EXERCISE 2.10

Q.1 Find two positive integers whose sum is 30 and their product will be maximum.

Solution:

Let x and $30-x$ be two positive integers. Their product is $x(30-x)$

Let $f(x)=x(30-x)$

$=30x-x^2$

Differentiate w.r.t "x".

$f'(x)=30-2x$ and $f''(x)=-2$

put $f'(x)=0$

$30-2x=0 \Rightarrow x=15$

At $x=15$ $f''(x)=-2<0$

so f will be maximum

So the required two positive integers are 15 and 15.

Q.2 Divide 20 into two parts so that the sum of their squares will be minimum.

Solution:

Let the required two parts are x and $20-x$ then the sum of their squares will be $x^2+(20-x)^2$

Let $f(x)=x^2+(20-x)^2$

Differentiate w.r.t "x".

$f'(x)=2x+2(20-x)(-1)$

Put $f'(x)=0$

$2x-2(20-x)=0$

$\cancel{2}x=\cancel{2}(20-x)$

$x=10$

at $x=10$ $f''(x)=4>0$

so $f(x)$ will be minimum

so one of the integers is 10

and other one is $20-10=10$.

So the required two integers are 10 & 10.

Q.3 Find two positive integers whose sum is 12 and the product of one with the square of the other will be maximum.

Solution:

Let two integers are x and $12-x$ then the product of $12-x$ and the square of x is $x^2(12-x)$

Let $f(x)=x^2(12-x)$

$f(x)=12x^2-x^3$

Differentiate w.r.t. 'x'

$f'(x)=24x-3x^2$ and $f''(x)=24-6x$

Put $f'(x)=0$

$x(24-3x)=0$

$x=0$, $x=8$

at $x=0$, $f''(0)=24>0$

So f will be minimum so we discard this possibility

At $x=8$, $f''(8)=24-48=-24<0$

So f will be maximum

Hence the required integers are

$x=8$

$12-x=12-8=4$

Q.4 The perimeter of a triangle is $16cm$. If one side is of length $6cm$, what are lengths of the other sides for maximum area of the triangle?

Solution:

Sum of lengths of unknown sides is $16-6=10cm$

Let the lengths of unknown sides be x and $10-x$.

If A is the area of triangle, then

$s=\dfrac{6+x+10-x}{2} \Rightarrow s=8$

$A=\sqrt{(8)(8-6)(8-x)(8-10+x)}$

By Hero's formula

$A=\sqrt{(8)(2)(8-x)(-2+x)}$

$A=4\sqrt{10x-x^2-16}$

The maximum value of A depends on the function

$f(x) = 10x - x^2 - 16$

Differentiate w.r.t "x"

$f'(x) = 4\dfrac{1}{2\sqrt{10x - x^2 - 16}}(10 - 2x)$

$\dfrac{dA}{dx} = \dfrac{20 - 4x}{\sqrt{10x - x^2 - 16}}$

Put $\dfrac{dA}{dx} = 0$ gives,

$\dfrac{20 - 4x}{\sqrt{10x - x^2 - 16}} = 0 \Rightarrow x = 5$

Now

$\dfrac{d^2A}{dx^2} = 4\dfrac{\sqrt{10x - x^2 - 16}(-1) - (5 - x)\dfrac{10 - 2x}{2\sqrt{10x - x^2 - 16}}}{10x - x^2 - 16}$

at $x = 5$,

$\left.\dfrac{d^2A}{dx^2}\right|_{x=5} = \dfrac{-36}{(50 - 25 - 16)^{3/2}} = -\dfrac{4}{3} < 0$

Since $\left.\dfrac{d^2A}{dx^2}\right|_{x=5} < 0$

So A is maximum

So the lengths of unknown sides are 5 and $10 - 5 = 5$

Q.5 Find the dimensions of a rectangle of largest area having perimeter 120cm.

Solution:

Let x be the length of rectangle and y be breadth of rectangle

then perimeter $2x + 2y = 120$

or $x + y = 60$

or $y = 60 - x$

Let A be the area of rectangle

then $A = xy$

or $A = x(60 - x)$

$A = 60x - x^2$

Differentiate w.r.t. x

$\dfrac{dA}{dx} = 60 - 2x$

Put $\dfrac{dA}{dx} = 0 \quad \Rightarrow \quad 60 - 2x = 0$

$x = 30$

And $\dfrac{d^2A}{dx^2} = -2$

At $x = 30$ $\dfrac{d^2A}{dx^2} = -2 < 0$,

So A is maximum

So length of rectangle is $x = 30$ and breadth $60 - x = 30$

Q.6 Find the lengths of the sides of a variable rectangle having area $36cm^2$ when its perimeter is minimum.

