EXERCISE 2.2

Q.1 Find from first principles, the derivatives of the following expressions w.r.t. their respective independent variables.

(i) $(ax+b)^3$

Solution:

Let $y=(ax+b)^3$

Taking increments on both sides,

$$y+\delta y=\left[a(x+\delta x)+b\right]^3$$

$$\delta y=\left[a(x+\delta x)+b\right]^3-y$$

$$\delta y=(ax+b+a\delta x)^3-(ax+b)^3$$

$$\delta y=\left[(ax+b)^3+3(ax+b)^2.a\delta x+3(ax+b)(a\delta x)^2+(a\delta x)^3\right]-(ax+b)^3$$

$$\delta y=3a(ax+b)^2\delta x+3a^2(ax+b)\delta x^2+a^3\delta x^3$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=3a(ax+b)^2+3a^2(ax+b)\delta x+a^3\delta x^2$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left[3a(ax+b)^2+3a^2(ax+b)\delta x+a^3\delta x^2\right]$$

$$\boxed{\frac{dy}{dx}=3a(ax+b)^2}$$

(ii) $(2x+3)^5$

Solution:

Let $y=(2x+3)^5$

Taking increments on both sides,

$$y+\delta y=\left[2(x+\delta x)+3\right]^5$$

$$\delta y=(2x+2\delta x+3)^5-(2x+3)^5$$

$$\delta y=\left[(2x+3)+2\delta x\right]^5-(2x+3)^5$$

$$\delta y=\binom{5}{0}(2x+3)^5+\binom{5}{1}(2x+3)^4.(2\delta x)+\binom{5}{2}(2x+3)^3.(2\delta x)^2$$

$$+\binom{5}{3}(2x+3)^2(2\delta x)^3+\binom{5}{4}(2x+3)(2\delta x)^4+\binom{5}{5}(2\delta x)^5-(2x+3)^5$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=\binom{5}{0}(2x+3)^5+\binom{5}{1}(2x+3)^4.2+\binom{5}{2}(2x+3)^3.4\delta x$$

$$+\binom{5}{3}(2x+3)^2.8\delta x^2+\binom{5}{4}(2x+3)16\delta x^3+\binom{5}{5}\delta x^4-\cancel{(2x+3)^5}$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=(2)(5)(2x+3)^4$$

$$\boxed{\frac{dy}{dx}=10(2x+3)^4}$$

(iii) $(3t+2)^{-2}$

Solution:

Let $y=(3t+2)^{-2}$

Taking increments on both sides,

$$y+\delta y=\left[3(t+\delta t)+2\right]^{-2}$$

$$\delta y=(3t+3\delta t+2)^{-2}-(3t+2)^{-2}$$

$$\delta y=(3t+2+3\delta t)^{-2}-(3t+2)^{-2}$$

$$\delta y=(3t+2)^{-2}\left(1+\frac{3\delta t}{3t+2}\right)^{-2}-(3t+2)^{-2}$$

$$\delta y=(3t+2)^{-2}\left[\left(1+\frac{3\delta t}{3t+2}\right)^{-2}-1\right]$$

$$\delta y=(3t+2)^{-2}\left[\cancel{1}+(-2)\left(\frac{3\delta t}{3t+2}\right)+\frac{\cancel{(-2)}(-3)}{2}\left(\frac{3\delta t}{3t+2}\right)^2+\ldots\cancel{-1}\right]$$

Dividing both sides by ‘δt’

$$\frac{\delta y}{\delta t}=(3t+2)^{-2}\left[\frac{-6}{3t+2}+\frac{27\delta t}{(3t+2)^2}+\ldots\right]$$

Taking limit as $\delta t\to 0$

$$\lim_{\delta t\to 0}\frac{\delta y}{\delta t}=\lim_{\delta t\to 0}(3t+2)^{-2}\left[\frac{-6}{3t+2}+\frac{27\delta t}{(3t+2)^2}+\ldots\right]$$

$$\boxed{\frac{dy}{dt}=\frac{-6}{(3t+2)^3}}$$

(iv) $(ax+b)^{-5}$

Solution:

Let $y=(ax+b)^{-5}$

Taking increments on both sides,

$$y+\delta y=\left[a(x+\delta x)+b\right]^{-5}$$

$$y+\delta y=\left[a\left(x+\delta x\right)+b\right]^{-5}$$
$$\delta y=\left(ax+a\delta x+b\right)^{-5}-\left(ax+b\right)^{-5}$$
$$\delta y=\left(ax+b+a\delta x\right)^{-5}-\left(ax+b\right)^{-5}$$
$$\delta y=\left(ax+b\right)^{-5}\left(1+\frac{a\delta x}{ax+b}\right)^{-5}-\left(ax+b\right)^{-5}$$
$$\delta y=\left(ax+b\right)^{-5}\left[\left(1+\frac{a\delta x}{ax+b}\right)^{-5}-1\right]$$
$$\delta y=\left(ax+b\right)^{-5}\left[1+\left(-5\right)\left(\frac{a\delta x}{ax+b}\right)+\frac{\left(-5\right)\left(-6\right)}{2}\left(\frac{a\delta x}{ax+b}\right)^{2}+\ldots-1\right]$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=\left(ax+b\right)^{-5}\left[\frac{-5a}{ax+b}+\frac{15a^{2}\delta x}{\left(ax+b\right)^{2}}+\ldots\right]$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left(ax+b\right)^{-5}\left[\frac{-5a}{ax+b}+\frac{15a^{2}\delta x}{\left(ax+b\right)^{2}}+\ldots\right]$$

$$\boxed{\frac{dy}{dt}=\frac{-5a}{\left(ax+b\right)^{6}}}$$

(v) $\dfrac{1}{\left(az-b\right)^{7}}$

Solution:

Let $y=\left(az-b\right)^{-7}$

Taking increments on both sides,

$$y+\delta y=\left[a\left(z+\delta z\right)-b\right]^{-7}$$
$$\delta y=\left(a\left(z+\delta z\right)-b\right)^{-7}-y$$
$$\delta y=\left(az+a\delta z-b\right)^{-7}-\left(az-b\right)^{-7}$$
$$\delta y=\left(az-b+a\delta z\right)^{-7}-\left(az-b\right)^{-7}$$
$$\delta y=\left(az-b\right)^{-7}\left[1+\frac{a\delta z}{az-b}\right]^{-7}-\left(az-b\right)^{-7}$$
$$\delta y=\left(az-b\right)^{-7}\left[\left(1+\frac{a\delta z}{az-b}\right)^{-7}-1\right]$$

$$\delta y=(az-b)^{-7}\left[\cancel{1}+\frac{(-7)a\delta z}{az-b}+\frac{(-7)(^{4}\cancel{-8})}{\cancel{2}}\frac{a^2\delta z^2}{(az-b)^2}+\ldots\cancel{-1}\right]$$

Dividing both sides by 'δz'

$$\frac{\delta y}{\delta z}=(az-b)^{-7}\left[\cancel{1}+\frac{-7a}{az-b}+\frac{28a^2\delta z}{(az-b)^2}+\ldots\right]$$

Taking limit as $\delta z\to 0$

$$\lim_{\delta z\to 0}\frac{\delta y}{\delta z}=\lim_{\delta z\to 0}(az-b)^{-7}\left[\frac{-7a}{(az-b)}+\frac{28a^2\delta z}{(az-b)^2}+\ldots\right]$$

$$\boxed{\frac{dy}{dz}=\frac{-7a}{(az-b)^8}}$$

<u>Theorems on differentiation</u>:

(1) $\frac{d}{dx}(c)=0$ (c is constant)

(2) $\frac{d}{dx}(x^n)=nx^{n-1},$ $n\in Q$

(3) $\frac{d}{dx}[cf(x)]=c.f'(x)$

Proof:

(3) Let $y=c.f(x)$

$$y+\delta y=cf(x+\delta x)$$

$$\delta y=cf(x+\delta x)-cf(x)$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=c\left[\frac{f(x+\delta x)-f(x)}{\delta x}\right]$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=c\left[\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}\right]$$

$$\frac{d}{dx}(y)=cf'(x)$$

$$\frac{d}{dx}(cf(x))=c.f'(x)$$

(4) $\frac{d}{dx}(f(x)+g(x))=f'(x)+g'(x)$

Let

$$y=f(x)+g(x)$$

$$y+\delta y=f(x+\delta x)+g(x+\delta x)$$

$\delta y = f(x+\delta x)+g(x+\delta x)-[f(x)+g(x)]$

$= f(x+\delta x)-f(x)+g(x+\delta x)-g(x)$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=\frac{f(x+\delta x)-f(x)}{\delta x}+\frac{g(x+\delta x)-g(x)}{\delta x}$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}+\lim_{\delta x\to 0}\frac{g(x+\delta x)-g(x)}{\delta x}$$

$$\frac{dy}{dx}=f'(x)+g'(x)$$

$$\frac{d}{dx}\left[f(x)+g(x)\right]=f'(x)+g'(x)$$

The proof for $\frac{d}{dx}\left[f(x)-g(x)\right]=f'(x)-g'(x)$ is similar.

