UNIT 2

DIFFERENTIATION

- Differentiation deals with the rate of change of a dependent variable with respect to one or more independent variables.
- In the function of the form $y = f(x)$ where $x \in Dom\, f$, x is called independent variable while y is called the dependent variable.

<u>Derivative of a Function</u>:

Let f be a real valued function continuous in the interval $(x, x_1) \subseteq D_f$ then

$\dfrac{f(x_1) - f(x)}{x_1 - x}$ is called average rate of change of the function.

If x_1 approaches to x then $\lim\limits_{x_1 \to x} \dfrac{f(x_1) - f(x)}{x_1 - x}$ is called the instantaneous rate of change of function with respect to x and is written as $f'(x)$ (read as "f prime of x")

Finding $f'(x)$ from definition of derivative:

$$\text{If } y = f(x) \qquad \text{(i)}$$

Step 1 $$y + \delta y = f(x + \delta x) \qquad \text{(ii)}$$

Subtracting equation (i) from equation (ii)

Step 2 $$\delta y = f(x + \delta x) - f(x)$$

Step 3 $$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

Step 4 $$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

and

$\lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$ is denoted by $\dfrac{dy}{dx}$, so

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

Note:

The symbol $\frac{dy}{dx}$ is used for the derivative of y with respect to x and is not a quotient of dy and dx.

Name of Mathematician	Leibniz	Newton	Lagrange	Cauchy
Notation used for derivative	$\frac{dy}{dx}$ or $\frac{df}{dx}$	$f^{\bullet}(x)$	$f'(x)$	$Df(x)$

The Power Rule:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Case I: when $n \in Z^+$

Proof:

Let $y = x^n$

$$y + \delta y = (x + \delta x)^n$$

$$\delta y = (x + \delta x)^n - y$$

$$\delta y = (x + \delta x)^n - x^n$$

Using the binomial theorem we have

$$\delta y = \left[x^n + nx^{n-1}\delta x + \frac{n(n-1)}{2!}x^{n-2}(\delta x)^2 + \ldots + (\delta x)^n \right] - x^n$$

$$\delta x = \cancel{x^n} + nx^{n-1}\delta x + \frac{n(n-1)}{2!}x^{n-2}(\delta x)^2 + \ldots + (\delta x)^n - \cancel{x^n}$$

$$\delta y = \delta x\left[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}\delta x + \ldots + (\delta x)^{n-1} \right]$$

$$\frac{\delta y}{\delta x} = \frac{\cancel{\delta x}}{\cancel{\delta x}}\left[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}\delta x + \ldots + (\delta x)^{n-1} \right]$$

Taking the limit as $\delta x \to 0$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\left[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}\delta x + \ldots + (\delta x)^{n-1} \right]$$

$$\frac{dy}{dx} = nx^{n-1}$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Case II: When $n \in Z^-$

Let

$n = -m$

$y = x^{-m}$

$y + \delta y = (x + \delta x)^{-m}$

$\delta y = (x + \delta x)^{-m} - y$

$= (x + \delta x)^{-m} - x^{-m}$

$$= \left[x\left(1 + \frac{\delta x}{x}\right)\right]^{-m} - x^{-m}$$

$$= x^{-m}\left(1 + \frac{\delta x}{x}\right)^{-m} - x^{-m}$$

$$= x^{-m}\left[\left(1 + \frac{\delta x}{x}\right)^{-m} - 1\right]$$

Using binomial series

$$\delta y = x^{-m}\left[1 - m\frac{\delta x}{x} + \frac{-m(-m-1)}{2!}\frac{(\delta x)^2}{x^2} + \ldots - 1\right]$$

$$= x^{-m}\left[\frac{-m\delta x}{x} + \frac{-m(-m-1)}{2!}\frac{(\delta x)^2}{x^2} + \ldots\right]$$

$$\delta y = \delta x\, x^{-m}\left[\frac{-m}{x} + \frac{-m(-m-1)}{2!}\frac{\delta x}{x^2} + \ldots\right]$$

$$\frac{\delta y}{\delta x} = \frac{\delta x}{\delta x}x^{-m}\left[\frac{-m}{x} + \frac{-m(-m-1)}{2!}\frac{\delta x}{x^2} + \ldots\right]$$

