

$\underline{-4a \pm 4c = -4}$

$-c = -3$

$\Rightarrow c = 3$

substituting the value of c in (iii)

$2a + 2(3) = 2$

$\Rightarrow 2a = -4$

$a = -2$

Multiply 2 to equation (v) and subtracting from (iv)

$4b + 3d = -1$

$\underline{-4b \pm 4d = -8}$

$-d = -9$

$\Rightarrow d = 9$

Substituting the value of *d* in (v) we get

$2b + 2(9) = 4$

$\Rightarrow 2b = -14$

$\Rightarrow b = -7$

So,

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$A = \begin{bmatrix} -2 & -7 \\ 3 & 9 \end{bmatrix}$ is the square matrix.

(ii) $\mathbf{A\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 5 \end{bmatrix}}$

$\Rightarrow A\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 5 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$

$\Rightarrow A\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 2-1 & 0+2 \\ -1+3 & 5+1 \end{bmatrix}$

$\Rightarrow A\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$ (i)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then (i) becomes

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$

$\begin{bmatrix} 3a+4b & a+2b \\ 3c+4d & c+2d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$

Now by definition of equal matrices

$3a + 4b = 1$ (ii)

$3c + 4d = 2$ (iii)

$a + 2b = 2$ (iv)

$c + 2d = 6$ (v)

Multiply 2 to equation (iv) and subtracting from (ii)

$3a + 4b = 1$

$\underline{-2a \pm 4b = -4}$

$a = -3$

Substituting the value of a in (iv) we get

$-3 + 2b = 2$

$\Rightarrow 2b = 5$

$\Rightarrow b = \frac{5}{2}$

Multiply 2 to equation (v) and subtracting from (iii)

$3c + 4d = 2$

$\underline{-2c \pm 4d = -12}$

$c = -10$

Substituting the value of c in (v) we get

$-10 + 2d = 6$

$\Rightarrow 2d = 16$

$\Rightarrow d = 8$

So,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$= \begin{bmatrix} -3 & \frac{5}{2} \\ -10 & 8 \end{bmatrix}$ is the required matrix.

Singular and Non Singular Matrices:

A square matrix A is said to be **singular** if $|A| = 0$ and if. $|A| \neq 0$ then A is said to be non singular.

Minor and Cofactor of an Element of a Matrix:

Minor:

Let us consider a square matrix A of order n then minor of an element a_{ij} denoted by M_{ij} is the determinant of the $(n-1)\times(n-1)$ matrix formed by deleting the ith row and jth column of A.

Cofactor:

The cofactor of a an element a_{ij} denoted by A_{ij} is defined by $A_{ij} = (-1)^{i+j} \times M_{ij}$.

Properties of Determinants:

1. For a square matrix $A, |A| = |A^t|$.

2. If in a square matrix A, two rows or two columns are interchanged, the determinant of the resulting matrix is $-|A|$.

3. If a square matrix A has two identical rows or two identical columns, then $|A| = 0$.

4. If all the entries of a row (or a column) of a square matrix A are zero, then $|A| = 0$.

5. If the entries of a row (or a column) in a square matrix A are multiplied by a number $k \in \mathbb{R}$, then the determinant of the resulting matrix is $k|A|$.

6. If each entry of a row (or a column) of a square matrix consists of two terms, then its determinant can be written as the sum of two determinants, i.e., if

$$B = \begin{bmatrix} a_{11}+b_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22} & A_{23} \\ a_{31}+b_{31} & a_{32} & a_{33} \end{bmatrix}, \text{then}$$

$$|B| = \begin{vmatrix} a_{11}+b_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22} & a_{23} \\ a_{31}+b_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{32} & a_{33} \end{vmatrix}$$

7. If to each entry of a row (or a column) of a square matrix A is added a non-zero multiple of the corresponding entry of another row (or column), then the determinant of the resulting matrix is $|A|$.

8. If a matrix is in triangular form, then the value of its determinant is the product of all the entries on its main diagonal.

EXERCISE 3.3

Q.1 Evaluate the determinants.

(i) $\begin{vmatrix} 5 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 1 & 2 \end{vmatrix}$

Solution:

$\begin{vmatrix} 5 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 1 & 2 \end{vmatrix}$

Expanding from R_1 we get

$=5\begin{vmatrix} -1 & -3 \\ 1 & 2 \end{vmatrix}-(-2)\begin{vmatrix} 3 & -3 \\ -2 & 2 \end{vmatrix}-4\begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix}$

$=5((-1)\times 2-1\times(-3))+2(3\times 2-(-2)\times(-3))-4(3\times 1-(-2)(-1))$

$=5(-2+3)+2(6-6)-4(3-2)$

$=5(1)+2(0)-4(1)$

$=5+0-4$

$=1$

Hence $\begin{vmatrix} 5 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 1 & 2 \end{vmatrix}=1$

(ii) $\begin{vmatrix} 5 & 2 & -3 \\ 3 & -1 & 1 \\ -2 & 1 & -2 \end{vmatrix}$

Solution:

$\begin{vmatrix} 5 & 2 & -3 \\ 3 & -1 & 1 \\ -2 & 1 & -2 \end{vmatrix}$

Expanding from R_1 we get

$=5\begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix}-2\begin{vmatrix} 3 & 1 \\ -2 & -2 \end{vmatrix}-3\begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix}$

$=5((-1)\times(-2)-(1)(1))-2(3\times(-2)-(-2)(1))-3(3\times 1-(-2)(-1))$

$=5(2-1)-2(-6+2)-3(3-2)$

$=5(1)-2(-4)-3(1)$

$=5+8-3$

$=10$

Hence $\begin{vmatrix} 5 & 2 & -3 \\ 3 & -1 & 1 \\ -2 & 1 & -2 \end{vmatrix}=10$

(iii) $\begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-3} \\ \mathbf{-1} & \mathbf{3} & \mathbf{4} \\ \mathbf{-2} & \mathbf{5} & \mathbf{6} \end{vmatrix}$

Solution:

$\begin{vmatrix} 1 & 2 & -3 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{vmatrix}$

Expanding from R_1 we get

$=1\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}-2\begin{vmatrix} -1 & 4 \\ -2 & 6 \end{vmatrix}-3\begin{vmatrix} -1 & 3 \\ -2 & 5 \end{vmatrix}$

$=1(18-20)-2(-6+8)-3(-5+6)$

$=1(-2)-2(2)-3(1)$

$=-2-4-3$

$=-9$

Hence $\begin{vmatrix} 1 & 2 & -3 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{vmatrix}=-9$

(iv) $\begin{vmatrix} \boldsymbol{a+l} & \boldsymbol{a-l} & \boldsymbol{a} \\ \boldsymbol{a} & \boldsymbol{a+l} & \boldsymbol{a-l} \\ \boldsymbol{a-l} & \boldsymbol{a} & \boldsymbol{a+l} \end{vmatrix}$

Solution:

$\begin{vmatrix} a+l & a-l & a \\ a & a+l & a-l \\ a-l & a & a+l \end{vmatrix}$

Expanding from R_1 we get

$=(a+l)\begin{vmatrix} a+l & a-l \\ a & a+l \end{vmatrix}-(a-l)\begin{vmatrix} a & a-l \\ a-l & a+l \end{vmatrix}+a\begin{vmatrix} a & a+l \\ a-l & a \end{vmatrix}$

$$=(a+l)\left((a+l)^2-a(a-l)\right)-(a-l)\left(a(a+l)-(a-l)^2\right)+a\left(a^2-(a+l)(a-l)\right)$$

$$=(a+l)\left(a^2+l^2+2al-a^2+al\right)-(a-l)\left(a^2+al-a^2-l^2+2al\right)+a\left(a^2-a^2+l^2\right)$$

$$=(a+l)\left(a^2+l^2+2al-a^2+al\right)-(a-l)\left(a^2+al-a^2-l^2+2al\right)+a\left(a^2-a^2+l^2\right)$$

$$=(a+l)\left(l^2+3al\right)-(a-l)\left(3al-l^2\right)+al^2$$

$$=al^2+3a^2l+l^3+3al^2-\left(3a^2l-al^2-3al^2+l^3\right)+al^2$$

$$=4al^2+3a^2l+l^3-3a^2l+4al^2-l^3+al^2$$

$$=9al^2$$

Hence $\begin{vmatrix} a+l & a-l & a \\ a & a+l & a-l \\ a-l & a & a+l \end{vmatrix}=9al^2$

(v) $\begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-2} \\ \mathbf{-1} & \mathbf{1} & \mathbf{-3} \\ \mathbf{2} & \mathbf{4} & \mathbf{-1} \end{vmatrix}$

Solution:

$$\begin{vmatrix} 1 & 2 & -2 \\ -1 & 1 & -3 \\ 2 & 4 & -1 \end{vmatrix}$$

Expanding from R_1 we get

$$=1\begin{vmatrix} 1 & -3 \\ 4 & -1 \end{vmatrix}-2\begin{vmatrix} -1 & -3 \\ 2 & -1 \end{vmatrix}-2\begin{vmatrix} -1 & 1 \\ 2 & 4 \end{vmatrix}$$

$$=1\left(1\times(-1)-4(-3)\right)-2\left((-1)\times(-1)-(2)(-3)\right)-2\left((-1)\times4-2\times1\right)$$

$$=1(-1+12)-2(1+6)-2(-4-2)$$

$$=1(11)-2(7)-2(-6)$$

$$=11-14+12$$

$$=9$$

Hence $\begin{vmatrix} 1 & 2 & -2 \\ -1 & 1 & -3 \\ 2 & 4 & -1 \end{vmatrix}=9$

(vi) $\begin{vmatrix} 2a & a & a \\ b & 2b & b \\ c & c & 2c \end{vmatrix}$

Solution:

$$\begin{vmatrix} 2a & a & a \\ b & 2b & b \\ c & c & 2c \end{vmatrix}$$

Expanding from R_1 we get

$$= 2a\begin{vmatrix} 2b & b \\ c & 2c \end{vmatrix} - a\begin{vmatrix} b & b \\ c & 2c \end{vmatrix} + a\begin{vmatrix} b & 2b \\ c & c \end{vmatrix}$$

$$= 2a(4bc - bc) - a(2bc - bc) + a(bc - 2bc)$$

$$= 2a(3bc) - a(bc) + a(-bc)$$

$$= 6abc - abc - abc$$

$$= 4abc$$

Hence $\begin{vmatrix} 2a & a & a \\ b & 2b & b \\ c & c & 2c \end{vmatrix} = 4abc$

Q.2 Without expansion show that

(i) $\begin{vmatrix} 6 & 7 & 8 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix} = 0$

Solution:

Consider L.H.S = $\begin{vmatrix} 6 & 7 & 8 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix}$

$= \begin{vmatrix} 6 & 1 & 1 \\ 3 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix}$ By $C_3 - C_2 \to C_3$ and $C_2 - C_1 \to C_2'$

$= 0 =$ R.H.S $\because C_2$ and C_3 are identical

Since L.H.S = R.H.S

So $\begin{vmatrix} 6 & 7 & 8 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix} = 0$

(ii) $\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix} = 0$

Solution:

Consider L.H.S $= \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix}$

$= \begin{vmatrix} 2 & 3 & -1+3 \\ 1 & 1 & 0+1 \\ 2 & -3 & 5-3 \end{vmatrix}$ By $C_3 + C_2 \to C_3'$

$= \begin{vmatrix} 2 & 3 & 2 \\ 1 & 1 & 1 \\ 2 & -3 & 2 \end{vmatrix}$

$= 0 =$ R.H.S $\because C_1$ and C_3 are identical.

Since L.H.S = R.H.S

So $\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix} = 0$

(iii) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$

Solution:

Consider L.H.S $= \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

$= \begin{vmatrix} 1 & 2-1 & 3-2 \\ 4 & 5-4 & 6-5 \\ 7 & 8-7 & 9-8 \end{vmatrix}$ By $C_3 - C_2 \to C_3'$ and $C_2 - C_1 \to C_2'$

$= \begin{vmatrix} 1 & 1 & 1 \\ 4 & 1 & 1 \\ 7 & 1 & 1 \end{vmatrix}$

$= 0 =$ R.H.S $\because C_2$ and C_3 are identical.

Since L.H.S = R.H.S

So $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$

Q.3 Show that

(i) $$\begin{vmatrix} a_{11} & a_{12} & a_{13}+\alpha_{13} \\ a_{21} & a_{22} & a_{23}+\alpha_{23} \\ a_{31} & a_{32} & a_{33}+\alpha_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \alpha_{13} \\ a_{21} & a_{22} & \alpha_{23} \\ a_{31} & a_{32} & \alpha_{33} \end{vmatrix}$$

Proof:

Consider L.H.S $= \begin{vmatrix} a_{11} & a_{12} & a_{13}+\alpha_{13} \\ a_{21} & a_{22} & a_{23}+\alpha_{23} \\ a_{31} & a_{32} & a_{33}+\alpha_{33} \end{vmatrix}$

Expanding from C_3 we get

$$= (a_{13}+\alpha_{13})\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - (a_{23}+\alpha_{23})\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + (a_{33}+\alpha_{33})\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} + \alpha_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} - \alpha_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \alpha_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= \left(a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \right) + \left(\alpha_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - \alpha_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + \alpha_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \right) \quad \text{(i)}$$

Now consider R.H.S $= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \alpha_{13} \\ a_{21} & a_{22} & \alpha_{23} \\ a_{31} & a_{32} & \alpha_{33} \end{vmatrix}$

Expanding both the determinants from C_3

$$= \left(a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \right) + \left(\alpha_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - \alpha_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + \alpha_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \right) \quad \text{(ii)}$$

From (i) and (ii) L.H.S = R.H.S

Hence $\begin{vmatrix} a_{11} & a_{12} & a_{13}+\alpha_{13} \\ a_{21} & a_{22} & a_{23}+\alpha_{23} \\ a_{31} & a_{32} & a_{33}+\alpha_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \alpha_{13} \\ a_{21} & a_{22} & \alpha_{23} \\ a_{31} & a_{32} & \alpha_{33} \end{vmatrix}$

(ii) $$\begin{vmatrix} 2 & 3 & 0 \\ 3 & 9 & 6 \\ 2 & 15 & 1 \end{vmatrix} = 9\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{vmatrix}$$

Proof:

Consider L.H.S $= \begin{vmatrix} 2 & 3 & 0 \\ 3 & 9 & 6 \\ 2 & 15 & 1 \end{vmatrix}$

$= 3\begin{vmatrix} 2 & 3 & 0 \\ 1 & 3 & 2 \\ 2 & 15 & 1 \end{vmatrix}$ By taking 3 common from R_2

$$=3\times 3\begin{vmatrix}2 & 1 & 0\\ 1 & 1 & 2\\ 2 & 5 & 1\end{vmatrix}$$ By taking 3 common from C_2

$$=9\begin{vmatrix}2 & 1 & 0\\ 1 & 1 & 2\\ 2 & 5 & 1\end{vmatrix}=\text{R.H.S}$$

Since L.H.S = R.H.S

So $\begin{vmatrix}2 & 3 & 0\\ 1 & 3 & 6\\ 2 & 15 & 1\end{vmatrix}=9\begin{vmatrix}2 & 1 & 0\\ 1 & 1 & 2\\ 2 & 5 & 1\end{vmatrix}$

(iii) $\begin{vmatrix}a+l & a & a\\ a & a+l & a\\ a & a & a+l\end{vmatrix}=l^2(3a+l)$

Proof:

Consider L.H.S $=\begin{vmatrix}a+l & a & a\\ a & a+l & a\\ a & a & a+l\end{vmatrix}$

$$=\begin{vmatrix}3a+l & a & a\\ 3a+l & a+l & a\\ 3a+l & a & a+l\end{vmatrix}$$ By $C_1+(C_2+C_3)$

$$=(3a+l)\begin{vmatrix}1 & a & a\\ 1 & a+l & a\\ 1 & a & a+l\end{vmatrix}$$ By taking $(3a+l)$ Common from C_1

$$=(3a+l)\begin{vmatrix}1 & a & a\\ 0 & l & 0\\ 0 & 0 & l\end{vmatrix}$$ By $R_2+(-R_1)$ and $R_3+(-R_1)$

By expanding from R_1

$$=(3a+l)\left[1\begin{vmatrix}l & 0\\ 0 & l\end{vmatrix}-a\begin{vmatrix}0 & 0\\ 0 & l\end{vmatrix}+a\begin{vmatrix}0 & l\\ 0 & 0\end{vmatrix}\right]$$

$$=(3a+l)\left[(l^2-0)-a(0)+a(0)\right]$$

$$=(3a+l)(l^2-0+0)$$

$$=l^2(3a+l)=\text{R.H.S}$$

Since L.H.S = R.H.S

So $\begin{vmatrix}a+l & a & a\\ a & a+l & a\\ a & a & a+l\end{vmatrix}=l^2(3a+l)$

(iv) $$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Proof:

Consider L.H.S $= \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix}$

Multiply and divided by x, y and z to C_1, C_2 and C_3 respectively

$$= \frac{1}{x}.\frac{1}{y}.\frac{1}{z}\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ xyz & xyz & xyz \end{vmatrix}$$

$$= \frac{xyz}{xyz}\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}$$ By taking xyz common from R_3

$$= (-1)\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ x^2 & y^2 & z^2 \end{vmatrix}$$ By $R_2 \leftrightarrow R_3$

$$= (-1)(-1)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$ By $R_1 \leftrightarrow R_2$

$$= \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = \text{R.H.S}$$

Since L.H.S = R.H.S

So $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$

(v) $\begin{vmatrix} \mathbf{b+c} & \mathbf{a} & \mathbf{a} \\ \mathbf{b} & \mathbf{c+a} & \mathbf{b} \\ \mathbf{c} & \mathbf{c} & \mathbf{a+b} \end{vmatrix} = \mathbf{4abc}$

Proof:

Consider L.H.S = $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

Expanding from R_1 we get

$= (b+c)\begin{vmatrix} c+a & b \\ c & a+b \end{vmatrix} - a\begin{vmatrix} b & b \\ c & a+b \end{vmatrix} + a\begin{vmatrix} b & c+a \\ c & c \end{vmatrix}$

$= (b+c)\left[(a+b)(a+c) - bc\right] - a\left[b(a+b) - bc\right] + a\left[bc - c(a+c)\right]$

$= (b+c)\left(a^2 + ac + ba + \cancel{bc} - \cancel{bc}\right) - a\left(ab + b^2 - bc\right) + a\left(bc - ac - c^2\right)$

$= a^2\cancel{b} + abc + \cancel{b^2a} + \cancel{a^2c} + \cancel{ac^2} + abc - \cancel{a^2b} - \cancel{ab^2} + abc + abc - \cancel{a^2c} - \cancel{ac^2}$