Solution:

Let x and y be the lengths of sides of rectangle, then

$\text{Area} = xy = 36 \Rightarrow y = \dfrac{36}{x}$

If p denotes the perimeter of rectangle then

$p = 2x + 2y$

$p = 2\left(x + \dfrac{36}{x}\right) \qquad \left(\because y = \dfrac{36}{x}\right)$

Differentiate w.r.t "x"

$\dfrac{dp}{dx} = 2\left(1 - \dfrac{36}{x^2}\right)$

put $\dfrac{dp}{dx} = 0$ gives

$x = 6$ (neglecting negative value)

now $\dfrac{d^2P}{dx^2} = 2\left[-\dfrac{(-2)36}{x^3}\right]$

at $x = 6$, $\left.\dfrac{d^2p}{dx^2}\right|_{x=6} = \dfrac{2}{3} > 0$

Which shows that perimeter is minimum,

For $x = 6$, we get $y = \dfrac{36}{6} \Rightarrow y = 6$

So the rectangle having area $36cm^2$ will have minimum perimeter if length and breadth are $6cm$ and $6cm$.

Q.7 A box with a square base and open top is to have a volume of 4 cubic dm. Find the dimensions of the box which will require the least material.

Solution:

Let h be the height of the box and x be the side of its square base, then

v(volume) = x^2h

$\Rightarrow \quad x^2h = 4$

or $\quad h = \dfrac{4}{x^2}$

if s is the surface area, then

$s = x^2 + 4xh$

$s = x^2 + \dfrac{16}{x} \qquad \left(\because h = \dfrac{4}{x^2}\right)$

Differentiate w.r.t "x",

$\dfrac{ds}{dx} = 2x - \dfrac{16}{x^2}$

Put $\dfrac{ds}{dx} = 0$ gives,

$2x - \dfrac{16}{x^2} = 0$

$2x^3 - 16 = 0$

$x^3 - 8 = 0$

$\Rightarrow \quad x = 2$ (real value only)

Now $\quad \dfrac{d^2s}{dx^2} = 2 + \dfrac{2(16)}{x^3}$

at $x = 2$

$\left.\dfrac{d^2s}{dx^2}\right|_{x=2} = 2 + \dfrac{2(16)}{2^3} > 0$

So the required surface area is minimum at $x = 2$.

So the required dimensions are $2dm$, $2dm$ and height $1dm$.

Q.8 Find the dimensions of a rectangular garden having perimeter 80, if its area is to be maximum.

Solution:

Let x, y be the length and breadth of that rectangular garden,

Then its perimeter $P = 2x + 2y = 80$

$\Rightarrow \quad x + y = 40$

$\quad y = 40 - x$

Now if A be the area of rectangular garden then $A = xy$

or $\quad A = x(40 - x)$

Differentiate w.r.t "x",

$\dfrac{dA}{dx} = 40 - 2x$ and $\dfrac{d^2A}{dx^2} = -2$

Put $\dfrac{dA}{dx} = 0$

$40 - 2x = 0$

$\Rightarrow x = 20$

at $x = 20 \quad \dfrac{d^2A}{dx^2} = -2 < 0$

so area will be maximum

So the dimensions are $20cm$ and $20cm$.

Q.9 An open tank of square base of side x and vertical side is to be constructed to contain a given quantity of water. Find the depth in terms of x if the expense of lining the inside of the tank with lead will be least.

Solution:

Let the given quantity of water be 'q' cubic unit and 'h' be the depth of the tank, having square base of length x.

So the volume of the tank = $= x^2h$

So $q = x^2h \Rightarrow h = \dfrac{q}{x^2}$

If S be the surface area of inside of the tank, then $s = x^2 + 4xh$

or $\quad s = x^2 + 4x\left(\dfrac{q}{x^2}\right)$

$\quad s = x^2 + \dfrac{4q}{x}$

Differentiate w.r.t "x"

$\dfrac{ds}{dx} = 2x - \dfrac{4q}{x^2}$ and $\dfrac{d^2s}{dx^2} = 2 + \dfrac{8q}{x^3}$

Put $\dfrac{ds}{dx} = 0$ gives $2x - \dfrac{q}{x^2} = 0$

$\Rightarrow q = \dfrac{x^3}{2}$

$x^3 = 2q \Rightarrow x = \sqrt[3]{2q}$

at $x = \sqrt[3]{2q}$

$\dfrac{d^2s}{dx^2} = 2 + \dfrac{8q}{x^3}$

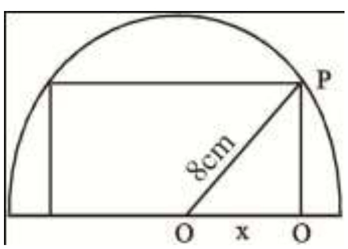
$\dfrac{d^2s}{dx^2} = 2 + \dfrac{8q}{2q} > 0 \qquad (\because x^3 = 2q)$

$\dfrac{d^2s}{dx^2} > 0$

so s is minimum (as required)

thus for least expense $h = \dfrac{q}{x^2} = \dfrac{x}{2}$

Q.10 Find the dimensions of the rectangle of maximum area which fit inside the semi-circle of radius 8cm as shown in the figure.