Taking the limit as $\delta x \to 0$

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x \to 0} x^{-m}\left[\frac{-m}{x} + \frac{-m(-m-1)}{2!}\frac{\delta x}{x^2} + \ldots\right]$$

$$\frac{dy}{dx} = x^{-m}\left[\frac{-m}{x} + 0 + \ldots\right]$$

$$\frac{d}{dx}\left(x^{-m}\right) = -mx^{-m-1}$$

Above rule also holds for $n \in Q - Z$

EXERCISE 2.1

Q.1 Find by definition, the derivatives w.r.t 'x' of the following functions defined as:

(i) $2x^2+1$

Solution:

Let $y=2x^2+1$

Taking increments on both sides,

$$y+\delta y=2(x+\delta x)^2+1$$

$$\delta y=2\left(x^2+\delta x^2+2x\delta x\right)+1-y$$

$$\delta y=\cancel{2x^2}+2\delta x^2+4x\delta x\cancel{+1}\cancel{-2x^2}\cancel{-1}$$

$$\delta y=2\delta x^2+4x\delta x$$

$$\delta y=\delta x\left(2\delta x+4x\right)$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=2\delta x+4x$$

Taking limit when $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left(4x+2\delta x\right)$$

$$\boxed{\frac{dy}{dx}=4x}$$

(ii) $2-\sqrt{x}$

Solution:

Let $y=2-\sqrt{x}$

Taking increments on both sides,

$$y+\delta y=2-\sqrt{x+\delta x}$$

$$\delta y=2-\sqrt{x+\delta x}-y$$

$$\delta y=\cancel{2}-\sqrt{x+\delta x}-\cancel{2}+\sqrt{x}$$

$$\delta y=\sqrt{x}-\sqrt{x+\delta x}$$

$$\delta y=x^{\frac{1}{2}}-(x+\delta x)^{\frac{1}{2}}$$

$$\delta y=x^{\frac{1}{2}}-x^{\frac{1}{2}}\left(1+\frac{\delta x}{x}\right)^{\frac{1}{2}}$$

$$\delta y=x^{\frac{1}{2}}\left[1-\left(1+\frac{\delta x}{x}\right)^{\frac{1}{2}}\right]$$

$$\delta y=x^{\frac{1}{2}}\left[1-\left(1+\frac{1}{2}\frac{\delta x}{x}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{|2}\left(\frac{\delta x}{x}\right)^2+\ldots\right)\right]$$

$$\delta y=x^{\frac{1}{2}}\left[\cancel{1}-\cancel{1}-\frac{1}{2}\frac{\delta x}{x}+\frac{1}{8}\frac{\delta x^2}{x^2}+\ldots\right]$$

$$\delta y=-x^{\frac{1}{2}}\delta x\left[\frac{1}{2x}-\frac{\delta x}{8x^2}+\ldots\right]$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=-x^{\frac{1}{2}}\left[\frac{1}{2x}-\frac{\delta x}{8x^2}+\ldots\right]$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left[-x^{\frac{1}{2}}\left(\frac{1}{2x}-\frac{\delta x}{8x^2}+\ldots\right)\right]$$

$$\boxed{\frac{dy}{dx}=-\frac{1}{2\sqrt{x}}}$$

(iii) $\dfrac{1}{\sqrt{x}}$

Solution:

