EXERCISE 2.3

Differentiate w.r.t "*x*"

Q.1 $x^4+2x^3+x^2$

Solution:

Let $y=x^4+2x^3+x^2$

Differentiating both sides w.r.t "*x*",

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^4+2x^3+x^2\right)$$

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^4\right)+2\frac{d}{dx}\left(x^3\right)+\frac{d}{dx}\left(x^2\right)$$

$$\frac{dy}{dx}=4x^3+2\left(3x^2\right)+2x$$

$$\boxed{\frac{dy}{dx}=4x^3+6x^2+2x}$$

Q.2 $x^{-3}+2x^{-\frac{3}{2}}+3$

Solution:

Let $y=x^{-3}+2x^{-\frac{3}{2}}+3$

Differentiating both sides w.r.t "*x*"

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^{-3}+2x^{-\frac{3}{2}}+3\right)$$

$$\frac{dy}{dx}=-3x^{-4}+\not{2}.\left(\frac{-3}{\not{2}}\right)x^{\frac{-5}{2}}+0$$

$$\boxed{\frac{dy}{dx}=\frac{-3}{x^4}-\frac{3}{x^{\frac{5}{2}}}}$$

Q.3 $\frac{a+x}{a-x}$

Solution:

Let $y=\frac{a+x}{a-x}$

Differentiating both sides w.r.t "*x*"

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{a+x}{a-x}\right)$$

$$\frac{dy}{dx}=\frac{(a-x)(1)-(a+x)(-1)}{(a-x)^2}$$

$$=\frac{a-\not{x}+a+\not{x}}{(a-x)^2}$$

$$\boxed{\frac{dy}{dx}=\frac{2a}{(a-x)^2}}$$

Q.4 $\frac{2x-3}{2x+1}$

Solution:

Let $y=\frac{2x-3}{2x+1}$

Differentiating both sides w.r.t "*x*"

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{2x-3}{2x+1}\right)$$

$$\frac{dy}{dx}=\frac{(2x+1)(2)-(2x-3)(2)}{(2x+1)^2}$$

$$\frac{dy}{dx}=\frac{4x+2-4x+6}{(2x+1)^2}$$

$$\boxed{\frac{dy}{dx}=\frac{8}{(2x+1)^2}}$$

Q.5 $(x-5)(3-x)$

Solution:

Let $y=(x-5)(3-x)$

$y=3x-x^2-15+5x$

$y=8x-x^2-15$

Differentiating both sides w.r.t "*x*"

$$\frac{d}{dx}(y)=\frac{d}{dx}\left(8x-x^2-15\right)$$

$$\boxed{\frac{dy}{dx}=8-2x}$$

Q.6 $\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2$

Solution:

Let $y=\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2$

$y=x+\frac{1}{x}-2$

Differentiating both sides w.r.t "*x*"

$$\frac{dy}{dx}=\frac{d}{dx}\left(x+x^{-1}-2\right)$$

$$\boxed{\frac{dy}{dx}=1-\frac{1}{x^2}}$$

Q.7 $\dfrac{\left(1+\sqrt{x}\right)\left(x-x^{\frac{3}{2}}\right)}{\sqrt{x}}$

Solution:

Let $y=\dfrac{\left(1+\sqrt{x}\right)\left(x-x^{\frac{3}{2}}\right)}{\sqrt{x}}$

$$y=\frac{\left(1+\sqrt{x}\right)(x)\left(1-\sqrt{x}\right)}{\sqrt{x}}$$

$$=\frac{x(1-x)}{\sqrt{x}}$$

$$=\sqrt{x}(1-x)$$

$$y=\sqrt{x}-x\sqrt{x}$$

$$y=x^{\frac{1}{2}}-x^{\frac{3}{2}}$$

Differentiating w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^{\frac{1}{2}}-x^{\frac{3}{2}}\right)$$

$$\frac{dy}{dx}=\frac{1}{2}x^{\frac{-1}{2}}-\frac{3}{2}x^{\frac{1}{2}}$$

$$\boxed{\frac{dy}{dx}=\frac{1}{2\sqrt{x}}-\frac{3}{2}\sqrt{x}}$$

Q.8 $\dfrac{\left(x^2+1\right)^2}{x^2-1}$

Solution:

Let $y=\dfrac{\left(x^2+1\right)^2}{x^2-1}$

Differentiating w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\left(x^2+1\right)^2}{x^2-1}\right)$$

$$\frac{dy}{dx}=\frac{\left(x^2-1\right).2\left(x^2+1\right).2x-\left(x^2+1\right)^2.(2x)}{\left(x^2-1\right)^2}$$

$$=\frac{4x\left(x^2+1\right)\left(x^2-1\right)-2x\left(x^2+1\right)^2}{\left(x^2-1\right)^2}$$

$$=\frac{2x\left(x^2+1\right)\left\{2\left(x^2-1\right)-\left(x^2+1\right)\right\}}{\left(x^2-1\right)^2}$$

$$=\frac{2x\left(x^2+1\right)\left(x^2-3\right)}{\left(x^2-1\right)^2}$$

Q.9 $\dfrac{x^2+1}{x^2-3}$

Solution:

Let $y=\dfrac{x^2+1}{x^2-3}$

Differentiating w.r.t "x",

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{x^2+1}{x^2-3}\right)$$

$$=\frac{\left(x^2-3\right)\frac{d}{dx}\left(x^2+1\right)-\left(x^2+1\right)\frac{d}{dx}\left(x^2-3\right)}{\left(x^2-3\right)^2}$$

$$\frac{dy}{dx}=\frac{\left(x^2-3\right)(2x)-\left(x^2+1\right)(2x)}{\left(x^2-3\right)^2}$$

$$\frac{dy}{dx}=\frac{\cancel{2x^3}-6x-\cancel{2x^3}-2x}{\left(x^2-3\right)^2}$$

$$\boxed{\frac{dy}{dx}=\frac{-8x}{\left(x^2-3\right)^2}}$$

Q.10 $\dfrac{\sqrt{1+x}}{\sqrt{1-x}}$

Solution:

Let $y=\sqrt{\dfrac{1+x}{1-x}}$

$$\Rightarrow y=\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{1+x}{1-x}\right)^{\frac{-1}{2}}\frac{d}{dx}\left(\frac{1+x}{1-x}\right)$$

$$=\frac{1}{2}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}\left[\frac{(1-x)\frac{d}{dx}(1+x)-(1+x)\frac{d}{dx}(1-x)}{(1-x)^2}\right]$$

$$=\frac{1}{2}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}\left[\frac{(1-x)(1)-(1+x)(-1)}{(1-x)^2}\right]$$

$$=\frac{1}{2}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}\left[\frac{1-x+1+x}{(1-x)^2}\right]$$

$$=\frac{1}{2}\frac{(1-x)^{\frac{1}{2}}(2)}{\sqrt{1+x}.(1-x)^2}$$

$$\boxed{\frac{dy}{dx}=\frac{1}{\sqrt{1+x}(1-x)^{\frac{3}{2}}}}$$

Q.11 $\frac{2x-1}{\sqrt{x^2+1}}$

Solution:

Let $y=\frac{2x-1}{\sqrt{x^2+1}}$

Differentiating w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{2x-1}{\sqrt{x^2+1}}\right)$$

$$=\frac{\sqrt{x^2+1}\frac{d}{dx}(2x-1)-(2x-1)\frac{d}{dx}\sqrt{x^2+1}}{\left(\sqrt{x^2+1}\right)^2}$$

$$=\frac{\sqrt{x^2+1}(2)-(2x-1)\left(\frac{1}{2\sqrt{x^2+1}}(2x)\right)}{x^2+1}$$

$$\frac{dy}{dx}=\frac{2\sqrt{x^2+1}-\frac{x(2x-1)}{\sqrt{x^2+1}}}{x^2+1}$$

$$\frac{dy}{dx}=\frac{\frac{2(x^2+1)-x(2x-1)}{\sqrt{x^2+1}}}{x^2+1}$$

$$\frac{dy}{dx}=\frac{2(x^2+1)-x(2x-1)}{(x^2+1)\sqrt{x^2+1}}$$

$$=\frac{2x^2+2-2x^2+x}{(x^2+1)^{\frac{3}{2}}}$$

$$\boxed{\frac{dy}{dx}=\frac{x+2}{(x^2+1)^{\frac{3}{2}}}}$$

Q.12 $\sqrt{\frac{a-x}{a+x}}$

Solution:

Let $y=\sqrt{\frac{a-x}{a+x}}$

Differentiating w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dx}\sqrt{\frac{a-x}{a+x}}$$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{a-x}{a+x}\right)^{\frac{-1}{2}}.\frac{d}{dx}\left(\frac{a-x}{a+x}\right)$$

$$=\frac{1}{2}\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}\left[\frac{(a+x)\frac{d}{dx}(a-x)-(a-x)\frac{d}{dx}(a+x)}{(a+x)^2}\right]$$

$$=\frac{1}{2}\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}\left[\frac{(a+x)(-1)-(a-x)(1)}{(a+x)^2}\right]$$

$$=\frac{1}{2}\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}\left[\frac{-a-x-a+x}{(a+x)^2}\right]$$

$$=\frac{1}{2}\frac{(a+x)^{\frac{1}{2}}(-2a)}{(a-x)^{\frac{1}{2}}(a+x)^2}$$

$$\boxed{\frac{dy}{dx}=\frac{-a}{\sqrt{a-x}.(a+x)^{\frac{3}{2}}}}$$

Q.13 $\dfrac{\sqrt{x^2+1}}{\sqrt{x^2-1}}$

Solution:

Let $y=\sqrt{\dfrac{x^2+1}{x^2-1}}$

Differentiating w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{\frac{x^2+1}{x^2-1}}\right)$$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{x^2+1}{x^2-1}\right)^{\frac{-1}{2}}.\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{x^2-1}{x^2+1}\right)^{\frac{1}{2}}\frac{(x^2-1)\frac{d}{dx}(x^2+1)-(x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$\frac{dy}{dx}=\frac{1}{2}\left(\frac{x^2-1}{x^2+1}\right)^{\frac{1}{2}}.\frac{(x^2-1)(2x)-(x^2+1)(2x)}{(x^2-1)^2}$$

$$=\frac{1}{2}\frac{(x^2-1)^{\frac{1}{2}}}{(x^2+1)^{\frac{1}{2}}}.\frac{2x^3-2x-2x^3-2x}{(x^2-1)^2}$$

$$=\frac{1}{2}\frac{-4x}{\sqrt{x^2+1}(x^2-1)^{\frac{3}{2}}}$$

$$\boxed{\frac{dy}{dx}=\frac{-2x}{\sqrt{x^2+1}(x^2-1)^{\frac{3}{2}}}}$$

Q.14 $\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$

Solution:

Let $y=\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$

Rationalizing the denominator, we get

$$y=\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}.\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$$

$$y=\frac{\left(\sqrt{1+x}-\sqrt{1-x}\right)^2}{\left(\sqrt{1+x}\right)^2-\left(\sqrt{1-x}\right)^2}$$

$$y=\frac{\left(\sqrt{1+x}\right)^2+\left(\sqrt{1-x}\right)^2-2\sqrt{1-x^2}}{1+x-1+x}$$

$$y=\frac{1+x+1-x-2\sqrt{1-x^2}}{2x}$$

$$y=\frac{1-\sqrt{1-x^2}}{x}$$

Differentiating w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{1-\sqrt{1-x^2}}{x}\right)$$

$$\frac{dy}{dx}=\frac{x\frac{d}{dx}\left(1-\sqrt{1-x^2}\right)-\left(1-\sqrt{1-x^2}\right)\frac{d}{dx}(x)}{x^2}$$

$$\frac{dy}{dx}=\frac{x\left[-\frac{1(-2x)}{2\sqrt{1-x^2}}\right]-\left(1-\sqrt{1-x^2}\right)(1)}{x^2}$$

$$=\frac{\frac{x^2}{\sqrt{1-x^2}}-1+\sqrt{1-x^2}}{x^2}$$

$$=\frac{\frac{x^2-\sqrt{1-x^2}+1-x^2}{\sqrt{1-x^2}}}{x^2}$$

$$=\frac{x^2-\sqrt{1-x^2}+1-x^2}{x^2\sqrt{1-x^2}}$$

$$\boxed{\frac{dy}{dx}=\frac{1-\sqrt{1-x^2}}{x^2\sqrt{1-x^2}}}$$

Q.15 $\dfrac{x\sqrt{a+x}}{\sqrt{a-x}}$

Solution:

Let $y = x\sqrt{\dfrac{a+x}{a-x}}$

Differentiating w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left(x\sqrt{\frac{a+x}{a-x}}\right)$$

$$\frac{dy}{dx} = x\cdot\frac{d}{dx}\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}} + \sqrt{\frac{a+x}{a-x}}\cdot\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = x\frac{1}{2}\left(\frac{a+x}{a-x}\right)^{\frac{-1}{2}}\cdot\frac{d}{dx}\left(\frac{a+x}{a-x}\right) + \sqrt{\frac{a+x}{a-x}}$$

$$\frac{dy}{dx} = x.\frac{1}{2}\sqrt{\frac{a-x}{a+x}}.\frac{(a-x)(1)-(a+x)(-1)}{(a-x)^2} + \sqrt{\frac{a+x}{a-x}}$$

$$\frac{dy}{dx} = \frac{x}{2}.\sqrt{\frac{a-x}{a+x}}.\frac{a-x+a+x}{(a-x)^2} + \sqrt{\frac{a+x}{a-x}}$$

$$\frac{dy}{dx} = \frac{x}{\cancel{2}}.\frac{\cancel{2}a}{\sqrt{a+x}.(a-x)^{\frac{3}{2}}} + \sqrt{\frac{a+x}{a-x}}$$

$$\frac{dy}{dx} = \frac{ax}{\sqrt{a+x}\sqrt{a-x}(a-x)} + \frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$\frac{dy}{dx} = \frac{ax+(a+x)(a-x)}{\sqrt{a+x}(a-x)^{\frac{3}{2}}}$$

$$\boxed{\frac{dy}{dx} = \frac{a^2-x^2+ax}{\sqrt{a+x}(a-x)^{\frac{3}{2}}}}$$

Q.16 If $y = \sqrt{x} - \dfrac{1}{\sqrt{x}}$ **show that**

$$2x\frac{dy}{dx} + y = 2\sqrt{x}$$

Solution:

$$y = \sqrt{x} - \frac{1}{\sqrt{x}}$$

Differentiating w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + \frac{1}{2x^{\frac{3}{2}}}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + \frac{1}{2x\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{x+1}{2x\sqrt{x}}$$

$$2x\frac{dy}{dx} = \frac{x+1}{\sqrt{x}}$$

$$2x\frac{dy}{dx} = \frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}$$

$$2x\frac{dy}{dx} + y = \frac{x}{\sqrt{x}} + \cancel{\frac{1}{\sqrt{x}}} + \sqrt{x} - \cancel{\frac{1}{\sqrt{x}}}$$

$$2x\frac{dy}{dx} = \sqrt{x} + \sqrt{x}$$

$$2x\frac{dy}{dx} + y = 2\sqrt{x}$$

Hence the proof.

Q.17 If $y = x^4 + 2x^2 + 2$**, prove that**

$$\frac{dy}{dx} = 4x\sqrt{y-1}$$

Solution:

$$y = x^4 + 2x^2 + 2$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = 4x^3 + 4x$$

$$\frac{dy}{dx} = 4x\left(x^2+1\right) \quad \dots\text{(i)}$$

Now $y = x^4 + 2x^2 + 2$

$$y = x^4 + 2x^2 + 1 + 1$$

$$y - 1 = x^4 + 2x^2 + 1$$

$$y - 1 = \left(x^2+1\right)^2$$

$$\sqrt{y-1} = x^2 + 1 \quad \dots\text{(ii)}$$

Now using equation (ii) and Putting the value of x^2+1 from equation (ii) into equation (i).

$$\frac{dy}{dx}=4x\sqrt{y-1}$$

Hence the proof

The Chain Rule:

Theorem:

If g is differentiable at the point x and f is differentiable at the point $g(x)$, then the composition function fog is differentiable at the point x and

$(fog)'x=f'[g(x)].g'(x)$

THE CHAIN RULE

(1) If $x=f(t)$ and $y=g(t)$ then $\frac{dx}{dt}=f'(t)$ and $\frac{dy}{dt}=g'(t)$ therefore $\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}$

(2) If $y=[g(x)]^n$ and $u=g(x)$ then $y=u^n$ and $\frac{dy}{du}=nu^{n-1}$

$$\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}=nu^{n-1}\frac{du}{dx}$$

or

$$\frac{d}{dx}[g(x)]^n=n[g(x)]^{n-1}g'(x)$$