UNIT 1
FUNCTIONS AND LIMITS

Function:

A **Function** f from a set X to a set Y is a rule or correspondence that assigns to each element x in X a unique element y in Y.

Symbolically we write it as $f : X \rightarrow Y$ and read as f is a function from X to Y.

Domain and Range of Function:

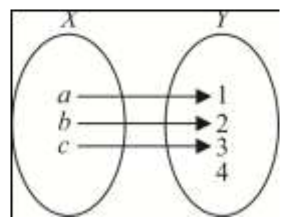
If f is a function from X to Y then

X is called the domain of f and the set of corresponding elements in Y is called range of f

For example:

Domain $= \{a, b, c\}$

Range $= \{1, 2, 3\}$



Example 1: **Given** $f(x) = x^3 - 2x^2 + 4x - 1$, **find**

(i) $f(0)$ **(ii)** $f(1)$ **(iii)** $f(-2)$ **(iv)** $f(1+x)$ **(v)** $f\left(\frac{1}{x}\right), x \neq 0$

Solution:

$$f(x) = x^3 - 2x^2 + 4x - 1$$

(i) $f(0) = (0)^3 - 2(0)^2 + 4(0) - 1 = 0 - 0 + 0 - 1 = -1$

(ii) $f(1) = (1)^3 - 2(1)^2 + 4(1) - 1 = 1 - 2 + 4 - 1 = 2$

(iii) $f(-2) = (-2)^3 - 2(-2)^2 + 4(-2) - 1 = -8 - 8 - 8 - 1 = -25$

(iv) $f(1+x) = (1+x)^3 - 2(1+x)^2 + 4(1+x) - 1$

$= 1 + 3x + 3x^2 + x^3 - 2 - 4x - 2x^2 + 4 + 4x - 1$

$= x^3 + x^2 + 3x + 2$

(v) $f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^3 - 2\left(\frac{1}{x}\right)^2 + 4\left(\frac{1}{x}\right) - 1 = \frac{1}{x^3} - \frac{2}{x^2} + \frac{4}{x} - 1, x \neq 0$

Example 2: **Let** $f(x) = x^2$. **Find the domain and range of** f.

Solution:

$f(x)$ is defined for every real number x.

Further for every real number x, $f(x) = x^2$ is a non- negative real number. So

Domain $f =$ Set of all real numbers.

Range $f =$ Set of all non-negative real numbers.

Example 3: **Let** $f(x)=\dfrac{x}{x^2-4}$. **Find the domain and range of** ***f*****.**

Solution:

$$f(x)=\frac{x}{x^2-4}$$

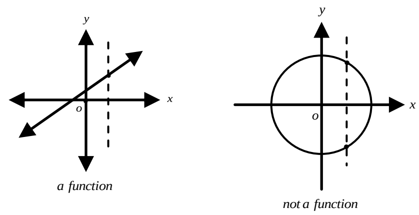
$f(x)$ is not defined if $x^2-4=0 \Rightarrow x^2=4$ or $x=\pm 2$

Domain f = Set of all real numbers except –2 and 2.

Range f = Set of all real numbers.

Vertical Line Test:

If a vertical line meets a graph in more than one point, then it is not a graph of a function.



Piece –Wise (Compound) Function:

A function which is defined by two or more than two rules is called Piece-wise function.

For example:

$$f(x)=\begin{cases} x & \text{if} \quad 0\le x\le 1 \\ x-1 & \text{if} \quad 1<x\le 2 \end{cases}$$

Algebraic Functions:

Algebraic functions are those functions which are defined by algebraic expressions.

For example:

$f(x)=3x+5,\ f(x)=x^2+3x+2$

We classify Algebraic functions as follows:

(i) Polynomial Function:

A function *P* of the form $P(x)=a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+...+a_2x^2+a_1x+a_0$ for all *x*, where the coefficients $a_n, a_{n-1}, ..., a_2, a_1, a_0$ are real numbers and the exponents are non-negative integers, is called a **polynomial function**. If $a_n \ne 0$ then $P(x)$ is called a **polynomial function** of degree *n* and a_n is the leading co-efficient of $P(x)$.

For example:

$P(x)=2x^4-3x^3+2x-1$ is a **polynomial function** of degree 4 with leading coefficient 2.

(ii) Linear Function:

If the degree of a polynomial function is 1, then it is called a **linear function**.

Symbolically we write $f(x)=ax+b$ where $a\ne 0$, a,b are real numbers.

For example:

$f(x)=3x+4,\ f(x)=x+2$ are **linear functions** of *x*.

(iii) Identity Function:

For any set X, a function $I : X \to X$ of the form $I(x) = x \quad \forall x \in X$ is called an **identity function**.

(iv) Constant Function:

Let X and Y be sets of real numbers. A function $C : X \to Y$ defined by $C(x) = a, \forall x \in X, a \in Y$ and fixed is called **constant function**. For example $C : R \to R$ defined by $C(x) = 2, \forall x \in R$ is a constant function.

(v) Rational Function:

A function $R(x)$ of the form $\dfrac{P(x)}{Q(x)}$, where both $P(x)$ and $Q(x)$ are polynomial functions and $Q(x) \neq 0$, is called a **rational function**.

Exponential Function:

A function, in which the variable appears as exponent (power), is called an **exponential function**. The functions $y = e^{ax}, y = e^{x}, y = 2^{x} = e^{x\ln 2}$, etc are exponential functions of x.

Logarithmic Function:

If $x = a^{y}$, then $y = \log_a x$, where $a > 0$, $a \neq 1$ is called **Logarithmic function** of x.

(i) If $a = 10,$ then we have $\log_{10} x$ (written as $\log x$) which is known as the **common logarithm** of x.

(ii) If $a = e,$ then we have $\log_e x$ (written as $\ln x$) which is known as the **natural logarithm** of x.

Hyperbolic Functions:

(i) $\sinh x = \dfrac{1}{2}\left(e^{x} - e^{-x}\right)$ is called **hyperbolic sine** function. Its domain and range are the set of all real numbers.

(ii) $\cosh x = \dfrac{1}{2}\left(e^{x} + e^{-x}\right)$ is called **hyperbolic cosine** function. Its domain is the set of all real numbers and the range is the set of all numbers in the interval $[1, +\infty)$.

(iii) The remaining four hyperbolic functions are defined in terms of the hyperbolic sine and the hyperbolic cosine function as follows:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} \quad ; \quad \sec hx = \frac{1}{\cosh x} = \frac{2}{e^{x} + e^{-x}}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}} \quad ; \quad \csc hx = \frac{1}{\sinh x} = \frac{2}{e^{x} - e^{-x}}$$

Inverse Hyperbolic Functions:

The inverse hyperbolic functions are expressed in terms of natural logarithms and we shall study them in higher classes.

(i) $\sinh^{-1}x=\ln\left(x+\sqrt{x^2+1}\right),$ for all x **(ii)** $\cosh^{-1}x=\ln\left(x+\sqrt{x^2-1}\right),\ x\geq 1$

(iii) $\tanh^{-1}x=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),\ |x|<1$ **(iv)** $\coth^{-1}x=\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right),\ |x|<1$

(v) $\operatorname{sech}^{-1}x=\ln\left(\frac{1}{x}+\frac{\sqrt{1-x^2}}{x}\right),\ 0<x\leq 1$ **(vi)** $\operatorname{cosech}^{-1}x=\ln\left(\frac{1}{x}+\frac{\sqrt{1+x^2}}{|x|}\right),\ x\neq 0$

Explicit Function:

If *y* is easily expressed in terms of the independent variable *x*, then *y* is called an **explicit function** of *x*.

For example:

$y=x^2+2x-1,\ y=\sqrt{x-1}$ are explicit functions of *x*.

Symbolically it can be written as $y=f(x)$.

Implicit Function:

If *x* and *y* are so mixed up and *y* cannot be expressed in terms of the independent variable *x*, then *y* is called an **implicit function** of *x*. For example,

$x^2+xy+y^2=2,\ \frac{xy^2-y+9}{xy}=1$ are implicit functions of *x* and *y*.

Symbolically it is written as $f(x,y)=0$.

Parametric Functions:

Sometimes a curve is described by expressing both *x* and *y* as function of a third variable "t" or "θ" which is called a parameter. The equations of the type $x=f(t)$ and $y=g(t)$ are called the parametric equations of the curve.

The functions of the form:

(i) $x=at^2$, $y=at$ **(ii)** $x=a\cos t$, $y=a\sin t$ **(iii)** $x=a\cos\theta$, $y=b\sin\theta$ **(iv)** $x=a\sec\theta$, $y=a\tan\theta$

are called **parametric functions**. Here the variable *t* or θ is called parameter.

Even Function:

A function *f* is said to be an **even function** if $f(-x)=f(x)$, for every number *x* in the domain of *f*.

For example:

$f(x)=x^2,\ f(x)=\cos x$ are even functions of *x*.

Odd Function:

A function *f* is said to be an **odd function** if $f(-x)=-f(x)$, for every number *x* in the domain of *f*.

For example:

$f(x)=\sin x,\ f(x)=x^3$ are odd functions of *x*.

Some Important Results

Hyperbolic Identities:

- $\cosh^2 x + \sinh^2 x = \cosh 2x$
- $\cosh^2 x - \sinh^2 x = 1$
- $2\sinh x \cosh x = \sinh 2x$
- $1 - \tanh^2 x = \text{sec}h^2 x$
- $\coth^2 x - 1 = \text{cosech}^2 x$

Parametric Equations:

- $x = a\cos\theta$, $y = a\sin\theta$ represent the equation of circle $x^2 + y^2 = a^2$
- $x = a\cos\theta$, $y = b\sin\theta$ represent the equation of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
- $x = a\sec\theta$, $y = b\tan\theta$ represent the equation of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
- $x = at^2$, $y = 2at$ represent the equation of parabola $y^2 = 4ax$.

EXERCISE 1.1

Q.1 Given that:

(a) $f(x) = x^2 - x$

(b) $f(x) = \sqrt{x+4}$

Find

(i) $f(-2)$

(ii) $f(0)$

(iii) $f(x-1)$

(iv) $f(x^2+4)$

(a) $f(x) = x^2 - x$

(i) $f(-2)$

Solution:

$f(x) = x^2 - x$

Put $x = -2$

$f(-2) = (-2)^2 - (-2)$

$= 4 + 2$

$\boxed{f(-2) = 6}$

(ii) $f(0)$

Solution:

$f(x) = x^2 - x$

Put $x = 0$

$f(0) = 0^2 - 0$

$\boxed{f(0) = 0}$

(iii) $f(x-1)$

Solution:

$f(x) = x^2 - x$

Replace x by $x-1$

$f(x-1) = (x-1)^2 - (x-1)$

$= x^2 - 2x + 1 - x + 1$

$\boxed{f(x-1) = x^2 - 3x + 2}$

(iv) $f(x^2+4)$

Solution:

$f(x) = x^2 - x$

Replace x by $x^2 + 4$

$$f\left(x^2+4\right)=\left(x^2+4\right)^2-\left(x^2+4\right)$$
$$=x^4+8x^2+16-x^2-4$$
$$\boxed{f\left(x^2+4\right)=x^4+7x^2+12}$$

(b) $f(x)=\sqrt{x+4}$

(i) $f(-2)$

Solution:

$$f(x)=\sqrt{x+4}$$
Put $x=-2$
$$f(-2)=\sqrt{-2+4}$$
$$\boxed{f(-2)=\sqrt{2}}$$

(ii) $f(0)$

Solution:

$$f(x)=\sqrt{x+4}$$
Put $x=0$
$$f(0)=\sqrt{0+4}=\sqrt{4}$$
$$\boxed{f(0)=2}$$

(iii) $f(x-1)$

Solution:

$$f(x)=\sqrt{x+4}$$
Replace x by $x-1$
$$f(x-1)=\sqrt{x-1+4}$$
$$\boxed{f(x-1)=\sqrt{x+3}}$$

(iv) $f\left(x^2+4\right)$

Solution:

$$f(x)=\sqrt{x+4}$$
Replace x by x^2+4
$$f\left(x^2+4\right)=\sqrt{x^2+4+4}$$
$$\boxed{f\left(x^2+4\right)=\sqrt{x^2+8}}$$

Q.2 Find $\dfrac{f(a+h)-f(a)}{h}$ and simplify where,

(i) $f(x)=6x-9$

(ii) $f(x)=\sin x$

(iii) $f(x)=x^3+2x^2-1$

(iv) $f(x)=\cos x$

(i) $f(x)=6x-9$

Solution:

$$f(x)=6x-9$$
$$\frac{f(a+h)-f(a)}{h}$$
$$=\frac{\left[6(a+h)-9\right]-(6a-9)}{h}$$
$$=\frac{6a+6h-9-6a+9}{h}$$
$$=\frac{6h}{h}$$
$$\boxed{\frac{f(a+h)-f(a)}{h}=6}$$

(ii) $f(x)=\sin x$

Solution:

$$f(x)=\sin x$$
$$\frac{f(a+h)-f(a)}{h}$$
$$=\frac{\sin(a+h)-\sin a}{h}$$
$$=\frac{1}{h}\left[\sin(a+h)-\sin a\right]$$
$$\because \sin P-\sin Q=2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$$
$$=\frac{1}{h}\left[2\cos\left(\frac{a+h+a}{2}\right)\sin\left(\frac{a+h-a}{2}\right)\right]$$
$$=\frac{2}{h}\left[\cos\left(\frac{2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$
$$\boxed{\frac{f(a+h)-f(a)}{h}=\frac{2}{h}\left[\cos\left(a+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)\right]}$$

(iii) $f(x)=x^3+2x^2-1$

Solution:

$$f(x)=x^3+2x^2-1$$
$$\frac{f(a+h)-f(a)}{h}$$

$$=\frac{\left[(a+h)^3+2(a+h)^2-1\right]-\left[a^3+2a^2-1\right]}{h}$$

$$=\frac{a^3+3a^2h+3ah^2+h^3+2\left(a^2+2ah+h^2\right)-1-a^3-2a^2+1}{h}$$

$$=\frac{3a^2h+3ah^2+h^3+2a^2+4ah+2h^2-2a^2}{h}$$

$$=\frac{3a^2h+3ah^2+h^3+4ah+2h^2}{h}$$

$$=\frac{h\left[3a^2+3ah+h^2+4a+2h\right]}{h}$$

$$\boxed{\frac{f(a+h)-f(a)}{h}=3a^2+3ah+h^2+4a+2h}$$

(iv) $f(x)=\cos x$

Solution:

$$f(x)=\cos x$$

$$\frac{f(a+h)-f(a)}{h}$$

$$=\frac{\cos(a+h)-\cos a}{h}$$

$$=\frac{1}{h}\left[\cos(a+h)-\cos a\right]$$

$$\because \cos P-\cos Q=-2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$$

$$=\frac{1}{h}\left[-2\sin\left(\frac{a+h+a}{2}\right)\sin\left(\frac{a+h-a}{2}\right)\right]$$

$$=-\frac{2}{h}\left[\sin\left(\frac{2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$\boxed{\frac{f(a+h)-f(a)}{h}=-\frac{2}{h}\left[\sin\left(a+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)\right]}$$

Q.3 Express the following:

(a) The perimeter P of square as a function of its area A.

Solution:

Let x be the length of each side of square then

$$A=x^2\Rightarrow\sqrt{A}=\sqrt{x^2}$$

$$x=\sqrt{A}$$

$$P=4x\ \ldots\text{(i)}$$

Put $x=\sqrt{A}$ in equation (i)

$$\boxed{P=4\sqrt{A}}$$

(b) The area A of a circle as a function of its circumference C.