Let $y=\dfrac{1}{\sqrt{x}}$

Taking increments on both sides,

$$y+\delta y=\frac{1}{\sqrt{x+\delta x}}$$

$$\delta y=\frac{1}{\sqrt{x+\delta x}}-y$$

$$\delta y=\frac{1}{\sqrt{x+\delta x}}-\frac{1}{\sqrt{x}}$$

$$\delta y=(x+\delta x)^{\frac{-1}{2}}-x^{\frac{-1}{2}}$$

$$\delta y=x^{\frac{-1}{2}}\left(1+\frac{\delta x}{x}\right)^{\frac{-1}{2}}-x^{\frac{-1}{2}}$$

$$= x^{\frac{-1}{2}}\left[\left(1+\frac{\delta x}{x}\right)^{\frac{-1}{2}} - 1\right]$$

$$= x^{\frac{-1}{2}}\left[1 + \left(\frac{-1}{2}\right)\left(\frac{\delta x}{x}\right) + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{\lfloor 2}\frac{\delta x^2}{x^2} + \dots - 1\right]$$

$$= \delta x . x^{\frac{-1}{2}}\left[\frac{-1}{2x} + \frac{3\delta x}{8x^2} + \dots\right]$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x} = -x^{\frac{-1}{2}}\left[\frac{1}{2x} - \frac{3}{8}\frac{\delta x}{x^2} + \dots\right]$$

Taking limit as $\delta x \to 0$ both sides.

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left[-x^{\frac{-1}{2}}\left(\frac{1}{2x} - \frac{3}{8}\frac{\delta x}{x^2} + \dots\right)\right]$$

$$\frac{dy}{dx} = -x^{\frac{-1}{2}} . \frac{1}{2x}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{2x^{\frac{3}{2}}}}$$

(iv) $\frac{1}{x^3}$

Solution:

Let $y = \frac{1}{x^3}$

Taking increments on both sides,

$$y + \delta y = (x + \delta x)^{-3}$$

$$\delta y = (x + \delta x)^{-3} - y$$

$$\delta y = (x + \delta x)^{-3} - x^{-3}$$

$$\delta y = x^{-3}\left(1 + \frac{\delta x}{x}\right)^{-3} - x^{-3}$$

$$\delta y = x^{-3}\left[\left(1 + \frac{\delta x}{x}\right)^{-3} - 1\right]$$

$$\delta y = x^{-3}\left[1 + (-3)\left(\frac{\delta x}{x}\right) + \frac{(-3)(-4)}{2}\frac{\delta x^2}{x^2} + \dots - 1\right]$$

$$\delta y = -x^{-3}\delta x\left[\frac{3}{x} - \frac{6\delta x}{x^2} + \dots\right]$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x} = -x^{-3}\left[\frac{3}{x} - \frac{6\delta x}{x^2} + \dots\right]$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left[-x^{-3}\left(\frac{3}{x} - \frac{6\delta x}{x^2} + \dots\right)\right]$$

$$\frac{\delta y}{\delta x} = -x^{-3} . \frac{3}{x}$$

$$\boxed{\frac{dy}{dx} = \frac{-3}{x^4}}$$

(v) $\frac{1}{x-a}$

Solution:

Let $y = \frac{1}{x-a}$

Taking increments on both sides,

$$y + \delta y = (x + \delta x - a)^{-1}$$

$$\delta y = (x + \delta x - a)^{-1} - (x-a)^{-1}$$

$$\delta y = (x - a + \delta x)^{-1} - (x-a)^{-1}$$

$$\delta y = (x-a)^{-1}\left(1 + \frac{\delta x}{x-a}\right)^{-1} - (x-a)^{-1}$$

$$= (x-a)^{-1}\left[\left(1 + \frac{\delta x}{x-a}\right)^{-1} - 1\right]$$

$$\delta y = (x-a)^{-1}\left[1 + (-1)\left(\frac{\delta x}{x-a}\right) + \frac{(-1)(-2)}{\lfloor 2}\frac{\delta x^2}{(x-a)^2} + \dots - 1\right]$$

$$\delta y = -(x-a)^{-1}\left[\frac{\delta x}{x-a} - \frac{\delta x^2}{(x-a)^2} + \dots\right]$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x} = -(x-a)^{-1}\left[\frac{1}{x-a} - \frac{\delta x}{(x-a)^2} + \dots\right]$$