$= 4abc =$ R.H.S

Since L.H.S = R.H.S

So $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

(vi) $\begin{vmatrix} \boldsymbol{b} & \mathbf{-1} & \boldsymbol{a} \\ \boldsymbol{a} & \boldsymbol{b} & \mathbf{0} \\ \mathbf{1} & \boldsymbol{a} & \boldsymbol{b} \end{vmatrix} = \boldsymbol{a^3 + b^3}$

Proof:

Consider L.H.S $= \begin{vmatrix} b & -1 & a \\ a & b & 0 \\ 1 & a & b \end{vmatrix}$

Expanding from C_3 we get

$= a\begin{vmatrix} a & b \\ 1 & a \end{vmatrix} - 0\begin{vmatrix} b & -1 \\ 1 & a \end{vmatrix} + b\begin{vmatrix} b & -1 \\ a & b \end{vmatrix}$

$= a\left(a^2 - b\right) - 0(ab+1) + b\left(b^2 + a\right)$

$= a^3 - ab - 0 + b^3 + ab$

$= a^3 + b^3 =$ R.H.S

Since L.H.S = R.H.S

So $\begin{vmatrix} b & -1 & a \\ a & b & 0 \\ 1 & a & b \end{vmatrix} = a^3 + b^3$

(vii) $\begin{vmatrix} r\cos\phi & 1 & -\sin\phi \\ 0 & 1 & 0 \\ r\sin\phi & 0 & \cos\phi \end{vmatrix} = r$

Proof:

Consider L.H.S $= \begin{vmatrix} r\cos\phi & 1 & -\sin\phi \\ 0 & 1 & 0 \\ r\sin\phi & 0 & \cos\phi \end{vmatrix}$

Expanding from R_1 we get

$= r\cos\phi \begin{vmatrix} 1 & 0 \\ 0 & \cos\phi \end{vmatrix} - 1\begin{vmatrix} 0 & 0 \\ r\sin\phi & \cos\phi \end{vmatrix} - \sin\phi \begin{vmatrix} 0 & 1 \\ r\sin\phi & 0 \end{vmatrix}$

$= r\cos\phi(\cos\phi - 0) - 1(0-0) - \sin\phi(0 - r\sin\phi)$

$= r\cos^2\phi + r\sin^2\phi$

$= r\left(\cos^2\phi + \sin^2\phi\right)$

$= r(1)$

$= r =$ R.H.S

Since L.H.S = R.H.S

So $\begin{vmatrix} r\cos\phi & 1 & -\sin\phi \\ 0 & 1 & 0 \\ r\sin\phi & 0 & \cos\phi \end{vmatrix} = r$

(viii) $\begin{vmatrix} a & b+c & a+b \\ b & c+a & b+c \\ c & a+b & c+a \end{vmatrix} = a^3 + b^3 + c^3 - 3abc$

Proof:

Consider L.H.S = $\begin{vmatrix} a & b+c & a+b \\ b & c+a & b+c \\ c & a+b & c+a \end{vmatrix}$

$= \begin{vmatrix} a+b+c & b+c & a+b \\ a+b+c & c+a & b+c \\ a+b+c & a+b & c+a \end{vmatrix}$ By $C_1 + C_2$

$= (a+b+c)\begin{vmatrix} 1 & b+c & a+b \\ 1 & c+a & b+c \\ 1 & a+b & c+a \end{vmatrix}$ By taking common $a+b+c$ from C_1

$$=(a+b+c)\begin{vmatrix}1 & b+c & a+b\\0 & a-b & c-a\\0 & a-c & c-b\end{vmatrix}$$ By R_2-R_1 and R_3-R_1

Expanding from C_1

$$=(a+b+c)\left[\begin{vmatrix}a-b & c-a\\a-c & c-b\end{vmatrix}-0\begin{vmatrix}b+c & a+b\\a-c & c-b\end{vmatrix}+0\begin{vmatrix}b+c & a+b\\a-b & c-a\end{vmatrix}\right]$$

$$=(a+b+c)\left[(a-b)(c-b)-(a-c)(c-a)-0+0\right]$$

$$=(a+b+c)\left[\cancel{ac}-ab-bc+b^2-ac+a^2+c^2-\cancel{ac}\right]$$

$$=(a+b+c)\left[a^2+b^2+c^2-ab-bc-ca\right]$$

$$\because a^3+b^3+c^3-3abc=(a+b+c)\left(a^2+b^2+c^2-ab-bc-ca\right)$$

$\Rightarrow a^3+b^3+c^3-3abc$ = R.H.S

Since L.H.S = R.H.S

So $\begin{vmatrix}a & b+c & a+b\\b & c+a & b+c\\c & a+b & c+a\end{vmatrix}=a^3+b^3+c^3-3abc$

(ix) $\begin{vmatrix}a+\lambda & b & c\\a & b+\lambda & c\\a & b & c+\lambda\end{vmatrix}=\lambda^2(a+b+c+\lambda)$

Proof:

Consider L.H.S = $\begin{vmatrix}a+\lambda & b & c\\a & b+\lambda & c\\a & b & c+\lambda\end{vmatrix}$

$$=\begin{vmatrix}a+b+c+\lambda & b & c\\a+b+c+\lambda & b+\lambda & c\\a+b+c+\lambda & b & c+\lambda\end{vmatrix}$$ By $C_1+(C_2+C_3)$

$$=(a+b+c+\lambda)\begin{vmatrix}1 & b & c\\1 & b+\lambda & c\\1 & b & c+\lambda\end{vmatrix}$$ Taking $(a+b+c+\lambda)$ common from C_1

$$=(a+b+c+\lambda)\begin{vmatrix}1 & b & c\\0 & \lambda & 0\\0 & 0 & \lambda\end{vmatrix}$$ By $R_2+(-R_1)$ and $R_3+(-R_1)$

Expanding from C_1

$$=(a+b+c+\lambda)\left[1\begin{vmatrix}\lambda & 0\\ 0 & \lambda\end{vmatrix}-b\begin{vmatrix}0 & 0\\ 0 & \lambda\end{vmatrix}+c\begin{vmatrix}0 & \lambda\\ 0 & 0\end{vmatrix}\right]$$

$$=(a+b+c+\lambda)\left[(\lambda^2-0)-b(0)+c(0)\right]$$

$$=(a+b+c+\lambda)(\lambda^2-0+0)$$

$$=\lambda^2(a+b+c+\lambda)=\text{R.H.S}$$

Since L.H.S = R.H.S

So $\begin{vmatrix}a+\lambda & b & c\\ a & b+\lambda & c\\ a & b & c+\lambda\end{vmatrix}=\lambda^2(a+b+c+\lambda)$

(x) $\begin{vmatrix}\mathbf{1} & \mathbf{1} & \mathbf{1}\\ \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c}\\ \boldsymbol{a^2} & \boldsymbol{b^2} & \boldsymbol{c^2}\end{vmatrix}=\boldsymbol{(a-b)(b-c)(c-a)}$

Proof:

Consider L.H.S = $\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}$

$$=\begin{vmatrix}1 & 0 & 0\\ a & b-a & c-a\\ a^2 & b^2-a^2 & c^2-a^2\end{vmatrix} \quad \text{By } C_2-C_1 \text{ and } C_3-C_1$$

Expanding from R_1, we get

$$=1\begin{vmatrix}b-a & c-a\\ b^2-a^2 & c^2-a^2\end{vmatrix}-0\begin{vmatrix}a & c-a\\ a^2 & c^2-a^2\end{vmatrix}+0\begin{vmatrix}a & b-a\\ a^2 & b^2-a^2\end{vmatrix}$$

$$=1\begin{vmatrix}b-a & c-a\\ (b-a)(b+a) & (c-a)(c+a)\end{vmatrix}-0+0$$

$=(b-a)(c-a)\begin{vmatrix}1 & 1\\ b+a & c+a\end{vmatrix}$ Taking common $b-a$ and $c-a$ from C_1 and C_2 respectively

$$=(b-a)(c-a)(c+a-b-a)$$

$$=(b-a)(c-a)(c-b)$$

$$=(-1)(a-b)\times(-1)(b-c)(c-a)$$

$$=(a-b)(b-c)(c-a)=\text{R.H.S}$$

Since L.H.S = R.H.S

So $\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}=(a-b)(b-c)(c-a)$

(xi) $\begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$

Proof:

Consider L.H.S $= \begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix}$

$= \begin{vmatrix} a+b+c & a & a^2 \\ a+b+c & b & b^2 \\ a+b+c & c & c^2 \end{vmatrix}$ By $C_1 + C_2$

$= (a+b+c)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

$= (a+b+c)\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$ By $R_2 - R_1$ and $R_3 - R_1$

Expanding from C_1 we get

$= (a+b+c)\left[1\begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} - 0 + 0\right]$

$= (a+b+c)\begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$

$= (a+b+c)\begin{vmatrix} b-a & (b-a)(b+a) \\ c-a & (c-a)(c+a) \end{vmatrix}$

$= (a+b+c)(b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$ Taking common $b-a$ and $c-a$ from R_1 and R_2 respectively

$= (a+b+c)(b-a)(c-a)(c+a-b-a)$

$= (a+b+c)(b-a)(c-a)(c-b)$

$= (a+b+c)\times(-1)(a-b)(-1)(b-c)(c-a)$

$= (a+b+c)(a-b)(b-c)(c-a) =$ R.H.S

Since L.H.S = R.H.S

So $\begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$

Q.4 If $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 1 \end{bmatrix}$ **and**

$B = \begin{bmatrix} 5 & -2 & 5 \\ 3 & -1 & 4 \\ -2 & 1 & -2 \end{bmatrix}$**, then find**

(i) A_{12}, A_{22}, A_{32} and $|A|$

Solution:

Consider $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 1 \end{bmatrix}$

Then by using formula

$A_{ij} = (-1)^{i+j} M_{ij}$

$A_{12} = (-1)^{1+2} M_{12}$

$= -\begin{vmatrix} 0 & 0 \\ -2 & 1 \end{vmatrix}$

$= -(0+0) = 0$

$A_{12} = 0$

$A_{22} = (-1)^{2+2} M_{22}$

$= \begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix}$

$= (1-6) = -5$

$A_{22} = -5$

$A_{32} = (-1)^{3+2} M_{32}$

$= -\begin{vmatrix} 1 & -3 \\ 0 & 0 \end{vmatrix}$

$= -(0-0)$

$A_{32} = 0$

Now $|A| = \begin{vmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 1 \end{vmatrix}$

Expanding from C_2 we get

$|A| = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$

$= 2(0) + (-2)(-5) + (-2)(0)$

Hence $|A| = 10$

(ii) **B** = $\begin{bmatrix} 5 & -2 & 5 \\ 3 & -1 & 4 \\ -2 & 1 & -2 \end{bmatrix}$,

Solution:

Consider $B = \begin{bmatrix} 5 & -2 & 5 \\ 3 & -1 & 4 \\ -2 & 1 & -2 \end{bmatrix}$

Then by using formula

$B_{ij} = (-1)^{i+j} M_{ij}$

$B_{21} = (-1)^{2+1} M_{21}$

$= -\begin{vmatrix} -2 & 5 \\ 1 & -2 \end{vmatrix}$

$= -(4-5) = 1$

$B_{21} = 1$

$B_{22} = (-1)^{2+2} M_{22}$

$= \begin{vmatrix} 5 & 5 \\ -2 & -2 \end{vmatrix}$

$= -10 + 10$

$B_{22} = 0$

$B_{23} = (-1)^{2+3} M_{23}$

$= -\begin{vmatrix} 5 & -2 \\ -2 & 1 \end{vmatrix}$

$= -(5-4)$

$B_{23} = -1$

Now $|B| = \begin{vmatrix} 5 & -2 & 5 \\ 3 & -1 & 4 \\ -2 & 1 & -2 \end{vmatrix}$

Expanding from R_2 we get

$|B| = b_{21}B_{21} + b_{22}B_{22} + b_{23}B_{23}$

$= 3(1) + (-1)(0) + 4(-1)$

$= 3 + 0 - 4$

$|B| = -1$

Q.5 Without expansion verify that

(i) $$\begin{vmatrix} \alpha & \beta+\gamma & 1 \\ \beta & \gamma+\alpha & 1 \\ \gamma & \alpha+\beta & 1 \end{vmatrix} = 0$$

Solution: Consider

$$\text{L.H.S} = \begin{vmatrix} \alpha & \beta+\gamma & 1 \\ \beta & \gamma+\alpha & 1 \\ \gamma & \alpha+\beta & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \alpha+\beta+\gamma & \beta+\gamma & 1 \\ \beta+\alpha+\gamma & \gamma+\alpha & 1 \\ \gamma+\alpha+\beta & \alpha+\beta & 1 \end{vmatrix} \quad \because C_1+C_2$$

$$= (\alpha+\beta+\gamma)\begin{vmatrix} 1 & \beta+\gamma & 1 \\ 1 & \gamma+\alpha & 1 \\ 1 & \alpha+\beta & 1 \end{vmatrix}$$ Taking $(\alpha+\beta+\gamma)$ common from C_1

$= 0$ $\quad \because C_1$ and C_3 are identical

Hence $$\begin{vmatrix} \alpha & \beta+\gamma & 1 \\ \beta & \gamma+\alpha & 1 \\ \gamma & \alpha+\beta & 1 \end{vmatrix} = 0$$

(ii) $$\begin{vmatrix} 1 & 2 & 3x \\ 2 & 3 & 6x \\ 3 & 5 & 9x \end{vmatrix} = 0$$

Solution: Consider

$$\text{L.H.S} = \begin{vmatrix} 1 & 2 & 3x \\ 2 & 3 & 6x \\ 3 & 5 & 9x \end{vmatrix}$$

$$= 3x\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 5 & 3 \end{vmatrix}$$ By taking $3x$ common from C_3

$= 3x(0)$ Because C_1 and C_3 are identical.

$= 0$

Hence $$\begin{vmatrix} 1 & 2 & 3x \\ 2 & 3 & 6x \\ 3 & 5 & 9x \end{vmatrix} = 0$$

(iii) $\begin{vmatrix} \mathbf{1} & \mathbf{a^2} & \mathbf{\frac{a}{bc}} \\ \mathbf{1} & \mathbf{b^2} & \mathbf{\frac{b}{ca}} \\ \mathbf{1} & \mathbf{c^2} & \mathbf{\frac{c}{ab}} \end{vmatrix} = \mathbf{0}$

Solution: Consider

$$\text{L.H.S} = \begin{vmatrix} 1 & a^2 & \frac{a}{bc} \\ 1 & b^2 & \frac{b}{ca} \\ 1 & c^2 & \frac{c}{ab} \end{vmatrix}$$

$$= \frac{abc}{abc}\begin{vmatrix} 1 & a^2 & \frac{a}{bc} \\ 1 & b^2 & \frac{b}{ca} \\ 1 & c^2 & \frac{c}{ab} \end{vmatrix}$$ Multiply and dividing by abc

$$= \frac{1}{abc}\begin{vmatrix} 1 & a^2 & abc\times\frac{a}{bc} \\ 1 & b^2 & abc\times\frac{b}{ca} \\ 1 & c^2 & abc\times\frac{c}{ab} \end{vmatrix}$$ Multiply C_3 by abc

$$= \frac{1}{abc}\begin{vmatrix} 1 & a^2 & a^2 \\ 1 & b^2 & b^2 \\ 1 & c^2 & c^2 \end{vmatrix}$$

$$= \frac{1}{abc}(0)$$ $\because C_2$ and C_3 are identical

$$= 0$$

Hence $\begin{vmatrix} 1 & a^2 & \frac{a}{bc} \\ 1 & b^2 & \frac{b}{ca} \\ 1 & c^2 & \frac{c}{ab} \end{vmatrix} = 0$

(iv) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

Solution: Consider

$$\text{L.H.S} = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

$$= \begin{vmatrix} a-b+b-c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c \end{vmatrix} \qquad \because C_1 + (C_2 + C_3)$$

$$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix}$$

$= 0$ $\qquad \because$ All the entries of C_1 are zero

Hence $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

(v) $\begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a & b & c \end{vmatrix} = 0$

Solution: Consider

$$\text{L.H.S} = \begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a & b & c \end{vmatrix}$$

$$= \frac{abc}{abc}\begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a & b & c \end{vmatrix} \qquad \text{Multiplying and dividing by } abc$$

$$= \frac{1}{abc}\begin{vmatrix} bc & ca & ab \\ \frac{1}{a}\times abc & \frac{1}{b}\times abc & \frac{1}{c}\times abc \\ a & b & c \end{vmatrix} \qquad \text{Multiplying by } abc \text{ to } R_2$$

$$= \frac{1}{abc}\begin{vmatrix} bc & ca & ab \\ bc & ac & ab \\ a & b & c \end{vmatrix}$$

$$= \frac{1}{abc}(0) \qquad \because R_1 \text{ and } R_2 \text{ are identical}$$

Hence $\begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a & b & c \end{vmatrix} = 0$

(vi) $\begin{vmatrix} \boldsymbol{mn} & \boldsymbol{l} & \boldsymbol{l^2} \\ \boldsymbol{nl} & \boldsymbol{m} & \boldsymbol{m^2} \\ \boldsymbol{lm} & \boldsymbol{n} & \boldsymbol{n^2} \end{vmatrix} = \begin{vmatrix} \boldsymbol{1} & \boldsymbol{l^2} & \boldsymbol{l^3} \\ \boldsymbol{1} & \boldsymbol{m^2} & \boldsymbol{m^3} \\ \boldsymbol{1} & \boldsymbol{n^2} & \boldsymbol{n^3} \end{vmatrix}$

Solution: Consider

L.H.S $= \begin{vmatrix} mn & l & l^2 \\ nl & m & m^2 \\ lm & n & n^2 \end{vmatrix}$

Multiplying R_1, R_2 and R_3 by l, m and n respectively and dividing the determined by lmn we get

$$= \frac{1}{l}.\frac{1}{m}.\frac{1}{n}\begin{vmatrix} lmn & l^2 & l^3 \\ lmn & m^2 & m^3 \\ lmn & n^2 & n^3 \end{vmatrix}$$

$$= \frac{1}{lmn}.lmn\begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix} \qquad \text{Taking common } lmn \text{ from } C_1 \text{ we get}$$

$$= \begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix} = \text{R.H.S}$$

L.H.S = R.H.S

Hence $\begin{vmatrix} mn & l & l^2 \\ nl & m & m^2 \\ lm & n & n^2 \end{vmatrix} = \begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}$

(vii) $\begin{vmatrix} 2a & 2b & 2c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix} = 0$