Solution:

Let r be the radius of circle, then

$$A=\pi r^2\ \ldots\text{(i)}$$

$$C=2\pi r\Rightarrow r=\frac{C}{2\pi}$$

Put $r=\frac{C}{2\pi}$ in equation (i)

$$A=\pi\left(\frac{C}{2\pi}\right)^2$$

$$A=\pi\times\frac{C^2}{4\pi^2}$$

$$\boxed{A=\frac{C^2}{4\pi}}$$

(c) The volume V of a cube as a function of the area A of its base.

Solution:

Let x be the length of each edge of a cube, then

$$V=x^3\ \ldots\text{(i)}$$

$$A=x^2\Rightarrow\sqrt{A}=\sqrt{x^2}$$

$$\sqrt{A}=x$$

Put $x=\sqrt{A}$ in equation (i)

$$V=\left(\sqrt{A}\right)^3$$

$$\boxed{V=A^{\frac{3}{2}}}$$

Q.4 Find the domain and the range of the function g defined below, and sketch the graph of g.

(i) $g(x)=2x-5$

Solution:

$$g(x)=2x-5$$

Domain $g=R$

Range $g=R$

(ii) $g\left(x\right)=\sqrt{x^2-4}$

Solution:

$g\left(x\right)=\sqrt{x^2-4}$

$g\left(x\right)$ is defined in real numbers if

$x^2-4\ge 0$

$x^2\ge 4$

$\pm x\ge 2$

$x\le -2$ or $x\ge 2$

Domain $g=(-\infty,-2]\cup[2,\infty)$

Range $g=[0,\infty)$

(iii) $g\left(x\right)=\sqrt{x+1}$

Solution:

$g\left(x\right)=\sqrt{x+1}$

$g\left(x\right)$ is defined in real numbers if

$x+1\ge 0$

$x\ge -1$

Domain $g=[-1,\infty)$

Range $g=[0,\infty)$

(iv) $g\left(x\right)=\left|x-3\right|$

Solution:

$g\left(x\right)=\left|x-3\right|$

Domain $g=R$

Range $g=[0,\infty)$

(v) $g\left(x\right)=\begin{cases}6x+7 & x\le -2\\ 4-3x & -2<x\end{cases}$

Solution:

$g\left(x\right)=\begin{cases}6x+7 & x\le -2\\ 4-3x & -2<x\end{cases}$

Domain $g=R$

For range:

If $x\le -2$

Multiplying both sides by 6

$6x\le -12$

Adding 7 on both sides

$6x+7\le -12+7$

$6x+7\le -5$

$g(x)\le -5$

$g\left(x\right)\in\left(-\infty,-5\right]$

If $x>-2$

Multiplying both sides by –3

$-3x<6$

Adding 4 on both sides

$4-3x<4+6$

$4-3x<10$

$g(x)<10$

$g\left(x\right)\in\left(-\infty,10\right)$

$g\left(x\right)\in\left(-\infty,-5\right]\cup\left(-\infty,10\right)$

Range $g=(-\infty,10)$

(vi) $g(x)=\begin{cases}x-1 & , \quad x<3\\ 2x+1 & , \quad 3\le x\end{cases}$

Solution:

$g(x)=\begin{cases}x-1 & , \quad x<3\\ 2x+1 & , \quad 3\le x\end{cases}$

Domain $g = R$

For range:

If $x<3$

Subtracting '1' on both sides

$x-1<3-1$

$x-1<2$

$g(x)<2$

$g\left(x\right)\in(-\infty,2)$

If $x\ge 3$

Multiplying both sides by 2

$2x\ge 6$

$2x+1\ge 7$

$g(x)\ge 7$

$g\left(x\right)\in\left[7,\infty\right)$

Range $g=(-\infty,2)\cup[7,\infty)$

(vii) $g(x)=\dfrac{x^2-3x+2}{x+1}, \quad x\ne -1$

Solution:

$g(x)=\dfrac{x^2-3x+2}{x+1}, \quad x\ne -1$

$g\left(x\right)$ is not defined if

$x+1=0\Rightarrow x=-1$

Domain $g=R-\left\{-1\right\}$

Note: After the correction

For range:

$g(x)=\dfrac{x^2+3x+2}{x+1}, \qquad x\ne -1$

$g(x)=\frac{x^2+2x+x+2}{x+1}, \quad x\neq -1$

$g(x)=\frac{x(x+2)+1(x+2)}{x+1}, \ x\neq -1$

$g(x)=\frac{(x+2)(x+1)}{x+1}, \quad x\neq -1$

$g(x)=x+2, \quad x\neq -1$

By putting $x=-1$

$g(-1)=-1+2=1$

Range $g=R-\{1\}$

(viii) $g(x)=\frac{x^2-16}{x-4}, \quad x\neq 4$

Solution:

$g(x)=\frac{x^2-16}{x-4}, \quad x\neq 4$

$g(x)$ is not defined if

$x-4=0 \Rightarrow x=4$

Domain $g=R-\{4\}$

For range:

$g(x)=\frac{(x-4)(x+4)}{x-4}, \quad x\neq 4$

$g(x)=x+4, \quad x\neq 4$

By putting $x=4$

$g(4)=4+4$

$g(4)=8$

Range $g=R-\{8\}$

Q.5 Given $f(x)=x^3-ax^2+bx+1$. If $f(2)=-3$ and $f(-1)=0$. Find the values of a and b.

Solution:

$f(x)=x^3-ax^2+bx+1$

Putting $x=2$ in $f(x)$

$f(2)=(2)^3-a(2)^2+b(2)+1$

$-3=8-4a+2b+1$

$-3=9-4a+2b$

$-3-9=-4a+2b$

$-12=-4a+2b$

$-12=-2(2a-b)$

$\frac{-12}{-2}=2a-b$

$6=2a-b$

$2a-b=6 \ \ldots$(i)

Putting $x=-1$ in $f(x)$

$f(-1)=(-1)^3-a(-1)^2+b(-1)+1$

$0=-1-a-b+1$

$a+b=0 \ldots$(ii)

Adding equation (i) and equation (ii)

$2a-b=6$

$a+b=0$

$3a=6$

$a=\frac{6}{3}$

$\boxed{a=2}$

Put $a=2$ in equation (ii)

$2+b=0$

$\boxed{b=-2}$

Q.6 A stone falls from a height of $60m$ on the ground, the height h after x second is approximately given by $h(x)=40-10x^2$.

(i) What is the height of the stone when:

(a) $x=1$ **sec.**

Solution:

$h(x)=40-10x^2$

Put $x=1$ in $h(x)$

$h(1)=40-10(1)^2$

$=40-10$

$=30m$

(b) $x=1.5$ **sec.**

Solution:

$h(x)=40-10x^2$

Put $x=1.5$ in $h(x)$

$h(1.5)=40-10(1.5)^2$

$=40-22.5$

$=17.5m$

(ii) When does the stone strike the ground?

Solution:

When stone strikes the ground, then $h(x)=0$

$40-10x^2=0$

$$40 = 10x^2$$
$$\frac{40}{10} = x^2$$
$$4 = x^2$$

By taking square root on both sides

$$x = \pm 2$$

As time is always a positive quantity, therefore,

$$x = 2 \text{ sec}$$

Q.7 Show that the parametric equations:

(i) $x = at^2, y = 2at$ **represent the equation of parabola** $y^2 = 4ax$

Solution:

$$x = at^2 \ldots\text{(i)}$$
$$y = 2at \Rightarrow t = \frac{y}{2a}$$

Put $t = \frac{y}{2a}$ in equation (i)

$$x = a\left(\frac{y}{2a}\right)^2$$
$$x = a \times \frac{y^2}{4a^2}$$
$$x = \frac{y^2}{4a}$$
$$\boxed{y^2 = 4ax}$$

(ii) $x = a\cos\theta, y = b\sin\theta$ **represent the equation of ellipse** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution:

$$x = a\cos\theta \Rightarrow \frac{x}{a} = \cos\theta$$
$$y = b\sin\theta \Rightarrow \frac{y}{b} = \sin\theta$$

Squaring and adding

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\theta + \sin^2\theta$$
$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

(iii) $x = a\sec\theta, y = b\tan\theta$ **represent the equation of hyperbola** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Solution:

$$x = a\sec\theta \Rightarrow \frac{x}{a} = \sec\theta$$
$$y = b\tan\theta \Rightarrow \frac{y}{b} = \tan\theta$$

Squaring and subtracting

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \sec^2\theta - \tan^2\theta$$
$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

Q.8 Prove the identities:

(i) $\sinh 2x = 2\sinh x\cosh x$

Solution:

$$\text{L.H.S} = \sinh 2x$$
$$= \frac{e^{2x} - e^{-2x}}{2} \ldots\text{(i)}$$
$$\text{R.H.S} = 2\sinh x\cosh x$$
$$= 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right)$$
$$= \frac{\left(e^x\right)^2 - \left(e^{-x}\right)^2}{2}$$
$$= \frac{e^{2x} - e^{-2x}}{2} \ldots\text{(ii)}$$

From equation (i) and equation (ii)

$$\boxed{\sinh 2x = 2\sinh x\cosh x}$$

(ii) $\text{sech}^2 x = 1 - \tanh^2 x$

Solution:

$$\text{L.H.S} = \text{sech}^2 x$$
$$= \left(\frac{2}{e^x + e^{-x}}\right)^2$$
$$= \frac{4}{\left(e^x + e^{-x}\right)^2} \ldots\text{(i)}$$
$$\text{R.H.S} = 1 - \tanh^2 x$$
$$= 1 - \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2$$

$$=1-\frac{\left(e^{x}-e^{-x}\right)^{2}}{\left(e^{x}+e^{-x}\right)^{2}}$$

$$=\frac{\left(e^{x}+e^{-x}\right)^{2}-\left(e^{x}-e^{-x}\right)^{2}}{\left(e^{x}+e^{-x}\right)^{2}}$$

$$=\frac{e^{2x}+e^{-2x}+2e^{x}.e^{-x}-\left(e^{2x}+e^{-2x}-2e^{x}.e^{-x}\right)}{\left(e^{x}+e^{-x}\right)^{2}}$$

$$=\frac{e^{2x}+e^{-2x}+2-e^{2x}-e^{-2x}+2}{\left(e^{x}+e^{-x}\right)^{2}}$$

$$=\frac{4}{\left(e^{x}+e^{-x}\right)^{2}}\quad\ldots\text{(ii)}$$

From equation (i) and equation (ii)

$$\boxed{\operatorname{sech}^2 x=1-\tanh^2 x}$$

(iii) $\operatorname{cosech}^2 x=\coth^2 x-1$

Solution:

$$\text{L.H.S}=\operatorname{cosech}^2 x$$

$$=\left(\frac{2}{e^{x}-e^{-x}}\right)^{2}$$

$$=\frac{4}{\left(e^{x}-e^{-x}\right)^{2}}\ldots\text{(i)}$$

$$\text{R.H.S}=\coth^2 x-1$$

$$=\left(\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}\right)^{2}-1$$

$$=\frac{\left(e^{x}+e^{-x}\right)^{2}}{\left(e^{x}-e^{-x}\right)^{2}}-1$$

$$=\frac{\left(e^{x}+e^{-x}\right)^{2}-\left(e^{x}-e^{-x}\right)^{2}}{\left(e^{x}-e^{-x}\right)^{2}}$$

$$=\frac{e^{2x}+e^{-2x}+2e^{x}.e^{-x}-\left(e^{2x}+e^{-2x}-2e^{x}.e^{-x}\right)}{\left(e^{x}-e^{-x}\right)^{2}}$$

$$=\frac{e^{2x}+e^{-2x}+2-e^{2x}-e^{-2x}+2}{\left(e^{x}-e^{-x}\right)^{2}}$$

$$=\frac{4}{\left(e^{x}-e^{-x}\right)^{2}}\ldots\text{(ii)}$$

From equation (i) and equation (ii)

$$\boxed{\operatorname{cosech}^2 x=\coth^2 x-1}$$

Q.9 Determine whether the given function *f* is even or odd.

(i) $f(x)=x^3+x$

Solution:

$$f(x)=x^3+x$$

Replace x by $-x$

$$f(-x)=(-x)^3+(-x)$$
$$=-x^3-x$$
$$=-\left(x^3+x\right)$$
$$f(-x)=-f(x)$$

Hence $f(x)$ is an odd function.

(ii) $f(x)=(x+2)^2$

Solution:

$$f(x)=(x+2)^2$$

Replace x by $-x$

$$f(-x)=(-x+2)^2$$
$$=\left[-(x-2)\right]^2$$
$$=(x-2)^2$$

As neither $f(-x)=f(x)$ nor $f(-x)=-f(x)$

Hence, $f(x)$ is neither even nor odd

(iii) $f(x)=x\sqrt{x^2+5}$

Solution:

$$f(x)=x\sqrt{x^2+5}$$

Replace x by $-x$

$$f(-x)=-x\sqrt{(-x)^2+5}$$
$$f(-x)=-x\sqrt{x^2+5}$$
$$f(-x)=-f(x)$$

Hence $f(x)$ is an odd function.

(iv) $f(x)=\dfrac{x-1}{x+1}$, $x\neq -1$

Solution:

$$f(x)=\frac{x-1}{x+1}$$

Replace x by $-x$

$$f(-x)=\frac{-x-1}{-x+1}$$

$$=\frac{-(x+1)}{-(x-1)}$$

$$f(-x)=\frac{x+1}{x-1}$$

As neither $f(-x)=f(x)$ nor $f(-x)=-f(x)$

Hence, $f(x)$ is neither even nor odd.

(v) $f(x)=x^{\frac{2}{3}}+6$

Solution:

$$f(x)=x^{\frac{2}{3}}+6$$

Replace x by $-x$

$$f(-x)=(-x)^{\frac{2}{3}}+6$$

$$=\left[(-x)^2\right]^{\frac{1}{3}}+6$$

$$=\left(x^2\right)^{\frac{1}{3}}+6$$

$$=x^{\frac{2}{3}}+6$$

$$f(-x)=f(x)$$

Hence, $f(x)$ is even function.

(vi) $f(x)=\dfrac{x^3-x}{x^2+1}$

Solution:

$$f(x)=\frac{x^3-x}{x^2+1}$$

Replace x by $-x$

$$f(-x)=\frac{(-x)^3-(-x)}{(-x)^2+1}$$

$$=\frac{-x^3+x}{x^2+1}$$

$$=\frac{-\left(x^3-x\right)}{x^2+1}$$

$$=-\frac{x^3-x}{x^2+1}$$

$$f(-x)=-f(x)$$

Hence, $f(x)$ is an odd function.

<u>Composition of Functions</u>:

Let f be a function from set X to set Y and g be a function from set Y to set Z.
The composition of f and g is a function, denoted by gof , from X to Z and is defined by

$$(gof)(x)=g\left(f(x)\right)=gf(x),\ \forall x\in X$$

Example 1: Let the real valued functions f and g be defined by $f(x)=2x+1$ and $g(x)=x^2-1$.

Obtained the expressions for (i) $fg(x)$ (ii) $gf(x)$ (iii) $f^2(x)$ (iv) $g^2(x)$

Solution:

(i) $fg(x)=f\left(g(x)\right)=f\left(x^2-1\right)=2\left(x^2-1\right)+1=2x^2-1$

(ii) $gf(x)=g\left(f(x)\right)=g(2x+1)=(2x+1)^2-1=4x^2+4x$

(iii) $f^2(x)=f\left(f(x)\right)=f(2x+1)=2(2x+1)+1=4x+3$

(iv) $g^2(x)=g\left(g(x)\right)=g\left(x^2-1\right)=\left(x^2-1\right)^2-1=x^4-2x^2$

We observe from (i) and (ii) that $fg(x)\neq gf(x)$

Note:

(i) It is important to note that in general, $gf(x) \neq fg(x)$, because $gf(x)$ means that f is applied first then followed by g, whereas $fg(x)$ means that g is applied first then followed by f.

