EXERCISE 2.5

Q.1 Differentiate the following trigonometric functions from the first principle.

(i) $\sin 2x$

Solution:

Let $y=\sin 2x$

Taking increments both sides,

$$y+\delta y=\sin 2\left(x+\delta x\right)$$

$$\delta y=\sin\left(2x+2\delta x\right)-\sin 2x$$

$$\delta y=2\cos\left(\frac{2x+2\delta x+2x}{2}\right)\sin\left(\frac{\cancel{2x}+2\delta x-\cancel{2x}}{2}\right)$$

$$\delta y=2\cos\left(2x+\delta x\right).\sin\delta x$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=2\cos\left(2x+\delta x\right).\frac{\sin\delta x}{\delta x}$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left[2\cos\left(2x+\delta x\right).\frac{\sin\delta x}{\delta x}\right]$$

$$\frac{dy}{dx}=2\cos\left(2x+0\right).1 \qquad \left(\because \lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x}=1\right)$$

$$\boxed{\frac{dy}{dx}=2\cos 2x}$$

(ii) $\tan 3x$

Solution:

Let $y=\tan 3x$

Taking increments both sides

$$y+\delta y=\tan 3\left(x+\delta x\right)$$

$$\delta y=\tan\left(3x+3\delta x\right)-\tan 3x$$

$$\delta y=\frac{\sin\left(3x+3\delta x\right)}{\cos\left(3x+3\delta x\right)}-\frac{\sin 3x}{\cos 3x}$$

$$\delta y=\frac{\cos 3x\sin\left(3x+3\delta x\right)-\sin 3x\cos\left(3x+3\delta x\right)}{\cos 3x.\cos\left(3x+3\delta x\right)}$$

$$\delta y=\frac{\sin\left(\cancel{3x}+3\delta x-\cancel{3x}\right)}{\cos 3x.\cos\left(3x+3\delta x\right)}$$

$$\delta y=\frac{\sin 3\delta x}{\cos 3x\cos\left(3x+3\delta x\right)}$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=\frac{1}{\cos(3x+3\delta x)\cos 3x}.\frac{3\sin(3\delta x)}{3\delta x}$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{3}{\cos(3x+3\delta x).\cos 3x}.\lim_{\delta x\to 0}\frac{\sin 3\delta x}{3\delta x}=\frac{3}{\cos 3x\cos 3x}$$

$$\boxed{\frac{dy}{dx}=3\sec^2 3x}$$

(iii) $\sin 2x+\cos 2x$

Solution:

Let $y=\sin 2x+\cos 2x$

Taking increments both sides,

$$y+\delta y=\sin(2x+2\delta x)+\cos(2x+2\delta x)$$

$$\delta y=\sin(2x+2\delta x)+\cos(2x+2\delta x)-\sin 2x-\cos 2x$$

$$\delta y=\left[\sin(2x+2\delta x)-\sin 2x\right]+\left[\cos(2x+2\delta x)-\cos 2x\right]$$

$$\delta y=2\cos\left(\frac{2x+2\delta x+2x}{2}\right)\sin\left(\frac{2x+2\delta x-2x}{2}\right)+\left[-2\sin\left(\frac{2x+2\delta x+2x}{2}\right)\sin\left(\frac{2x+2\delta x-2x}{2}\right)\right]$$

$$\delta y=\left[2\cos(2x+\delta x)\sin\delta x\right]-\left[2\sin(2x+\delta x)\sin\delta x\right]$$

Dividing both sides by ‘ δx ’

$$\frac{\delta y}{\delta x}=2\cos(2x+\delta x).\frac{\sin\delta x}{\delta x}-2\sin(2x+\delta x).\frac{\sin\delta x}{\delta x}$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=2\lim_{\delta x\to 0}\cos(2x+\delta x)\lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x}-2\lim_{\delta x\to 0}\sin(2x+\delta x)\lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x}$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=2\cos(2x+0).1-2\sin(2x+0).1$$

$$\boxed{\frac{dy}{dx}=2\cos 2x-2\sin 2x}$$

(iv) $\cos x^2$

Solution:

Let $y=\cos x^2$

Taking increments both sides,

$$y+\delta y=\cos(x+\delta x)^2$$

$$\delta y=\cos(x+\delta x)^2-\cos x^2$$

$$\delta y=-2\sin\left(\frac{(x+\delta x)^2+x^2}{2}\right)\sin\left(\frac{(x+\delta x)^2-x^2}{2}\right)$$

$$= -2\sin\left(x^2 + \frac{\delta x^2}{2} + x\delta x\right)\sin\left(\frac{\delta x^2 + 2x\delta x}{2}\right)$$

Dividing both sides by ‘ δx ’

$$= -2\sin\left(x^2 + \frac{\delta x^2}{2} + x\delta x\right)\sin\frac{\left(\frac{\delta x(\delta x + 2x)}{2}\right)}{\left(\frac{\delta x(\delta x + 2x)}{2}\right)}\frac{(\delta x + 2x)}{2}$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = -2\lim_{\delta x \to 0}\sin\left(x^2 + \frac{\delta x^2}{2} + x\delta x\right)\lim_{\delta x \to 0}\sin\frac{\left(\frac{\delta x(\delta x + 2x)}{2}\right)}{\left(\frac{\delta x(\delta x + 2x)}{2}\right)}\lim_{\delta x \to 0}\frac{(\delta x + 2x)}{2}$$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = -2x\sin\left(x^2 + 0\right).1$$

$$\boxed{\frac{dy}{dx} = -2x\sin x^2}$$

(v) $\tan^2 x$

Solution:

Let $y = \tan^2 x$

Taking increments on both sides,

$$y + \delta y = \tan^2(x + \delta x)$$

$$\delta y = \tan^2(x + \delta x) - y$$

$$\delta y = \left[\tan(x + \delta x)\right]^2 - (\tan x)^2$$

$$\delta y = \left[\tan(x + \delta x) + \tan x\right]\left[\tan(x + \delta x) - \tan x\right]$$

$$\delta y = \left[\tan(x + \delta x) + \tan x\right]\left[\frac{\sin(x + \delta x)}{\cos(x + \delta x)} - \frac{\sin x}{\cos x}\right]$$

$$\delta y = \left[\tan(x + \delta x) + \tan x\right]\left[\frac{\sin(x + \delta x)\cos x - \cos(x + \delta x)\sin x}{\cos(x + \delta x)\cos x}\right]$$

$$\delta y = \left[\tan(x + \delta x) + \tan x\right]\left[\frac{\sin(x + \delta x - x)}{\cos(x + \delta x)\cos x}\right]$$

$$\delta y = \left[\tan(x + \delta x) + \tan x\right]\left[\frac{\sin\delta x}{\cos(x + \delta x)\cos x}\right]$$

Dividing both sides by ‘ δx ’

$$\frac{\delta y}{\delta x} = \left[\tan(x + \delta x) + \tan x\right]\left(\frac{1}{\cos(x + \delta x)\cos x}\right).\frac{\sin\delta x}{\delta x}$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left[\tan(x+\delta x)+\tan x\right]\left(\lim_{\delta x\to 0}\frac{1}{\cos(x+\delta x)\cos x}\right)\lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x}$$

$$\frac{dy}{dx}=2\tan x.\frac{1}{\cos^2 x}(1)$$

$$\boxed{\frac{dy}{dx}=2\tan x\sec^2 x}$$

(vi) $\sqrt{\tan x}$

Solution:

Let $y=\sqrt{\tan x}$

Taking increments on both sides,

$$y+\delta y=\sqrt{\tan(x+\delta x)}$$

$$\delta y=\sqrt{\tan(x+\delta x)}-y$$

$$\delta y=\sqrt{\tan(x+\delta x)}-\sqrt{\tan x}$$

$$\delta y=\left[\sqrt{\tan(x+\delta x)}-\sqrt{\tan x}\right].\left[\frac{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}}{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}}\right]$$

$$\delta y=\frac{\tan(x+\delta x)-\tan x}{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}}$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=\frac{1}{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}}.\left[\frac{1}{\delta x}\frac{\sin(x+\delta x)}{\cos(x+\delta x)}-\frac{\sin x}{\cos x}\right]$$