Taking $\lim_{\delta x\to 0}$ both sides,

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left[-\frac{1}{x-a}\left(\frac{1}{x-a}-\frac{\delta x}{(x-a)^2}+\ldots\right)\right]$$

$$\frac{dy}{dx}=\frac{-1}{x-a}\left(\frac{1}{x-a}\right)$$

$$\boxed{\frac{dy}{dx}=\frac{-1}{(x-a)^2}}$$

(vi) $x(x-3)$

Solution:

Let $y=x(x-3)$

$$\Rightarrow y=x^2-3x$$

Taking increments both sides

$$y+\delta y=(x+\delta x)^2-3(x+\delta x)$$

$$\delta y=x^2+\delta x^2+2x\delta x-3x-3\delta x-x^2+3x$$

$$\delta y=\delta x^2+2x\delta x-3\delta x$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=\delta x+2x-3$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}[\delta x+2x-3]$$

$$\boxed{\frac{dy}{dx}=2x-3}$$

(vii) $\frac{2}{x^4}$

Solution:

Let $y=\frac{2}{x^4}$

Taking increments on both sides,

$$y+\delta y=2(x+\delta x)^{-4}$$

$$\delta y=2x^{-4}\left(1+\frac{\delta x}{x}\right)^{-4}-2x^{-4}$$

$$\delta y=2x^{-4}\left[1+(-4)\left(\frac{\delta x}{x}\right)+\frac{(-4)(-5)}{\lfloor 2}\frac{\delta x^2}{x^2}+\ldots-1\right]$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=2x^{-4}\left[1-\frac{4}{x}+\frac{10\delta x}{x^2}+\ldots-1\right]$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left[2x^{-4}\left(-\frac{4}{x}+10\frac{\delta x}{x^2}+\ldots\right)\right]$$

$$\boxed{\frac{dy}{dx}=\frac{-8}{x^5}}$$

(viii) $(x+4)^{\frac{1}{3}}$

Solution:

Let $y=(x+4)^{\frac{1}{3}}$

Taking increments on both sides,

$$y+\delta y=(x+\delta x+4)^{\frac{1}{3}}$$

$$\delta y=(x+4+\delta x)^{\frac{1}{3}}-(x+4)^{\frac{1}{3}}$$

$$\delta y=(x+4)^{\frac{1}{3}}\left[\left(1+\frac{\delta x}{x+4}\right)^{\frac{1}{3}}-1\right]$$

$$\delta y=(x+4)^{\frac{1}{3}}\left[1+\frac{1}{3}\frac{\delta x}{x+4}+\frac{\frac{1}{3}\left(\frac{-2}{3}\right)}{\lfloor 2}\frac{\delta x^2}{(x+4)^2}+\ldots-1\right]$$

$$\delta y=\delta x(x+4)^{\frac{1}{3}}\left[\frac{1}{3(x+4)}-\frac{1}{9}\cdot\frac{\delta x}{(x+4)^2}+\ldots\right]$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=(x+4)^{\frac{1}{3}}\left[\frac{1}{3(x+4)}-\frac{\delta x}{9(x+4)^2}+\ldots\right]$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}(x+4)^{\frac{1}{3}}\left[\frac{1}{3(x+4)}-\frac{1}{9}\cdot\frac{\delta x}{(x+4)^2}+\ldots\right]$$

$$\frac{dy}{dx}=\frac{(x+4)^{\frac{1}{3}}}{3(x+4)}$$

$$\boxed{\frac{dy}{dx}=\frac{1}{3(x+4)^{\frac{2}{3}}}}$$

(ix) $x^{\frac{3}{2}}$

Solution:

Let $y=x^{\frac{3}{2}}$

Taking increments on both sides,

$$y+\delta y=(x+\delta x)^{\frac{3}{2}}$$

$$\delta y=x^{\frac{3}{2}}\left(1+\frac{\delta x}{x}\right)^{\frac{3}{2}}-y$$

$$\delta y=x^{\frac{3}{2}}\left[\left(1+\frac{\delta x}{x}\right)^{\frac{3}{2}}-1\right]$$

$$\delta y=x^{\frac{3}{2}}\left[\cancel{1}+\frac{3}{2}.\frac{\delta x}{x}+\frac{\frac{3}{2}\left(\frac{1}{2}\right)}{2}\frac{\delta x^2}{x^2}+\ldots\cancel{-1}\right]$$

$$\delta y=\delta x.x^{\frac{3}{2}}\left[\frac{3}{2x}+\frac{3}{8}\frac{\delta x}{x^2}+\ldots\right]$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=x^{\frac{3}{2}}\left[\frac{3}{2x}+\frac{3\delta x}{8x^2}+\ldots\right]$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\frac{3}{2}.\frac{x^{\frac{3}{2}}}{x}$$

$$\boxed{\frac{dy}{dx}=\frac{3}{2}x^{\frac{1}{2}}}$$

(x) $x^{\frac{5}{2}}$

Solution:

Let $y=x^{\frac{5}{2}}$

Taking increments on both sides,

$$y+\delta y=(x+\delta x)^{\frac{5}{2}}$$

$$\delta y=x^{\frac{5}{2}}\left(1+\frac{\delta x}{x}\right)^{\frac{5}{2}}-x^{\frac{5}{2}}$$

$$\delta y=x^{\frac{5}{2}}\left[\left(1+\frac{\delta x}{x}\right)^{\frac{5}{2}}-1\right]$$

$$\delta y=x^{\frac{5}{2}}\left[\cancel{1}+\frac{5}{2}\left(\frac{\delta x}{x}\right)+\frac{\frac{5}{2}\left(\frac{3}{2}\right)}{2}\frac{\delta x^2}{x^2}+\ldots\cancel{-1}\right]$$

$$\delta y=\delta x.x^{\frac{5}{2}}\left[\frac{5}{2x}+\frac{15}{8}\frac{\delta x}{x^2}+\ldots\right]$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=x^{\frac{5}{2}}\left[\frac{5}{2x}+\frac{15}{8}\frac{\delta x}{x^2}+\ldots\right]$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}x^{\frac{5}{2}}\left[\frac{5}{2x}+\frac{15}{8}\frac{\delta x}{x^2}+\ldots\right]$$

$$\frac{dy}{dx}=x^{\frac{5}{2}}.\frac{5}{2x}$$

$$\boxed{\frac{dy}{dx}=\frac{5}{2}x^{\frac{3}{2}}}$$

(xi) $x^m,\ m\in$ □

Solution:

Let $y=x^m$

Taking increments on both sides,

$$y+\delta y=(x+\delta x)^m$$

$$\delta y=(x+\delta x)^m-x^m$$

$$\delta y=x^m\left(1+\frac{\delta x}{x}\right)^m-x^m$$

$$\delta y = x^m\left[\left(1+\frac{\delta x}{x}\right)^m - 1\right]$$

$$\delta y = x^m\left[1 + m\frac{\delta x}{x} + \frac{m(m-1)}{2}\frac{\delta x^2}{x^2} + \ldots - 1\right]$$

$$\delta y = x^m \delta x\left[\frac{m}{x} + \frac{m(m-1)}{\lfloor 2}\frac{\delta x}{x^2} + \ldots\right]$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x} = x^m\left[\frac{m}{x} + \frac{m(m-1)}{\lfloor 2}\frac{\delta x}{x^2} + \ldots\right]$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0} x^m\left[\frac{m}{x} + \frac{m(m-1)}{2}\frac{\delta x}{x^2} + \ldots\right]$$

$$\boxed{\frac{dy}{dx} = mx^{m-1}}$$

(xii) $\frac{1}{x^m}, \quad m \in \square$

Solution:

Let $y = \frac{1}{x^m}$

Taking increments on both sides,

$$y + \delta y = (x+\delta x)^{-m}$$

$$\delta y = x^{-m}\left(1+\frac{\delta x}{x}\right)^{-m}$$

$$\delta y = x^{-m}\left[\left(1+\frac{\delta x}{x}\right)^{-m} - 1\right]$$

$$\delta y = x^{-m}\left[1 + (-m)\left(\frac{\delta x}{x}\right) + \frac{(-m)(-m-1)}{2}\frac{\delta x^2}{x^2} + \ldots - 1\right]$$

$$\delta y = \delta x . x^{-m}\left[\frac{-m}{x} + \frac{-m(-m-1)}{2}\frac{\delta x}{x^2} + \ldots\right]$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x} = -x^{-m}\left[\frac{m}{x} + \frac{m(m+1)}{2}\frac{\delta x}{x^2} + \ldots\right]$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left[-x^{-m}\left(\frac{m}{x} + \frac{m(m+1)}{2}\frac{\delta x}{x^2} + \ldots\right)\right]$$

$$\boxed{\frac{dy}{dx} = \frac{-m}{x^{m+1}}}$$

(xiii) x^{40}

Solution:

Let $y = x^{40}$

Taking increments on both sides,

$$y + \delta y = (x+\delta x)^{40}$$

$$\delta y = (x+\delta x)^{40} - x^{40}$$

$$\delta y = x^{40}\left[\left(1+\frac{\delta x}{x}\right)^{40} - 1\right]$$

$$\delta y = x^{40}\left[1 + 40.\frac{\delta x}{x} + \frac{40(39)}{2}\frac{\delta x^2}{x^2} + \ldots - 1\right]$$

$$\delta y = \delta x . x^{40}\left[\frac{40}{x} + 780\frac{\delta x}{x^2} + \ldots\right]$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x} = x^{40}\left[\frac{40}{x} + 780\frac{\delta x}{x^2} + \ldots\right]$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left[x^{40}.\left(\frac{40}{x} + 780\frac{\delta x}{x^2} + \ldots\right)\right]$$

$$\boxed{\frac{dy}{dx} = 40x^{39}}$$

(xiv) x^{-100}

Solution:

Let $y = x^{-100}$

$$y + \delta y = (x+\delta x)^{-100}$$

$$\delta y = (x+\delta x)^{-100} - x^{-100}$$

$$\delta y = x^{-100}\left[\left(1+\frac{\delta x}{x}\right)^{-100} - 1\right]$$

$$\delta y = x^{-100}\left[1 + (-100)\frac{\delta x}{x} + \frac{(-100)(-101)}{2}\frac{\delta x^2}{x^2} + \ldots - 1\right]$$

$$\delta y = \delta x . x^{-100}\left[\frac{-100}{x} + 780\frac{\delta x}{x^2} + \ldots\right]$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x} = x^{-100}\left[\frac{-100}{x} + 780\frac{\delta x}{x^2} + \dots\right]$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left[x^{-100}.\left(\frac{-100}{x} - 5050\frac{\delta x}{x^2} + \dots\right)\right]$$

$$\boxed{\frac{dy}{dx} = -100x^{-101}}$$

Q.2 Find $\frac{dy}{dx}$ from first principle if

(i) $\sqrt{x+2}$

Solution:

Let $y = \sqrt{x+2}$

Taking increments on both sides,

$$y + \delta y = (x + \delta x + 2)^{\frac{1}{2}}$$

$$\delta y = (x + 2 + \delta x)^{\frac{1}{2}} - (x+2)^{\frac{1}{2}}$$

$$\delta y = (x+2)^{\frac{1}{2}}\left[1 + \frac{\delta x}{x+2}\right]^{\frac{1}{2}} - (x+2)^{\frac{1}{2}}$$

$$\delta y = (x+2)^{\frac{1}{2}}\left[\left(1 + \frac{\delta x}{x+2}\right)^{\frac{1}{2}} - 1\right]$$

$$\delta y = (x+2)^{\frac{1}{2}}\left[1 + \frac{1}{2}\frac{\delta x}{x+2} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\frac{\delta x^2}{(x+2)^2} + \dots - 1\right]$$

$$\delta y = (x+2)^{\frac{1}{2}}.\delta x\left[\frac{1}{2(x+2)} - \frac{1}{8}\frac{\delta x}{(x+2)^2} + \dots\right]$$