Solution:

Let P be the point on rectangle as shown in figure. Taking O as centre of semi-circle on the origin, the point P will be $\left(x, \sqrt{64-x^2}\right)$, if the length of rectangle is taken to be $2x$.

Let A be the area of rectangle, then

$A = 2x\sqrt{64-x^2}$

Differentiate w.r.t 'x".

$$\frac{dA}{dx} = 2x\frac{1(-\cancel{2}x)}{\cancel{2}\sqrt{64-x^2}} + \sqrt{64-x^2}.(2)$$

$$= 2\sqrt{64-x^2} - \frac{2x^2}{\sqrt{64-x^2}}$$

$$= \frac{2(64-x^2) - 2x^2}{\sqrt{64-x^2}}$$

Put $\dfrac{dA}{dx} = 0 \Rightarrow \dfrac{128-2x^2-2x^2}{\sqrt{64-x^2}} = 0$

$128 - 4x^2 = 0$

$x^2 = 32$

$x = 4\sqrt{2}$

(neglecting negative value of x)

Now $\dfrac{d^2A}{dx^2} = \dfrac{4x\left(x^2-96\right)}{\left(64-x^2\right)^{\frac{3}{2}}}$

(after simplification)

At $x = 4\sqrt{2}$

$$\left.\frac{d^2A}{dx^2}\right|_{x=4\sqrt{2}} = \frac{16\sqrt{2}(-64)}{(32)^{\frac{3}{2}}} < 0$$

So the area of rectangle will be maximum if $x = 4\sqrt{2}$.

Q.11 Find the point on the curve $y = x^2 - 1$ that is closest to the point $(3,-1)$

Solution:

Let ℓ be the distance between the point $(3,-1)$ and a point (x, y) on the curve $y = x^2 - 1$

Then $l = \sqrt{(x-3)^2 + (y+1)^2}$

$= \sqrt{(x-3)^2 + \left(x^2-1+1\right)^2}$

$l = \sqrt{(x-3)^2 + x^4}$

The minimum value of l depends on the function

$l = (x-3)^2 + x^4$

Differentiate w.r.t "x"

$$\frac{dl}{dx} = 2(x-3) + 4x^3 \text{ and } \frac{d^2l}{dx^2} = 12x^2 + 2$$

Put $\frac{dl}{dx} = 0$,

$4x^3 + 2x - 6 = 0$

$(2x^2 + 2x - 3)(x - 1) = 0$

$2x^2 + 2x - 3 = 0$ gives no real roots

So $x - 1 = 0 \Rightarrow x = 1$

At $x = 1$ $\frac{d^2l}{dx^2} = 12(1)^2 + 2 = 14$

$$\frac{d^2l}{dx^2} > 0$$

So l has minimum value at $x = 1$

Put $x = 1$ in $y = x^2 - 1$

$$y = 0$$

Hence the required point on the curve is $(1,0)$

Q.12 Find the point on the curve $y = x^2 + 1$ that is closest to the point $(18,1)$

Solution:

let l be the distance between the point $(18,1)$ and a point (x, y) on the curve $y = x^2 + 1$ then.

$$l = \sqrt{(x-18)^2 + (y-1)^2}$$

$$= \sqrt{(x-18)^2 + (y-1)^2}$$

$$(\because x^2 + 1 = y)$$

$$= \sqrt{(x-18)^2 + x^4}$$

The minimum value of l depends on the function

$$l = (x-18)^2 + x^4$$

Differentiate w.r.t "x"

$$\frac{dl}{dx} = 2(x-18) + 4x^3 \text{ and } \frac{d^2l}{dx^2} = 12x^2 + 2$$

Put $\frac{dl}{dx} = 0$

$2x^3 + x - 18 = 0$

$(x-2)(2x^2 + 4x + 9) = 0$

$2x^2 + 4x + 9 = 0$ gives complex roots

so, we must have $x - 2 = 0 \Rightarrow x = 2$

At $x = 2$ $\frac{d^2l}{dx^2} = 12(2)^2 + 2 = 50$

So l has minimum value at $x = 2$

put $x = 2$ in $y = x^2 + 1$ we get

$y = 2^2 + 1 = 5$

Hence the required point on the curve is $(2,5)$.