Solution: Consider

$$\text{L.H.S} = \begin{vmatrix} 2a & 2b & 2c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix}$$

$$= 2\begin{vmatrix} a & b & c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix}$$ Taking 2 common from R_1

$$= 2\begin{vmatrix} a & b & c \\ b & b & b \\ c & c & c \end{vmatrix}$$ By $R_2 - R_1$ and $R_3 - R_1$

$$= 2bc\begin{vmatrix} a & b & c \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$ By taking common in b from R_2 and c from R_3

$= 2bc(0)$ $\because R_2$ *and* R_3 are identical

$0 =$ R.H.S

Since L.H.S = R.H.S so $\begin{vmatrix} 2a & 2b & 2c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix} = 0$

(viii) $\begin{vmatrix} 7 & 2 & 6 \\ 6 & 3 & 2 \\ -3 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 7 & 2 & 7 \\ 6 & 3 & 5 \\ -3 & 5 & -3 \end{vmatrix} + \begin{vmatrix} 7 & 2 & -1 \\ 6 & 3 & -3 \\ -3 & 5 & 4 \end{vmatrix}$

Solution: Consider

$$\text{L.H.S} = \begin{vmatrix} 7 & 2 & 6 \\ 6 & 3 & 2 \\ -3 & 5 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 7 & 2 & (7-1) \\ 6 & 3 & (5-3) \\ -3 & 5 & (-3+4) \end{vmatrix}$$

$$=\begin{vmatrix} 7 & 2 & 7 \\ 6 & 3 & 5 \\ -3 & 5 & -3 \end{vmatrix}+\begin{vmatrix} 7 & 2 & -1 \\ 6 & 3 & -3 \\ -3 & 5 & 4 \end{vmatrix}$$

By using the property of each entry if a row (or column) of a square matrix consist of two terms, then its determinant can be written as the sum of two determinants

= R.H.S

L.H.S = R.H.S

Hence

$$\begin{vmatrix} 7 & 2 & 6 \\ 6 & 3 & 2 \\ -3 & 5 & 1 \end{vmatrix}=\begin{vmatrix} 7 & 2 & 7 \\ 6 & 3 & 5 \\ -3 & 5 & -3 \end{vmatrix}+\begin{vmatrix} 7 & 2 & -1 \\ 6 & 3 & -3 \\ -3 & 5 & 4 \end{vmatrix}$$

(ix) $\begin{vmatrix} -a & 0 & c \\ 0 & a & -b \\ b & -c & 0 \end{vmatrix}=0$

Solution: Consider

$$\text{L.H.S} = \begin{vmatrix} -a & 0 & c \\ 0 & a & -b \\ b & -c & 0 \end{vmatrix}$$

Multiply and divided by b, c and a to R_1, R_2 and R_3 respectively

$$=\frac{1}{a}.\frac{1}{b}.\frac{1}{c}\begin{vmatrix} -ab & 0 & bc \\ 0 & ac & -bc \\ ab & -ac & 0 \end{vmatrix}$$

$$=\frac{1}{abc}\begin{vmatrix} -ab+0+ab & 0+ac-ac & bc-bc+0 \\ 0 & ac & -bc \\ ab & -ac & 0 \end{vmatrix} \qquad \text{By } R_1+\left(R_2+R_3\right)$$

$$=\frac{1}{abc}\begin{vmatrix} 0 & 0 & 0 \\ 0 & ac & -bc \\ ab & -ac & 0 \end{vmatrix}$$

$$=\frac{1}{abc}(0) \qquad \because \text{ Each entry of } R_1 \text{ is zero}$$

$= 0 =$ R.H.S

Since L.H.S. = R.H.S so $\begin{vmatrix} -a & 0 & c \\ 0 & a & -b \\ b & -c & 0 \end{vmatrix}=0$

Q.6 Find the values of *x* if

(i) $\begin{vmatrix} 3 & 1 & x \\ -1 & 3 & 4 \\ x & 1 & 0 \end{vmatrix} = -30$

Solution:

Consider $\begin{vmatrix} 3 & 1 & x \\ -1 & 3 & 4 \\ x & 1 & 0 \end{vmatrix} = -30$

Expanding from R_1, we get

$$3\begin{vmatrix} 3 & 4 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & 4 \\ x & 0 \end{vmatrix} + x\begin{vmatrix} -1 & 3 \\ x & 1 \end{vmatrix} = -30$$

$$3(0-4) - 1(0-4x) + x(-1-3x) = -30$$

$$-12 + 4x - x - 3x^2 = -30$$

$$-3x^2 + 3x - 12 + 30 = 0$$

$$-3x^2 + 3x + 18 = 0$$

$$-3(x^2 - x - 6) = 0$$

$$\Rightarrow x^2 - x - 6 = 0$$

$$x^2 - 3x + 2x - 6 = 0$$

$$x(x-3) + 2(x-3) = 0$$

$$(x-3)(x+2) = 0$$

$\Rightarrow$ Either $x-3=0$ or $x+2=0$

$\Rightarrow x=3$ or $x=-2$

So $x=3$ and $x=-2$ are the required values of x

(ii) $\begin{vmatrix} 1 & x-1 & 3 \\ -1 & x+1 & 2 \\ 2 & -2 & x \end{vmatrix} = 0$

Solution:

Consider $\begin{vmatrix} 1 & x-1 & 3 \\ -1 & x+1 & 2 \\ 2 & -2 & x \end{vmatrix} = 0$

$$\begin{vmatrix} 1 & x-1 & 3 \\ 0 & 2x & 5 \\ 0 & -2x & x-6 \end{vmatrix} = 0 \qquad \text{By } R_2 + R_1,\ R_3 - 2R_1$$

Expanding from C_1 we get

$$1\begin{vmatrix} 2x & 5 \\ -2x & x-6 \end{vmatrix} - 0\begin{vmatrix} x-1 & 3 \\ -2x & x-6 \end{vmatrix} + 0\begin{vmatrix} x-1 & 3 \\ 2x & 5 \end{vmatrix} = 0$$

$2x(x-6)+10x=0$

$2x^2-12x+10x=0$

$2x^2-2x=0$

$2x(x-1)=0$

$\Rightarrow$ Either $2x=0$ or $x-1=0$

$\Rightarrow x=0$ or $x=1$

So $x=0$ and $x=1$ are the required values of *x*.

(iii) $\begin{vmatrix} 1 & 2 & 1 \\ 2 & x & 2 \\ 3 & 6 & x \end{vmatrix} = 0$

Solution:

Consider $\begin{vmatrix} 1 & 2 & 1 \\ 2 & x & 2 \\ 3 & 6 & x \end{vmatrix} = 0$

Expanding from R_1 we get

$$1\begin{vmatrix} x & 2 \\ 6 & x \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 3 & x \end{vmatrix} + 1\begin{vmatrix} 2 & x \\ 3 & 6 \end{vmatrix} = 0$$

$x^2-12-2(2x-6)+12-3x=0$

$x^2-12-4x+12+12-3x=0$

$x^2-7x+12=0$

$x^2-3x-4x+12=0$

$x(x-3)-4(x-3)=0$

$(x-3)(x-4)=0$

$\Rightarrow$ Either $x-3=0$ or $x-4=0$

$x=3$ or $x=4$

So $x=3$ and $x=4$ are required values of x

Q.7 Evaluate the following determinants

(i) $\begin{vmatrix} 3 & 4 & 2 & 7 \\ 2 & 5 & 0 & 3 \\ 1 & 2 & -3 & 5 \\ 4 & 1 & -2 & 6 \end{vmatrix}$

Solution:

Consider $\begin{vmatrix} 3 & 4 & 2 & 7 \\ 2 & 5 & 0 & 3 \\ 1 & 2 & -3 & 5 \\ 4 & 1 & -2 & 6 \end{vmatrix}$

$$= \begin{vmatrix} 1 & -1 & 2 & 4 \\ 2 & 5 & 0 & 3 \\ 1 & 2 & -3 & 5 \\ 4 & 1 & -2 & 6 \end{vmatrix} \qquad \text{By } R_1 - R_2$$

$$= \begin{vmatrix} 1 & -1 & 2 & 4 \\ 0 & 7 & -4 & -5 \\ 0 & 3 & -5 & 1 \\ 0 & 5 & -10 & -10 \end{vmatrix} \qquad \text{By } R_2 - 2R_1,\ R_3 - R_1,\ R_4 - 4R_1$$

Expanding from C_1

$$= 1\begin{vmatrix} 7 & -4 & -5 \\ 3 & -5 & 1 \\ 5 & -10 & -10 \end{vmatrix} - 0\begin{vmatrix} -1 & 2 & 4 \\ 3 & -5 & 1 \\ 5 & -10 & -10 \end{vmatrix} + 0\begin{vmatrix} -1 & 2 & 4 \\ 7 & -4 & -5 \\ 5 & -10 & -10 \end{vmatrix} - 0\begin{vmatrix} -1 & 2 & 4 \\ 7 & -4 & -5 \\ 3 & -5 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 7 & -4 & -5 \\ 3 & -5 & 1 \\ 5 & -10 & -10 \end{vmatrix}$$

Expanding from R_1

$$= 7(50+10) + 4(-30-5) - 5(-30+25)$$

$$= 7(60) + 4(-35) - 5(-5)$$

$$= 420 - 140 + 25$$

$$= 280 + 25$$

$$= 305$$

Hence $\begin{vmatrix} 3 & 4 & 2 & 7 \\ 2 & 5 & 0 & 3 \\ 1 & 2 & -3 & 5 \\ 4 & 1 & -2 & 6 \end{vmatrix} = 305$

(ii) $\begin{vmatrix} 2 & 3 & 1 & -1 \\ 4 & 0 & 2 & 1 \\ 5 & 2 & -1 & 6 \\ 3 & -7 & 2 & -2 \end{vmatrix}$

Solution:

Consider $\begin{vmatrix} 2 & 3 & 1 & -1 \\ 4 & 0 & 2 & 1 \\ 5 & 2 & -1 & 6 \\ 3 & -7 & 2 & -2 \end{vmatrix}$

$= \begin{vmatrix} 2 & 3 & 1 & -1 \\ 0 & -6 & 0 & 3 \\ 7 & 5 & 0 & 5 \\ -1 & -13 & 0 & 0 \end{vmatrix}$ By $R_2 - 2R_1,\ R_3 + R_1,\ R_4 - 2R_1$

Expanding from C_3

$= 1\begin{vmatrix} 0 & -6 & 3 \\ 7 & 5 & 5 \\ -1 & -13 & 0 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 & -1 \\ 7 & 5 & 5 \\ -1 & -13 & 0 \end{vmatrix} + 0\begin{vmatrix} 2 & 3 & -1 \\ 0 & -6 & 3 \\ -1 & -13 & 0 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 & -1 \\ 0 & -6 & 3 \\ 7 & 5 & 5 \end{vmatrix}$

$= \begin{vmatrix} 0 & -6 & 3 \\ 7 & 5 & 5 \\ -1 & -13 & 0 \end{vmatrix}$

Expanding from R_1

$= 0(0+65) + 6(0+5) + 3(-91+5)$

$= 0 + 30 + 3(-86)$

$= 30 - 258$

$= -228$

Hence $\begin{vmatrix} 2 & 3 & 1 & -1 \\ 4 & 0 & 2 & 1 \\ 5 & 2 & -1 & 6 \\ 3 & -7 & 2 & -2 \end{vmatrix} = -228$

(iii) $\begin{vmatrix} -3 & 9 & 1 & 1 \\ 0 & 3 & -1 & 2 \\ 9 & 7 & -1 & 1 \\ -2 & 0 & 1 & -1 \end{vmatrix}$

Solution:

Consider $\begin{vmatrix} -3 & 9 & 1 & 1 \\ 0 & 3 & -1 & 2 \\ 9 & 7 & -1 & 1 \\ -2 & 0 & 1 & -1 \end{vmatrix}$

$= \begin{vmatrix} -3 & 9 & 1 & 1 \\ -3 & 12 & 0 & 3 \\ 6 & 16 & 0 & 2 \\ 1 & -9 & 0 & -2 \end{vmatrix}$ By $R_2 + R_1$, $R_3 + R_1$, $R_4 - R_1$

Expanding from C_3

$$=1\begin{vmatrix} -3 & 12 & 3 \\ 6 & 16 & 2 \\ 1 & -9 & -2 \end{vmatrix} - 0\begin{vmatrix} -3 & 9 & 1 \\ 6 & 16 & 2 \\ 1 & -9 & -2 \end{vmatrix} + 0\begin{vmatrix} -3 & 9 & 1 \\ -3 & 12 & 3 \\ 1 & -9 & -2 \end{vmatrix} - 0\begin{vmatrix} -3 & 9 & 1 \\ -3 & 12 & 3 \\ 6 & 16 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} -3 & 12 & 3 \\ 6 & 16 & 2 \\ 1 & -9 & -2 \end{vmatrix}$$

Expanding from R_1

$= -3(-32+18) - 12(-12-2) + 3(-54-16)$

$= 42 + 168 + 3(-70)$

$= 0$

Hence $\begin{vmatrix} -3 & 9 & 1 & 1 \\ 0 & 3 & -1 & 2 \\ 9 & 7 & -1 & 1 \\ -2 & 0 & 1 & -1 \end{vmatrix} = 0$

Q.8 Show that $\begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix} = (x+3)(x-1)^3$

Proof:

L.H.S = $\begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix}$

$= \begin{vmatrix} x+3 & 1 & 1 & 1 \\ x+3 & x & 1 & 1 \\ x+3 & 1 & x & 1 \\ x+3 & 1 & 1 & x \end{vmatrix}$ By $C_1 + (C_2 + C_3 + C_4)$

$= (x+3)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix}$ By taking common $(x+3)$ from C_1

$= (x+3)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & x-1 \end{vmatrix}$ By $R_2 - R_1$, $R_3 - R_1$, $R_4 - R_1$

$= (x+3)\left[1\times(x-1)\times(x-1)\times(x-1)\right]$ $\because$ Determinant of a diagonal matrix is the product of diagonal entries

$= (x+3)(x-1)^3 =$ R.H.S

Hence $\begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix} = (x+3)(x-1)^3$

Q.9 Find $|AA^t|$ and $|A^tA|$

(i) $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

Solution:

if $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$ then

$A^t = \begin{bmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}$

Now

$$AA^t = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9+4+1 & 6+2-3 \\ 6+2-3 & 4+1+9 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 5 \\ 5 & 14 \end{bmatrix}$$

Now $\left|AA^t\right| = \begin{vmatrix} 14 & 5 \\ 5 & 14 \end{vmatrix}$

$= 196 - 25$

$\left|AA^t\right| = 171$

Now $A^t A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 6+4 & 9+2 & -3+6 \\ 4+2 & 6+1 & -2+3 \\ -2+6 & -3+3 & 1+9 \end{bmatrix}$$

$$= \begin{vmatrix} 10 & 11 & 3 \\ 6 & 7 & 1 \\ 4 & 0 & 10 \end{vmatrix}$$

$= 10(70-0) - 11(60-4) + 3(0-28)$

$= 700 - 11(56) - 84$

$= 700 - 616 - 84$

$700 - 700 = 0$

Hence $\left|A^t A\right| = 0$

(ii) **If A =** $\begin{bmatrix} \mathbf{3} & \mathbf{4} \\ \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \\ \mathbf{2} & \mathbf{3} \end{bmatrix}$**, then**

Solution:

$$A^t = \begin{bmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{bmatrix}$$

Now $AA^t = \begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{bmatrix}$

$$=\begin{bmatrix} 9+16 & 6+4 & 3+4 & 6+12 \\ 6+4 & 4+1 & 2+1 & 4+3 \\ 3+4 & 2+1 & 1+1 & 2+3 \\ 6+12 & 4+3 & 2+3 & 4+9 \end{bmatrix}$$

$$AA^t=\begin{bmatrix} 25 & 10 & 7 & 18 \\ 10 & 5 & 3 & 7 \\ 7 & 3 & 2 & 5 \\ 18 & 7 & 5 & 13 \end{bmatrix}$$

$$\left|AA^t\right|=\begin{vmatrix} 25 & 10 & 7 & 18 \\ 10 & 5 & 3 & 7 \\ 7 & 3 & 2 & 5 \\ 18 & 7 & 5 & 13 \end{vmatrix}$$

$$=\begin{vmatrix} 25 & 10 & 7 & 18 \\ 3 & 2 & 1 & 2 \\ 7 & 3 & 2 & 5 \\ 18 & 7 & 5 & 13 \end{vmatrix}$$ By R_2-R_3

$$=\begin{vmatrix} 4 & -4 & 0 & 4 \\ 3 & 2 & 1 & 2 \\ 1 & -1 & 0 & 1 \\ 3 & -3 & 0 & 3 \end{vmatrix}$$ By R_1-7R_2 , R_3-2R_2 , R_4-5R_2

$$=4\begin{vmatrix} 1 & -1 & 0 & 1 \\ 3 & 2 & 1 & 2 \\ 1 & -1 & 0 & 1 \\ 3 & -3 & 0 & 3 \end{vmatrix}$$

$=4(0)$ $\because$ R_1 and R_3 are identical

$=0$

Hence $\left|AA^t\right|=0$

Now $A^tA=\begin{bmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$

$$=\begin{bmatrix} 9+4+1+4 & 12+2+1+6 \\ 12+2+1+6 & 16+1+1+9 \end{bmatrix}$$

$$A^t A=\begin{bmatrix}18 & 21\\ 21 & 27\end{bmatrix}$$

Now $\left|A^t A\right|=\begin{vmatrix}18 & 21\\ 21 & 27\end{vmatrix}$

$=486-441$

$=45$

Hence $\left|A^t A\right|=45$

Q.10 If A is a square matrix of order 3, then show that $|\mathbf{kA}|=\mathbf{k}^3|\mathbf{A}|$

Proof: Let $A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$ be a square matrix of order 3 and k be any scalar then by scalar multiplication

$$kA=k\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$$

$$kA=\begin{bmatrix}ka_{11} & ka_{12} & ka_{13}\\ ka_{21} & ka_{22} & ka_{23}\\ ka_{31} & ka_{32} & ka_{33}\end{bmatrix}$$

Now

$$|kA|=\begin{vmatrix}ka_{11} & ka_{12} & ka_{13}\\ ka_{21} & ka_{22} & ka_{23}\\ ka_{31} & ka_{32} & ka_{33}\end{vmatrix}$$

$$=k.k.k\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$$ Taking common k from R_1, R_2 and R_3, respectively

$$=k^3\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}=k^3|A|$$

Hence $|kA|=k^3|A|$

Q.11 Find the values of λ if A and B are singular

(i) $\mathbf{A} = \begin{bmatrix} 4 & \lambda & 3 \\ 7 & 3 & 6 \\ 2 & 3 & 1 \end{bmatrix}$