(ii) We usually write ff as f^2 and fff as f^3 and so on.

(iii) $f^n(x) \neq [f(x)]^n$, $n \in Z^+ \cup \{0\}$

Inverse of a Function:

Let f be a one-one function from X onto Y. The **inverse function** of f, denoted by f^{-1}, is a function from Y onto X and is defined by $x = f^{-1}(y)$, $\forall\, y \in Y$ iff $y = f(x), \forall x \in X$.

Example 2: **Let $f: R \to R$ be the function defined by $f(x) = 2x+1$. find $f^{-1}(x)$**

Solution:

We find the inverse of f as follows:

Write $f(x) = 2x+1 = y$

So that y is the image of x under f.

Now solve this equation for x as follows:

$$y = 2x+1$$

$$\Rightarrow \quad 2x = y-1$$

$$\Rightarrow \quad x = \frac{y-1}{2}$$

$$\therefore \quad f^{-1}(y) = \frac{1}{2}(y-1) \qquad [\because x = f^{-1}(y)]$$

To find $f^{-1}(x)$, replace y by x.

$$\therefore \quad \boxed{f^{-1}(x) = \frac{1}{2}(x-1)}$$

Example 3: **Without finding the inverse, state the domain and range of f^{-1}, where $f(x) = 2+\sqrt{x-1}$**

Solution:

We see that f is not defined when $x < 1$.

$\therefore$ Domain $f = [1, +\infty)$

As x varies over the interval $[1, +\infty)$, the value of $\sqrt{x-1}$ varies over the interval $[0, +\infty)$. So the value of $f(x) = 2+\sqrt{x-1}$ varies over the interval $[2, +\infty)$.

Therefore range $f = [2, +\infty)$

By definition of inverse function f^{-1}, we have

Domain $f^{-1} =$ range $f = [2, +\infty)$

Range $f^{-1} =$ domain $f = [1, +\infty)$.

EXERCISE 1.2

Q.1 The real valued functions *f* and *g* are defined below. Find

(a) $fog(x)$

(b) $gof(x)$

(c) $fof(x)$

(d) $gog(x)$

(i) $f(x)=2x+1,\quad g(x)=\dfrac{3}{x-1},\quad x\neq 1$

Solution:

$$f(x)=2x+1,\quad g(x)=\frac{3}{x-1},\quad x\neq 1$$

(a) $fog(x)$

$$fog(x)=f\left(g(x)\right)$$
$$=f\left(\frac{3}{x-1}\right)$$

Replace x by $\dfrac{3}{x-1}$ in $f(x)$.

$$=2\left(\frac{3}{x-1}\right)+1$$
$$=\frac{6}{x-1}+1$$
$$=\frac{6+x-1}{x-1}$$
$$\boxed{fog(x)=\frac{x+5}{x-1}}$$

(b) $gof(x)$

$$gof(x)=g\left(f(x)\right)$$
$$=g\left(2x+1\right)$$

Replace x by $2x+1$ in $g(x)$

$$=\frac{3}{2x+1-1}$$
$$\boxed{gof(x)=\frac{3}{2x}}$$

(c) $fof(x)$

$$fof(x)=f\left(f(x)\right)$$
$$=f\left(2x+1\right)$$

Replace x by $2x+1$ in $f(x)$

$$=2(2x+1)+1$$
$$=4x+2+1$$
$$\boxed{fof(x)=4x+3}$$

(d) $gog(x)$

$$gog(x)=g\left(g(x)\right)$$
$$=g\left(\frac{3}{x-1}\right)$$

Replace x by $\dfrac{3}{x-1}$ in $g(x)$

$$=\frac{3}{\frac{3}{x-1}-1}$$
$$=\frac{3}{\frac{3-(x-1)}{x-1}}$$
$$=\frac{3(x-1)}{3-x+1}$$
$$\boxed{gog(x)=\frac{3(x-1)}{4-x}}$$

(ii) $f(x)=\sqrt{x+1},\quad g(x)=\dfrac{1}{x^2},\, x\neq 0$

(a) $fog(x)$

Solution:

$$fog(x)=f\left(g(x)\right)$$
$$=f\left(\frac{1}{x^2}\right)$$

Replace x by $\dfrac{1}{x^2}$ in $f(x)$

$$=\sqrt{\frac{1}{x^2}+1}$$
$$=\sqrt{\frac{1+x^2}{x^2}}$$
$$\boxed{fog(x)=\frac{\sqrt{1+x^2}}{x}}$$

(b) $gof(x)$

Solution:

$$gof(x) = g(f(x))$$
$$= g\left(\sqrt{x+1}\right)$$

Replace x by $\sqrt{x+1}$ in $g(x)$

$$= \frac{1}{\left(\sqrt{x+1}\right)^2}$$

$$\boxed{gof(x) = \frac{1}{x+1}}$$

(c) $fof(x)$

Solution:

$$fof(x) = f(f(x))$$
$$= f\left(\sqrt{x+1}\right)$$

Replace x by $\sqrt{x+1}$ in $f(x)$

$$\boxed{fof(x) = \sqrt{\sqrt{x+1}+1}}$$

(d) $gog(x)$

Solution:

$$gog(x) = g(g(x))$$
$$= g\left(\frac{1}{x^2}\right)$$

Replace x by $\frac{1}{x^2}$ in $g(x)$

$$= \frac{1}{\left(\frac{1}{x^2}\right)^2}$$
$$= \frac{1}{\left(\frac{1}{x^4}\right)}$$

$$\boxed{gog(x) = x^4}$$

(iii) $f(x) = \frac{1}{\sqrt{x-1}},\ x \neq 1,\ g(x) = \left(x^2+1\right)^2$

(a) $fog(x)$

Solution:

$$fog(x) = f(g(x))$$
$$= f\left(\left(x^2+1\right)^2\right)$$

Replace x by $\left(x^2+1\right)^2$ in $f(x)$

$$= \frac{1}{\sqrt{\left(x^2+1\right)^2-1}}$$
$$= \frac{1}{\sqrt{x^4+2x^2+1-1}}$$
$$= \frac{1}{\sqrt{x^4+2x^2}}$$
$$= \frac{1}{\sqrt{x^2\left(x^2+2\right)}}$$

$$\boxed{fog(x) = \frac{1}{x\sqrt{x^2+2}}}$$

(b) $gof(x)$

Solution:

$$gof(x) = g(f(x))$$
$$= g\left(\frac{1}{\sqrt{x-1}}\right)$$

Replace x by $\frac{1}{\sqrt{x-1}}$ in $g(x)$.

$$= \left[\left(\frac{1}{\sqrt{x-1}}\right)^2+1\right]^2$$
$$= \left(\frac{1}{x-1}+1\right)^2$$
$$= \left(\frac{1+x-1}{x-1}\right)^2$$
$$= \left(\frac{x}{x-1}\right)^2$$

$$\boxed{gof(x) = \frac{x^2}{(x-1)^2}}$$

(c) $fof(x)$

Solution:

$$fof(x) = f(f(x))$$
$$= f\left(\frac{1}{\sqrt{x-1}}\right)$$

Replace x by $\frac{1}{\sqrt{x-1}}$ in $f(x)$

$$= \frac{1}{\sqrt{\frac{1}{\sqrt{x-1}} - 1}}$$

$$= \frac{1}{\sqrt{\frac{1-\sqrt{x-1}}{\sqrt{x-1}}}}$$

$$\boxed{fof(x) = \sqrt{\frac{\sqrt{x-1}}{1-\sqrt{x-1}}}}$$

(d) $gog(x)$

Solution:

$$gog(x) = g(g(x))$$

$$= g\left(\left(x^2+1\right)^2\right)$$

Replace x by $\left(x^2+1\right)^2$ in $g(x)$

$$= \left[\left(\left(x^2+1\right)^2\right)^2 + 1\right]^2$$

$$\boxed{gog(x) = \left(\left(x^4+2x^2+1\right)^2+1\right)^2}$$

(iv) $f(x) = 3x^4 - 2x^2,\ g(x) = \frac{2}{\sqrt{x}}, x \neq 0$

(a) $fog(x)$

Solution:

$$fog(x) = f(g(x))$$

$$= f\left(\frac{2}{\sqrt{x}}\right)$$

Replace x by $\frac{2}{\sqrt{x}}$ in $f(x)$

$$fog(x) = 3\left(\frac{2}{\sqrt{x}}\right)^4 - 2\left(\frac{2}{\sqrt{x}}\right)^2$$

$$= 3\left(\frac{16}{x^2}\right) - 2\left(\frac{4}{x}\right)$$

$$= \frac{48}{x^2} - \frac{8}{x}$$

$$= \frac{48-8x}{x^2}$$

$$\boxed{fog(x) = \frac{8(6-x)}{x^2}}$$

(b) $gof(x)$

Solution:

$$gof(x) = g(f(x))$$

$$= g\left(3x^4 - 2x^2\right)$$

Replace x by $3x^4 - 2x^2$ in $g(x)$

$$= \frac{2}{\sqrt{3x^4 - 2x^2}}$$

$$= \frac{2}{\sqrt{x^2\left(3x^2-2\right)}}$$

$$\boxed{gof(x) = \frac{2}{x\sqrt{3x^2-2}}}$$

(c) $fof(x)$

Solution:

$$fof(x) = f(f(x))$$

$$= f\left(3x^4 - 2x^2\right)$$

Replace x by $3x^4 - 2x^2$ in $f(x)$

$$\boxed{fof(x) = 3\left(3x^4-2x^2\right)^4 - 2\left(3x^4-2x^2\right)^2}$$

(d) $gog(x)$

Solution:

$$gog(x) = g(g(x))$$

$$= g\left(\frac{2}{\sqrt{x}}\right)$$

Replace x by $\frac{2}{\sqrt{x}}$ in $g(x)$

$$= \frac{2}{\sqrt{\frac{2}{\sqrt{x}}}}$$

$$= \frac{2}{\frac{\sqrt{2}}{\sqrt{\sqrt{x}}}}$$

$$= \frac{2\sqrt{\sqrt{x}}}{\sqrt{2}}$$

$$= \sqrt{2}\sqrt{\sqrt{x}}$$

$$\boxed{gog(x) = \sqrt{2\sqrt{x}}}$$

Q.2 For the real valued function f defined below find

(a) $f^{-1}(x)$

(b) $f^{-1}(-1)$ **and verify** $f\left(f^{-1}(x)\right)=f^{-1}\left(f(x)\right)=x$

(i) $f(x)=-2x+8$

(a) $f^{-1}(x)$

Solution:

$y=f(x)=-2x+8$

$y=-2x+8$

$2x=8-y$

$x=\dfrac{8-y}{2}$

$\because y=f(x)\Rightarrow f^{-1}(y)=x$

$f^{-1}(y)=\dfrac{8-y}{2}$

Replacing y by x

$$\boxed{f^{-1}(x)=\frac{8-x}{2}}$$

(b) $f^{-1}(-1)$

Solution:

$f^{-1}(x)=\dfrac{8-x}{2}$

Putting $x=-1$

$f^{-1}(-1)=\dfrac{8-(-1)}{2}$

$$\boxed{f^{-1}(-1)=\frac{9}{2}}$$

Verification:

$f\left(f^{-1}(x)\right)=f\left(\dfrac{8-x}{2}\right)$

Replace x by $\dfrac{8-x}{2}$ in $f(x)$

$=-2\left(\dfrac{8-x}{2}\right)+8$

$=-8+x+8$

$=x$

Now $f^{-1}\left(f(x)\right)=f^{-1}(-2x+8)$

Replace x by $-2x+8$ in $f^{-1}(x)$

$=\dfrac{8-(-2x+8)}{2}$

$=\dfrac{8+2x-8}{2}$

$=\dfrac{2x}{x}$

$=x$

$$\boxed{f\left(f^{-1}(x)\right)=f^{-1}\left(f(x)\right)=x}$$

(ii) $f(x)=3x^3+7$

(a) $f^{-1}(x)$

Solution:

$y=f(x)=3x^3+7$

$y=3x^3+7$

$y-7=3x^3$

$\dfrac{y-7}{3}=x^3$

By taking cube root on both sides.

$\left(\dfrac{y-7}{3}\right)^{\frac{1}{3}}=x$

$\because y=f(x)\Rightarrow f^{-1}(y)=x$

$f^{-1}(y)=\left(\dfrac{y-7}{3}\right)^{\frac{1}{3}}$

Replacing y by x

$$\boxed{f^{-1}(x)=\left(\frac{x-7}{3}\right)^{\frac{1}{3}}}$$

(b) $f^{-1}(-1)$

Solution:

$f^{-1}(x)=\left(\dfrac{x-7}{3}\right)^{\frac{1}{3}}$

By putting $x=-1$

$f^{-1}(-1)=\left(\dfrac{-1-7}{3}\right)^{\frac{1}{3}}$

$=\left(\dfrac{-8}{3}\right)^{\frac{1}{3}}$

Verification:

$$f\left(f^{-1}(x)\right)=f\left(\left(\frac{x-7}{3}\right)^{\frac{1}{3}}\right)$$

Replace x by $\left(\frac{x-7}{3}\right)^{\frac{1}{3}}$ in $f(x)$

$$=3\left[\left(\frac{x-7}{3}\right)^{\frac{1}{3}}\right]^3+7$$

$$=3\left(\frac{x-7}{3}\right)+7$$

$$=x-7+7$$

$$=x$$

Now $f^{-1}\left(f(x)\right)=f^{-1}\left(3x^3+7\right)$

Replace x by $3x^3+7$ in $f^{-1}(x)$

$$=\left(\frac{3x^3+7-7}{3}\right)^{\frac{1}{3}}$$

$$=\left(\frac{3x^3}{3}\right)^{\frac{1}{3}}$$

$$=\left(x^3\right)^{\frac{1}{3}}$$

$$=x$$

$$\boxed{f\left(f^{-1}(x)\right)=f^{-1}\left(f(x)\right)=x}$$

(iii) $f(x)=(-x+9)^3$

(a) $f^{-1}(x)$

Solution:

$$y=f(x)=(-x+9)^3$$

$$y=(-x+9)^3$$

By taking cube root on both sides.

$$y^{\frac{1}{3}}=\left[(-x+9)^3\right]^{\frac{1}{3}}$$

$$y^{\frac{1}{3}}=-x+9$$

$$x=9-y^{\frac{1}{3}}$$

$$\because y=f(x)\Rightarrow f^{-1}(y)=x$$

$$f^{-1}(y)=9-y^{\frac{1}{3}}$$

Replacing y by x

$$\boxed{f^{-1}(x)=9-x^{\frac{1}{3}}}$$

(b) $f^{-1}(-1)$

Solution:

$$f^{-1}(x)=9-x^{\frac{1}{3}}$$

Putting $x=-1$

$$f^{-1}(-1)=9-(-1)^{\frac{1}{3}}$$

Verification:

$$f\left(f^{-1}(x)\right)=f\left(9-x^{\frac{1}{3}}\right)$$

Replace x by $9-x^{\frac{1}{3}}$ in $f(x)$

$$=\left(-\left(9-x^{\frac{1}{3}}\right)+9\right)^3$$

$$=\left(-9+x^{\frac{1}{3}}+9\right)^3$$

$$=\left(x^{\frac{1}{3}}\right)^3$$

$$=x$$

Now $f^{-1}\left(f(x)\right)=f^{-1}\left((-x+9)^3\right)$

Replace x by $(-x+9)^3$ in $f(x)$

$$=9-\left((-x+9)^3\right)^{\frac{1}{3}}$$

$$=9-(-x+9)$$

$$=9+x-9$$

$$=x$$

$$\boxed{f\left(f^{-1}(x)\right)=f^{-1}\left(f(x)\right)=x}$$

(iv) $f(x)=\dfrac{2x+1}{x-1}, x>1$

(a) $f^{-1}(x)$

Solution:

$$y=f(x)=\frac{2x+1}{x-1}$$
$$y=\frac{2x+1}{x-1}$$
$$y(x-1)=2x+1$$
$$xy-y=2x+1$$
$$xy-2x=y+1$$
$$x(y-2)=y+1$$
$$x=\frac{y+1}{y-2}$$
$$\because y=f(x)\Rightarrow f^{-1}(y)=x$$
$$f^{-1}(y)=\frac{y+1}{y-2}$$

Replacing y by x

$$\boxed{f^{-1}(x)=\frac{x+1}{x-2}}$$

(b) $f^{-1}(-1)$

Solution:

$$f^{-1}(x)=\frac{x+1}{x-2}$$

Putting $x=-1$

$$f^{-1}(-1)=\frac{-1+1}{-1-2}$$
$$f^{-1}(-1)=0$$

Verification:

$$f\left(f^{-1}(x)\right)=f\left(\frac{x+1}{x-2}\right)$$

Replace x by $\dfrac{x+1}{x-2}$ in $f(x)$

$$=\frac{2\left(\frac{x+1}{x-2}\right)+1}{\frac{x+1}{x-2}-1}$$
$$=\frac{\frac{2(x+1)+x-2}{x-2}}{\frac{(x+1)-(x-2)}{x-2}}$$
$$=\frac{2x+2+x-2}{x+1-x+2}$$
$$=\frac{3x}{3}$$
$$=x$$

Now $f^{-1}\left(f(x)\right)=f^{-1}\left(\dfrac{2x+1}{x-1}\right)$

Replace x by $\dfrac{2x+1}{x-1}$ in $f^{-1}(x)$

$$f^{-1}\left(f(x)\right)=\frac{\frac{2x+1}{x-1}+1}{\frac{2x+1}{x-1}-2}$$
$$=\frac{2x+1+x-1}{(2x+1)-2(x-1)}$$
$$=\frac{3x}{2x+1-2x+2}$$
$$=\frac{3x}{3}$$
$$=x$$
$$\boxed{f\left(f^{-1}(x)\right)=f^{-1}\left(f(x)\right)=x}$$

Q.3 Without finding the inverse, state the domain and range of f^{-1}.