Dividing both sides by ‘δx’

$$\frac{\delta y}{\delta x} = (x+2)^{\frac{1}{2}}\left[\frac{1}{2(x+2)} - \frac{\delta x}{8(x+2)^2} + \dots\right]$$ Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 10}\left[(x+2)^{\frac{1}{2}}\left(\frac{1}{2(x+2)} - \frac{\delta x}{8(x+2)^2} + \dots\right)\right]$$

$$\boxed{\frac{dy}{dx} = \frac{1}{2\sqrt{x+2}}}$$

(ii) $\frac{1}{\sqrt{x+a}}$

Solution:

Let $y = \frac{1}{\sqrt{x+a}}$

Taking increments on both sides,

$$y + \delta y = (x + \delta x + a)^{\frac{-1}{2}}$$

$$\delta y = (x + a + \delta x)^{\frac{-1}{2}} - (x+a)^{\frac{-1}{2}}$$

$$\delta y = (x+a)^{\frac{-1}{2}}\left[1 + \frac{\delta x}{x+a}\right]^{\frac{-1}{2}} - (x+a)^{\frac{-1}{2}}$$

$$\delta y=(x+a)^{\frac{-1}{2}}\left[\left(1+\frac{\delta x}{x+a}\right)^{\frac{-1}{2}}-1\right]$$

$$\delta y=(x+a)^{\frac{-1}{2}}\left[1+\frac{-1}{2}\left(\frac{\delta x}{x+a}\right)+\frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{2}\frac{\delta x^2}{(x+a)^2}+\dots-1\right]$$

$$\delta y=-(x+a)^{\frac{-1}{2}}\left[\frac{\delta x}{2(x+a)}-\frac{3}{8}\frac{\delta x^2}{(x+a)^2}+\dots\right]$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=-\frac{1}{(x+a)^{\frac{1}{2}}}\left[\frac{1}{2(x+a)}-\frac{3}{8}\frac{\delta x}{(x+a)^2}+\dots\right]$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left[\frac{-1}{(x+a)^{\frac{1}{2}}}\left(\frac{1}{2(x+a)}-\frac{3}{8}\frac{\delta x}{(x+a)^2}+\dots\right)\right]$$

$$\boxed{\frac{dy}{dx}=\frac{-1}{2(x+a)^{\frac{3}{2}}}}$$

Differentiation of expressions of the types $(ax+b)^n$ and $\frac{1}{(ax+b)^n}$, $n=1,2,3,....$:

Let

$y=(ax+b)^n$, where n is a positive integer

$$y+\delta y=\left[a(x+\delta x)+b\right]^n=(ax+a\delta x+b)^n$$

$$\delta y=\left[(ax+b)+a\delta x\right]^n-(ax+b)^n$$

Using the binomial theorem

$$\delta y=(ax+b)^n+\binom{n}{1}(ax+b)^{n-1}(a\delta x)+\binom{n}{2}(ax+b)^{n-2}(a\delta x)^2+\dots+(a\delta x)^n-(ax+b)^n$$

$$\delta y=\binom{n}{1}(ax+b)^{n-1}(a\delta x)+\binom{n}{2}(ax+b)^{n-2}a^2(\delta x)^2+\dots+a^n(\delta x)^n$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=\frac{\delta x}{\delta x}\left[\binom{n}{1}(ax+b)^{n-1}a+\binom{n}{2}(ax+b)^{n-2}a^2\delta x+\dots+a^n(\delta x)^{n-1}\right]$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \left[\binom{n}{1} (ax+b)^{n-1} a + \binom{n}{2} (ax+b)^{n-2} a^2 \delta x + \ldots + a^n (\delta x)^{n-1} \right]$$

$$\frac{dy}{dx} = n(ax+b)^{n-1} a + 0 + \ldots + 0$$

$$\frac{d}{dx}(ax+b)^n = na(ax+b)^{n-1}$$