Solution: If A is singular, then $|A| = 0$

$$|A| = \begin{vmatrix} 4 & \lambda & 3 \\ 7 & 3 & 6 \\ 2 & 3 & 1 \end{vmatrix} = 0$$

Expanding from R_1, we get

$$4\begin{vmatrix} 3 & 6 \\ 3 & 1 \end{vmatrix} - \lambda\begin{vmatrix} 7 & 6 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} 7 & 3 \\ 2 & 3 \end{vmatrix} = 0$$

$$\Rightarrow 4(3-18) - \lambda(7-12) + 3(21-6) = 0$$

$$\Rightarrow -60 + 5\lambda + 45 = 0$$

$$\Rightarrow 5\lambda - 15 = 0$$

$$\Rightarrow \lambda = 3$$

Hence A will be singular if

$\lambda = 3$

(ii) $\mathbf{B} = \begin{bmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ 3 & 2 & 0 & 1 \\ 2 & \lambda & -1 & 3 \end{bmatrix}$

Solution: If B is singular, then $|B| = 0$

$$\begin{vmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ 3 & 2 & 0 & 1 \\ 2 & \lambda & -1 & 3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ -5 & 0 & -5 & 0 \\ -22 & \lambda-6 & -16 & 0 \end{vmatrix} = 0 \quad \text{By } R_3 - R_2,\ R_4 - 3R_2$$

Expanding from C_4, we get

$$-0\begin{vmatrix} 8 & 2 & 5 \\ -5 & 0 & -5 \\ -22 & \lambda-6 & -16 \end{vmatrix} + 1\begin{vmatrix} 5 & 1 & 2 \\ -5 & 0 & -5 \\ -22 & \lambda-6 & -16 \end{vmatrix} - 0\begin{vmatrix} 5 & 1 & 2 \\ 8 & 2 & 5 \\ -22 & \lambda-6 & -16 \end{vmatrix} + 0\begin{vmatrix} 5 & 1 & 2 \\ 8 & 2 & 5 \\ -5 & 0 & -5 \end{vmatrix} = 0$$

$$\begin{vmatrix} 5 & 1 & 2 \\ -5 & 0 & -5 \\ -22 & \lambda-6 & -16 \end{vmatrix} = 0$$

Expanding from R_1 we get

$$5\begin{vmatrix} 0 & -5 \\ \lambda-6 & -16 \end{vmatrix} - 1\begin{vmatrix} -5 & -5 \\ -22 & -16 \end{vmatrix} + 2\begin{vmatrix} -5 & 0 \\ -22 & \lambda-6 \end{vmatrix} = 0$$

$$5\big(0-(-5)(\lambda-6)\big)-1(80-110)+2\big((-5)(\lambda-6)-0\big)=0$$

$$5(5\lambda-30)-1(-30)+2(-5\lambda+30)=0$$

$$25\lambda-150+30-10\lambda+60=0$$

$$15\lambda-60=0$$

$$15\lambda=60$$

$$\lambda=4$$

Hence B is singular if $\lambda=4$

Q.12 Which of the following matrices are singular and which of them are non-singular.

(i) $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{3} \\ \mathbf{3} & \mathbf{1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{2} & \mathbf{4} \end{bmatrix}$

Solution:

Consider $\begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$, then

$$\begin{vmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{vmatrix}$$

Expanding from R_1 , we get

$$=1\begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} - 0\begin{vmatrix} 3 & -1 \\ 0 & 4 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix}$$

$$=1(4+2)-0(12-0)+3(6-0)$$

$$=6-0+18$$

$$=24\neq 0$$

Since $\begin{vmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{vmatrix} = 24\neq 0$ so

$\begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$ is non-singular matrix.

(ii) $\begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{-1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{2} & \mathbf{-3} & \mathbf{5} \end{bmatrix}$

Solution:

Consider $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$ then

$$\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix}$$

Expanding from R_1 we get

$$= 2\begin{vmatrix} 1 & 0 \\ -3 & 5 \end{vmatrix} - 3\begin{vmatrix} 1 & 0 \\ 2 & 5 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}$$

$$= 2(5) - 3(5-0) - 1(-3-2)$$

$$= 10 - 15 + 5$$

$$= 0$$

Since $\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix} = 0$ so

$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$ is a singular matrix

(iii) $\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{-3} \\ \mathbf{2} & \mathbf{3} & \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{-1} & \mathbf{3} & \mathbf{4} \end{bmatrix}$

Solution:

Consider $\begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & 2 & -1 & -3 \\ 2 & 3 & 1 & 2 \\ 3 & -1 & 3 & 4 \end{bmatrix}$ then

$$\begin{vmatrix} 1 & 1 & 2 & -1 \\ 1 & 2 & -1 & -3 \\ 2 & 3 & 1 & 2 \\ 3 & -1 & 3 & 4 \end{vmatrix}$$

$$=\begin{vmatrix}1&1&2&-1\\0&1&-3&-2\\0&1&-3&4\\0&-4&-3&7\end{vmatrix}$$ By R_2-R_1 , R_3-2R_1 , R_4-3R_1

Expanding from C_1 we get

$$=1\begin{vmatrix}1&-3&-2\\1&-3&4\\-4&-3&7\end{vmatrix}-0\begin{vmatrix}1&2&-1\\1&-3&4\\-4&-3&7\end{vmatrix}+0\begin{vmatrix}1&2&-1\\1&-3&-2\\-4&-3&7\end{vmatrix}-0\begin{vmatrix}1&2&-1\\1&-3&-2\\1&-3&4\end{vmatrix}$$

$$=\begin{vmatrix}1&-3&-2\\1&-3&4\\-4&-3&7\end{vmatrix}$$

Expanding from R_1 , we get

$$=1\begin{vmatrix}-3&4\\-3&7\end{vmatrix}+3\begin{vmatrix}1&4\\-4&7\end{vmatrix}-2\begin{vmatrix}1&-3\\-4&-3\end{vmatrix}$$

$$=1(-21+12)+3(7+16)-2(-3-12)$$

$$=-9+69+30$$

$$=90\neq 0$$

Since $\begin{vmatrix}1&1&2&-1\\1&2&-1&-3\\2&3&1&2\\3&-1&3&4\end{vmatrix}=90\neq 0$ so

$\begin{bmatrix}1&1&2&-1\\1&2&-1&-3\\2&3&1&2\\3&-1&3&4\end{bmatrix}$ is non-singular matrix

Q.13 Find the inverse of $\mathbf{A}=\begin{bmatrix}2&1&0\\1&1&0\\2&-3&5\end{bmatrix}$ and show that $\mathbf{A^{-1}A = I}$.

Solution: Since $A=\begin{bmatrix}2&1&0\\1&1&0\\2&-3&5\end{bmatrix}$

We know that

$$A^{-1}=\frac{AdjA}{|A|},\ |A|\neq 0$$

So,

$$|A| = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix}$$

$$= 2(5-0) - 1(5-0) + 0(-3-2)$$

$$|A| = 5 \neq 0$$

Now $Adj A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$ and $A_{ij} = (-1)^{i+j} M_{ij}$ so

$$A_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} 1 & 0 \\ -3 & 5 \end{vmatrix} = 5 - 0 = 5$$

$$A_{12} = (-1)^{1+2} M_{12} = -\begin{vmatrix} 1 & 0 \\ 2 & 5 \end{vmatrix} = -(5-0) = -5$$

$$A_{13} = (-1)^{1+3} M_{13} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -3 - 2 = -5$$

$$A_{21} = (-1)^{2+1} M_{21} = -\begin{vmatrix} 1 & 0 \\ -3 & 5 \end{vmatrix} = -(5-0) = -5$$

$$A_{22} = (-1)^{2+2} M_{22} = \begin{vmatrix} 2 & 0 \\ 2 & 5 \end{vmatrix} = 10 - 0 = 10$$

$$A_{23} = (-1)^{2+3} M_{23} = -\begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix} = -(-6-2) = 8$$

$$A_{31} = (-1)^{3+1} M_{31} = \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$A_{32} = (-1)^{3+2} M_{32} = -\begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$A_{33} = (-1)^{3+3} M_{33} = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2 - 1 = 1$$

Now $Adj A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$

$$AdjA=\begin{bmatrix}5 & -5 & -5\\ -5 & 10 & 8\\ 0 & 0 & 1\end{bmatrix}^t$$

$$AdjA=\begin{bmatrix}5 & -5 & 0\\ -5 & 10 & 0\\ -5 & 8 & 1\end{bmatrix}$$

Now $A^{-1}=\dfrac{AdjA}{|A|}$

$$A^{-1}=\frac{\begin{bmatrix}5 & -5 & 0\\ -5 & 10 & 0\\ -5 & 8 & 1\end{bmatrix}}{5}=\frac{1}{5}\begin{bmatrix}5 & -5 & 0\\ -5 & 10 & 0\\ -5 & 8 & 1\end{bmatrix}$$

Now $A^{-1}A=\dfrac{1}{5}\begin{bmatrix}5 & -5 & 0\\ -5 & 10 & 0\\ -5 & 8 & 1\end{bmatrix}\begin{bmatrix}2 & 1 & 0\\ 1 & 1 & 0\\ 2 & -3 & 5\end{bmatrix}$

$$=\frac{1}{5}\begin{bmatrix}10-5+0 & 5-5+0 & 0+0+0\\ -10+10+0 & -5+10+0 & 0+0+0\\ -10+8+2 & -5+8-3 & 0+0+5\end{bmatrix}$$

$$=\frac{1}{5}\begin{bmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 5\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=I_3$$

Hence

$$A^{-1}A=I_3$$

Q.14 Verify that $(AB)^{-1}=B^{-1}A^{-1}$ if

(i) $\mathbf{A}=\begin{bmatrix}\mathbf{1} & \mathbf{2}\\ \mathbf{-1} & \mathbf{0}\end{bmatrix}, \mathbf{B}=\begin{bmatrix}\mathbf{-3} & \mathbf{1}\\ \mathbf{4} & \mathbf{-1}\end{bmatrix}$

Solution:

$|A|=\begin{vmatrix}1 & 2\\ -1 & 0\end{vmatrix}$ and $|B|=\begin{vmatrix}-3 & 1\\ 4 & -1\end{vmatrix}$

$|A| = 2$ and $|B| = -1$

$AdjA = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$ and $AdjB = \begin{bmatrix} -1 & -1 \\ -4 & -3 \end{bmatrix}$

So

$$A^{-1} = \frac{AdjA}{|A|} \quad ; \quad B^{-1} = \frac{AdjB}{|B|}$$

$$A^{-1} = \frac{\begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}}{2} \quad ; \quad B^{-1} = \frac{\begin{bmatrix} -1 & -1 \\ -4 & -3 \end{bmatrix}}{-1}$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} \quad ; \quad B^{-1} = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0+1 & -2+1 \\ 0+3 & -8+3 \end{bmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 3 & -5 \end{bmatrix} \qquad \text{(i)}$$

Now

$$AB = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -3+8 & 1-2 \\ 3+0 & -1+0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & -1 \\ 3 & -1 \end{bmatrix}$$

So

$$|AB| = \begin{vmatrix} 5 & -1 \\ 3 & -1 \end{vmatrix} = -5+3$$

$$|AB| = -2$$

$$Adj(AB) = \begin{bmatrix} -1 & 1 \\ -3 & 5 \end{bmatrix}$$

So

$$(AB)^{-1} = \frac{Adj(AB)}{|AB|}$$

$$= \frac{\begin{bmatrix} -1 & 1 \\ -3 & 5 \end{bmatrix}}{-2}$$

$$(AB)^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 3 & -5 \end{bmatrix} \qquad \text{(ii)}$$

From (i) and (ii)

$$(AB)^{-1} = B^{-1}A^{-1}$$

(ii) $\mathbf{A = \begin{bmatrix} 5 & 1 \\ 2 & 2 \end{bmatrix}, B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}}$

Solution:

$$A = \begin{bmatrix} 5 & 1 \\ 2 & 2 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 5 & 1 \\ 2 & 2 \end{vmatrix} \quad ; \quad |B| = \begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix}$$

$$|A| = 8 \quad ; \quad |B| = -2$$

$$AdjA = \begin{bmatrix} 2 & -1 \\ -2 & 5 \end{bmatrix} \quad ; \quad AdjB = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$$

So

$$A^{-1} = \frac{AdjA}{|A|} \quad ; \quad B^{-1} = \frac{AdjB}{|B|}$$

$$A^{-1} = \frac{1}{8}\begin{bmatrix} 2 & -1 \\ -2 & 5 \end{bmatrix} \quad ; \quad B^{-1} = -\frac{1}{2}\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$$

So

$$B^{-1}A^{-1} = \left(-\frac{1}{2}\right)\left(\frac{1}{8}\right)\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -2 & 5 \end{bmatrix}$$

$$= -\frac{1}{16}\begin{bmatrix} 2+6 & -1-15 \\ -4-8 & 2+20 \end{bmatrix}$$

$$= -\frac{1}{16}\begin{bmatrix} 8 & -16 \\ -12 & 22 \end{bmatrix}$$

$$B^{-1}A^{-1} = -\frac{1}{8}\begin{bmatrix} 4 & -8 \\ -6 & 11 \end{bmatrix} \qquad \text{(i)}$$

Consider

$$AB = \begin{bmatrix} 5 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 20+2 & 15+1 \\ 8+4 & 6+2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 22 & 16 \\ 12 & 8 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 22 & 16 \\ 12 & 8 \end{vmatrix} = 176 - 192$$

$$|AB| = -16$$

$$Adj(AB) = \begin{bmatrix} 8 & -16 \\ -12 & 22 \end{bmatrix}$$

So

$$(AB)^{-1} = \frac{Adj(AB)}{|AB|}$$

$$= -\frac{1}{16}\begin{bmatrix} 8 & -16 \\ -12 & 22 \end{bmatrix}$$

$$= -\frac{1}{8}\begin{bmatrix} 4 & -8 \\ -6 & 11 \end{bmatrix} \qquad \text{(ii)}$$

From (i) and (ii)

$$(AB)^{-1} = B^{-1}A^{-1}$$

Q.15 Verify that $(AB)^t = B^tA^t$, if $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 3 & 2 \\ 0 & -1 \end{bmatrix}$

Solution:

If $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$, then $A^t = \begin{bmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 1 \end{bmatrix}$

and $B = \begin{bmatrix} 1 & 1 \\ 3 & 2 \\ 0 & -1 \end{bmatrix}$, then $B^t = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

Consider $AB = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & 2 \\ 0 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 1-3+0 & 1-2-2 \\ 0+9+0 & 0+6-1 \end{bmatrix}$$

$$AB = \begin{bmatrix} -2 & -3 \\ 9 & 5 \end{bmatrix}$$

$$(AB)^t = \begin{bmatrix} -2 & 9 \\ -3 & 5 \end{bmatrix} \qquad \text{(i)}$$

Consider $B^t A^t = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1-3+0 & 0+9+0 \\ 1-2-2 & 0+6-1 \end{bmatrix}$$

$$B^t A^t = \begin{bmatrix} -2 & 9 \\ -3 & 5 \end{bmatrix} \qquad \text{(ii)}$$

From (i) and (ii)

$$(AB)^t = B^t A^t$$

Q.16 If A = $\begin{bmatrix} \mathbf{2} & \mathbf{-1} \\ \mathbf{3} & \mathbf{1} \end{bmatrix}$ verify that $\left(A^{-1}\right)^t = \left(A^t\right)^{-1}$

Solution:

If $A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$

$$\Rightarrow A^t = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$$

And $A^{-1} = \dfrac{AdjA}{|A|} = \dfrac{\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix}}{2+3} = \dfrac{1}{5}\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix}$

$$\left(A^{-1}\right)^t = \frac{1}{5}\begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} \qquad \text{(i)}$$

Consider $\left(A^t\right)^{-1} = \dfrac{AdjA^t}{|A^t|} = \dfrac{\begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}}{2+3}$

$$\left(A^t\right)^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} \qquad \text{(ii)}$$

From (i) and (ii)

$$\left(A^{-1}\right)^t = \left(A^t\right)^{-1}$$

Q.17 If A and B are non-singular matrices, then show that

(i) $(AB)^{-1} = B^{-1}A^{-1}$

Solution:

$(AB)^{-1} = B^{-1}A^{-1}$

Since A and B are non singular matrices, so they are invertible now consider

$(AB)(B^{-1}A^{-1})$

$= A(BB^{-1})A^{-1}$ $\because$ Using associative property

$= A(I)A^{-1}$ $\because BB^{-1} = B^{-1}B = I$

$= AA^{-1}$

$= I$

$(AB)(B^{-1}A^{-1}) = I$ (i)

Again consider

$(B^{-1}A^{-1})(AB)$

$= B^{-1}(A^{-1}A)B$ $\because$ Using associative property

$= B^{-1}(I)B$

$= B^{-1}B$ $\because IB = B$

$= I$

$(B^{-1}A^{-1})(AB) = I$ (ii)

From (i) and (ii) it is clear that AB and $B^{-1}A^{-1}$ are multiplicative inverse of each other

So, we can say that $B^{-1}A^{-1}$ is the multiplicative inverse of AB hence

$(AB)^{-1} = B^{-1}A^{-1}$

(ii) $(A^{-1})^{-1} = A$

Consider

$(A)(A^{-1})$

$= AA^{-1}$

$= I$

$AA^{-1} = I$ (i)

Again $(A^{-1})(A)$

$= A^{-1}A$

$= I$

$A^{-1}A = I$ (ii)

From (i) and (ii) it is clear that

$AA^{-1} = A^{-1}A = I$

A^{-1} is the multiplicative inverse of A

$(A^{-1})^{-1} = A$

Elementary row and Column operations on a Matrix:

Following are the row or column operations.

(i) Interchanging of two rows or columns.

(ii) Multiplying a row or column by a non zero number.

(iii) Adding a multiple of one row or column to another row or column respectively.

Upper Triangular Matrix:

A square matrix $A=\left[a_{ij}\right]$ is called **upper triangular** if all elements below the principal diagonal are zero, that is, $a_{ij}=0$ for all $i>j$.

Lower Triangular Matrix:

A square matrix $A=\left[a_{ij}\right]$ is said to be **lower triangular** if all elements above the principal diagonal are zero, that is, $a_{ij}=0$ for all $i<j$

Triangular Matrix:

A square matrix A is named as **triangular** whether it is upper triangular or lower triangular matrix. For example the matrices.

$$\begin{bmatrix}1 & 2 & 3\\0 & 1 & 4\\0 & 0 & 6\end{bmatrix} \text{ and } \begin{bmatrix}1 & 0 & 0 & 0\\3 & 2 & 0 & 0\\4 & 1 & 0 & 0\\-1 & 2 & 3 & 1\end{bmatrix}$$ are triangular matrices.