$$\frac{\delta y}{\delta x}=\frac{1}{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}}.\frac{1}{\delta x}\left[\frac{\sin(x+\delta x)\cos x-\cos(x+\delta x)\sin x}{\cos(x+\delta x)\cos x}\right]$$

$$\frac{\delta y}{\delta x}=\frac{1}{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}}.\frac{1}{\delta x}\left[\frac{\sin(x+\delta x-x)}{\cos(x+\delta x)\cos x}\right]$$

$$\frac{\delta y}{\delta x}=\frac{1}{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}}.\frac{1}{\delta x}\left[\frac{\sin\delta x}{\cos(x+\delta x)\cos x}\right]$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{1}{\sqrt{\tan(x+\delta x)}+\sqrt{\tan x}}\lim_{\delta x\to 0}\frac{1}{\delta x}\left[\frac{\sin\delta x}{\cos(x+\delta x)\cos x}\right]$$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{\tan x}}\cdot\frac{1}{\cos^2 x}$$

$$\boxed{\frac{dy}{dx}=\frac{\sec^2 x}{2\sqrt{\tan x}}}$$

(vii) $\cos\sqrt{x}$

Solution:

Let $y=\cos\sqrt{x}$

Taking increments on both sides,

$$y+\delta y=\cos\sqrt{x+\delta x}$$

$$\delta y=\cos\sqrt{x+\delta x}-y$$

$$\delta y=\cos\sqrt{x+\delta x}-\cos\sqrt{x}$$

$$\delta y=-2\sin\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\sin\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=-2\sin\frac{\left(\sqrt{x+\delta x}+\sqrt{x}\right)}{2}\cdot\frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\delta x}$$

$$\boxed{\begin{aligned}\because \delta x&=x+\delta x-x\\&=\left(\sqrt{x+\delta x}\right)^2-\left(\sqrt{x}\right)^2\\&=\left(\sqrt{x+\delta x}+\sqrt{x}\right)\left(\sqrt{x+\delta x}-\sqrt{x}\right)\end{aligned}}$$

$$\frac{\delta y}{\delta x}=-\sin\frac{\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)}{\sqrt{x+\delta x}+\sqrt{x}}\cdot\frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=-\lim_{\delta x\to 0}\sin\frac{\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)}{\sqrt{x+\delta x}+\sqrt{x}}\lim_{\delta x\to 0}\frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}$$

$$\frac{dy}{dx}=\frac{-\sin\left(\frac{2\sqrt{x}}{2}\right)}{2\sqrt{x}}$$

$$\boxed{\frac{dy}{dx}=-\frac{\sin\sqrt{x}}{2\sqrt{x}}}$$

Q.2 Differentiate the following w.r.t. the variable involved.

(i) $x^2\sec 4x$

Solution:

Let $y=x^2\sec 4x$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^2\sec 4x\right)$$

$$=x^2\frac{d}{dx}(\sec 4x)+\sec 4x\frac{d}{dx}\left(x^2\right)$$

$$=x^2\left[\sec 4x\tan 4x.4\right]+\sec 4x(2x)$$

$$=4x^2\sec 4x\tan 4x+2x\sec 4x$$

$$\boxed{\frac{dy}{dx}=2x\sec 4x(2x\tan 4x+1)}$$

(ii) $\tan^3\theta\sec^2\theta$

Solution:

Let $y=\tan^3\theta\sec^2\theta$

Differentiate w.r.t "θ"

$$\frac{dy}{d\theta}=\frac{d}{d\theta}\left(\tan^3\theta\sec^2\theta\right)$$

$$=\tan^3\theta\frac{d}{d\theta}\left(\sec^2\theta\right)+\sec^2\theta\frac{d}{d\theta}\left(\tan^3\theta\right)$$

$$=\tan^3\theta.\left[2\sec\theta.(\sec\theta\tan\theta)\right]+\sec^2\theta\left[3\tan^2\theta.\sec^2\theta\right]$$

$$=2\sec^2\theta\tan^4\theta+3\tan^2\theta\sec^4\theta$$

$$\boxed{\frac{dy}{d\theta}=\sec^2\theta\tan^2\theta\left(2\tan^2\theta+3\sec^2\theta\right)}$$