(i) $f(x)=\sqrt{x+2}$

Solution:

As $f(x)=\sqrt{x+2}$

$f(x)$ is defined on real numbers if

$x+2\geq 0$

$x\geq -2$

Domain $f=[-2,\infty)$

Range $f=[0,\infty)$

By the definition of inverse function f^{-1}, we have

Domain $f^{-1}=$ Range $f=[0,\infty)$

Range $f^{-1}=$ Domain $f=[-2,\infty)$

(ii) $f(x)=\dfrac{x-1}{x-4}, x\neq 4$

Solution:

$f(x)=\dfrac{x-1}{x-4}, x\neq 4$

Domain $f=R-\{4\}$

Range $f=R-\{1\}$

By the definition of inverse function f^{-1}, we have

Domain $f^{-1}=$ Range $f=R-\{1\}$

Range $f^{-1}=$ Domain $f=R-\{4\}$

(iii) $f(x)=\dfrac{1}{x+3}, x\neq -3$

Solution:

$f(x)=\dfrac{1}{x+3}$

Domain $f=R-\{-3\}$

Range $f=R-\{0\}$

By the definition of inverse function f^{-1}, we have

Domain $f^{-1}=$ Range $f=R-\{0\}$

Range $f^{-1}=$ Domain $f=R-\{-3\}$

(iv) $f(x)=(x-5)^2, x\geq 5$

Solution:

$f(x)=(x-5)^2, x\geq 5$

Domain $f=[5,\infty)$

Range $f=[0,\infty)$

By the definition of inverse function f^{-1}, we have

Domain f^{-1} =Range $f=[0,\infty)$

Range $f^{-1}=$ Domain $f=[5,\infty)$

Limit of a Function:

Let a function $f(x)$ be defined in an open interval near the number *a* (need not at *a*). If, as *x* approaches *a* from both left and right side of *a*, $f(x)$ approaches a specific number *L*, then *L* is called the limit of $f(x)$ as *x* approaches *a*. Symbolically it is written as: $\lim\limits_{x\to a} f(x)=L$ (read as "limit of $f(x)$, as $x\to a$, is *L*")

Example: If $P(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ is a polynomial function of degree *n*, then show that: $\lim\limits_{x\to c}P(x)=P(c)$

Solution:

Using the theorems on limits, we have

$$\lim_{x\to c}P(x)=\lim_{x\to c}\left(a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0\right)$$

$$=a_n\lim_{x\to c}x^n+a_{n-1}\lim_{x\to c}x^{n-1}+\ldots+a_1\lim_{x\to c}x+\lim_{x\to c}a_0$$

$$=a_nc^n+a_{n-1}c^{n-1}+\ldots+a_1c+a_0$$

$$\therefore \lim_{x\to c}P(x)=P(c)$$

Limits Of Important Functions:

Theorem: Prove that $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=na^{n-1}$, where *n* is an integer and $a>0$.

Proof: Case-I: Suppose *n* is a positive integer.

By substituting $x=a$, we get $\left(\dfrac{0}{0}\right)$ form, so we make factors as follows:

$$x^n-a^n=(x-a)(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\ldots\ldots+a^{n-1})$$

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{x\to a}\frac{(x-a)(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\dots\dots\dots+a^{n-1})}{(x-a)}$$

$$=\lim_{x\to a}(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\dots\dots\dots+a^{n-1})$$

$$=a^{n-1}+aa^{n-2}+a^2a^{n-3}+\dots\dots\dots\dots\dots+a^{n-1}$$

$$=a^{n-1}+a^{n-1}+a^{n-1}+\dots\dots\dots\dots+a^{n-1}\quad (n \text{ terms})$$

$$=n\,a^{n-1}$$

Case-II: Suppose n is a negative integer (say $n=-m$), where m is a positive integer

Now $\dfrac{x^n-a^n}{x-a}=\dfrac{x^{-m}-a^{-m}}{x-a}$

$$=\frac{\frac{1}{x^m}-\frac{1}{a^m}}{x-a}$$

$$=\frac{\frac{a^m-x^m}{x^ma^m}}{x-a}$$

$$=\frac{-1}{x^ma^m}\left(\frac{x^m-a^m}{x-a}\right),\ (a\neq 0)$$

Now, $\displaystyle\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{x\to a}\frac{-1}{x^ma^m}\left(\frac{x^m-a^m}{x-a}\right)$

$$=\left(\lim_{x\to a}\frac{-1}{x^ma^m}\right)\times\left(\lim_{x\to a}\frac{x^m-a^m}{x-a}\right)$$

$$=\frac{-1}{a^ma^m}ma^{m-1}\quad \text{(By Case-I)}$$

$$=-m\,a^{m-1-m-m}$$

$$=(-m)a^{-m-1}$$

$$=n\,a^{n-1}\qquad (n=-m)$$

Example 1: Evaluate $\displaystyle\lim_{x\to-\infty}\frac{4x^4-5x^3}{3x^5+2x^2+1}$

Solution:

Since $x<0$, so dividing up and down by $(-x)^5=-x^5$, we get

$$\lim_{x\to-\infty}\frac{4x^4-5x^3}{3x^5+2x^2+1}=\lim_{x\to-\infty}\frac{-\frac{4}{x}+\frac{5}{x^2}}{-3-\frac{2}{x^3}-\frac{1}{x^5}}$$

$$=\frac{0+0}{-3-0-0}=0$$

Example 2: Evaluate $\displaystyle\lim_{x\to-\infty}\frac{2-3x}{\sqrt{3+4x^2}}$

Solution:

Here $\sqrt{x^2}=|x|=-x$ as $x<0$

$\therefore$ Dividing up and down by $-x$, we get

$$\lim_{x\to-\infty}\frac{2-3x}{\sqrt{3+4x^2}}=\lim_{x\to-\infty}\frac{-\frac{2}{x}+3}{\sqrt{\frac{3}{x^2}+4}}$$

$$=\frac{0+3}{\sqrt{0+4}}=\frac{3}{2}$$

Example 3: Evaluate $\lim_{x\to+\infty}\frac{2-3x}{\sqrt{3+4x^2}}$

Solution:

Here $\sqrt{x^2}=|x|=x$ as $x>0$

$\therefore$ Dividing up and down by x, we get

$$\lim_{x\to+\infty}\frac{2-3x}{\sqrt{3+4x^2}}=\lim_{x\to+\infty}\frac{\frac{2}{x}-3}{\sqrt{\frac{3}{x^2}+4}}$$

$$=\frac{0-3}{\sqrt{0+4}}=-\frac{3}{2}$$

Theorem: **Prove that** $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$

Proof:

By the binomial theorem, we have

$$\left(1+\frac{1}{n}\right)^n=1+n\left(\frac{1}{n}\right)+\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3+\dots$$

$$=1+1+\frac{1}{2!}n(n-1)\times\frac{1}{n^2}+\frac{1}{3!}n(n-1)(n-2)\times\frac{1}{n^3}+\dots$$

$$=2+\frac{1}{2!}n^2\left(1-\frac{1}{n}\right).\frac{1}{n^2}+\frac{1}{3!}n^3\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{1}{n^3}+\dots$$

$$=2+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\dots$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=\lim_{n\to\infty}\left[2+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\dots\right]$$

As $n\to\infty$, $\frac{1}{n},\frac{2}{n},\frac{3}{n}\dots$ all tend to zero.

$$=2+\frac{1}{2!}(1-0)+\frac{1}{3!}(1-0)(1-0)+\dots$$

$$=2+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\dots$$

$$=2+0.5+0.166667+0.0416667+\dots$$

$$=2.718281\dots$$

As approximate value of *e* is 2.718281, so

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=e}$$

Deduction: $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$

Proof:

We know that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e$

Put $n=\frac{1}{x}$, then $\frac{1}{n}=x$

When $n\to\infty, x\to 0$

As $\lim_{x\to\infty}\left(1+\frac{1}{n}\right)^{n}=e$

$$\boxed{\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e}$$

Theorem: **Prove that** $\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a$

Proof:

Put $a^x-1=y$ (i)

Then $a^x=1+y$

Taking logarithm on both sides with base *a*.

$\log_a a^x=\log_a(1+y)$

$x.\log_a a=\log_a(1+y)$

So $x=\log_a(1+y)$

From (i) when $x\to 0, y\to 0$

$$\lim_{x\to 0}\frac{a^x-1}{x}=\lim_{y\to 0}\frac{y}{\log_a(1+y)}$$

$$=\lim_{y\to 0}\frac{1}{\frac{1}{y}\log_a(1+y)}$$

$$=\lim_{y\to 0}\frac{1}{\log_a(1+y)^{\frac{1}{y}}}$$

$$=\frac{1}{\log_a\left(\lim_{y\to 0}(1+y)^{\frac{1}{y}}\right)}$$

$$=\frac{1}{\log_a e}\qquad\left(\because \lim_{y\to 0}(1+y)^{\frac{1}{y}}=e\right)$$

$$\boxed{\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a}$$

Deduction: $\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)=\log_e e=1$

We know that $\lim_{x\to 0}\left(\frac{a^x-1}{x}\right)=\log_e a$ (i)

Put $a=e$ in (i)

$$\boxed{\lim_{x\to 0}\frac{e^x-1}{x}=\log_e e=1}$$

<u>The Sandwitch Theorem</u>:

Let *f*, *g* and *h* be functions such that $f(x)\le g(x)\le h(x)$ for all numbers *x* in some open interval containing *c*, except possibly at *c* itself.

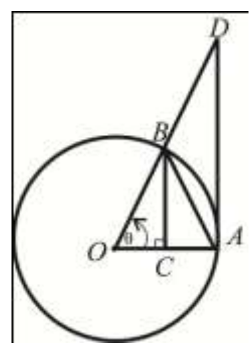
If $\lim_{x\to c} f(x)=L$ and $\lim_{x\to c} h(x)=L$, then $\lim_{x\to c} g(x)=L$

Theorem: **If θ is measured in radian, then** $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$

Proof: Take θ a positive acute central angle of a circle with radius $r = 1$.

Produce $\overline{OB}$ to D, so that $\overline{AD}\perp\overline{OA}$.

Draw $\overline{BC}\perp\overline{OA}$. Join *A* and *B*. As show in figure, OAB represent a sector of the circle.



Given $|\overline{OA}|=|\overline{OB}|=1$ (radii of unit circle)

In right ΔOCB, $\sin\theta=\frac{|\overline{BC}|}{|\overline{OB}|}=|\overline{BC}|$ $\left(|\overline{OB}|=1\right)$

In right ΔOAD, $\tan\theta=\frac{|\overline{AD}|}{|\overline{OA}|}=|\overline{AD}|$ $\left(|\overline{OA}|=1\right)$

In terms of θ, the areas are expressed as:

(i) Area of $\Delta OAB=\frac{1}{2}|\overline{OA}||\overline{BC}|=\frac{1}{2}(1)\sin\theta=\frac{1}{2}\sin\theta$

(ii) Area of sector $OAB=\frac{1}{2}r^2\theta=\frac{1}{2}(1)^2\theta=\frac{1}{2}\theta$ $(r=1)$

(iii) Area of $\Delta OAD=\frac{1}{2}|\overline{OA}||\overline{AD}|=\frac{1}{2}(1)\tan\theta$

From the figure, we see that

Area of $\Delta OAB<$ Area of sector $OAB<$ Area of ΔOAD

$$\frac{1}{2}\sin\theta<\frac{\theta}{2}<\frac{1}{2}\tan\theta$$

$$\frac{1}{2}\sin\theta<\frac{\theta}{2}<\frac{1}{2}\frac{\sin\theta}{\cos\theta}$$

As $\sin\theta$ is positive, so on division by $\frac{1}{2}\sin\theta$, we get

$$\frac{\frac{1}{2}\sin\theta}{\frac{1}{2}\sin\theta} < \frac{\frac{\theta}{2}}{\frac{1}{2}\sin\theta} < \frac{\frac{1}{2}\frac{\sin\theta}{\cos\theta}}{\frac{1}{2}\sin\theta}$$

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta} \qquad \left(\because 0<\theta<\frac{\pi}{2}\right)$$

i.e., $1 > \frac{\sin\theta}{\theta} > \cos\theta$ or $\cos\theta < \frac{\sin\theta}{\theta} < 1$

when $\theta \to 0, \cos\theta \to 1$

since $\frac{\sin\theta}{\theta}$ is sandwitched between 1 and a quantity approaching 1 itself.

So, by the sandwitch theorem, it must also approach 1.

i.e., $\boxed{\underset{\theta\to 0}{Lim}\frac{\sin\theta}{\theta}=1}$

Limits of Important Functions:

$\underset{x\to a}{Lim}\frac{x^n-a^n}{x-a}=na^{n-1}$, n is an integer, $a>0$	
$\underset{x\to a}{Lim}\frac{x^n-a^n}{x^m-a^m}=\frac{n}{m}a^{n-m}$	
$\underset{x\to\pm\infty}{Lim}\frac{1}{x}=0,\ x\neq 0$	
$\underset{x\to\pm\infty}{Lim}\frac{a}{x^p}=0$ where $p\in Q^+, a\in R$	
$\underset{n\to\infty}{Lim}\left(1+\frac{1}{n}\right)^n=e$	$\underset{x\to+\infty}{Lim}\left(e^x\right)=\infty$
$\underset{x\to 0}{Lim}(1+x)^{\frac{1}{x}}=e$	$\underset{x\to-\infty}{Lim}\left(e^x\right)=\underset{x\to-\infty}{Lim}\left(\frac{1}{e^{-x}}\right)=0$
$\underset{x\to 0}{Lim}\frac{a^x-1}{x}=\log_e a$	$\underset{x\to\pm\infty}{Lim}\left(\frac{a}{x}\right)=0,\ a\in R$
$\underset{x\to 0}{Lim}\left(\frac{e^x-1}{x}\right)=\log_e e=1$	$\underset{\theta\to 0}{Lim}\frac{\sin\theta}{\theta}=1$, where θ is in radians.