Symmetric Matrix:

A square matrices $A=\left[a_{ij}\right]_{n\times n}$ is called **symmetric matrix** if $A^t=A$.

Skew Symmetric Matrix:

A square matrix $A=\left[a_{ij}\right]_{n\times n}$ is called **skew symmetric** or anti-symmetric if $A^t=-A$.

Hermitian Matrix:

A square matrix $A=\left[a_{ij}\right]_{n\times n}$ with complex entries, is called **hermitian** if $\left(\overline{A}\right)^t = A$.

Skew Hermitian Matrix:

A square matrix $A=\left[a_{ij}\right]_{n\times n}$ with complex entries, is called **skew – hermitian** or **anti - hermitian** if $\left(\overline{A}\right)^t = -A$.

Echelon and Reduced Echelon Forms of Matrix:

An $m\times n$ matrix A is called in **(row) echelon form** if

(i) In each successive non-zero row, the number of zeros before the leading entry is greater than the number of such zeros in the preceding row,

(ii) Every non-zero row in A precedes every zero row (if any).

Reduced Echelon Form of a Matrix:

A matrix is in reduced (row) echelon form if is satisfies the following conditions.

- It is in Row echelon form.
- Every leading entry is 1 and is the only non-zero entry in its column.

The matrices $\begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *and* $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are in (row) reduced echelon form.

Rank of Matrix:

Let A be a non-zero matrix, if r is the number of non-zero rows when it is reduced to the reduced echelon form, then r is called the (row) **rank** of the matrix A.

EXERCISE 3.4

Q.1 If $A = \begin{bmatrix} 1 & -2 & 5 \\ -2 & 3 & -1 \\ 5 & -1 & 0 \end{bmatrix}$ **and** $B = \begin{bmatrix} -3 & 1 & -2 \\ 1 & 0 & -1 \\ -2 & -1 & 2 \end{bmatrix}$**, then show that** $A+B$ **is symmetric.**

Solution:

$$A+B = \begin{bmatrix} 1 & -2 & 5 \\ -2 & 3 & -1 \\ 5 & -1 & 0 \end{bmatrix} + \begin{bmatrix} -3 & 1 & -2 \\ 1 & 0 & -1 \\ -2 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3 & -2+1 & 5-2 \\ -2+1 & 3+0 & -1-1 \\ 5-2 & -1-1 & 0+2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -1 & 3 \\ -1 & 3 & -2 \\ 3 & -2 & 2 \end{bmatrix}$$

Now

$$(A+B)^t = \begin{bmatrix} -2 & -1 & 3 \\ -1 & 3 & -2 \\ 3 & -2 & 2 \end{bmatrix}^t$$

$$= \begin{bmatrix} -2 & -1 & 3 \\ -1 & 3 & -2 \\ 3 & -2 & 2 \end{bmatrix} = A+B$$

$$(A+B)^t = A+B$$

Hence $A+B$ is symmetric

Q.2 If $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}$**, show that**

(i) $A+A^t$ **is symmetric**

Solution:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}, \text{then } A^t = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$$

Now let

$$U = A + A^t = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\Rightarrow U = \begin{bmatrix} 1+1 & 2+3 & 0-1 \\ 3+2 & 2+2 & -1+3 \\ -1+0 & 3-1 & 2+2 \end{bmatrix}$$

$$U = \begin{bmatrix} 2 & 5 & -1 \\ 5 & 4 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

Now

$$U^t = \left(A + A^t\right)^t = \begin{bmatrix} 2 & 5 & -1 \\ 5 & 4 & 2 \\ -1 & 2 & 4 \end{bmatrix}^t$$

$$U^t = \begin{bmatrix} 2 & 5 & -1 \\ 5 & 4 & 2 \\ -1 & 2 & 4 \end{bmatrix} = U$$

$$\left(A + A^t\right)^t = A + A^t$$

Hence $A + A^t$ is symmetric

(ii) $A - A^t$ **is skew symmetric**

Solution:

Let $V = A - A^t = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$

$$V = A - A^t = \begin{bmatrix} 1-1 & 2-3 & 0-(-1) \\ 3-2 & 2-2 & -1-3 \\ -1-0 & 3-(-1) & 2-2 \end{bmatrix}$$

$$V = A - A^t = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -4 \\ -1 & 4 & 0 \end{bmatrix}$$

Now $V^t = \left(A - A^t\right)^t = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -4 \\ -1 & 4 & 0 \end{bmatrix}^t = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 4 \\ 1 & -4 & 0 \end{bmatrix}$

$$V^t = \left(A - A^t\right)^t = -\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -4 \\ -1 & 4 & 0 \end{bmatrix} = -\left(A - A\right)^t$$

Hence $A - A^t$ is skew symmetric

Q.3 If A is any matrix of order 3, show that

(i) $A+A^t$ is symmetric

Solution:

Let $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix of order 3. then

$$A^t=\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

Now $A+A^t=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}+\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

$$=\begin{bmatrix} a_{11}+a_{11} & a_{12}+a_{21} & a_{13}+a_{31} \\ a_{21}+a_{12} & a_{22}+a_{22} & a_{23}+a_{32} \\ a_{31}+a_{32} & a_{32}+a_{23} & a_{33}+a_{33} \end{bmatrix}$$

$$A+A^t=\begin{bmatrix} 2a_{11} & a_{12}+a_{21} & a_{13}+a_{31} \\ a_{21}+a_{12} & 2a_{22} & a_{23}+a_{32} \\ a_{31}+a_{32} & a_{32}+a_{23} & 2a_{33} \end{bmatrix}$$

Now

$$(A+A^t)^t=\begin{bmatrix} 2a_{11} & a_{12}+a_{21} & a_{13}+a_{31} \\ a_{21}+a_{12} & 2a_{22} & a_{23}+a_{32} \\ a_{31}+a_{32} & a_{32}+a_{23} & 2a_{33} \end{bmatrix}^t$$

$$=\begin{bmatrix} 2a_{11} & a_{12}+a_{21} & a_{13}+a_{31} \\ a_{21}+a_{12} & 2a_{22} & a_{23}+a_{32} \\ a_{31}+a_{32} & a_{32}+a_{23} & 2a_{33} \end{bmatrix}=A+A^t$$

$$\left(A+A^t\right)^t=A+A^t$$

Hence $A+A^t$ is symmetric.

(ii) $A-A^t$ is skew symmetric

Solution:

Now $A-A^t=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}-\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

$$=\begin{bmatrix} a_{11}-a_{11} & a_{12}-a_{21} & a_{13}-a_{31} \\ a_{21}-a_{12} & a_{22}-a_{22} & a_{23}-a_{32} \\ a_{31}-a_{32} & a_{32}-a_{23} & a_{33}-a_{33} \end{bmatrix}$$

$$A-A^t=\begin{bmatrix} 0 & a_{12}-a_{21} & a_{13}-a_{31} \\ a_{21}-a_{12} & 0 & a_{23}-a_{32} \\ a_{31}-a_{32} & a_{32}-a_{23} & 0 \end{bmatrix}$$

Now

$$\left(A-A^t\right)^t=\begin{bmatrix} 0 & a_{12}-a_{21} & a_{13}-a_{31} \\ a_{21}-a_{12} & 0 & a_{23}-a_{32} \\ a_{31}-a_{32} & a_{32}-a_{23} & 0 \end{bmatrix}^t=\begin{bmatrix} 0 & a_{21}-a_{12} & a_{31}-a_{32} \\ a_{12}-a_{21} & 0 & a_{32}-a_{23} \\ a_{13}-a_{31} & a_{23}-a_{32} & 0 \end{bmatrix}$$

$$=-\begin{bmatrix} 0 & a_{21}-a_{12} & a_{31}-a_{32} \\ a_{12}-a_{21} & 0 & a_{32}-a_{23} \\ a_{13}-a_{31} & a_{23}-a_{32} & 0 \end{bmatrix}=A-A^t$$

Since $\left(A-A^t\right)^t=-\left(A-A^t\right)$ so $A-A^t$ is skew symmetric.

Q.4 If the matrices A and B are symmetric and $AB=BA$, show that AB is symmetric.

Solution: Let A and B are symmetric , then

$A^t=A$ ——— (i)

$B^t=B$ ——— (ii)

Now we have to show that AB is symmetric ,for this

Consider $\left(AB\right)^t=B^tA^t$ $\quad\therefore\left(AB\right)^t=B^tA^t$

$=BA$ $\quad\therefore B^t=B, A^t=A$

$=AB$ $\quad\therefore AB=BA$

$\left(AB\right)^t=AB$

Hence AB is symmetric

Q.5 Show that AA^t and A^tA are symmetric for any matrix of order 2×3

Solution: Let $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ be a matrix of order 2×3,then $A^t=\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$

Now

$$AA^t=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

$$=\begin{bmatrix} a_{11}^2+a_{12}^2+a_{13}^2 & a_{11}a_{21}+a_{12}a_{22}+a_{13}a_{23} \\ a_{21}a_{11}+a_{22}a_{12}+a_{23}a_{13} & a_{21}^2+a_{22}^2+a_{23}^2 \end{bmatrix}$$

$$\left(AA^t\right)^t = \begin{bmatrix} a_{11}^2+a_{12}^2+a_{13}^2 & a_{11}a_{21}+a_{12}a_{22}+a_{13}a_{23} \\ a_{21}a_{11}+a_{22}a_{12}+a_{23}a_{13} & a_{21}^2+a_{22}^2+a_{23}^2 \end{bmatrix}^t$$

$$= \begin{bmatrix} a_{11}^2+a_{12}^2+a_{13}^2 & a_{21}a_{11}+a_{22}a_{12}+a_{23}a_{13} \\ a_{11}a_{21}+a_{12}a_{22}+a_{13}a_{23} & a_{21}^2+a_{22}^2+a_{23}^2 \end{bmatrix}$$

$$= AA^t$$

Hence AA^t is symmetric

Now

Consider $A^tA = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

$$A^tA = \begin{bmatrix} a_{11}^2+a_{21}^2 & a_{11}a_{21}+a_{21}a_{22} & a_{11}a_{13}+a_{21}a_{23} \\ a_{21}a_{11}+a_{22}a_{21} & a_{12}^2+a_{22}^2 & a_{12}a_{13}+a_{22}a_{23} \\ a_{13}a_{11}+a_{23}a_{21} & a_{13}a_{12}+a_{23}a_{22} & a_{13}^2+a_{23}^2 \end{bmatrix}$$

$$\left(A^tA\right)^t = \begin{bmatrix} a_{11}^2+a_{21}^2 & a_{11}a_{21}+a_{21}a_{22} & a_{11}a_{13}+a_{21}a_{23} \\ a_{21}a_{11}+a_{22}a_{21} & a_{12}^2+a_{22}^2 & a_{12}a_{13}+a_{22}a_{23} \\ a_{13}a_{11}+a_{23}a_{21} & a_{13}a_{12}+a_{23}a_{22} & a_{13}^2+a_{23}^2 \end{bmatrix}^t$$

$$= \begin{bmatrix} a_{11}^2+a_{21}^2 & a_{21}a_{11}+a_{22}a_{21} & a_{13}a_{11}+a_{23}a_{21} \\ a_{11}a_{21}+a_{21}a_{22} & a_{12}^2+a_{22}^2 & a_{13}a_{12}+a_{23}a_{22} \\ a_{11}a_{13}+a_{21}a_{23} & a_{12}a_{13}+a_{22}a_{23} & a_{13}^2+a_{23}^2 \end{bmatrix}$$

$$\left(A^tA\right)^t = A^tA$$

Hence A^tA is symmetric

Q.6 If $A = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}$, show that

(i) $A+\left(\overline{A}\right)^t$ **is hermitian**

Solution: Consider $A = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}$

then $\overline{A} = \begin{bmatrix} -i & 1-i \\ 1 & i \end{bmatrix}$

$$\left(\overline{A}\right)^t = \begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}$$

Let $U = A+\left(\overline{A}\right)^t = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix} + \begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}$

$$U=\begin{bmatrix}0 & 2+i\\ 2-i & 0\end{bmatrix}$$

$$\bar{U}=\begin{bmatrix}0 & 2-i\\ 2+i & 0\end{bmatrix}$$

$$\left(\bar{U}\right)^t=\begin{bmatrix}0 & 2-i\\ 2+i & 0\end{bmatrix}^t$$

$$\left(\bar{U}\right)^t=\begin{bmatrix}0 & 2+i\\ 2-i & 0\end{bmatrix}=U$$

$$\left(\bar{U}\right)^t=U$$

$$\left(\overline{A+\left(\bar{A}\right)^t}\right)^t=A+\left(\bar{A}\right)^t \qquad \because U=A+\left(\bar{A}\right)^t$$

So $A+\left(\bar{A}\right)^t$ is a hermitian matrix.

(ii) $A-\left(\bar{A}\right)^t$ **is skew hermitian**

Solution: Let $V=A-\left(\bar{A}\right)^t$ so

$$V=A-\left(\bar{A}\right)^t=\begin{bmatrix}i & 1+i\\ 1 & -i\end{bmatrix}-\begin{bmatrix}-i & 1\\ 1-i & i\end{bmatrix}$$

$$V=\begin{bmatrix}2i & i\\ i & -2i\end{bmatrix}$$

So $\bar{V}=\begin{bmatrix}-2i & -i\\ -i & 2i\end{bmatrix}$

$$\left(\bar{V}\right)^t=\begin{bmatrix}-2i & -i\\ -i & 2i\end{bmatrix}^t$$

$$=\begin{bmatrix}-2i & -i\\ -i & 2i\end{bmatrix}$$

$$=-\begin{bmatrix}2i & i\\ i & -2i\end{bmatrix}=-V$$

$$\left(\bar{V}\right)^t=-V$$

$$\left(\overline{A-\left(\bar{A}\right)^t}\right)^t=-\left(A-\left(\bar{A}\right)^t\right) \qquad \because V=A-\left(\bar{A}\right)^t$$

Hence $A-\left(\bar{A}\right)^t$ is skew hermitian

Q.7 If A is symmetric or skew symmetric show that A^2 is symmetric.

Proof: Case I: when A is symmetric

Since A is symmetric so $A^t = A$

Now $A^2 = A.A$

$(A^2)^t = (A.A)^t$

$= A^t.A^t$

$= A.A \qquad \because A^t = A$

$= A^2$

$(A^2)^t = A^2$

Hence if A is symmetric, then A^2 is symmetric.

Case II: when A is skew symmetric

Since A is skew symmetric so $A^t = -A$

$(A^2)^t = (A.A)^t$

$= A^t.A^t \qquad \because A^t = -A$

$= (-A)(-A)$

$= A^2$

$(A^2)^t = A^2$

Hence if A is skew symmetric, then A^2 is symmetric

Q.8 If $\mathbf{A} = \begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix}$ find $\mathbf{A}\left(\overline{\mathbf{A}}\right)^t$

Solution: Consider $A = \begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix}$

$\overline{A} = \begin{bmatrix} 1 \\ 1-i \\ -i \end{bmatrix}$

$\left(\overline{A}\right)^t = \begin{bmatrix} 1 & 1-i & -i \end{bmatrix}$

$A\left(\overline{A}\right)^t = \begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix}\begin{bmatrix} 1 & 1-i & -i \end{bmatrix}$

$$A\left(\bar{A}\right)^t=\begin{bmatrix}1 & 1-i & -i\\ 1+i & 1-i^2 & -i(1+i)\\ i & i(1-i) & -i^2\end{bmatrix}$$

$$=\begin{bmatrix}1 & 1-i & -i\\ 1+i & 1+1 & -i-i^2\\ i & i-i^2 & -i^2\end{bmatrix} \qquad \because i^2=-1$$

$$A\left(\bar{A}\right)^t=\begin{bmatrix}1 & 1-i & -i\\ 1+i & 2 & -i+1\\ i & 1+i & 1\end{bmatrix}$$

Q.9 Find the inverses of the following matrices. Also find their inverses by row and column operations.

(i) $\begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{-3}\\ \mathbf{0} & \mathbf{-2} & \mathbf{0}\\ \mathbf{-2} & \mathbf{-2} & \mathbf{2}\end{bmatrix}$

Solution: Let $A=\begin{bmatrix}1 & 2 & -3\\ 0 & -2 & 0\\ -2 & -2 & 2\end{bmatrix}$

Inverse of any square matrix A is given by

$$A^{-1}=\frac{AdjA}{|A|}$$

Therefore $|A|=\begin{vmatrix}1 & 2 & -3\\ 0 & -2 & 0\\ -2 & -2 & 2\end{vmatrix}$

$$=1\begin{vmatrix}-2 & 0\\ -2 & 2\end{vmatrix}-2\begin{vmatrix}0 & 0\\ -2 & 2\end{vmatrix}-3\begin{vmatrix}0 & -2\\ -2 & -2\end{vmatrix}$$

$$=1(-4-0)-2(0-0)-3(0-4)$$

$$|A|=-4+12=8\neq 0$$

Since $|A|\neq 0$ so A^{-1} exists

Now

$$AdjA=\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}^t$$

$$AdjA=\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

From $A_{ij}=(-1)^{i+j}M_{ij}$ we get

$$A_{11}=(-1)^{1+1}M_{11}=M_{11}=\begin{vmatrix} -2 & 0 \\ -2 & 2 \end{vmatrix}=-4$$

$$A_{12}=(-1)^{1+2}M_{12}=-M_{12}=-\begin{vmatrix} 0 & 0 \\ -2 & 2 \end{vmatrix}=0$$

$$A_{13}=(-1)^{1+3}M_{13}=M_{13}=\begin{vmatrix} 0 & -2 \\ -2 & -2 \end{vmatrix}=-4$$

$$A_{21}=(-1)^{2+1}M_{21}=-M_{21}=-\begin{vmatrix} 2 & -3 \\ -2 & 2 \end{vmatrix}=-(4-6)=2$$

$$A_{22}=(-1)^{2+2}M_{22}=\begin{vmatrix} 1 & -3 \\ -2 & 2 \end{vmatrix}=2-6=-4$$

$$A_{23}=(-1)^{2+3}M_{23}=-\begin{vmatrix} 1 & 2 \\ -2 & -2 \end{vmatrix}=-(-2+4)=-2$$

$$A_{31}=(-1)^{3+1}M_{31}=\begin{vmatrix} 2 & -3 \\ -2 & 0 \end{vmatrix}=0-6=-6$$

$$A_{32}=(-1)^{3+2}M_{32}=-\begin{vmatrix} 1 & -3 \\ 0 & 0 \end{vmatrix}=0$$

$$A_{33}=(-1)^{3+3}M_{33}=\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix}=-2-0=-2$$

So $$AdjA=\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$=\begin{bmatrix} -4 & 2 & -6 \\ 0 & -4 & 0 \\ -4 & -2 & -2 \end{bmatrix}$$