(iii) $(\sin 2\theta-\cos 3\theta)^2$

Solution:

Let $y=(\sin 2\theta-\cos 3\theta)^2$

Differentiate w.r.t "θ"

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\sin 2\theta - \cos 3\theta)^2$$

$$= 2(\sin 2\theta - \cos 3\theta)\frac{d}{d\theta}(\sin 2\theta - \cos 3\theta)$$

$$\boxed{\frac{dy}{d\theta} = 2(\sin 2\theta - \cos 3\theta)[2\cos 2\theta + 3\sin 3\theta]}$$

(iv) $\cos\sqrt{x} + \sqrt{\sin x}$

Solution:

Let $y = \cos\sqrt{x} + \sqrt{\sin x}$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cos\sqrt{x} + \sqrt{\sin x}\right)$$

$$= -\sin\sqrt{x}\,\frac{1}{2\sqrt{x}} + \frac{(\cos x)}{2\sqrt{\sin x}}$$

$$\boxed{\frac{dy}{dx} = \frac{-\sin\sqrt{x}}{2\sqrt{x}} + \frac{\cos x}{2\sqrt{\sin x}}}$$

Q.3 Find $\frac{dy}{dx}$ if

(i) $y = x\cos y$

Solution:

$y = x\cos y$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}(x\cos y)$$

$$\frac{dy}{dx} = x\left(-\sin y\frac{dy}{dx}\right) + \cos y(1)$$

$$\frac{dy}{dx} + x\sin y\frac{dy}{dx} = \cos y$$

$$(1 + x\sin y)\frac{dy}{dx} = \cos y$$

$$\boxed{\frac{dy}{dx} = \frac{\cos y}{1 + x\sin y}}$$

(ii) $x = y\sin y$

Solution:

$x = y\sin y$

Differentiate w.r.t "x"

$$\frac{d}{dx}(x)=\frac{d}{dx}(y\sin y)$$

$$1=y\cos y\frac{dy}{dx}+\sin y\frac{dy}{dx}$$

$$(y\cos y+\sin y)\frac{dy}{dx}=1$$

$$\boxed{\frac{dy}{dx}=\frac{1}{y\cos y+\sin y}}$$

Q.4 Find the derivative w.r.t "x",

(i) $\cos\sqrt{\dfrac{1+x}{1+2x}}$

Solution:

Let $y=\cos\sqrt{\dfrac{1+x}{1+2x}}$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dx}\cos\sqrt{\frac{1+x}{1+2x}}$$

$$=-\sin\sqrt{\frac{1+x}{1+2x}}\frac{d}{dx}\sqrt{\frac{1+x}{1+2x}}$$

$$=-\sin\sqrt{\frac{1+x}{1+2x}}.\frac{1}{2}\sqrt{\frac{1+2x}{1+x}}\frac{d}{dx}\left(\frac{1+x}{1+2x}\right)$$

$$=-\sin\sqrt{\frac{1+x}{1+2x}}.\frac{1}{2}\sqrt{\frac{1+2x}{1+x}}\left[\frac{(1+2x)(1)-(1+x)(2)}{(1+2x)^2}\right]$$

$$=-\sin\sqrt{\frac{1+x}{1+2x}}.\frac{1}{2}\sqrt{\frac{1+2x}{1+x}}\left[\frac{(1+2x-2-2x)}{(1+2x)^2}\right]$$

$$\boxed{\frac{dy}{dx}=\frac{\sin\sqrt{\frac{1+x}{1+2x}}}{2\sqrt{1+x}\,(1+2x)^{\frac{3}{2}}}}$$

(ii) $\sin\sqrt{\dfrac{1+2x}{1+x}}$

Solution:

Let $y=\sin\sqrt{\frac{1+2x}{1+x}}$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sin\sqrt{\frac{1+2x}{1+x}}\right)$$

$$=\cos\sqrt{\frac{1+2x}{1+x}}.\frac{d}{dx}\sqrt{\frac{1+2x}{1+x}}$$

$$=\cos\sqrt{\frac{1+2x}{1+x}}.\frac{1}{2}\sqrt{\frac{1+x}{1+2x}}.\frac{d}{dx}\left(\frac{1+2x}{1+x}\right)$$

$$=\cos\sqrt{\frac{1+2x}{1+x}}.\frac{1}{2}\sqrt{\frac{1+x}{1+2x}}.\left[\frac{(1+x)(2)-(1+2x)(1)}{(1+x)^2}\right]$$

$$\frac{dy}{dx}=\cos\sqrt{\frac{1+2x}{1+x}}.\frac{1}{2}\sqrt{\frac{1+x}{1+2x}}.\left[\frac{2+2x-1-2x}{(1+x)^2}\right]$$

$$\boxed{\frac{dy}{dx}=\frac{\cos\sqrt{\frac{1+2x}{1+x}}}{2\sqrt{1+2x}.(1+x)^{\frac{3}{2}}}}$$

Q.5 Differentiate

(i) $\sin x$ w.r.t $\cot x$

Solution:

Let $y=\sin x$, and $t=\cot x$

We have to find $\frac{dy}{dt}$

$y=\sin x$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\cos x\ \ldots\text{(i)}$$

also $t=\cot x$

Differentiate w.r.t "x"

$$\frac{dt}{dx}=-\operatorname{cosec}^2 x\ \ldots\text{(ii)}$$

Applying chain rule on equation (i) and (ii)

$$\frac{dy}{dt}=\frac{dy}{dx}.\frac{dx}{dt}$$

$$\frac{dy}{dt}=\cos x.\left(\frac{-1}{\operatorname{cosec}^2 x}\right)$$

$$\boxed{\frac{dy}{dt}=-\cos x\sin^2 x}$$

(ii) $\sin^2 x$ w.r.t $\cos^4 x$

Solution:

Let $y=\sin^2 x$ and $t=\cos^4 x$

We have to find $\frac{dy}{dt}$

$y=\sin^2 x$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=2\sin x\cos x\ \ldots\text{(i)}$$

also $t=\cos^4 x$

Differentiate w.r.t "x"

$$\frac{dt}{dx}=4\cos^3 x(-\sin x)$$

$$\frac{dt}{dx}=-4\sin x\cos^3 x\ \ldots\text{(ii)}$$

Applying chain rule on equation (i) and (ii)

$$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$$

$$\frac{dy}{dt}=2\sin x\cos x\cdot\frac{-1}{4\sin x\cos^3 x}$$

$$\boxed{\frac{dy}{dt}=-\frac{1}{2}\sec^2 x}$$

Q.6 If $\tan y(1+\tan x)=1-\tan x$**, Show that** $\frac{dy}{dx}=-1$

Solution:

$$\tan y(1+\tan x)=1-\tan x$$

$$\Rightarrow \tan y=\frac{1-\tan x}{1+\tan x}$$

$$\tan y=\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\cdot\tan x} \qquad \because \tan\frac{\pi}{4}=1$$

$$\tan y=\tan\left(\frac{\pi}{4}-x\right) \qquad \because \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

$$\Rightarrow y=\frac{\pi}{4}-x$$

Differentiate w.r.t "x"

$$\boxed{\frac{dy}{dx}=-1}$$

Alternate method:

$$\Rightarrow \tan y=\frac{1-\tan x}{1+\tan x}$$

Differentiate w.r.t "x"

$$\sec^2 y\frac{dy}{dx}=\frac{(1+\tan x)(-\sec^2 x)-(1-\tan x)(\sec^2 x)}{(1+\tan x)^2}$$

$$(1+\tan^2 y)\frac{dy}{dx}=\frac{-\sec^2 x-\sec^2 x\tan x-\sec^2 x+\sec^2 x\tan x}{(1+\tan x)^2}$$

$$\left(1+\left(\frac{1-\tan x}{1+\tan x}\right)^2\right)\frac{dy}{dx}=\frac{-2\sec^2 x}{(1+\tan x)^2}$$