EXERCISE 1.3

Q.1 Evaluate each limit by using theorems of limits:

(i) $\lim_{x\to 3}(2x+4)$

Solution:

$$\lim_{x\to 3}(2x+4)$$

$$\because \lim_{x\to a}\left[f(x)+g(x)\right]=\lim_{x\to a}f(x)+\lim_{x\to a}g(x)$$

$$=\lim_{x\to 3}(2x)+\lim_{x\to 3}(4)$$

$$\because \lim_{x\to a}\left[kf(x)\right]=k\left[\lim_{x\to a}f(x)\right]$$

$$=2\lim_{x\to 3}(x)+4$$

$$=2(3)+4$$

$$=6+4$$

$$=10$$

$$\boxed{\lim_{x\to 3}(2x+4)=10}$$

(ii) $\lim_{x\to 1}(3x^2-2x+4)$

Solution:

$$\lim_{x\to 1}(3x^2-2x+4)$$

$$\because \lim_{x\to a}\left[f(x)\pm g(x)\right]=\lim_{x\to a}\left[f(x)\right]\pm\lim_{x\to a}\left[g(x)\right]$$

$$=\lim_{x\to 1}(3x^2)-\lim_{x\to 1}(2x)+\lim_{x\to 1}(4)$$

$$\because \lim_{x\to a}\left[kf(x)\right]=k\left[\lim_{x\to a}f(x)\right]$$

$$=3\lim_{x\to 1}(x^2)-2\lim_{x\to 1}(x)+4$$

$$=3(1)^2-2(1)+4$$

$$=3-2+4$$

$$=5$$

$$\boxed{\lim_{x\to 1}(3x^2-2x+4)=5}$$

(iii) $\lim_{x\to 3}\sqrt{x^2+x+4}$

Solution:

$$\lim_{x\to 3}\sqrt{x^2+x+4}$$

$$\because \lim_{x\to a}\left[f(x)\right]^n=\left[\lim_{x\to a}f(x)\right]^n$$

$$=\sqrt{\lim_{x\to 3}(x^2+x+4)}$$

$$\because \lim_{x\to a}\left[f(x)+g(x)\right]=\lim_{x\to a}\left[f(x)\right]+\lim_{x\to a}\left[g(x)\right]$$

$$=\sqrt{\lim_{x\to 3}x^2+\lim_{x\to 3}x+\lim_{x\to 3}4}$$

$$=\sqrt{3^2+3+4}$$

$$=\sqrt{9+3+4}$$

$$=\sqrt{16}$$

$$=4$$

$$\boxed{\lim_{x\to 3}\sqrt{x^2+x+4}=4}$$

(iv) $\lim_{x\to 2}x\sqrt{x^2-4}$

Solution:

$$\lim_{x\to 2}x\sqrt{x^2-4}$$

$$\because \lim_{x\to a}\left[f(x)g(x)\right]=\left[\lim_{x\to a}f(x)\right].\left[\lim_{x\to a}g(x)\right]$$

$$=\left[\lim_{x\to 2}x\right].\left[\lim_{x\to 2}\sqrt{x^2-4}\right]$$

$$\because \lim_{x\to a}\left[f(x)\right]^n=\left[\lim_{x\to a}f(x)\right]^n$$

$$=2.\sqrt{\lim_{x\to 2}(x^2-4)}$$

$$\because \lim_{x\to a}\left[f(x)-g(x)\right]=\lim_{x\to a}f(x)-\lim_{x\to a}g(x)$$

$$=2\sqrt{\lim_{x\to 2}x^2-\lim_{x\to 2}4}$$

$$=2\sqrt{2^2-4}$$

$$=2\sqrt{4-4}$$

$$=2\sqrt{0}$$

$$=2(0)$$

$$=0$$

$$\boxed{\lim_{x\to 2}x\sqrt{x^2-4}=0}$$

(v) $\lim_{x\to 2}\left(\sqrt{x^3+1}-\sqrt{x^2+5}\right)$

Solution:

$$\lim_{x\to 2}\left(\sqrt{x^3+1}-\sqrt{x^2+5}\right)$$

$$\because \lim_{x\to a}\left[f(x)-g(x)\right]=\lim_{x\to a}f(x)-\lim_{x\to a}g(x)$$

$$=\lim_{x\to 2}\left(\sqrt{x^3+1}\right)-\lim_{x\to 2}\left(\sqrt{x^2+5}\right)$$

$\because \lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$

$= \sqrt{\lim_{x\to 2}(x^3+1)} - \sqrt{\lim_{x\to 2}(x^2+5)}$

$\because \lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

$= \sqrt{\lim_{x\to 2} x^3 + \lim_{x\to 2} 1} - \sqrt{\lim_{x\to 2} x^2 + \lim_{x\to 2} 5}$

$= \sqrt{2^3+1} - \sqrt{2^2+5}$

$= \sqrt{9} - \sqrt{9}$

$$\boxed{\lim_{x\to 2}\left(\sqrt{x^3+1} - \sqrt{x^2+5}\right) = 0}$$

(vi) $\lim_{x\to -2} \dfrac{2x^3+5x}{3x-2}$

Solution:

$\lim_{x\to -2} \dfrac{2x^3+5x}{3x-2}$

$\because \lim_{x\to a}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

$= \dfrac{\lim_{x\to -2}(2x^3+5x)}{\lim_{x\to -2}(3x-2)}$

$\because \lim_{x\to a}[f(x)\pm g(x)] = \lim_{x\to a}[f(x)] \pm \lim_{x\to a}[g(x)]$

$= \dfrac{\lim_{x\to -2}(2x^3) + \lim_{x\to -2}(5x)}{\lim_{x\to -2}(3x) - \lim_{x\to -2}(2)}$

$\because \lim_{x\to a}[kf(x)] = k\left[\lim_{x\to a} f(x)\right]$

$= \dfrac{2\lim_{x\to -2}(x^3) + 5\lim_{x\to -2}(x)}{3\lim_{x\to -2}(x) - \lim_{x\to -2}(2)}$

$= \dfrac{2(-2)^3 + 5(-2)}{3(-2)-2}$

$= \dfrac{2(-8)+5(-2)}{3(-2)-2}$

$= \dfrac{-16-10}{-6-2}$

$= \dfrac{-26}{-8}$

$= \dfrac{13}{4}$

Hence $\boxed{\lim_{x\to -2} \dfrac{2x^3+5x}{3x-2} = \dfrac{13}{4}}$

Q.2 Evaluate each limit by using algebraic techniques.

(i) $\lim_{x\to -1} \dfrac{x^3-x}{x+1}$

Solution:

$\lim_{x\to -1} \dfrac{x^3-x}{x+1}$

$= \lim_{x\to -1} \dfrac{x(x^2-1)}{x+1}$

$= \lim_{x\to -1} \dfrac{x(x-1)(x+1)}{x+1}$

$= \lim_{x\to -1} x(x-1)$

$= -1(-1-1)$

$= -1(-2)$

$= 2$

$$\boxed{\lim_{x\to -1} \frac{x^3-x}{x+1} = 2}$$

(ii) $\lim_{x\to 0}\left(\dfrac{3x^3+4x}{x^2+x}\right)$

Solution:

$\lim_{x\to 0}\left(\dfrac{3x^3+4x}{x^2+x}\right)$

$= \lim_{x\to 0} \dfrac{x(3x^2+4)}{x(x+1)}$

$= \lim_{x\to 0} \dfrac{3x^2+4}{x+1}$

$= \dfrac{3(0)^2+4}{0+1}$

$= \dfrac{0+4}{0+1}$

$= 4$

$$\boxed{\lim_{x\to 0} \frac{3x^3+4x}{x^2+x} = 4}$$

(iii) $\lim_{x\to2}\frac{x^3-8}{x^2+x-6}$

Solution:

$$\lim_{x\to2}\frac{x^3-8}{x^2+x-6}$$

$$=\lim_{x\to2}\frac{x^3-2^3}{x^2+3x-2x-6}$$

$$\because a^3-b^3=(a-b)(a^2+ab+b^2)$$

$$=\lim_{x\to2}\frac{(x-2)(x^2+2x+2^2)}{x(x+3)-2(x+3)}$$

$$=\lim_{x\to2}\frac{(x-2)(x^2+2x+4)}{(x+3)(x-2)}$$

$$=\lim_{x\to2}\frac{x^2+2x+4}{x+3}$$

$$=\frac{2^2+2(2)+4}{2+3}$$

$$=\frac{4+4+4}{5}$$

$$=\frac{12}{5}$$

$$\lim_{x\to2}\frac{x^3-8}{x^2+x-6}=\frac{12}{5}$$

(iv) $\lim_{x\to1}\frac{x^3-3x^2+3x-1}{x^3-x}$

Solution:

$$\lim_{x\to1}\frac{x^3-3x^2+3x-1}{x^3-x}$$

$$=\lim_{x\to1}\frac{(x^3)-3(x)^2(1)+3(x)(1)^2-(1)^3}{x(x^2-1)}$$

$$\because (a-b)^3=a^3-3a^2b+3ab^2-b^3$$

$$=\lim_{x\to1}\frac{(x-1)^3}{x(x^2-1)}$$

$$=\lim_{x\to1}\frac{(x-1)^3}{x(x-1)(x+1)}$$

$$=\lim_{x\to1}\frac{(x-1)^2}{x(x+1)}$$

$$=\frac{(1-1)^2}{1(1+1)}$$

$$=\frac{0}{2}$$

$$=0$$

$$\boxed{\lim_{x\to1}\frac{x^3-3x^2+3x-1}{x^3-x}=0}$$

(v) $\lim_{x\to-1}\frac{x^3+x^2}{x^2-1}$

Solution:

$$\lim_{x\to-1}\frac{x^3+x^2}{x^2-1}$$

$$=\lim_{x\to-1}\frac{x^2(x+1)}{(x-1)(x+1)}$$

$$=\lim_{x\to-1}\frac{x^2}{x-1}$$

$$=\frac{(-1)^2}{-1-1}$$

$$=-\frac{1}{2}$$

$$\boxed{\lim_{x\to-1}\left(\frac{x^3+x^2}{x^2-1}\right)=-\frac{1}{2}}$$

(vi) $\lim_{x\to4}\frac{2x^2-32}{x^3-4x^2}$

Solution:

$$\lim_{x\to4}\frac{2x^2-32}{x^3-4x^2}$$

$$=\lim_{x\to4}\frac{2(x^2-16)}{x^2(x-4)}$$

$$=\lim_{x\to4}\frac{2(x^2-4^2)}{x^2(x-4)}$$

$$=\lim_{x\to4}\frac{2(x-4)(x+4)}{x^2(x-4)}$$

$$=\lim_{x\to4}\frac{2(x+4)}{x^2}$$

$$=\frac{2(4+4)}{4^2}$$

$$=\frac{2\times 8}{16}$$
$$=1$$
$$\lim_{x\to 4}\left(\frac{2x^2-32}{x^3-4x^2}\right)=1$$

(vii) $\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x-2}$

Solution:

$$\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x-2}$$

By Rationalizing the numerator

$$=\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x-2}\times\frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}}$$
$$=\lim_{x\to 2}\frac{\left(\sqrt{x}\right)^2-\left(\sqrt{2}\right)^2}{(x-2)\left(\sqrt{x}+\sqrt{2}\right)}$$
$$=\lim_{x\to 2}\frac{x-2}{(x-2)\left(\sqrt{x}+\sqrt{2}\right)}$$
$$=\lim_{x\to 2}\frac{1}{\sqrt{x}+\sqrt{2}}$$
$$=\frac{1}{\sqrt{2}+\sqrt{2}}$$
$$=\frac{1}{2\sqrt{2}}$$
$$\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x-2}=\frac{1}{2\sqrt{2}}$$

(viii) $\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$

Solution:

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

By rationalizing the numerator

$$=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$
$$=\lim_{h\to 0}\frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left[\sqrt{x+h}+\sqrt{x}\right]}$$
$$=\lim_{h\to 0}\frac{x+h-x}{h\left[\sqrt{x+h}+\sqrt{x}\right]}$$
$$=\lim_{h\to 0}\frac{h}{h\left[\sqrt{x+h}+\sqrt{x}\right]}$$
$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}$$
$$=\frac{1}{\sqrt{x+0}+\sqrt{x}}$$
$$=\frac{1}{\sqrt{x}+\sqrt{x}}$$
$$=\frac{1}{2\sqrt{x}}$$
$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\frac{1}{2\sqrt{x}}$$

(ix) $\lim_{x\to a}\frac{x^n-a^n}{x^m-a^m}$

Solution:

$$\lim_{x\to a}\frac{x^n-a^n}{x^m-a^m}$$
$$=\lim_{x\to a}\frac{(x-a)\left(x^{n-1}+x^{n-2}a+x^{n-3}a^2+......+a^{n-1}\right)}{(x-a)\left(x^{m-1}+x^{m-2}a+x^{m-3}a^2+......+a^{m-1}\right)}$$
$$=\lim_{x\to a}\frac{x^{n-1}+x^{n-2}a+x^{n-3}.a^2+......+a^{n-1}}{x^{m-1}+x^{m-2}.a+x^{m-3}a^2+......+a^{m-1}}$$
$$=\frac{a^{n-1}+a^{n-2}a+a^{n-3}.a^2+......+a^{n-1}}{a^{m-1}+a^{m-2}.a+a^{m-3}a^2+.....+a^{m-1}}$$
$$=\frac{a^{n-1}+a^{n-1}+a^{n-1}+......+a^{n-1}}{a^{m-1}+a^{m-1}+a^{m-1}+......+a^{m-1}}$$
$$=\frac{na^{n-1}}{ma^{m-1}}$$
$$=\frac{na^{n-1-(m-1)}}{m}$$
$$=\frac{na^{n-1-m+1}}{m}$$
$$=\frac{n}{m}a^{n-m}$$
$$\lim_{x\to a}\frac{x^n-a^n}{x^m-a^m}=\frac{n}{m}a^{n-m}$$

Q.3 Evaluate the following limits:

(i) $\lim_{x\to 0}\frac{\sin 7x}{x}$

Solution:

$\lim_{x\to 0}\frac{\sin(7x)}{x}$

Multiply and divide by 7

$=\lim_{x\to 0} 7\times\frac{\sin 7x}{7x}$

$=7\lim_{x\to 0}\frac{\sin 7x}{7x}$

As $x\to 0$, then $7x\to 0$

$=7\lim_{7x\to 0}\frac{\sin 7x}{7x}$

$=7(1)$

$$\boxed{\lim_{x\to 0}\frac{\sin 7x}{x}=7}$$

(ii) $\lim_{x\to 0}\frac{\sin x^\circ}{x}$

Solution:

$\lim_{x\to 0}\frac{\sin x^\circ}{x}$

As $1^\circ=\frac{\pi}{180}$ radian

$x^\circ=\frac{\pi x}{180}$ radian

$\lim_{x\to 0}\frac{\sin x^\circ}{x}=\lim_{x\to 0}\frac{\sin\left(\frac{\pi x}{180}\right)}{x}$

Multiply and divide by $\frac{\pi}{180}$

$=\lim_{x\to 0}\frac{\pi}{180}\times\frac{\sin\left(\frac{\pi x}{180}\right)}{\frac{\pi}{180}x}$