Now $A^{-1}=\dfrac{AdjA}{|A|}=\dfrac{1}{8}\begin{bmatrix} -4 & 2 & -6 \\ 0 & -4 & 0 \\ -4 & -2 & -2 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Inverse of A by Row operations

Appending I_3 on the right of matrix A

$$\begin{bmatrix} 1 & 2 & -3 & : & 1 & 0 & 0 \\ 0 & -2 & 0 & : & 0 & 1 & 0 \\ -2 & -2 & 2 & : & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{R}{\sim}\begin{bmatrix} 1 & 2 & -3 & : & 1 & 0 & 0 \\ 0 & -2 & 0 & : & 0 & 1 & 0 \\ 0 & 2 & -4 & : & 2 & 0 & 1 \end{bmatrix} \qquad \text{By } R_3 + 2R_1 \to R_3'$$

$$\underset{R}{\sim}\begin{bmatrix} 1 & 2 & -3 & : & 1 & 0 & 0 \\ 0 & 1 & 0 & : & 0 & -\frac{1}{2} & 0 \\ 0 & 2 & -4 & : & 2 & 0 & 1 \end{bmatrix} \qquad \text{By } -\frac{1}{2}R_2 \to R_2'$$

$$\underset{R}{\sim}\begin{bmatrix} 1 & 0 & -3 & : & 1 & 1 & 0 \\ 0 & 1 & 0 & : & 0 & -\frac{1}{2} & 0 \\ 0 & 2 & -4 & : & 2 & 0 & 1 \end{bmatrix} \qquad \text{By } R_1 - 2R_2 \to R_1'$$

$$\underset{R}{\sim}\begin{bmatrix} 1 & 0 & -3 & : & 1 & 1 & 0 \\ 0 & 1 & 0 & : & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & : & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} \qquad \text{By } \left(\frac{-1}{4}\right)R_3 \to R_3'$$

$$\underset{R}{\sim}\begin{bmatrix} 1 & 0 & 0 & : & -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & 1 & 0 & : & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & : & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} \qquad \text{By } R_1 + 3R_3 \to R_1'$$

Thus the inverse of A is

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Inverse of A by column operation,

Appending I_3 below the matrix A

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \\ : & : & : \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overset{C}{\sim}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ -2 & 2 & -4 \\ : & : & : \\ 1 & -2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{By } C_2 - 2C_1 \to C_2', C_3 + 3C_1 \to C_3'$$

$$\overset{C}{\sim}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & -4 \\ : & : & : \\ 1 & 1 & 3 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{By } -\frac{1}{2}C_2 \to C_2'$$

$$\overset{C}{\sim}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & 1 \\ : & : & : \\ 1 & 1 & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix} \qquad \text{By } \left(-\frac{1}{4}\right)C_3 \to C_3'$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \vdots & \vdots & \vdots \\ -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} \quad \text{By } C_1 + 2C_3 \to C_1', C_2 + C_3 \to C_2'$$

Hence inverse of A is

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

(ii) $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{0} & \mathbf{-1} & \mathbf{3} \\ \mathbf{1} & \mathbf{0} & \mathbf{2} \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$

Inverse of any square matrix A is given by

$$A^{-1} = \frac{AdjA}{|A|}$$

Therefore $|A| = \begin{vmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{vmatrix}$

$$= 1\begin{vmatrix} -1 & 3 \\ 0 & 2 \end{vmatrix} - 2\begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$$

$$= 1(-2-0) - 2(0-3) - 1(0+1)$$

$$= -2 + 6 - 1$$

$$|A| = 3 \neq 0$$

Since $|A| \neq 0$ so A^{-1} exists

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Now

$$AdjA=\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$$

$$AdjA=\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Now by using $A_{ij}=(-1)^{i+j}M_{ij}$

$$A_{11}=(-1)^{1+1}M_{11}=M_{11}=\begin{vmatrix} -1 & 3 \\ 0 & 2 \end{vmatrix}=-2$$

$$A_{12}=(-1)^{1+2}M_{12}=-M_{12}=-\begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix}=(0-3)=3$$

$$A_{13}=(-1)^{1+3}M_{13}=M_{13}=\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}=(0+1)=1$$

$$A_{21}=(-1)^{2+1}M_{21}=-M_{21}\begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix}=-(4-0)=-4$$

$$A_{22}=(-1)^{2+2}M_{22}=M_{22}=\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix}=(2+1)=3$$

$$A_{23}=(-1)^{2+3}M_{23}=-\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix}=-(0-2)=2$$

$$A_{31}=(-1)^{3+1}M_{31}=M_{31}=\begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix}=(6-1)=5$$

$$A_{32}=(-1)^{3+2}M_{32}=-M_{32}=-\begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix}=-(3-0)=-3$$

$$A_{33}=(-1)^{3+3}M_{33}=M_{33}=\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix}=(-1-0)=-1$$

Now $AdjA=\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

$$=\begin{bmatrix} -2 & -4 & 5 \\ 3 & 3 & -3 \\ 1 & 2 & -1 \end{bmatrix}$$

$$A^{-1}=\frac{AdjA}{|A|}=\frac{1}{3}\begin{bmatrix}-2 & -4 & 5\\ 3 & 3 & -3\\ 1 & 2 & -1\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}-\frac{2}{3} & -\frac{4}{3} & \frac{5}{3}\\ 1 & 1 & -1\\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3}\end{bmatrix}$$

Now inverse of A by row operation Appending I_3 to the right of matrix A

$$\begin{bmatrix}1 & 2 & -1 & : & 1 & 0 & 0\\ 0 & -1 & 3 & : & 0 & 1 & 0\\ 1 & 0 & 2 & : & 0 & 0 & 1\end{bmatrix}$$

$$\underset{R}{\sim}\begin{bmatrix}1 & 2 & -1 & : & 1 & 0 & 0\\ 0 & -1 & 3 & : & 0 & 1 & 0\\ 0 & -2 & 3 & : & -1 & 0 & 1\end{bmatrix}$$ By $R_3-R_1\to R_3'$

$$\underset{R}{\sim}\begin{bmatrix}1 & 2 & -1 & : & 1 & 0 & 0\\ 0 & 1 & -3 & : & 0 & -1 & 0\\ 0 & -2 & 3 & : & -1 & 0 & 1\end{bmatrix}$$ By $(-1)R_2\to R_2'$

$$\underset{R}{\sim}\begin{bmatrix}1 & 0 & 5 & : & 1 & 2 & 0\\ 0 & 1 & -3 & : & 0 & -1 & 0\\ 0 & 0 & -3 & : & -1 & -2 & 1\end{bmatrix}$$ By

$R_1-2R_2\to R_1', R_3+2R_2\to R_3'$

$$\underset{R}{\sim}\begin{bmatrix}1 & 0 & 5 & : & 1 & 2 & 0\\ 0 & 1 & -3 & : & 0 & -1 & 0\\ 0 & 0 & 1 & : & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3}\end{bmatrix}$$ By $-\frac{1}{3}R_3\to R_3'$

$$\underset{R}{\sim}\begin{bmatrix}1 & 0 & 0 & : & -\frac{2}{3} & -\frac{4}{3} & \frac{5}{3}\\ 0 & 1 & 0 & : & 1 & 1 & -1\\ 0 & 0 & 1 & : & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3}\end{bmatrix}$$ By

$R_2+3R_3\to R_2', R_1-5R_3\to R_1'$

Thus inverse of A is

$$A^{-1}=\begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \\ 1 & 1 & -1 \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

Inverse by column operation:

Appending I_3 below the given matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \\ : & : & : \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{C}{\sim}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 3 \\ 1 & -2 & 3 \\ : & : & : \\ 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{By}$$

$C_2-2C_1\to C_2', C_3+C_1\to C_3'$

$$\underset{C}{\sim}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 1 & 2 & 3 \\ : & : & : \\ 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{By } (-1)C_2\to C_2'$$

$$\underset{C}{\sim}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & -3 \\ : & : & : \\ 1 & 2 & -5 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{By } C_3-3C_2\to C_3'$$

$$C\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \\ : & : & : \\ 1 & 2 & \frac{5}{3} \\ 0 & -1 & -1 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} \quad \text{By } \left(-\frac{1}{3}\right)C_3 \to C_3'$$

$$C\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ : & : & : \\ -\frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \\ 1 & 1 & -1 \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \quad \text{By}$$

$C_1 - C_3 \to C_1', C_2 - 2C_3 \to C_2'$

Thus inverse of A is $\begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \\ 1 & 1 & -1 \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$

(iii) $\begin{bmatrix} \mathbf{1} & \mathbf{-3} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

Inverse of any square matrix A is given by

$$A^{-1} = \frac{Adj\,A}{|A|}$$

Therefore $|A| = \begin{vmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix}$

$$=1\begin{vmatrix}1 & 0\\ -1 & 1\end{vmatrix}+3\begin{vmatrix}2 & 0\\ 0 & 1\end{vmatrix}+2\begin{vmatrix}2 & 1\\ 0 & -1\end{vmatrix}$$

$$=1(1-0)+3(2-0)+2(-2-0)$$

$$|A|=1+6-4=3\neq 0$$

Since $|A|\neq 0$ so A^{-1} exists

Now

$$AdjA=\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}^t$$

$$AdjA=\begin{bmatrix}A_{11} & A_{21} & A_{31}\\ A_{12} & A_{22} & A_{32}\\ A_{13} & A_{23} & A_{33}\end{bmatrix}$$

Now by using $A=(-1)^{i+j}M_{ij}$ we get

$$A_{11}=(-1)^{1+1}M_{11}=M_{11}=\begin{vmatrix}1 & 0\\ -1 & 1\end{vmatrix}=(1-0)=1$$

$$A_{12}=(-1)^{1+2}M_{12}=-M_{12}=-\begin{vmatrix}2 & 0\\ 0 & 1\end{vmatrix}=-(2-0)=-2$$

$$A_{13}(-1)^{1+3}M_{13}=M_{13}=\begin{vmatrix}2 & 1\\ 0 & -1\end{vmatrix}=-2-0=-2$$

$$A_{21}=(-1)^{2+1}M_{21}=-M_{21}=-\begin{vmatrix}-3 & 2\\ -1 & 1\end{vmatrix}=-(-3+2)=1$$

$$A_{22}=(-1)^{2+2}M_{22}=M_{22}=\begin{vmatrix}1 & 2\\ 0 & 1\end{vmatrix}=(1-0)=1$$

$$A_{23}=(-1)^{2+3}M_{23}=M_{23}=-\begin{vmatrix}1 & -3\\ 0 & -1\end{vmatrix}=-(-1-0)=1$$

$$A_{31}=(-1)^{3+1}M_{31}=M_{31}=\begin{vmatrix}-3 & 2\\ 1 & 0\end{vmatrix}=(0-2)=-2$$

$$A_{32}=(-1)^{3+2}M_{32}=-M_{32}=-\begin{vmatrix}1 & 2\\ 2 & 0\end{vmatrix}=-(0-4)=4$$

$$A_{33}=(-1)^{3+3}M_{33}=M_{33}=\begin{vmatrix}1 & -3\\ 2 & 1\end{vmatrix}=(1+6)=7$$

$$\text{Thus } AdjA=\begin{bmatrix} A_{11} & A_{21} & A_{31}\\ A_{12} & A_{22} & A_{32}\\ A_{13} & A_{23} & A_{33}\end{bmatrix}$$

$$=\begin{bmatrix} 1 & 1 & -2\\ -2 & 1 & 4\\ -2 & 1 & 7\end{bmatrix}$$

Now

$$A^{-1}=\frac{AdjA}{|A|}=\frac{1}{3}\begin{bmatrix} 1 & 1 & -2\\ -2 & 1 & 4\\ -2 & 1 & 7\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{2}{3}\\ -\frac{2}{3} & \frac{1}{3} & \frac{4}{3}\\ -\frac{2}{3} & \frac{1}{3} & \frac{7}{3}\end{bmatrix}$$

Inverse of the matrix by row operation

Appending I_3 to the right of the matrix

$$\begin{bmatrix} 1 & -3 & 2 & : & 1 & 0 & 0\\ 2 & 1 & 0 & : & 0 & 1 & 0\\ 0 & -1 & 1 & : & 0 & 0 & 1\end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -3 & 2 & : & 1 & 0 & 0\\ 0 & 7 & -4 & : & -2 & 1 & 0\\ 0 & -1 & 1 & : & 0 & 0 & 1\end{bmatrix} \qquad \text{By } R_2-2R_1\to R_2'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -3 & 2 & : & 1 & 0 & 0\\ 0 & 1 & 2 & : & -2 & 1 & 6\\ 0 & -1 & 1 & : & 0 & 0 & 1\end{bmatrix} \qquad \text{By } R_2+6R_3\to R_2'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 8 & : & -5 & 3 & 18\\ 0 & 1 & 2 & : & -2 & 1 & 6\\ 0 & 0 & 3 & : & -2 & 1 & 7\end{bmatrix} \qquad \text{By}$$

$R_1+3R_2\to R_1', R_3+R_2\to R_3'$

$$\underset{R}{\sim}\begin{bmatrix} 1 & 0 & 8 & : & -5 & 3 & 18 \\ 0 & 1 & 2 & : & -2 & 1 & 6 \\ 0 & 0 & 1 & : & -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix} \quad \text{By } \left(\frac{1}{3}R_3\right) \to R_3'$$

$$\underset{R}{\sim}\begin{bmatrix} 1 & 0 & 0 & : & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 0 & : & -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ 0 & 0 & 1 & : & -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix} \quad \text{By } R_2 - 2R_3 \to R_2', R_1 - 8R_3 \to R_1'$$

This the inverse of given matrix is

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix}$$

Inverse by column operation:

Appending I_3 below the given matrix

$$\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \\ : & : & : \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{C}{\sim}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 7 & -4 \\ 0 & -1 & 1 \\ : & : & : \\ 1 & 3 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{By } C_2 + 3C_1 \to C_2', C_3 - 2C_1 \to C_3'$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & -4 \\ 0 & 1 & 1 \\ \vdots & \vdots & \vdots \\ 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \qquad \text{By } C_2 + 2C_3 \rightarrow C_2'$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -4 \\ 0 & -1 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 1 & -2 \\ 0 & -1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \qquad \text{By } (-1)C_2 \rightarrow C_2'$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & -3 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 2 \\ 2 & -1 & -4 \\ 4 & -2 & -7 \end{bmatrix} \qquad \text{By}$$

$C_1 - 2C_2 \rightarrow C_1', C_3 + 4C_2 \rightarrow C_3'$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & -0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \\ \vdots & \vdots & \vdots \\ -1 & 1 & -\frac{2}{3} \\ 2 & -1 & \frac{4}{3} \\ 4 & -2 & \frac{7}{3} \end{bmatrix} \qquad \text{By } \left(\frac{-1}{3}\right)C_3 \rightarrow C_3'$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ : & : & : \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix} \quad \text{By } C_2 + C_3 \to C_2', C_1 - 2C_3 \to C_1'$$

Thus inverse of A is $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix}$

Q.10 Find the rank of the following matrices

(i) $\begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{-6} & \mathbf{5} & \mathbf{1} \\ \mathbf{3} & \mathbf{5} & \mathbf{4} & \mathbf{-3} \end{bmatrix}$

Solution: Consider $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -6 & 5 & 1 \\ 3 & 5 & 4 & -3 \end{bmatrix}$ we reduced it to reduced echelon form by using elementary row operation.

$$\underset{\sim}{R}\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & -4 & 1 & -1 \\ 0 & 8 & -2 & -6 \end{bmatrix} \quad \text{By } R_2 - 2R_1 \to R_2' \text{ , } R_3 - 3R_1 \to R_3'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 8 & -2 & -6 \end{bmatrix} \quad \text{By } \left(\frac{-1}{4}\right)R_2 \to R_2'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & \frac{7}{4} & \frac{5}{4} \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & -8 \end{bmatrix} \quad \text{By } R_1 + R_2 \to R_1 \text{ , } R_3 - 8R_2 \to R_3'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & \frac{7}{4} & \frac{5}{4} \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{By } \left(-\frac{1}{8}\right)R_3 \to R_3'$$

Number of non-zero rows are 3 hence rank of the given matrix is 3.

(ii) $\begin{bmatrix} \mathbf{1} & \mathbf{-4} & \mathbf{-7} \\ \mathbf{2} & \mathbf{-5} & \mathbf{1} \\ \mathbf{1} & \mathbf{-2} & \mathbf{3} \\ \mathbf{3} & \mathbf{-7} & \mathbf{4} \end{bmatrix}$

Solution: Consider $\begin{bmatrix} 1 & -4 & -7 \\ 2 & -5 & 1 \\ 1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix}$ we reduced it to reduced echelon form by using elementary row operation.

$$\underset{\sim}{R}\begin{bmatrix} 1 & -4 & -7 \\ 0 & 3 & 15 \\ 0 & 2 & 10 \\ 0 & 5 & 25 \end{bmatrix} \quad \text{By } R_2 - 2R_1 \to R_2 \text{ , } R_3 - R_1 \to R_3' \text{ , } R_4 - 3R_1 \to R_4'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -4 & -7 \\ 0 & 1 & 5 \\ 0 & 2 & 10 \\ 0 & 5 & 25 \end{bmatrix} \quad \text{By } R_2 - R_3 \to R_2'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 13 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{By } R_1 + 4R_2 \to R_1' \text{ , } R_3 - 2R_2 \to R_3' \text{ , } R_4 - 5R_2 \to R_4'$$

As the number of non-zero rows in reduce echelon form are 2, hence rank of the matrix is 2.