$$\left(\frac{(1+\tan x)^2+(1-\tan x)^2}{(1+\tan x)^2}\right)\frac{dy}{dx}=\frac{-2\sec^2 x}{(1+\tan x)^2}$$

$$\left(\frac{1+2\tan x+\tan^2 x+1-2\tan x+\tan^2 x}{\cancel{(1+\tan x)^2}}\right)\frac{dy}{dx}=\frac{-2\sec^2 x}{\cancel{(1+\tan x)^2}}$$

$$2(1+\tan^2 x)\frac{dy}{dx}=-2\sec^2 x$$

$$2\sec^2 x\frac{dy}{dx}=-2\sec^2 x$$

$$\boxed{\frac{dy}{dx}=-1}$$

Q.7 If $y=\sqrt{\tan x+\sqrt{\tan x+\sqrt{\tan x+\ldots\infty}}}$ **, Prove that** $(2y-1)\frac{dy}{dx}=\sec^2 x$

Solution:

$$y=\sqrt{\tan x+\sqrt{\tan x+\sqrt{\tan x+\ldots\infty}}}$$

Squaring both sides,

$$y^2=\tan x+\sqrt{\tan x+\sqrt{\tan x+\ldots\infty}}$$

$$y^2=\tan x+y$$

Differentiate w.r.t "x"

$$2y\frac{dy}{dx}=\sec^2 x+\frac{dy}{dx}$$

$$2y\frac{dy}{dx}-\frac{dy}{dx}=\sec^2 x$$

$$\boxed{(2y-1)\frac{dy}{dx}=\sec^2 x}$$

Hence proved

Q.8 If $x=a\cos^3\theta, y=b\sin^3\theta$ **Show that** $a\frac{dy}{dx}+b\tan\theta=0$

Solution:

$x=a\cos^3\theta$

Differentiate w.r.t "θ"

$$\frac{dx}{d\theta}=a\left[3\cos^2\theta\sin\theta\right]$$

$$\frac{dx}{d\theta}=-3a\cos^2\theta\sin\theta\ldots\text{(i)}$$

and $y=b\sin^3\theta$

Differentiate w.r.t "θ"

$$\frac{dy}{d\theta}=b\left[3\sin^2\theta\cos\theta\right]$$

$$\frac{dy}{d\theta}=3b\sin^2\theta\cos\theta\ldots\text{(ii)}$$

Applying chain rule on equation (i) and (ii)

$$\frac{dy}{dx}=\frac{dy}{d\theta}.\frac{d\theta}{dx}$$

$$\frac{dy}{dx}=\cancel{3}b\sin^2\theta\cos\theta.\frac{-1}{\cancel{3}a\cos^2\theta\sin\theta}$$

$$\frac{dy}{dx}=\frac{-b\sin\theta}{a\cos\theta}$$

$$a\frac{dy}{dx}=-b\tan\theta$$

$$\boxed{a\frac{dy}{dx}+b\tan\theta=0}$$

Hence proved

Q.9 Find $\frac{dy}{dx}$ **if** $x=a(\cos t+\sin t),\quad y=a(\sin t-t\cos t)$

Solution:

$x=a\cos t+a\sin t$

Differentiate w.r.t "t"

$$\frac{dx}{dt}=-a\sin t+a\cos t$$

$$\frac{dx}{dt}=a(\cos t-\sin t)\ldots\text{(i)}$$

And $y=a\sin t-at\cos t$

Differentiate w.r.t "t"

$$\frac{dy}{dt} = a\cos t - a[-t\sin t + \cos t]$$

$$\frac{dy}{dt} = a\cos t + at\sin t - a\cos t$$

$$\frac{dy}{dt} = at\sin t \quad \ldots\text{(ii)}$$

Applying chain rule on equation (i) and (ii)

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$\frac{dy}{dx} = at\sin t\,\frac{1}{a(\cos t - \sin t)}$$

$$\boxed{\frac{dy}{dx} = \frac{t\sin t}{\cos t - \sin t}}$$

Q.10 Differentiate w.r.t. "*x*",

(i) $\cos^{-1}\left(\frac{x}{a}\right)$

Solution:

Let $y = \cos^{-1}\left(\frac{x}{a}\right)$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}\cos^{-1}\left(\frac{x}{a}\right)$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\ \frac{d}{dx}\left(\frac{x}{a}\right)$$

$$= \frac{-1}{\sqrt{1-\frac{x^2}{a^2}}}\frac{1}{a}$$

$$= \frac{-1}{\frac{\sqrt{a^2-x^2}}{a}}\frac{1}{a}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{\sqrt{a^2-x^2}}}$$

(ii) $\cot^{-1}\left(\frac{x}{a}\right)$

Solution:

Let $y = \cot^{-1}\left(\frac{x}{a}\right)$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{-1}{1+\left(\frac{x}{a}\right)^2}\frac{d}{dx}\left(\frac{x}{a}\right)$$

$$= -\frac{1}{\frac{a^2+x^2}{a^2}}\cdot\frac{1}{a}$$

$$= \frac{-a^2}{a^2+x^2}\cdot\frac{1}{a}$$

$$\boxed{\frac{dy}{dx} = -\frac{a}{a^2+x^2}}$$

(iii) $\frac{1}{a}\sin^{-1}\left(\frac{a}{x}\right)$

Solution:

Let $y = \frac{1}{a}\sin^{-1}\left(\frac{a}{x}\right)$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{1}{a}\frac{1}{\sqrt{1-\left(\frac{a}{x}\right)^2}}\left(\frac{-a}{x^2}\right)$$

$$\frac{dy}{dx} = \frac{-1}{\cancel{a}}\frac{\cancel{x}}{\sqrt{x^2-a^2}}\left(\frac{\cancel{a}}{x^{\cancel{2}}}\right)$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-a^2}}}$$

(iv) $\sin^{-1}\sqrt{1-x^2}$

Solution:

Let $y = \sin^{-1}\sqrt{1-x^2}$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}}\cdot\frac{d}{dx}\sqrt{1-x^2}$$

$$= \frac{1}{\sqrt{1-1+x^2}}\cdot\frac{1(-2x)}{2\sqrt{1-x^2}}$$

$$= \frac{-1}{\cancel{x}}\frac{\cancel{x}}{\sqrt{1-x^2}}$$

$$\boxed{\frac{dy}{dx}=\frac{-1}{\sqrt{1-x^2}}}$$

(v) $\sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$

Solution:

Let $y=\sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{1}{\left(\frac{x^2+1}{x^2-1}\right)\sqrt{\left(\frac{x^2+1}{x^2-1}\right)^2-1}}\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$=\frac{x^2-1}{\left(x^2+1\right)\sqrt{\frac{x^4+2x^2+1-x^4+2x^2-1}{\left(x^2-1\right)^2}}}\left[\frac{\left(x^2-1\right)(2x)-\left(x^2+1\right)(2x)}{\left(x^2-1\right)^2}\right]$$

$$=\frac{x^2-1}{\left(x^2+1\right)\frac{2x}{x^2-1}}.\left[\frac{2x^3-2x-2x^3-2x}{\left(x^2-1\right)^2}\right]$$

$$\frac{dy}{dx}=\frac{-4x}{2x\left(x^2+1\right)}$$

$$\boxed{\frac{dy}{dx}=\frac{-2}{x^2+1}}$$

(vi) $\cot^{-1}\left(\frac{2x}{1-x^2}\right)$

Solution:

Let $y=\cot^{-1}\left(\frac{2x}{1-x^2}\right)$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{-1}{1+\left(\frac{2x}{1-x^2}\right)^2}.\frac{d}{dx}\left(\frac{2x}{1-x^2}\right)$$

$$=\frac{-1}{1+\frac{4x^2}{1+x^4-2x^2}}.\left[\frac{\left(1-x^2\right)(2)-(2x)(-2x)}{\left(1-x^2\right)^2}\right]$$

$$= \frac{-1}{\dfrac{1+x^4-2x^2+4x^2}{1+x^4-2x^2}} \cdot \left[\frac{2-2x^2+4x^2}{\left(1-x^2\right)^2}\right]$$

$$= -\frac{\left(1+x^4-2x^2\right)}{1+x^4+2x^2} \cdot \frac{2+2x^2}{\left(1-x^2\right)^2}$$

$$= -\frac{\left(1-x^2\right)^2}{\left(1+x^2\right)^2} \cdot \frac{2\left(1+x^2\right)}{\left(1-x^2\right)^2}$$

$$\boxed{\frac{dy}{dx} = -\frac{2}{1+x^2}}$$

(vii) $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

Solution:

Let $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\dfrac{1-x^2}{1+x^2}\right)^2}} \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

$$= \frac{-1}{\sqrt{1-\dfrac{1+x^4-2x^2}{1+x^4+2x^2}}} \left[\frac{\left(1+x^2\right)(-2x)-\left(1-x^2\right)(2x)}{\left(1+x^2\right)^2}\right]$$

$$\frac{dy}{dx} = -\frac{\left(1+x^2\right)}{\sqrt{4x^2}} \cdot \left[\frac{-2x-2x^3-2x+2x^3}{\left(1+x^2\right)^2}\right]$$

$$= \frac{-1}{2x} \cdot \frac{-4x}{1+x^2}$$

$$\boxed{\frac{dy}{dx} = \frac{2}{1+x^2}}$$

Q.11 Show that $\dfrac{dy}{dx} = \dfrac{y}{x}$, If $\dfrac{y}{x} = \tan^{-1}\left(\dfrac{x}{y}\right)$

Solution:

$$\frac{y}{x} = \tan^{-1}\left(\frac{x}{y}\right)$$

$$y = x\tan^{-1}\frac{x}{y}$$

Differentiate w.r.t "*x*"

$$\frac{dy}{dx} = x\frac{d}{dx}\tan^{-1}\frac{x}{y} + \tan^{-1}\frac{x}{y}\frac{d}{dx}(x) \qquad \because \tan^{-1}\frac{x}{y} = \frac{y}{x}$$

$$= x\frac{1}{1+\frac{x^2}{y^2}}\frac{d}{dx}\left(\frac{x}{y}\right) + \frac{y}{x}(1)$$

$$= \frac{x}{\frac{x^2+y^2}{y^2}}\left(\frac{y(1) - x\frac{dy}{dx}}{y^2}\right) + \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{xy}{x^2+y^2} - \frac{x^2}{x^2+y^2}\frac{dy}{dx} + \frac{y}{x}$$

$$\frac{dy}{dx} + \frac{x^2}{x^2+y^2}\frac{dy}{dx} = \frac{xy}{x^2+y^2} + \frac{y}{x}$$

$$\frac{dy}{dx}\left(1 + \frac{x^2}{x^2+y^2}\right) = \frac{x^2y + y\left(x^2+y^2\right)}{x\left(x^2+y^2\right)}$$

$$\frac{dy}{dx}\left(\frac{x^2+y^2+x^2}{x^2+y^2}\right) = \frac{y\left(x^2+x^2+y^2\right)}{x\left(x^2+y^2\right)}$$

$$\boxed{\frac{dy}{dx} = \frac{y}{x}}$$

Q.12 If $y = \tan\left(p\tan^{-1}x\right)$, **show that** $\left(1+x^2\right)y_1 - p\left(1+y^2\right) = 0$

Solution:

$$y = \tan\left(p\tan^{-1}x\right)$$

$$\Rightarrow \tan^{-1}y = p\tan^{-1}x$$

Differentiate w.r.t "*x*",

$$\frac{d}{dx}\left(\tan^{-1}y\right) = p\frac{d}{dx}\left(\tan^{-1}x\right)$$

$$\frac{1}{1+y^2}\frac{dy}{dx} = p\frac{1}{1+x^2}$$

$$\frac{1}{1+y^2}(y_1) = \frac{p}{1+x^2} \qquad \left(\because \frac{dy}{dx} = y_1\right)$$

$$\left(1+x^2\right)y_1 = p\left(1+y^2\right)$$

$$\left(1+x^2\right)y_1 - p\left(1+y^2\right) = 0$$

Hence proved.