$=\frac{\pi}{180}\lim_{x\to 0}\frac{\sin\left(\frac{\pi x}{180}\right)}{\frac{\pi}{180}x}$

When $x\to 0$, then $\frac{\pi}{180}x\to 0$

$=\frac{\pi}{180}\lim_{\frac{\pi}{180}x\to 0}\frac{\sin\left(\frac{\pi}{180}x\right)}{\left(\frac{\pi}{180}x\right)}$

$=\frac{\pi}{180}\times 1$

$$\boxed{\lim_{x\to 0}\frac{\sin x^\circ}{x}=\frac{\pi}{180}}$$

(iii) $\lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta}$

Solution:

$\lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta}$

By rationalizing the numerator

$=\lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta}\times\frac{1+\cos\theta}{1+\cos\theta}$

$=\lim_{\theta\to 0}\frac{1^2-\cos^2\theta}{\sin\theta(1+\cos\theta)}$

$=\lim_{\theta\to 0}\frac{1-\cos^2\theta}{\sin\theta(1+\cos\theta)}$

$=\lim_{\theta\to 0}\frac{\sin^2\theta}{\sin\theta(1+\cos\theta)}$

$=\lim_{\theta\to 0}\frac{\sin\theta}{1+\cos\theta}$

$=\frac{\sin 0}{1+\cos 0}=\frac{0}{1+1}=\frac{0}{2}=0$

$$\boxed{\lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta}=0}$$

(iv) $\lim_{x\to \pi}\frac{\sin x}{\pi-x}$

Solution:

$\lim_{x\to \pi}\frac{\sin x}{\pi-x}$

Let $\theta=\pi-x$

$x=\pi-\theta$

When $x\to\pi$,we have $\theta\to 0$ then

$\lim_{x\to \pi}\frac{\sin x}{\pi-x}=\lim_{\theta\to 0}\frac{\sin(\pi-\theta)}{\theta}$

$\because \sin(\pi-\theta)=\sin\theta$

$$= \lim_{\theta \to 0} \frac{\sin \theta}{\theta}$$

$$= 1$$

$$\boxed{\lim_{x \to \pi} \frac{\sin x}{\pi - x} = 1}$$

(v) $\lim_{x \to 0} \frac{\sin ax}{\sin bx}$

Solution:

$$\lim_{x \to 0} \frac{\sin ax}{\sin bx}$$

$$= \lim_{x \to 0} \frac{\left(\frac{\sin ax}{ax}\right) \times ax}{\left(\frac{\sin bx}{bx}\right) \times bx}$$

$$= \frac{a}{b} \lim_{x \to 0} \frac{\left(\frac{\sin ax}{ax}\right)}{\left(\frac{\sin bx}{bx}\right)}$$

$$= \frac{a}{b} \frac{\lim_{x \to 0}\left(\frac{\sin ax}{ax}\right)}{\lim_{x \to 0}\left(\frac{\sin bx}{bx}\right)}$$

$$= \frac{a}{b}\left(\frac{1}{1}\right)$$

$$= \frac{a}{b}$$

$$\boxed{\lim_{x \to 0} \frac{\sin ax}{\sin bx} = \frac{a}{b}}$$

(vi) $\lim_{x \to 0} \frac{x}{\tan x}$

Solution:

$$\lim_{x \to 0} \frac{x}{\tan x}$$

$$= \lim_{x \to 0} \frac{x}{\left(\frac{\sin x}{\cos x}\right)}$$

$$= \lim_{x \to 0} \frac{\cos x}{\left(\frac{\sin x}{x}\right)}$$

$$= \frac{\lim_{x \to 0} \cos x}{\lim_{x \to 0}\left(\frac{\sin x}{x}\right)}$$

$$= \frac{\cos 0}{1}$$

$$= 1$$

$$\boxed{\lim_{x \to 0} \frac{x}{\tan x} = 1}$$

(vii) $\lim_{x \to 0} \frac{1 - \cos 2x}{x^2}$

Solution:

$$\lim_{x \to 0} \frac{1 - \cos 2x}{x^2}$$

$$\because \cos 2x = 1 - 2\sin^2 x$$

$$= \lim_{x \to 0} \frac{2\sin^2 x}{x^2}$$

$$= 2 \lim_{x \to 0} \frac{\sin^2 x}{x^2}$$

$$= 2 \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2$$

$$= 2\left(\lim_{x \to 0} \frac{\sin x}{x}\right)^2$$

$$= 2(1)^2$$

$$= 2$$

$$\boxed{\lim_{x \to 0} \frac{1 - \cos 2x}{x^2} = 2}$$

(viii) $\lim_{x \to 0} \frac{1 - \cos x}{\sin^2 x}$

Solution:

$$\lim_{x \to 0} \frac{1 - \cos x}{\sin^2 x}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{1 - \cos^2 x}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{(1 - \cos x)(1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{1}{1 + \cos x}$$

$$= \frac{1}{1 + \cos 0}$$

$$= \frac{1}{1+1}$$
$$= \frac{1}{2}$$
$$\boxed{\lim_{x\to 0}\frac{1-\cos x}{\sin^2 x} = \frac{1}{2}}$$

(ix) $\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta}$

Solution:

$$\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta}$$
$$= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}.\sin\theta$$
$$= \left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)\left(\lim_{\theta\to 0}\sin\theta\right)$$
$$= 1\times\sin 0$$
$$= 1\times(0)$$
$$= 0$$
$$\boxed{\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta} = 0}$$

(x) $\lim_{x\to 0}\frac{\sec x-\cos x}{x}$

Solution:

$$\lim_{x\to 0}\frac{\sec x-\cos x}{x}$$
$$= \lim_{x\to 0}\frac{\frac{1}{\cos x}-\cos x}{x}$$
$$= \lim_{x\to 0}\frac{\frac{1-\cos^2 x}{\cos x}}{x}$$
$$= \lim_{x\to 0}\frac{\sin^2 x}{x\cos x}$$
$$= \lim_{x\to 0}\frac{\sin x}{x}.\frac{\sin x}{\cos x}$$
$$= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\tan x\right)$$
$$= 1\times\tan(0)$$
$$= 1\times 0$$
$$= 0$$
$$\boxed{\lim_{x\to 0}\frac{\sec x-\cos x}{x} = 0}$$

(xi) $\lim_{\theta\to 0}\frac{1-\cos p\theta}{1-\cos q\theta}$

Solution:

$$\lim_{\theta\to 0}\frac{1-\cos p\theta}{1-\cos q\theta}$$
$$\because 1-\cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$$
$$= \lim_{\theta\to 0}\frac{2\sin^2\left(\frac{p\theta}{2}\right)}{2\sin^2\left(\frac{q\theta}{2}\right)}$$
$$= \lim_{\theta\to 0}\left[\frac{\sin\left(\frac{p\theta}{2}\right)}{\sin\left(\frac{q\theta}{2}\right)}\right]^2$$
$$= \lim_{\theta\to 0}\left[\frac{\frac{\sin\left(\frac{p\theta}{2}\right)\times\frac{p\theta}{2}}{\frac{p\theta}{2}}}{\frac{\sin\left(\frac{q\theta}{2}\right)\times\frac{q\theta}{2}}{\frac{q\theta}{2}}}\right]^2$$
$$= \left[\lim_{\theta\to 0}\frac{\frac{\sin\left(\frac{p\theta}{2}\right)}{\frac{p\theta}{2}}\times p}{\frac{\sin\left(\frac{q\theta}{2}\right)}{\frac{q\theta}{2}}\times q}\right]^2$$
$$= \frac{p^2}{q^2}\left[\frac{\lim_{\theta\to 0}\frac{\sin\left(\frac{p\theta}{2}\right)}{\frac{p\theta}{2}}}{\lim_{\theta\to 0}\frac{\sin\left(\frac{q\theta}{2}\right)}{\frac{q\theta}{2}}}\right]^2$$

$$= \frac{p^2}{q^2}\left(\frac{1}{1}\right)^2$$

$$= \frac{p^2}{q^2}$$

$$\boxed{\lim_{\theta\to 0}\frac{1-\cos p\theta}{1-\cos q\theta} = \frac{p^2}{q^2}}$$

(xii) $\lim_{\theta\to 0}\frac{\tan\theta-\sin\theta}{\sin^3\theta}$

Solution:

$$\lim_{\theta\to 0}\frac{\tan\theta-\sin\theta}{\sin^3\theta}$$

$$= \lim_{\theta\to 0}\frac{\frac{\sin\theta}{\cos\theta}-\sin\theta}{\sin^3\theta}$$

$$= \lim_{\theta\to 0}\frac{\sin\theta\left(\frac{1}{\cos\theta}-1\right)}{\sin^3\theta}$$

$$= \lim_{\theta\to 0}\frac{\frac{1-\cos\theta}{\cos\theta}}{\sin^2\theta}$$

$$= \lim_{\theta\to 0}\frac{1-\cos\theta}{\cos\theta\left(1-\cos^2\theta\right)}$$

$$= \lim_{\theta\to 0}\frac{1-\cos\theta}{\cos\theta(1-\cos\theta)(1+\cos\theta)}$$

$$= \lim_{\theta\to 0}\frac{1}{\cos\theta(1+\cos\theta)}$$

$$= \frac{1}{\cos 0(1+\cos 0)}$$

$$= \frac{1}{1(1+1)}$$

$$= \frac{1}{2}$$

$$\boxed{\lim_{\theta\to 0}\frac{\tan\theta-\sin\theta}{\sin^3\theta} = \frac{1}{2}}$$

Q.4 Express each limit in terms of e:

(i) $\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{2n}$

Solution:

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{2n}$$

$$= \lim_{n\to+\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^2$$

$$= \left[\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n\right]^2$$

$$= e^2$$

$$\boxed{\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{2n} = e^2}$$

(ii) $\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{\frac{n}{2}}$

Solution:

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{\frac{n}{2}}$$

$$= \lim_{n\to+\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^{\frac{1}{2}}$$

$$= \left[\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n\right]^{\frac{1}{2}}$$

$$= e^{\frac{1}{2}}$$

$$\boxed{\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{2n} = \sqrt{e}}$$

(iii) $\lim_{n\to+\infty}\left(1-\frac{1}{n}\right)^{n}$

Solution:

$$\lim_{n\to+\infty}\left(1-\frac{1}{n}\right)^{n}$$

$$= \lim_{n\to\infty}\left[\left(1-\frac{1}{n}\right)^{-n}\right]^{-1}$$

$$= \left[\lim_{n\to\infty} \left(1 + \left(-\frac{1}{n} \right) \right)^{-n} \right]^{-1}$$

$$= e^{-1}$$

$$= \frac{1}{e}$$

$$\boxed{\lim_{n\to+\infty} \left(1 - \frac{1}{n} \right)^{n} = \frac{1}{e}}$$

(iv) $\lim_{n\to+\infty} \left(1 + \frac{1}{3n} \right)^{n}$

Solution:

$$\lim_{n\to+\infty} \left(1 + \frac{1}{3n} \right)^{n}$$

$$= \lim_{n\to+\infty} \left[\left(1 + \frac{1}{3n} \right)^{n} \right]^{\frac{3}{3}}$$

$$= \left[\lim_{n\to\infty} \left(1 + \frac{1}{3n} \right)^{3n} \right]^{\frac{1}{3}}$$

$$\boxed{\lim_{n\to+\infty} \left(1 + \frac{1}{3n} \right)^{n} = e^{\frac{1}{3}}}$$

(v) $\lim_{n\to+\infty} \left(1 + \frac{4}{n} \right)^{n}$

Solution:

$$\lim_{n\to+\infty} \left(1 + \frac{4}{n} \right)^{n}$$

$$= \lim_{n\to\infty} \left[\left(1 + \frac{4}{n} \right)^{n} \right]^{\frac{4}{4}}$$

$$= \left(\lim_{n\to\infty} \left(1 + \frac{4}{n} \right)^{\frac{n}{4}} \right)^{4}$$

$$\boxed{\lim_{n\to+\infty} \left(1 + \frac{4}{n} \right)^{n} = e^{4}}$$

(vi) $\lim_{x\to0} (1+3x)^{\frac{2}{x}}$

Solution:

$$\lim_{x\to0} (1+3x)^{\frac{2}{x}}$$

$$= \lim_{x\to0} \left[(1+3x)^{\frac{2}{x}} \right]^{\frac{3}{3}}$$

$$= \left[\lim_{x\to0} (1+3x)^{\frac{1}{3x}} \right]^{6}$$

$$\boxed{\lim_{x\to0} (1+3x)^{\frac{2}{x}} = e^{6}}$$

(vii) $\lim_{x\to0} (1+2x^2)^{\frac{1}{x^2}}$

Solution:

$$\lim_{x\to0} (1+2x^2)^{\frac{1}{x^2}}$$

$$= \lim_{x\to0} \left[(1+2x^2)^{\frac{1}{x^2}} \right]^{\frac{2}{2}}$$

$$\boxed{\lim_{x\to0} (1+2x^2)^{\frac{1}{x^2}} = e^{2}}$$

(viii) $\lim_{h\to0} (1-2h)^{\frac{1}{h}}$

Solution:

$$\lim_{h\to0} (1-2h)^{\frac{1}{h}}$$

$$= \lim_{h\to0} \left[(1-2h)^{\frac{1}{h}} \right]^{\frac{-2}{-2}}$$

$$= \lim_{h\to0} \left[(1-2h)^{-\frac{1}{2h}} \right]^{-2}$$

$$= \left[\lim_{h\to0} (1-2h)^{-\frac{1}{2h}} \right]^{-2}$$

$$= e^{-2}$$

$$\boxed{\lim_{h\to0} (1-2h)^{\frac{1}{h}} = \frac{1}{e^{2}}}$$

(ix) $\lim\limits_{x\to\infty}\left(\dfrac{x}{1+x}\right)^{x}$

Solution:

$$\lim_{x\to\infty}\left(\frac{x}{1+x}\right)^{x}$$

$$=\lim_{x\to\infty}\left(\frac{1+x}{x}\right)^{-x}$$

$$=\lim_{x\to\infty}\left(\frac{1}{x}+1\right)^{-x}$$

$$=\lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^{x}\right]^{-1}$$

$$=\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right]^{-1}$$

$$=e^{-1}$$

$$=\frac{1}{e}$$

$$\boxed{\lim_{x\to\infty}\left(\frac{x}{1+x}\right)^{x}=\frac{1}{e}}$$

(x) $\lim\limits_{x\to 0}\dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1},\ x<0$

Solution:

$$\lim_{x\to 0}\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1},\ x<0$$

Replace x by $-y$ where $y>0$

$$=\lim_{y\to 0}\frac{e^{-\frac{1}{y}}-1}{e^{-\frac{1}{y}}+1}$$

$$=\lim_{y\to 0}\frac{\frac{1}{e^{\frac{1}{y}}}-1}{\frac{1}{e^{\frac{1}{y}}}+1}$$

$$=\frac{\frac{1}{e^{\frac{1}{0}}}-1}{\frac{1}{e^{\frac{1}{0}}}+1}$$

$$=\frac{\frac{1}{e^{\infty}}-1}{\frac{1}{e^{\infty}}+1}$$

$$=\frac{\frac{1}{\infty}-1}{\frac{1}{\infty}+1}$$

$$=\frac{0-1}{0+1}$$

$$\boxed{\lim_{x\to 0}\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1},\ x<0=-1}$$

(xi) $\lim\limits_{x\to 0}\dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1},\ x>0$

Solution:

$$\lim_{x\to 0}\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$$

As $x>0$, so $x=x$

$$=\lim_{x\to 0}\frac{e^{\frac{1}{x}}\left[1-\frac{1}{e^{\frac{1}{x}}}\right]}{e^{\frac{1}{x}}\left[1+\frac{1}{e^{\frac{1}{x}}}\right]}$$

$$=\lim_{x\to 0}\frac{1-\frac{1}{e^{\frac{1}{x}}}}{1+\frac{1}{e^{\frac{1}{x}}}}$$

$$=\frac{1-\frac{1}{e^{\frac{1}{0}}}}{1+\frac{1}{e^{\frac{1}{0}}}}$$

$$= \frac{1-\frac{1}{e^{\infty}}}{1+\frac{1}{e^{\infty}}}$$

$$= \frac{1-\frac{1}{\infty}}{1+\frac{1}{\infty}}$$

$$= \frac{1-0}{1+0}$$

$$\boxed{\lim_{x \to 0} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}, x>0 = 1}$$

Left Hand Limit:

$\lim_{x \to c^-} f(x) = L$ is read as the limit of $f(x)$ is equal to L as x approaches c from the left i.e., For all x sufficiently close to c, but less than c, the value of $f(x)$ can be made as close as we please to L.