(iii) $\begin{bmatrix} \mathbf{3} & \mathbf{-1} & \mathbf{3} & \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{-3} & \mathbf{-2} \\ \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{2} & \mathbf{5} \\ \mathbf{2} & \mathbf{5} & \mathbf{-2} & \mathbf{-3} & \mathbf{3} \end{bmatrix}$

Solution: Consider $\begin{bmatrix} 3 & -1 & 3 & 0 & -1 \\ 1 & 2 & -1 & -3 & -2 \\ 2 & 3 & 4 & 2 & 5 \\ 2 & 5 & -2 & -3 & 3 \end{bmatrix}$ we reduced it to reduced echelon form by using elementary row operation.

$$\underset{\sim}{R}\begin{bmatrix}1 & 2 & -1 & -3 & -2\\ 3 & -1 & 3 & 0 & -1\\ 2 & 3 & 4 & 2 & 5\\ 2 & 5 & -2 & -3 & 3\end{bmatrix} \quad \text{By } R_1 \leftrightarrow R_2$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 2 & -1 & -3 & -2\\ 0 & -7 & 6 & 9 & 5\\ 0 & -1 & 6 & 8 & 9\\ 0 & 1 & 0 & 3 & 7\end{bmatrix} \quad \text{By } R_2 - 3R_1 \to R_2',\ R_3 - 2R_1 \to R_3',\ R_4 - 2R_1 \to R_4'$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 2 & -1 & -3 & -2\\ 0 & 1 & 0 & 3 & 7\\ 0 & -1 & 6 & 8 & 9\\ 0 & -7 & 6 & 9 & 5\end{bmatrix} \quad \text{By } R_2 \leftrightarrow R_4$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & -1 & -9 & -16\\ 0 & 1 & 0 & 3 & 7\\ 0 & 0 & 6 & 11 & 16\\ 0 & 0 & 6 & 30 & 54\end{bmatrix} \quad \text{By } R_1 - 2R_2 \to R_1',\ R_3 + R_2 \to R_3',\ R_4 + 7R_2 \to R_4'$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & -1 & -9 & -16\\ 0 & 1 & 0 & 3 & 7\\ 0 & 0 & 1 & \frac{11}{6} & \frac{8}{3}\\ 0 & 0 & 6 & 30 & 54\end{bmatrix} \quad \text{By } \frac{1}{6}R_3 \to R_3'$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & -1 & -9 & -16\\ 0 & 1 & 0 & 3 & 7\\ 0 & 0 & 1 & \frac{11}{6} & \frac{8}{3}\\ 0 & 0 & 0 & 19 & 38\end{bmatrix} \quad \text{By } R_4 - 6R_3 \to R_4'$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & -1 & -9 & -16\\ 0 & 1 & 0 & 3 & 7\\ 0 & 0 & 1 & \frac{11}{6} & \frac{8}{3}\\ 0 & 0 & 0 & 1 & 2\end{bmatrix} \quad \text{By } \frac{1}{19}R_4 \to R_4'$$

As the number of non-zero rows are 4 in reduce echelon form. So, the rank of the matrix is 4.

Exercise No 3.5

Q.1 Solve the following systems of linear equations by Cramer's Rule.

(i) $$\left.\begin{aligned} 2x+2y+z&=3\\ 3x-2y-2z&=1\\ 5x+y-3z&=2 \end{aligned}\right\}$$

Solution: Here $|A| = \begin{vmatrix} 2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3 \end{vmatrix}$

$$= 2(6+2) - 2(-9+10) + 1(3+10)$$

$$= 16 - 2 + 13 = 27 \neq 0$$

Since $|A| \neq 0$ so solution of system is possible. Now by Cramer's Rule

$$x = \frac{\begin{vmatrix} 3 & 2 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & -3 \end{vmatrix}}{|A|} = \frac{3(6+2) - 2(-3+4) + 1(1+4)}{27}$$

$$= \frac{24-2+5}{27}$$

$$= \frac{27}{27}$$

$$\Rightarrow \quad x = 1$$

$$y = \frac{\begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & -2 \\ 5 & 2 & -3 \end{vmatrix}}{|A|} = \frac{2(-3+4) - 3(-9+10) + 1(6-5)}{27}$$

$$= \frac{2-3+1}{27} = \frac{0}{27}$$

$$\Rightarrow \quad y = 0$$

$$z = \frac{\begin{vmatrix} 2 & 2 & 3 \\ 3 & -2 & 1 \\ 5 & 1 & 2 \end{vmatrix}}{|A|} = \frac{2(-4-1) - 2(6-5) + 3(3+10)}{27}$$

$$= \frac{-10-2+39}{27}$$

$$= \frac{27}{27}$$

$\Rightarrow \quad z = 1$

Hence $x = 1, y = 0, z = 1$

(ii) $\left.\begin{array}{l} \mathbf{2x_1 - x_2 + x_3 = 5} \\ \mathbf{4x_1 + 2x_2 + 3x_3 = 8} \\ \mathbf{3x_1 - 4x_2 - x_3 = 3} \end{array}\right|$

Solution: Here $|A| = \begin{vmatrix} 2 & -1 & 1 \\ 4 & 2 & 3 \\ 3 & -4 & -1 \end{vmatrix} = 2(-2+12)+1(-4-9)+1(-16-6)$

$$= 2(10)+1(-13)+1(-22)$$

$$= 20-13-22$$

$\Rightarrow \quad |A| = -15 \neq 0$

So, solution is possible.

Now by Cramer's rule

$$x_1 = \frac{\begin{vmatrix} 5 & -1 & 1 \\ 8 & 2 & 3 \\ 3 & -4 & -1 \end{vmatrix}}{|A|} = \frac{5(-2+12)+1(-8-9)+1(-32-6)}{-15}$$

$$= \frac{50-17-38}{-15}$$

$$x_1 = \frac{1}{3}$$

$$x_2 = \frac{\begin{vmatrix} 2 & 5 & 1 \\ 4 & 8 & 3 \\ 3 & 3 & -1 \end{vmatrix}}{|A|} = \frac{2(-8-9)-5(-4-9)+1(12-24)}{-15}$$

$$= \frac{-34+65-12}{-15}$$

$$x_2 = \frac{-19}{15}$$

$$x_3 = \frac{\begin{vmatrix} 2 & -1 & 5 \\ 4 & 2 & 8 \\ 3 & -4 & 3 \end{vmatrix}}{|A|} = \frac{2(6+32)+1(12-24)+5(-16-6)}{-15}$$

$$= \frac{76 - 12 - 110}{-15}$$

$$x_3 = \frac{46}{15}$$

Hence $x_1 = \frac{1}{3}, x_2 = \frac{-19}{15}, x_3 = \frac{46}{15}$

(iii)
$$\left.\begin{aligned} 2x_1 - x_2 + x_3 &= 8 \\ x_1 + 2x_2 + 2x_3 &= 6 \\ x_1 - 2x_2 - x_3 &= 1 \end{aligned}\right\}$$

Solution: Here $|A| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix}$

$$= 2(-2+4) + 1(-1-2) + 1(-2-2)$$

$$= 4 - 3 - 4$$

$$|A| = -3 \neq 0$$

Since $|A| \neq 0$ so solution is possible now by Cramer's Rule

$$x_1 = \frac{\begin{vmatrix} 8 & -1 & 1 \\ 6 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix}}{|A|} = \frac{8(-2+4) + 1(-6-2) + 1(-12-2)}{-3}$$

$$= \frac{16 - 8 - 14}{-3}$$

$$= \frac{-6}{-3} = 2$$

$$x_1 = 2$$

$$x_2 = \frac{\begin{vmatrix} 2 & 8 & 1 \\ 1 & 6 & 2 \\ 1 & 1 & -1 \end{vmatrix}}{|A|} = \frac{2(-6-2) - 8(-1-2) + 1(1-6)}{-3}$$

$$= \frac{-16 + 24 - 5}{-3} = \frac{3}{-3} = -1$$

$$x_2 = -1$$

$$x_3 = \frac{\begin{vmatrix} 2 & -1 & 8 \\ 1 & 2 & 6 \\ 1 & -2 & 1 \end{vmatrix}}{|A|} = \frac{2(2+12)+1(1-6)+8(-2-2)}{|A|}$$

$$= \frac{28-5-32}{-3}$$

$$= \frac{-9}{-3} = 3$$

$x_3 = 3$

Hence $x_1 = 2, x_2 = -1, x_3 = 3$

Q.2 Use matrices to solve the following systems.

(i) $\left.\begin{array}{r} \mathbf{x - 2y + z = -1} \\ \mathbf{3x + y - 2z = 4} \\ \mathbf{y - z = 1} \end{array}\right\}$

Solution:

The matrix form of the given system is

$$\begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

$AX = B$

$X = A^{-1}B$ (i)

Where $A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= 1(-1+2) + 2(-3-0) + 1(3-0)$$

$$|A| = 1(1) - 6 + 3 = -2 \neq 0$$

Since $|A| \neq 0$ so inverse of A is possible and given by $A^{-1} = \frac{AdjA}{|A|}$

Now $AdjA = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$

$$= \begin{bmatrix} \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & -2 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} \end{bmatrix}^t$$

$$= \begin{bmatrix} 1 & 3 & 3 \\ -1 & -1 & -1 \\ 3 & 5 & 7 \end{bmatrix}^t$$

$$= \begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & 5 \\ 3 & -1 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{AdjA}{|A|} = \frac{-1}{2}\begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & 5 \\ 3 & -1 & 7 \end{bmatrix}$$

Put the values of A^{-1} and B in (i) we get

$$X = A^{-1}B = -\frac{1}{2}\begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & 5 \\ 3 & -1 & 7 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

$$= \frac{-1}{2}\begin{bmatrix} -1-4+3 \\ -3-4+5 \\ -3-4+7 \end{bmatrix}$$

$$= \frac{-1}{2}\begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Hence by definition of equal matrices $x = 1, y = 1, z = 0$

(ii)
$$\left.\begin{aligned} 2x_1 + x_2 + 3x_3 &= 3 \\ x_1 + x_2 - 2x_3 &= 0 \\ -3x_1 - x_2 + 2x_3 &= -4 \end{aligned}\right\}$$

Solution: The matrix form of given system of linear equation is

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & -2 \\ -3 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B \qquad \text{(i)}$$

Where

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & -2 \\ -3 & -1 & 2 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

Now

$$|A| = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 1 & -2 \\ -3 & -1 & 2 \end{vmatrix} = 2(2-2) - 1(2-6) + 3(-1+3)$$

$$= 0 + 4 + 6$$

$$|A| = 10 \neq 0$$

Since $|A| \neq 0$ so inverse of A is possible and given by $A^{-1} = \dfrac{AdjA}{|A|}$

Now $AdjA = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$

$$AdjA = \begin{bmatrix} \begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ -3 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -3 & -1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ -3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ -3 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \end{bmatrix}^t = \begin{bmatrix} 0 & 4 & 2 \\ -5 & 13 & -1 \\ -5 & 7 & 1 \end{bmatrix}^t$$

$$AdjA = \begin{bmatrix} 0 & -5 & -5 \\ 4 & 13 & 7 \\ 2 & -1 & 1 \end{bmatrix}$$

Now $A^{-1} = \dfrac{AdjA}{|A|} = \dfrac{1}{10}\begin{bmatrix} 0 & -5 & -5 \\ 4 & 13 & 7 \\ 2 & -1 & 1 \end{bmatrix}$

Put the values of A^{-1} and B in (i) we get

$$X = A^{-1}B$$

$$= \frac{1}{10}\begin{bmatrix} 0 & -5 & -5 \\ 4 & 13 & 7 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

$$=\frac{1}{10}\begin{bmatrix}0+0+20\\12+0-28\\6+0-4\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\frac{1}{10}\begin{bmatrix}20\\-16\\2\end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\-\frac{8}{5}\\\frac{1}{5}\end{bmatrix}$$

Hence by definition of equal matrices $x_1=2, x_2=-\frac{8}{5}$ and $x_3=\frac{1}{5}$

Note: Answer of the question does not match with the answer of the text book.

(iii) $\left.\begin{aligned}\mathbf{x+y=2}\\\mathbf{2x-z=1}\\\mathbf{2y-3z=-1}\end{aligned}\right\}$

The matrix form of the given system of linear equation is

$$\begin{bmatrix}1&1&0\\2&0&-1\\0&2&-3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\1\\-1\end{bmatrix}$$

$$AX=B$$

$$X=A^{-1}B \qquad \text{(i)}$$

Where

$$A=\begin{bmatrix}1&1&0\\2&0&-1\\0&2&-3\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix} \text{ and } B=\begin{bmatrix}2\\1\\-1\end{bmatrix}$$

So $|A|=\begin{vmatrix}1&1&0\\2&0&-1\\0&2&-3\end{vmatrix}=1(0+2)-1(-6-0)+0(4-0)$

$=2+6=8\neq 0$

$|A|=8\neq 0$

Since $|A|\neq 0$ so inverse of A is possible and given by $A^{-1}=\frac{AdjA}{|A|}$

Now $AdjA=\begin{bmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\A_{31}&A_{32}&A_{33}\end{bmatrix}^t$

$$\text{Now}\quad AdjA=\begin{bmatrix} \begin{vmatrix}0 & -1\\ 2 & -3\end{vmatrix} & -\begin{vmatrix}2 & -1\\ 0 & -3\end{vmatrix} & \begin{vmatrix}2 & 0\\ 0 & 2\end{vmatrix}\\ -\begin{vmatrix}1 & 0\\ 2 & -3\end{vmatrix} & \begin{vmatrix}1 & 0\\ 0 & -3\end{vmatrix} & -\begin{vmatrix}1 & 1\\ 0 & 2\end{vmatrix}\\ \begin{vmatrix}1 & 0\\ 0 & -1\end{vmatrix} & -\begin{vmatrix}1 & 0\\ 2 & -1\end{vmatrix} & \begin{vmatrix}1 & 1\\ 2 & 0\end{vmatrix}\end{bmatrix}^t$$

$$AdjA=\begin{bmatrix}2 & 6 & 4\\ 3 & -3 & -2\\ -1 & 1 & -2\end{bmatrix}^t$$

$$AdjA=\begin{bmatrix}2 & 3 & -1\\ 6 & -3 & 1\\ 4 & -2 & -2\end{bmatrix}$$

$$A^{-1}=\frac{AdjA}{|A|}=\frac{1}{8}\begin{bmatrix}2 & 3 & -1\\ 6 & -3 & 1\\ 4 & -2 & -2\end{bmatrix}$$

Put the values of A^{-1} and B in (i) we get

$$X=A^{-1}B$$

$$=\frac{1}{8}\begin{bmatrix}2 & 3 & -1\\ 6 & -3 & 1\\ 4 & -2 & -2\end{bmatrix}\begin{bmatrix}2\\ 1\\ -1\end{bmatrix}$$

$$=\frac{1}{8}\begin{bmatrix}4+3+1\\ 12-3-1\\ 8-2+2\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{8}\begin{bmatrix}8\\ 8\\ 8\end{bmatrix}=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

Hence by definition of equal matrices $x=1, y=1$ and $z=1$

Q.3 Solve the following system by reducing their augmented matrix to the echelon form and reduced echelon form:

(i)
$$\left.\begin{aligned} x_1 - 2x_2 - 2x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 1 \\ 5x_1 - 4x_2 - 3x_3 &= 1 \end{aligned}\right\}$$

Solution: The augmented matrix of the given system is

$$\begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 2 & 3 & 1 & : & 1 \\ 5 & -4 & -3 & : & 1 \end{bmatrix}$$

We reduce the above matrix to echelon and reduced echelon form by applying elementary row operations, that is,

$$\begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 2 & 3 & 1 & : & 1 \\ 5 & -4 & -3 & : & 1 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 0 & 7 & 5 & : & 3 \\ 0 & 6 & 7 & : & 6 \end{bmatrix} \qquad \text{By } R_2 + (-2)R_1 \to R_2', R_3 + (-5)R_1 \to R_3'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 0 & 1 & -2 & : & -3 \\ 0 & 6 & 7 & : & 6 \end{bmatrix} \qquad \text{By } R_2 + (-1)R_3 \to R_2'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -6 & : & -7 \\ 0 & 1 & -2 & : & -3 \\ 0 & 0 & 19 & : & 24 \end{bmatrix} \qquad \text{By } R_1 + 2R_2 \to R_1', R_3 + (-6)R_2 \to R_3'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -6 & : & -7 \\ 0 & 1 & -2 & : & -3 \\ 0 & 0 & 1 & : & \frac{24}{19} \end{bmatrix} \qquad \text{(I)} \quad \text{By } \frac{1}{19}(R_3) \to R_3'$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & : & \frac{11}{19} \\ 0 & 1 & 0 & : & -\frac{9}{19} \\ 0 & 0 & 1 & : & \frac{24}{19} \end{bmatrix} \qquad \text{(II)} \quad \text{By } R_2 + 2R_3 \to R_2', R_1 + 6R_3 \to R_1'$$

(I) is in echelon form the equivalent system is

$x_1 - 6x_3 = -7$ (i)

$x_2 - 2x_3 = -3$ (ii)

$x_3 = \frac{24}{19}$ (iii)

From (ii)

$x_2 - 2\left(\frac{24}{19}\right) = -3$

$\Rightarrow x_2 = -3 + \frac{48}{19} = \frac{-9}{19}$

$\Rightarrow x_2 = \frac{-9}{19}$

From (i)

$x_1 - 6\left(\frac{24}{19}\right) = -7$

$\Rightarrow x_1 = -7 + \frac{144}{19}$

$\Rightarrow x_1 = \frac{-133+144}{19}$

$\Rightarrow x_1 = \frac{11}{19}$

Hence $x_1 = \frac{11}{19}, x_2 = \frac{-9}{19}$ and $x_3 = \frac{24}{19}$

Note: Answer in the text books is different

$x_1 = \frac{11}{19}, x_2 = \frac{-9}{19}$ and $x_3 = \frac{-24}{19}$

(II) is in reduced echelon form the equivalent system in (row) reduced echelon form

$x_1 = \frac{11}{19}$

$x_2 = -\frac{9}{19}$

$x_3 = \frac{24}{19}$

(ii)
$$\left.\begin{aligned} \mathbf{x + 2y + z} &= \mathbf{2} \\ \mathbf{2x + y + 2z} &= \mathbf{-1} \\ \mathbf{2x + 3y - z} &= \mathbf{9} \end{aligned}\right\}$$

Solution: The augmented matrix of the given system is

$$\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 2 & 1 & 2 & : & -1 \\ 2 & 3 & -1 & : & 9 \end{bmatrix}$$

We reduce the above matrix to echelon and reduced echelon form by applying elementary row operations, that is,

$$\overset{R}{\sim}\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & -3 & 0 & : & -5 \\ 0 & -1 & -3 & : & 5 \end{bmatrix} \qquad \text{By } R_2 + (-2)R_1 \to R_2', R_3 + (-2)R_1 \to R_3'$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 1 & 0 & : & \frac{5}{3} \\ 0 & -1 & -3 & : & 5 \end{bmatrix} \qquad \text{By } -\frac{1}{3}(R_2) \to R_2'$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 1 & : & -\frac{4}{3} \\ 0 & 1 & 0 & : & \frac{5}{3} \\ 0 & 0 & -3 & : & \frac{20}{3} \end{bmatrix} \qquad \text{By } R_1 + (-2)R_2 \to R_1', R_3 + R_2 \to R_3'$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 1 & : & -\frac{4}{3} \\ 0 & 1 & 0 & : & \frac{5}{3} \\ 0 & 0 & 1 & : & -\frac{20}{9} \end{bmatrix} \qquad \text{(I)} \quad \text{By } \left(\frac{-1}{3}\right)R_3 \to R_3'$$

$$\overset{R}{\sim}\begin{bmatrix} 1 & 0 & 0 & : & \frac{8}{9} \\ 0 & 1 & 0 & : & \frac{5}{3} \\ 0 & 0 & 1 & : & -\frac{20}{9} \end{bmatrix} \qquad \text{(II)} \quad \text{By } R_1 + (-1)R_3 \to R_1'$$