(5) The Product Rule:

$$\frac{d}{dx}\left[f(x)g(x)\right]=f'(x)g(x)+f(x)g'(x)$$

Proof:

Let $y=f(x)g(x)$

$y+\delta y=f(x+\delta x)g(x+\delta x)$

$= f(x+\delta x)g(x-\delta x)-y$

$\delta y = f(x+\delta x)g(x+\delta x)-f(x)g(x)$

Subtracting and adding $f(x)\text{g}(x+\delta x)$

$$\delta y = f(x+\delta x)g(x+\delta x)-f(x)g(x+\delta x)+f(x)g(x+\delta x)-f(x)g(x)$$

$$\delta y=\left[f(x+\delta x)-f(x)\right]g(x+\delta x)+f(x)\left[g(x+\delta x)-g(x)\right]$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=\left[\frac{f(x+\delta x)-f(x)}{\delta x}\right]g(x+\delta x)+f(x)\left[\frac{g(x+\delta x)-g(x)}{\delta x}\right]$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\left[\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}\right]\lim_{\delta x\to 0}g(x+\delta x)+f(x)\lim_{\delta x\to 0}\left[\frac{g(x+\delta x)-g(x)}{\delta x}\right]$$

$$\frac{dy}{dx}=f'(x)g(x)+f(x)g'(x)$$

(6) The Quotient Rule:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{f'(x)g(x)-f(x)g'(x)}{\left[g(x)\right]^2},g\left(x\right)\neq 0$$

Proof:

Let $y=\frac{f(x)}{g(x)}$

$$y+\delta y=\frac{f(x+\delta x)}{g(x+\delta x)}$$

$$\delta y=\frac{f(x+\delta x)}{g(x+\delta x)}-y$$

$$\delta y=\frac{f(x+\delta x)}{g(x+\delta x)}-\frac{f(x)}{g(x)}$$

$$=\frac{f(x+\delta x)g(x)-f(x)g(x+\delta x)}{g(x+\delta x)g(x)}$$

Subtracting and adding $f(x)\text{g}(x)$

$$=\frac{f(x+\delta x)g(x)-f(x)g(x)-f(x)g(x+\delta x)+f(x)g(x)}{g(x+\delta x)g(x)}$$

$$=\frac{1}{g(x+\delta x)g(x)}\left[\left\{f(x+\delta x)g(x)-f(x)g(x)\right\}-\left\{f(x)g(x+\delta x)-f\left(x\right)g(x)\right\}\right]$$

Dividing both sides by ‘ δx ’

$$\frac{\delta y}{\delta x}=\frac{1}{g(x+\delta x)g(x)}\left[g(x)\frac{f(x+\delta x)-f(x)}{\delta x}-f(x)\frac{g(x+\delta x)-g(x)}{\delta x}\right]$$

Taking limit as $\delta x\rightarrow 0$

$$\lim_{\delta x\rightarrow 0}\frac{\delta y}{\delta x}=\lim_{\delta x\rightarrow 0}\frac{1}{g(x+\delta x)g(x)}\left[g(x)\lim_{\delta x\rightarrow 0}\frac{f(x+\delta x)-f(x)}{\delta x}-f(x)\lim_{\delta x\rightarrow 0}\frac{g(x+\delta x)-g(x)}{\delta x}\right]$$

$$\frac{dy}{dx}=\frac{1}{g(x)g(x)}\left[g(x)f'(x)-f(x)g'(x)\right]$$

$$=\frac{g(x)f'(x)-f(x)g'(x)}{\left[g^2(x)\right]^2}$$

(7) The reciprocal rule:

If g is differentiable at x and $g\left(x\right)\neq 0$ then $\frac{1}{g\left(x\right)}$ is differentiable and

$$\frac{d}{dx}\left[\frac{1}{g(x)}\right]=-\frac{g'(x)}{\left[g(x)\right]^2}$$