Right Hand Limit:

$\lim_{x \to c^+} f(x) = M$ is read as the limit of $f(x)$ is equal to M as x approaches from the right i.e., for all x sufficiently close to c, but greater than c, the value of $f(x)$ can be made as close as we please to M.

Criterion for Existence of Limit of a Function:

$\lim_{x \to c} f(x) = L$ iff $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$

Continuous Function:

A function f is said to be **continuous** at a number "c" iff the following three conditions are satisfied:

(i) $f(c)$ is defined.

(ii) $\lim_{x \to c} f(x)$ exists.

(iii) $\lim_{x \to c} f(x) = f(c)$

Discontinuous Function:

If one or more of these three conditions fail to hold at c then the function f is said to be **discontinuous** at c.

Example: Discuss the continuity of the function $f(x)$ and $g(x)$ at $x=3$.

(a) $f(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{if} \quad x \neq 3 \\ 6 & \text{if} \quad x = 3 \end{cases}$ **(b)** $g(x) = \frac{x^2-9}{x-3}$ if $x \neq 3$.

Solution:

(a) Given $f(3) = 6$

$\therefore$ The function f is defined at $x = 3$.

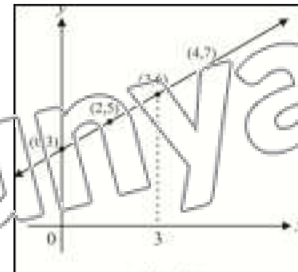


Fig (i)

Now $\lim_{x\to3} f(x) = \lim_{x\to3}\dfrac{x^2-9}{x-3}$

$= \lim_{x\to3}\dfrac{(x+3)(x-3)}{x-3}$

$= \lim_{x\to3}(x+3) = 6$

As $\lim_{x\to3} f(x) = 6 = f(3)$

$\therefore f(x)$ is continuous at $x = 3$

It is noted that there is no break in the graph. (See figure (i))

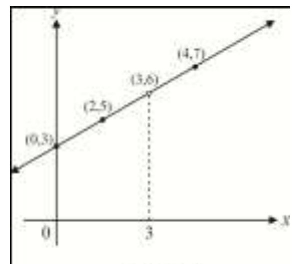


Fig (ii)

(b) $g(x) = \dfrac{x^2-9}{x-3}$ **if** $x \neq 3$

As $g(x)$ is not defined at $x = 3$

$g(x)$ is discontinuous at $x = 3$

It is noted that there is a break in the graph at $x = 3$. (See figure (ii))

EXERCISE 1.4

Q.1 Determine the left hand limit and the right hand limit and then, find the limit of the following functions when $x \to c$.

(i) $f(x) = 2x^2 + x - 5,\ c = 1$

Solution:

$f(x) = 2x^2 + x - 5$

Left hand limit:

$\lim_{x\to1^-} f(x) = \lim_{x\to1^-}(2x^2 + x - 5)$

$= 2(1)^2 + 1 - 5$

$= 2 + 1 - 5$

$= -2$

Right hand limit:

$\lim_{x\to1^+} f(x) = \lim_{x\to1^+}(2x^2 + x - 5)$

$= 2(1)^2 + 1 - 5$

$= 2 + 1 - 5$

$= -2$

As $\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x)$

Hence $\lim_{x\to1} f(x)$ exists and

$\lim_{x\to1} f(x) = \lim_{x\to1}(2x^2 + x - 5) = -2$

(ii) $f(x) = \dfrac{x^2-9}{x-3}, \quad c = -3$

Solution:

$f(x) = \dfrac{x^2-9}{x-3}, \quad c = -3$

Left hand limit:

$\lim_{x\to-3^-} f(x) = \lim_{x\to-3^-}\dfrac{x^2-9}{x-3}$

$= \dfrac{(-3)^2-9}{-3-3}$

$= \dfrac{9-9}{-6}$

$= \dfrac{0}{-6}$

$= 0$

Right hand limit:

$\lim_{x\to-3^+} f(x) = \lim_{x\to-3^+}\dfrac{x^2-9}{x-3}$

$$= \frac{(-3)^2 - 9}{-3-3}$$
$$= \frac{9-9}{-6}$$
$$= \frac{0}{-6}$$
$$= 0$$

As $\lim_{x\to -3^-} f(x) = \lim_{x\to -3^+} f(x)$

Hence $\lim_{x\to c} f(x)$ exists

and $\lim_{x\to -3} f(x) = \lim_{x\to -3} \frac{x^2-9}{x-3} = 0$

(iii) $f(x) = |x-5|, \quad c = 5$

Solution:

$f(x) = |x-5|, \quad c = 5$

Left hand limit:

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} |x-5|$$
$$= \lim_{x\to 5^-} [-(x-5)]$$
$$= -(5-5)$$
$$= 0$$

Right hand limit:

$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} |x-5|$$
$$= \lim_{x\to 5^+} (x-5)$$
$$= 5-5$$
$$= 0$$

As $\lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x)$

Hence $\lim_{x\to 5} f(x)$ exists and

$$\lim_{x\to 5} f(x) = \lim_{x\to 5} |x-5| = 0$$

Q.2 Discuss the continuity of $f(x)$ at $x = c$:

(i) $f(x) = \begin{cases} 2x+5 & \text{if } x \le 2 \\ 4x+1 & \text{if } x > 2 \end{cases}, \quad c = 2$

Solution:

At $x = 2$

$f(x) = 2x+5$

$f(2) = 2(2)+5 = 9$

$f(2) = 9$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2x+5)$$
$$= 2(2)+5$$
$$= 9$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (4x+1)$$
$$= 4(2)+1$$
$$= 9$$

As $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$, so

$\lim_{x\to 2} f(x)$ exists.

As $f(2) = \lim_{x\to 2} f(x)$

Hence, $f(x)$ is continuous at $x = 2$.

(ii) $f(x) = \begin{cases} 3x-1 & \text{if } x < 1 \\ 4 & \text{if } x = 1, \quad c = 1 \\ 2x & \text{if } x > 1 \end{cases}$

Solution:

At $x = 1$

$f(x) = 4$

$f(1) = 4$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (3x-1)$$
$$= 3(1) - 1 = 2$$
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x)$$
$$= 2\lim_{x\to 1^+} (x) = 2$$

As $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$,

so $\lim_{x\to 2} f(x)$ exists.

As, $f(1) \ne \lim_{x\to 1} f(x)$

$4 \ne 2$

Hence function $f(x)$ is discontinuous at $x = 1$

Q.3 If $f(x) = \begin{cases} 3x & \text{if } x \le -2 \\ x^2 - 1 & \text{if } -2 < x < 2 \\ 3 & \text{if } x \ge 2 \end{cases}$.

Discuss continuity at $x = 2$ and $x = -2$.

Solution:

Continuity at $x = 2$

At $x = 2$

$f(2) = 3$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2 - 1)$$
$$= 2^2 - 1 = 4 - 1$$

$= 3$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(3)$

$= 3$

As $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$, so

$\lim_{x\to 2} f(x)$ exists.

As $f(2) = \lim_{x\to 2} f(x)$

Hence, $f(x)$ is continuous at $x = 2$.

continuity at $x = -2$

at $x = -2$

$f(x) = 3x$

$f(-2) = 3(-2) = -6$

$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-}(3x)$

$= 3(-2)$

$= -6$

$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+}(x^2 - 1)$

$= (-2)^2 - 1$

$= 4 - 1$

$= 3$

As $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$,

so $\lim_{x\to 2} f(x)$ does not exists.

Hence, $f(x)$ is discontinuous at $x = -2$.

Q.4 If $f(x) = \begin{cases} x+2 & , \quad x \le -1 \\ c+2 & , \quad x > -1 \end{cases}$

Find "c" so that $\lim_{x\to -1} f(x)$ exists.

Solution:

As $\lim_{x\to -1} f(x)$ exists.

So $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$

$\lim_{x\to -1^-}(x+2) = \lim_{x\to -1^+}(c+2)$

$-1 + 2 = c + 2$

$-1 = c$

$\boxed{c = -1}$

Q.5 Find the values m and n, so that given function f is continuous at $x = 3$.

(i) $f(x) = \begin{cases} mx & \text{if} \quad x < 3 \\ n & \text{if} \quad x = 3 \\ -2x+9 & \text{if} \quad x > 3 \end{cases}$

Solution:

As $f(x)$ is continuous at $x = 3$

So $f(3) = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$

At $x = 3$

$f(3) = n$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(mx)$

$= m(3)$

$= 3m$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(-2x+9)$

$= -2(3) + 9$

$= -6 + 9$

$= 3$

As $f(3) = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$

$n = 3m = 3$

$\boxed{n = 3}$, $3m = 3$

$\boxed{m = 1}$

(ii) $f(x) = \begin{cases} mx & \text{if} \quad x < 3 \\ x^2 & \text{if} \quad x \ge 3 \end{cases}$

Solution:

As $f(x)$ is continuous at $x = 3$

So $f(3) = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$

At $x = 3$

$f(x) = x^2$

$f(3) = 3^2 = 9$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(mx)$

$= 3m$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(x^2)$

$= 3^2$

$= 9$

As $f(3) = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$

$$9=3m=9$$
$$3m=3$$
$$\boxed{m=1}$$

Q.6 If

$$f(x)=\begin{cases}\dfrac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}, & x\neq 2\\ k & x=2\end{cases}$$

find the value of k so that f is continuous at $x=2$.

Solution:

As $f(x)$ is continuous at $x=2$

So $f(2)=\lim\limits_{x\to 2} f(x)$

At $x=2$

$f(2)=k$

Now,

$$\lim_{x\to 2} f(x)$$
$$=\lim_{x\to 2}\frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}$$

By rationalization of numerator

$$=\lim_{x\to 2}\frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}\times\frac{\sqrt{2x+5}+\sqrt{x+7}}{\sqrt{2x+5}+\sqrt{x+7}}$$
$$=\lim_{x\to 2}\frac{\left(\sqrt{2x+5}\right)^2-\left(\sqrt{x+7}\right)^2}{(x-2)\left(\sqrt{2x+5}+\sqrt{x+7}\right)}$$
$$=\lim_{x\to 2}\frac{2x+5-(x+7)}{(x-2)\left(\sqrt{2x+5}+\sqrt{x+7}\right)}$$
$$=\lim_{x\to 2}\frac{2x+5-x-7}{(x-2)\left(\sqrt{2x+5}+\sqrt{x+7}\right)}$$
$$=\lim_{x\to 2}\frac{x-2}{(x-2)\left(\sqrt{2x+5}+\sqrt{x+7}\right)}$$
$$=\lim_{x\to 2}\frac{1}{\sqrt{2x+5}+\sqrt{x+7}}$$
$$=\frac{1}{\sqrt{2(2)+5}+\sqrt{2+7}}$$
$$=\frac{1}{\sqrt{9}+\sqrt{9}}$$
$$=\frac{1}{3+3}$$
$$\lim_{x\to 2} f(x)=\frac{1}{6}$$

As $f(2)=\lim\limits_{x\to 2} f(x)$

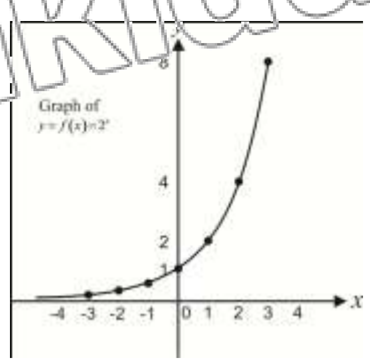
$$\boxed{k=\frac{1}{6}}$$

<u>Graph of the Exponential Function $f(x)=a^x$:</u>

Let us draw the graph of $y=2^x$, here $a=2$.

We prepare the following table for different values of x and $f(x)$ near the origin:

x	–4	–3	–2	–1	0	1	2	3	4
y	0.0625	0.125	0.25	0.5	1	2	4	8	16



Plotting the points (x, y) and joining them with smooth curve as shown in the figure, we get the graph of $y = 2^x$

From the graph of 2^x, the characteristics of the graph of $y = a^x$ are observed as follows:

If $a > 1$,

(i) a^x is always positive for all real values of x.

(ii) a^x increases as x increases.

(iii) $a^x = 1$ when $x = 0$

(iv) $a^x \to 0$ as $x \to -\infty$

Graph of Common Logarithmic Function $f(x) = \log x$**:**

If $x = 10^y$, then $y = \log x$

Now for all real values of $y, 10^y > 0 \Rightarrow x > 0$

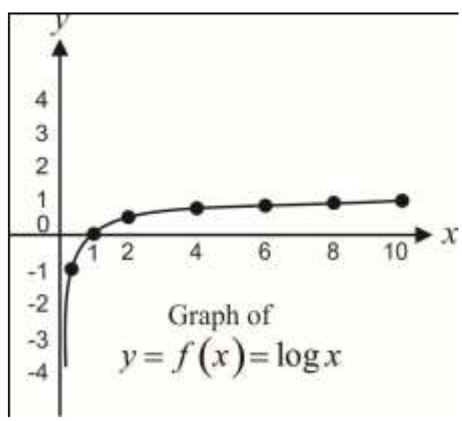
This means $\log x$ exists only when $x > 0$

$\Rightarrow$ Domain of the $\log x$ is positive real numbers. It is undefined at $x = 0$.

For graph of $f(x) = \log x$, we find the values of $\lg x$ from the common logarithmic table for various values of $x > 0$.

Table of some of the corresponding values of x and $f(x)$ is as under.

x	$\to 0$	0.1	1	2	4	6	8	10	$\to +\infty$
$y = f(x) = \log x$	$\to -\infty$	–1	0	0.30	0.60	0.77	0.90	1	$\to +\infty$



Plotting the points (x, y) and joining them with a smooth curve we get the graph as shown in the figure.

Note:

(i) If we replace (x, y) with $(x, -y)$ and there is no change in the equation then the graph is symmetric with respect to x-axis.

(ii) If we replace (x, y) with $(-x, y)$ and there is no change in the equation then the graph is symmetric with respect to y-axis.

(iii) If we replace (x, y) with $(-x, -y)$ and there is no change in the equation then the graph is symmetric with respect to origin.