(I) is in echelon form, the equivalent system is

$x+z=-\frac{4}{3}$ (i)

$y=\frac{5}{3}$ (ii)

$z=-\frac{20}{9}$ (iii)

Put the value of z in (i)

$x-\frac{20}{9}=-\frac{4}{3}$

$\Rightarrow x=-\frac{4}{3}+\frac{20}{9}=\frac{-12+20}{9}=\frac{8}{9}$

Hence $x=\frac{8}{9}, y=\frac{5}{3}$ and $z=-\frac{20}{9}$

(II) is in reduce echelon form, the equivalent system is

$x=\frac{8}{9}, y=\frac{5}{3}, z=-\frac{20}{9}$

(iii) $\left.\begin{aligned} \mathbf{x_1+4x_2+2x_3=2} \\ \mathbf{2x_1+x_2-2x_3=9} \\ \mathbf{3x_1+2x_2-2x_3=12} \end{aligned}\right\}$

Solution: The augmented matrix of the given system is

$$\begin{bmatrix} 1 & 4 & 2 & : & 2 \\ 2 & 1 & -2 & : & 9 \\ 3 & 2 & -2 & : & 12 \end{bmatrix}$$

We reduce the above matrix to echelon and reduced echelon form by applying elementary row operation, that is

$$\underset{R}{\sim}\begin{bmatrix} 1 & 4 & 2 & : & 2 \\ 0 & -7 & -6 & : & 5 \\ 0 & -10 & -8 & : & 6 \end{bmatrix} \qquad \text{By } R_2+(-2)R_1\to R_2', R_3+(-3)R_1\to R_3'$$

$$\underset{R}{\sim}\begin{bmatrix} 1 & 4 & 2 & : & 2 \\ 0 & 1 & \frac{6}{7} & : & -\frac{5}{7} \\ 0 & -10 & -8 & : & 6 \end{bmatrix} \qquad \text{By } -\frac{1}{7}(R_2)\to R_2'$$

$$\underset{R}{\sim}\begin{bmatrix} 1 & 0 & -\frac{10}{7} & : & \frac{34}{7} \\ 0 & 1 & \frac{6}{7} & : & -\frac{5}{7} \\ 0 & 0 & \frac{4}{7} & : & -\frac{8}{7} \end{bmatrix} \qquad \text{By } R_1+(-4)R_2\to R_1', R_3+10R_2\to R_3'$$

$$\underset{R}{\sim}\begin{bmatrix} 1 & 0 & -\frac{10}{7} & : & \frac{34}{7} \\ 0 & 1 & \frac{6}{7} & : & -\frac{5}{7} \\ 0 & 0 & 1 & : & -2 \end{bmatrix} \quad \text{(I)} \quad \text{By } \frac{7}{4}(R_3) \to R_3'$$

$$\underset{R}{\sim}\begin{bmatrix} 1 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & -2 \end{bmatrix} \quad \text{(II) By } R_2 + \left(\frac{-6}{7}\right)R_3 \to R_2', R_1 + \frac{10}{7}R_3 \to R_1'$$

(I) is in echelon form. The equivalent system is

$$x_1 + \left(-\frac{10}{7}\right)x_3 = \frac{34}{7} \qquad \text{(i)}$$

$$x_2 + \frac{6}{7}x_3 = \frac{-5}{7} \qquad \text{(ii)}$$

$$x_3 = -2 \qquad \text{(iii)}$$

Put $x_3 = -2$ in (ii) we get

$$x_2 + \frac{6}{7}(-2) = -\frac{5}{7}$$

$$\Rightarrow x_2 = \frac{-5}{7} + \frac{12}{7} = \frac{7}{7} = 1$$

$$\Rightarrow x_2 = 1$$

Put $x_3 = -2$ in (i) we get

$$x_1 + \left(-\frac{10}{7}\right)(-2) = \frac{34}{7}$$

$$\Rightarrow x_1 = \frac{34}{7} - \frac{20}{7} = \frac{14}{7} = 2$$

Hence $x_1 = 2, x_2 = 1$ and $x_3 = -2$ are the required values.

(II) is in reduce echelon form. The equivalent system is $x_1 = 2, x_2 = 1$ and $x_3 = -2$

Q.4 Solve the following system of homogenous linear equations.

(i) $\left.\begin{aligned} x+2y-2z&=0\\ 2x+y+5z&=0\\ 5x+4y+8z&=0\end{aligned}\right\}$

Solution: Coefficient matrix 'A' is

$$A=\begin{bmatrix}1 & 2 & -2\\ 2 & 1 & 5\\ 5 & 4 & 8\end{bmatrix}$$

We convert 'A' into echelon form.

$$\underset{\sim}{R}\begin{bmatrix}1 & 2 & -2\\ 0 & -3 & 9\\ 0 & -6 & 18\end{bmatrix} \qquad \text{By } R_2-2R_1\to R_2',\ R_3-5R_1\to R_3'$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 2 & -2\\ 0 & 1 & -3\\ 0 & -6 & 18\end{bmatrix} \qquad \text{By } -\frac{1}{3}R_2\to R_2'$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 2 & -2\\ 0 & 1 & -3\\ 0 & 0 & 0\end{bmatrix} \qquad \text{By } R_3+6R_2\to R_3'$$

Since matrix is in echelon form so equivalent system of equations is

$$x+2y-2z=0 \qquad \text{(i)}$$

$$y-3z=0 \qquad \text{(ii)}$$

Let $z=t$ then (ii) becomes $y=3t$. Put the value of $y=3t$ and $z=t$ in (i) we get

$$x+6t-2t=0$$
$$x=-4t$$

Hence solution set is $x=-4t, y=3t, z=t,\ t\in R,\ t\neq 0$

(ii)
$$\left.\begin{aligned} \mathbf{x_1 + 4x_2 + 2x_3} &= \mathbf{0} \\ \mathbf{2x_1 + x_2 - 3x_3} &= \mathbf{0} \\ \mathbf{3x_1 + 2x_2 - 4x_3} &= \mathbf{0} \end{aligned}\right\}$$

Solution: Coefficient matrix 'A' is

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 1 & -3 \\ 3 & 2 & -4 \end{bmatrix}$$

Now we convert 'A' into echelon form ; so here

$$\underset{R}{\sim}\begin{bmatrix} 1 & 4 & 2 \\ 0 & -7 & -7 \\ 0 & -10 & -10 \end{bmatrix} \qquad \text{By } R_2 - 2R_1 \to R_2', R_3 - 3R_1 \to R_3'$$

$$\underset{R}{\sim}\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & -10 & -10 \end{bmatrix} \qquad \text{By } -\frac{1}{7}R_2 \to R_2'$$

$$\underset{R}{\sim}\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{By } R_3 + 10R_2 \to R_3'$$

Since matrix is in echelon form so equivalent system of equations is

$x_1 + 4x_2 + 2x_3 = 0$ (i)

$x_2 + x_3 = 0$ (ii)

Let $x_3 = t$ then (ii) becomes $x_2 = -t$

Put the value of $x_2 = -t$ and $x_3 = t$ in (i) we get

$x_1 + 4(-t) + 2t = 0$

$x_1 = 2t$

Hence solution set is $x_1 = 2t, x_2 = -t, x_3 = t, \; t \in R, \; t \neq 0$

(iii) $\left.\begin{aligned} \mathbf{x_1 - 2x_2 - x_3 = 0} \\ \mathbf{x_1 + x_2 + 5x_3 = 0} \\ \mathbf{2x_1 - x_2 + 4x_3 = 0} \end{aligned}\right\}$

Solution: The coefficient matrix 'A' is

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 1 & 1 & 5 \\ 2 & -1 & 4 \end{bmatrix}$$

We convert 'A' into echelon form

$$\underset{R}{\sim}\begin{bmatrix} 1 & -2 & -1 \\ 0 & 3 & 6 \\ 0 & 3 & 6 \end{bmatrix} \qquad \text{By } R_2 - R_1 \to R_2', R_3 - 2R_1 \to R_3'$$

$$\underset{R}{\sim}\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \end{bmatrix} \qquad \text{By } \frac{1}{3}R_2 \to R_2'$$

$$\underset{R}{\sim}\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{By } R_3 - 3R_2 \to R_3'$$

Since matrix is in echelon form so equivalent system of equations is

$x_1 - 2x_2 - x_3 = 0$ (i)

$x_2 + 2x_3 = 0$ (ii)

Let $x_3 = t$ then (ii) becomes $x_2 = -2t$

Put the value of $x_2 = -2t$ and $x_3 = t$ in (i) we get

$x_1 - 2(-2t) - t = 0$

$x_1 + 4t - t = 0$

$x_1 = -3t$

Hence solution set $x_1 = -3t, x_2 = -2t, x_3 = t, \quad t \in R \ , \ t \neq 0$

Q.5 Find the value of λ for which the following system have non trivial solutions. Also solve the system for the value of λ

(i)
$$\left.\begin{aligned} x+y+z&=0\\ 2x+y-\lambda z&=0\\ x+2y-2z&=0\end{aligned}\right\}$$

Solution: Let coefficient matrix represented is by A so

$$A=\begin{bmatrix}1&1&1\\2&1&-\lambda\\1&2&-2\end{bmatrix}$$

For non-trivial solution take $|A|=0$

$$\begin{vmatrix}1&1&1\\2&1&-\lambda\\1&2&-2\end{vmatrix}=0$$

Expanding form R_1 we get;

$$1\begin{vmatrix}1&-\lambda\\2&-2\end{vmatrix}-1\begin{vmatrix}2&-\lambda\\1&-2\end{vmatrix}+1\begin{vmatrix}2&1\\1&2\end{vmatrix}=0$$

$$-2+2\lambda-1(-4+\lambda)+1(4-1)=0$$

$$-2+2\lambda+4-\lambda+3=0$$

$$\lambda+5=0$$

$$\lambda=-5$$

Put $\lambda=-5$ in given system of equations we get;

$$\left.\begin{aligned} x+y+z&=0\\ 2x+y+5z&=0\\ x+2y-2z&=0\end{aligned}\right\}$$

Here $A=\begin{bmatrix}1&1&1\\2&1&5\\1&2&-2\end{bmatrix}$

Now we convert 'A' into echelon form, so here

$$R\begin{bmatrix}1&1&1\\0&-1&3\\0&1&-3\end{bmatrix}$$ By $R_2-2R_1\rightarrow R_2',\ R_3-R_1\rightarrow R_3'$

$$\underset{R}{\sim}\begin{bmatrix}1 & 1 & 1\\0 & 1 & -3\\0 & 1 & -3\end{bmatrix}$$ By $(-1)\times R_2 \rightarrow R_2'$

$$\underset{R}{\sim}\begin{bmatrix}1 & 1 & 1\\0 & 1 & -3\\0 & 0 & 0\end{bmatrix}$$ By $R_3 - R_2 \rightarrow R_3'$

Since matrix is in echelon form so equivalent system of equation is

$x+y+z=0$ (i)

$y-3z=0$ (ii)

Let $z=t$ then (ii) becomes $y=3t$

Put the values of $y=3t$ and $z=t$ in (i) we get

$x+3t+t=0 \Rightarrow x=-4t$

Hence solution set $x=-4t, y=3t, z=t, \quad t\in R$, $t\neq 0$

(ii) $$\left.\begin{aligned}\mathbf{x_1+4x_2+\lambda x_3=0}\\ \mathbf{2x_1+x_2-3x_3=0}\\ \mathbf{3x_1+\lambda x_2-4x_3=0}\end{aligned}\right\}$$

Solution: Let coefficient matrix represented is by A so

$$A=\begin{bmatrix}1 & 4 & \lambda\\2 & 1 & -3\\3 & \lambda & -4\end{bmatrix}$$

For non-trivial solution put $|A|=0$

$$\begin{vmatrix}1 & 4 & \lambda\\2 & 1 & -3\\3 & \lambda & -4\end{vmatrix}=0$$

$$1\begin{vmatrix}1 & -3\\\lambda & -4\end{vmatrix}-4\begin{vmatrix}2 & -3\\3 & -4\end{vmatrix}+\lambda\begin{vmatrix}2 & 1\\3 & \lambda\end{vmatrix}=0$$

$$(-4+3\lambda)-4(-8+9)+\lambda(2\lambda-3)=0$$

$$-4+3\lambda-4+2\lambda^2-3\lambda=0$$

$2\lambda^2 = 8$

$\lambda^2 = 4$

$\lambda = \pm 2$

When $\lambda = 2$ given system of equation becomes;

$$\left.\begin{aligned} x_1 + 4x_2 + 2x_3 &= 0 \\ 2x_1 + x_2 - 3x_3 &= 0 \\ 3x_1 + 2x_2 - 4x_3 &= 0 \end{aligned}\right\}$$

Here

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 1 & -3 \\ 3 & 2 & -4 \end{bmatrix}$$

Now we convert 'A' into echelon form;

$$\underset{R}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ 0 & -7 & -7 \\ 0 & -10 & -10 \end{bmatrix} \qquad \text{By } R_2 - 2R_1 \to R_2', R_3 - 3R_1 \to R_3'$$

$$\underset{R}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & -10 & -10 \end{bmatrix} \qquad \text{By } \frac{-R_2}{7} \to R_2'$$

$$\underset{R}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{By } R_3 + 10R_2 \to R_3'$$

Since matrix is in echelon form so equivalent system of equations is

$x_1 + 4x_2 + 2x_3 = 0$ (i)

$x_2 + x_3 = 0$ (ii)

Let $x_3 = t$ then (ii) becomes $x_2 = -t$

Put $x_2 = -t$ and $x_3 = t$ in (i) we get

$x_1 + 4(-t) + 2t = 0$

$x_1 - 2t = 0 \Rightarrow x_1 = 2t$

So when $\lambda = 2$ then solution set $x_1 = 2t, x_2 = -t, x_3 = t, \quad t \in R \;, \; t \neq 0$

When $\lambda = -2$ given system of equation becomes

$x_1 + 4x_2 - 2x_3 = 0$

$2x_1 + x_2 - 3x_3 = 0$

$3x_1 - 2x_2 - 4x_3 = 0$

Here

$$A = \begin{bmatrix} 1 & 4 & -2 \\ 2 & 1 & -3 \\ 3 & -2 & -4 \end{bmatrix}$$

Now we convert 'A' into echelon form

$$\underset{R}{\sim} \begin{bmatrix} 1 & 4 & -2 \\ 0 & -7 & 1 \\ 0 & -14 & 2 \end{bmatrix} \qquad \text{By } R_2 - 2R_1 \to R_2', R_3 - 3R_1 \to R_3'$$

$$\underset{R}{\sim} \begin{bmatrix} 1 & 4 & -2 \\ 0 & 1 & -\frac{1}{7} \\ 0 & -14 & 2 \end{bmatrix} \qquad \text{By } \frac{R_2}{-7} \to R_2'$$

$$\underset{R}{\sim} \begin{bmatrix} 1 & 4 & -2 \\ 0 & 1 & -\frac{1}{7} \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{By } R_3 + 14R_2 \to R_2'$$

Since matrix is in echelon form so equivalent system of equations is

$x_1 + 4x_2 - 2x_3 = 0$ (i)

$x_2 - \frac{1}{7}x_3 = 0$ (ii)

Let $x_3 = 7t$ then (ii) becomes $x_2 = t$

Put $x_2 = t$ and $x_3 = 7t$ in (i) we get

$x_1 + 4t - 14t = 0$

$x_1 = 10t$

Hence solutions set $x_1 = 10t, x_2 = t, x_3 = 7t,\ t \in R\ , t \neq 0$

Q.6 Find the value of λ for which the following system does not possess a unique solution. Also solve the system for the value of λ.

$$\left.\begin{aligned} x_1 + 4x_2 + \lambda x_3 &= 2 \\ 2x_1 + x_2 - 2x_3 &= 11 \\ 3x_1 + 2x_2 - 2x_3 &= 16 \end{aligned}\right\}$$

Solution: Let coefficient matrix be represented by A then

$$A = \begin{bmatrix} 1 & 4 & \lambda \\ 2 & 1 & -2 \\ 3 & 2 & -2 \end{bmatrix}$$

Non-homogeneous system of linear equation doesn't possess unique solution when

$|A| = 0$

$$|A| = \begin{vmatrix} 1 & 4 & \lambda \\ 2 & 1 & -2 \\ 3 & 2 & -2 \end{vmatrix} = 0$$

Expanding form R_1

$$1\begin{vmatrix} 1 & -2 \\ 2 & -2 \end{vmatrix} - 4\begin{vmatrix} 2 & -2 \\ 3 & -2 \end{vmatrix} + \lambda\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 0$$

$$1(-2+4) - 4(-4+6) + \lambda(4-3) = 0$$

$$2 - 8 + \lambda = 0$$

$$-6 + \lambda = 0$$

$$\lambda = 6$$

Put $\lambda = 6$ in give system of equation we get;

$$x_1 + 4x_2 + 6x_3 = 2$$
$$2x_1 + x_2 - 2x_3 = 11$$
$$3x_1 + 2x_2 - 2x_3 = 16$$

Augmented matrix A_b is

$$A_b = \left[\begin{array}{ccc|c} 1 & 4 & 6 & 2 \\ 2 & 1 & -2 & 11 \\ 3 & 2 & -2 & 16 \end{array}\right]$$

Now we convert into echelon form

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 4 & 6 & 2 \\ 0 & -7 & -14 & 7 \\ 0 & -10 & -20 & 10 \end{array}\right] \quad \text{By } R_2 - 2R_1 \to R_2', R_3 - 3R_1 \to R_3'$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 4 & 6 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & -10 & -20 & 10 \end{array}\right] \quad \text{By } -\frac{1}{7}R_2 \to R_2'$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 4 & 6 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{By } R_3 + 10R_2 \to R_3'$$

Since matrix is in echelon form so equivalent system of equation is

$x_1 + 4x_2 + 6x_3 = 2$ (i)

$x_2 + 2x_3 = -1$ (ii)

Let $x_3 = t$ then equation (ii) becomes

$x_2 + 2t = -1$

$x_2 = -2t - 1$

Put value of x_2 and x_3 in equation (i)

$x_1 + 4(-2t - 1) + 6(t) = 2$

$x_1 - 8t - 4 + 6t = 2$

$x_1 = 6 + 2t$

$\Rightarrow$ $x_1 = 2t + 6$

Hence

Solution set $x_1 = 2t + 6, x_2 = -2t - 1, x_3 = t$, $t \in R$, $t \neq 0$