EXERCISE 1.5

Q.1 Draw the graphs of the following equations

(i) $x^2+y^2=9$ **(ii)** $\dfrac{x^2}{16}+\dfrac{y^2}{4}=1$ **(iii)** $y=e^{2x}$ **(iv)** $y=3^x$

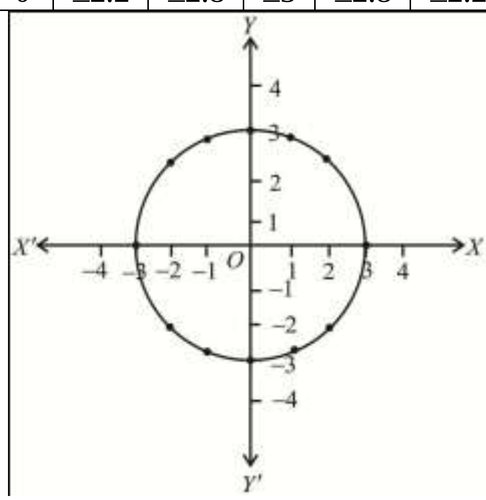
(i) $x^2+y^2=9$

Solution:

$$x^2+y^2=9$$
$$y^2=9-x^2$$
$$y=\pm\sqrt{9-x^2}$$

Here domain $=[-3,3]$

X	–3	–2	–1	0	1	2	3
Y	0	±2.2	±2.8	±3	±2.8	±2.2	0



(ii) $\dfrac{x^2}{16}+\dfrac{y^2}{4}=1$

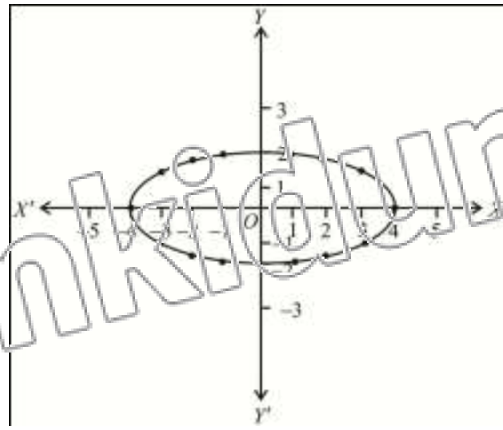
Solution:

Given $\dfrac{x^2}{16}+\dfrac{y^2}{4}=1$

$$\frac{y^2}{4}=1-\frac{x^2}{16}$$
$$\frac{y^2}{4}=\frac{16-x^2}{16}$$
$$y^2=4\left(\frac{16-x^2}{16}\right)=\frac{16-x^2}{4}$$
$$y=\pm\frac{\sqrt{16-x^2}}{2}$$

Here domain $=[-4,4]$

x	–4	–3	–2	–1	0	1	2	3	4
y	0	±1.3	±1.7	±1.9	±2	±1.9	±1.7	±1.3	0

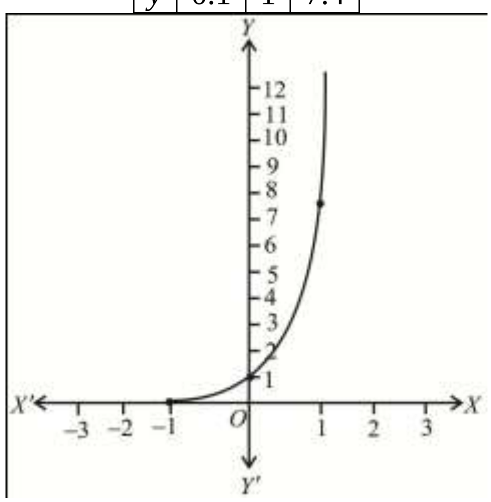


(iii) $y = e^{2x}$

Solution:

Given $y = e^{2x}$

x	–1	0	1
y	0.1	1	7.4

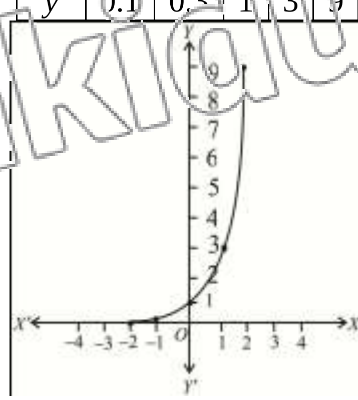


(iv) $y = 3^x$

Solution:

$y = 3^x$

x	–2	–1	0	1	2
y	0.1	0.3	1	3	9



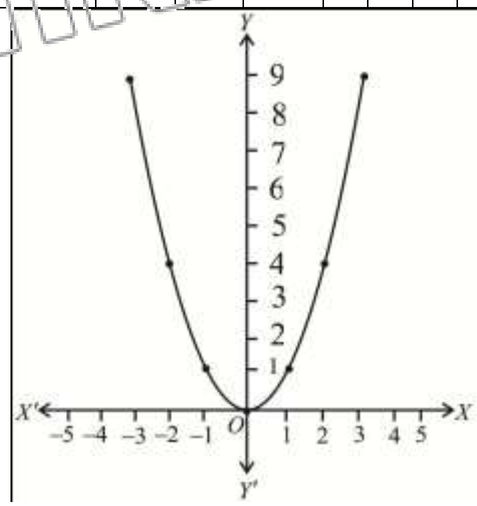
Q.2 Graph the curves that has the parametric equations given below

(i) $x = t, y = t^2, -3 \le t \le 3$ **where "t" is a parameter**

Solution:

$x = t, y = t^2, -3 \le t \le 3$

t	–3	–2	–1	0	1	2	3
$x = t$	–3	–2	–1	0	1	2	3
$y = t^2$	9	4	1	0	1	4	9

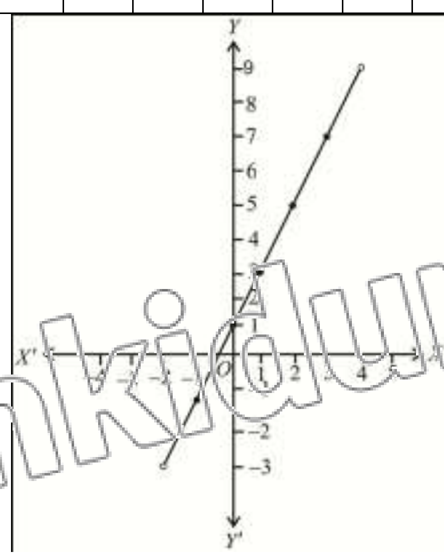


(i) $x = t - 1, y = 2t - 1, -1 < t < 5$ **where "t' is a parameter**

Solution:

$x = t - 1, y = 2t - 1, -1 < t < 5$

t	–1	0	1	2	3	4	5
$x = t - 1$	–2	–1	0	1	2	3	4
$y = 2t - 1$	–3	–1	1	3	5	7	9



(ii) $x = \sec\theta, y = \tan\theta$ **where "θ" is a parameter**

Solution:

$x = \sec\theta, y = \tan\theta$

$\Rightarrow x^2 - y^2 = \sec^2\theta - \tan^2\theta$

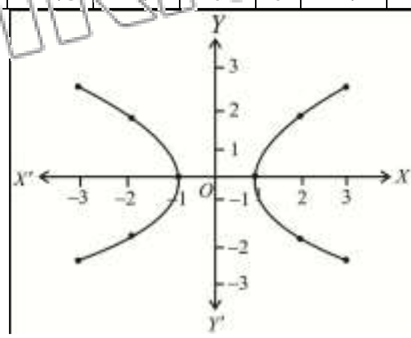
$$x^2 - y^2 = 1$$

$$x^2 - 1 = y^2$$

$$y = \pm\sqrt{x^2 - 1}$$

Here domain $= (-\infty, -1] \cup [1, \infty)$

x	–3	–2	–1	1	2	3
y	±2.8	±1.7	0	0	±1.7	±2.8



Q.3 Draw the graphs of the functions defined below and find whether they are continuous.

(i) $y = \begin{cases} x - 1 & \text{if} \quad x < 3 \\ 2x + 1 & \text{if} \quad x \geq 3 \end{cases}$

Solution:

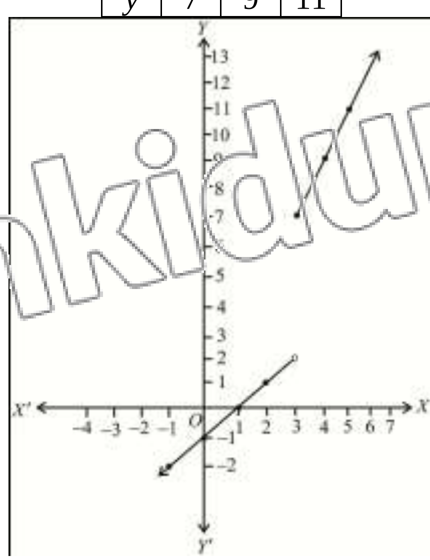
$$y = \begin{cases} x - 1 & \text{if} \quad x < 3 \\ 2x + 1 & \text{if} \quad x \geq 3 \end{cases}$$

Table for $y = x - 1,\ x < 3$

x	–1	0	1	2	3
y	–2	–1	0	1	2

Table for $y = 2x + 1,\ x \geq 3$

x	3	4	5
y	7	9	11



(ii) $y = \dfrac{x^2 - 4}{x - 2}, x \neq 2$

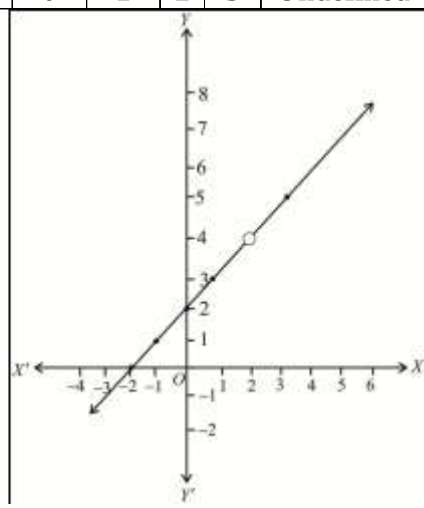
Solution:

$$y = \frac{x^2 - 4}{x - 2}, x \neq 2$$

$$y = \frac{(x+2)(x-2)}{(x-2)}$$

$$y = x + 2, x \neq 2$$

x	–2	–1	0	1	2	3
y	0	1	2	3	Undefined	5

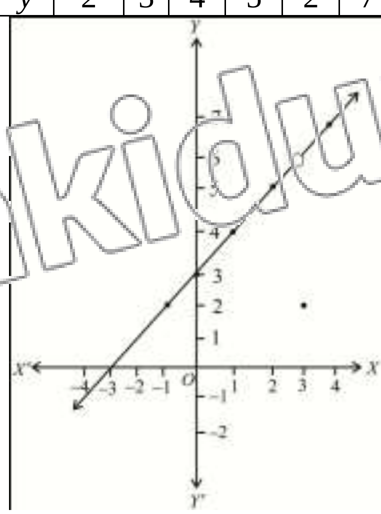


(iii) $y = \begin{cases} x+3 & \text{if} \quad x \neq 3 \\ 2 & \text{if} \quad x = 3 \end{cases}$

Solution:

$$y = \begin{cases} x+3 & \text{if} \quad x \neq 3 \\ 2 & \text{if} \quad x = 3 \end{cases}$$

x	–1	0	1	2	3	4
y	2	3	4	5	2	7



(iv) $y=\dfrac{x^2-16}{x-4}, x\neq 4$

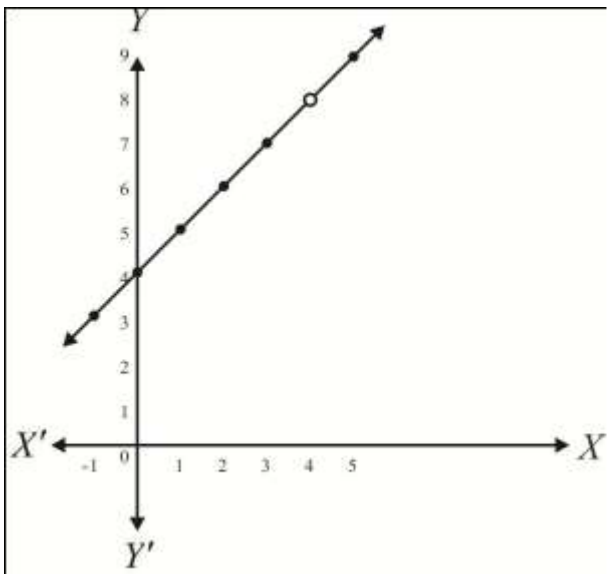
Solution:

$$y=\frac{x^2-16}{x-4}, x\neq 4$$

$$y=\frac{(x+4)(x-4)}{(x-4)}, x\neq 4$$

$$y=x+4, x\neq 4$$

x	–1	0	1	2	3	4	5
y	3	4	5	6	7	Undefined	9



Q.4 Find the graphical solution of the following equations:

(i) $x=\sin 2x$

Solution:

Let $y=x=\sin 2x$

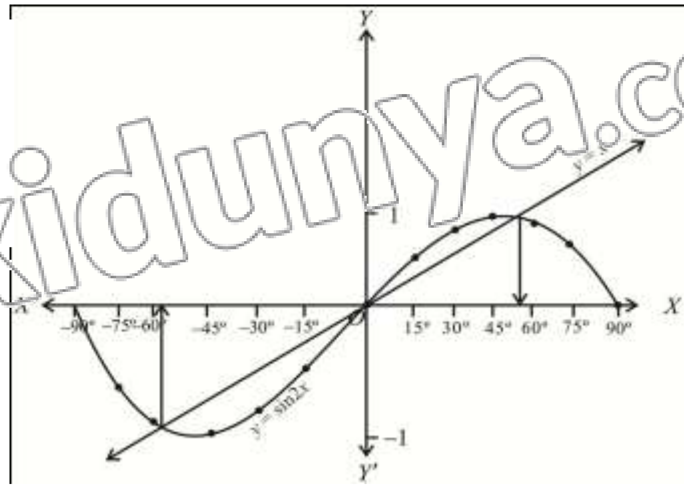
So $y=x$

x	0°	90°
$y=x$ (radian)	0	1.6

Also $y=\sin 2x$

X	–90°	–75°	–60°	–45°	–30°	–15°	0°	15°	30°	45°	60°	75°	90°
$y=\sin 2x$	0	–0.5	–0.6	–1	–0.9	–0.5	0	0.5	0.9	1	0.9	0.5	0

From two graphs, solutions are
$x = -55°, 0°, 55°$
Solution set $= \{-55°, 0°, 55°\}$



(ii) $\dfrac{x}{2} = \cos x$

Solution:

Let $y = \dfrac{x}{2} = \cos x$

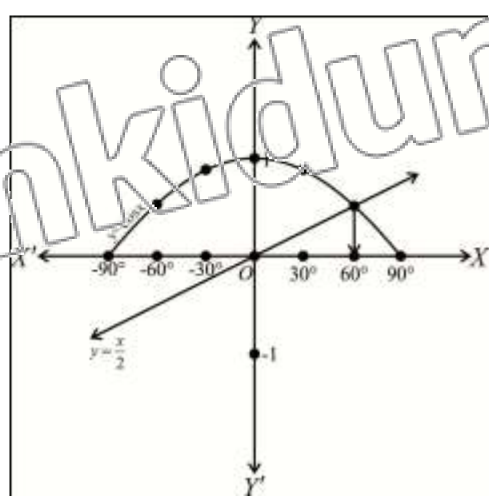
So $y = \dfrac{x}{2}$

x	0°	60°
$y = \dfrac{x}{2}$ (radian)	0	0.5

Also $y = \cos x$

x	–90°	–60°	–30°	0°	30°	60°	90°
$y = \cos x$	0	0.5	0.9	1	0.9	0.5	0

From two graphs, solution is
$x = 60°$
Solution set = $\{60°\}$



(iii) $2x = \tan x$

Solution:

Lets $y = 2x = \tan x$

So $y = 2x$

x	0°	60°
$y = 2x$ (radian)	0	2.1

Also $y = \tan x$

x	–90°	–60°	–30°	0°	30°	60°	90°
$y = \tan x$	Undefined	–1.7	–0.6	0	0.6	1.7	Undefined

From two graphs, solution is $x = 0°$

Solution set = $\{0°\}$