

SUCCESSIVE DIFFERENTIATION (OR HIGHER DERIVATIVES):

We state different notations used for derivatives of higher orders.

1 st Derivative	y' , $\frac{dy}{dx}$, y_1 , Dy , $f'(x)$, $\frac{df}{dx}$
2 nd Derivative	y'' , $\frac{d^2y}{dx^2}$, y_2 , D^2y , $f''(x)$, $\frac{d^2f}{dx^2}$
3 rd Derivative	y''' , $\frac{d^3y}{dx^3}$, y_3 , D^3y , $f'''(x)$, $\frac{d^3f}{dx^3}$
⋮	⋮
⋮	⋮
⋮	⋮
n^{th} Derivative	$y^{(n)}$, $\frac{d^n y}{dx^n}$, y_n , $D^n y$, $f^{(n)}(x)$, $\frac{d^n f}{dx^n}$

EXERCISE 2.7**Q.1 Find y_2 if**

(i) $y = 2x^5 - 3x^4 + 4x^3 + x - 2$

Solution:

$$y = 2x^5 - 3x^4 + 4x^3 + x - 2$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}(2x^5 - 3x^4 + 4x^3 + x - 2)$$

$$y_1 = 10x^4 - 12x^3 + 12x^2 + 1$$

Differentiate again w.r.t "x"

$$\boxed{y_2 = 40x^3 - 36x^2 + 24x}$$

(ii) $y = (2x + 5)^{\frac{3}{2}}$

Solution:

$$y = (2x + 5)^{\frac{3}{2}}$$

Differentiate w.r.t "x"

$$y_1 = \frac{3}{2}(2x + 5)^{\frac{3}{2} - 1} \cdot 2$$

$$y_1 = \frac{3}{2}(2x + 5)^{\frac{1}{2}} \cdot 2$$

$$y_1 = 3(2x + 5)^{\frac{1}{2}}$$

Differentiate again w.r.t "x"

$$y_2 = 3\left(\frac{1}{2}\right)(2x + 5)^{-\frac{1}{2}}(2)$$

$$\boxed{y_2 = \frac{3}{\sqrt{2x + 5}}}$$

(iii) $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

Solution:

$$y = \sqrt{x} + \frac{1}{\sqrt{x}}$$

Differentiate w.r.t "x"

$$y_1 = \frac{1}{2\sqrt{x}} + \left(\frac{-1}{2}x^{-\frac{3}{2}}\right)$$

$$y_1 = \frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}$$

Differentiate again w.r.t "x"

$$y_2 = \frac{1}{2}\left[\frac{-1}{2}x^{-\frac{3}{2}} + \frac{3}{2}x^{-\frac{5}{2}}\right]$$

$$y_2 = \frac{1}{4}\left[\frac{-1}{x^{\frac{3}{2}}} + \frac{3}{x^{\frac{5}{2}}}\right]$$

$$\boxed{y_2 = \frac{3 - x}{4x^{\frac{5}{2}}}}$$

Q.2 Find y_2 if

(i) $y = x^2 e^{-x}$

Solution:

$$y = x^2 e^{-x}$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = x^2 \cdot \frac{d}{dx}(e^{-x}) + e^{-x} \cdot \frac{d}{dx}(x^2)$$

$$= x^2 e^{-x}(-1) + e^{-x}(2x)$$

$$y_1 = e^{-x}(2x - x^2)$$

Differentiate again w.r.t "x"

$$y_2 = e^{-x} \frac{d}{dx}(2x - x^2) + (2x - x^2) \frac{d}{dx}(e^{-x})$$

$$= e^{-x}(2 - 2x) + (2x - x^2)(e^{-x})(-1)$$

$$= e^{-x}[2 - 2x - 2x + x^2]$$

$$\boxed{y_2 = e^{-x}(x^2 - 4x + 2)}$$

(ii) $y = \ln\left(\frac{2x+3}{3x+2}\right)$

Solution:

$$y = \ln\left(\frac{2x+3}{3x+2}\right)$$

$$y = \ln(2x+3) - \ln(3x+2)$$

Differentiate w.r.t "x"

$$y_1 = \frac{1}{2x+3}(2) - \frac{1}{3x+2}(3)$$

$$y_1 = 2(2x+3)^{-1} - 3(3x+2)^{-1}$$

Differentiate again w.r.t "x"

$$y_2 = 2(-1)(2x+3)^{-2}(2) - 3(-1)(3x+2)^{-2}(3)$$

$$y_2 = \frac{-4}{(2x+3)^2} + \frac{9}{(3x+2)^2}$$

$$= \frac{9}{9x^2 + 12x + 4} - \frac{4}{4x^2 - 12x + 9}$$

$$= \frac{9(4x^2 + 12x + 9) - 4(9x^2 + 12x + 4)}{(2x+3)^2(3x+2)^2}$$

$$= \frac{36x^2 + 108x + 81 - 36x^2 - 48x - 16}{(2x+3)^2(3x+2)^2}$$

$$\boxed{y_2 = \frac{60x + 65}{(2x+3)^2(3x+2)^2}}$$

Q.3 Find y_2 if,

(i) $x^2 + y^2 = a^2$

Solution:

$$x^2 + y^2 = a^2$$

Differentiate w.r.t "x"

$$2x + 2yy_1 = 0$$

$$\cancel{2}yy_1 = -\cancel{2}x$$

$$\Rightarrow y_1 = -\frac{x}{y}$$

Differentiate again w.r.t "x"

$$y_2 = -\frac{y(1) - x(y_1)}{y^2}$$

$$y_2 = -\frac{y - x\left(-\frac{x}{y}\right)}{y^2} \quad \left(\because y_1 = -\frac{x}{y}\right)$$

$$y_2 = -\frac{x^2 + y^2}{y^3}$$

$$\boxed{y_2 = -\frac{a^2}{y^3}} \quad (\because x^2 + y^2 = a^2)$$

(ii) $x^3 - y^3 = a^3$

Solution:

$$x^3 - y^3 = a^3$$

Differentiate w.r.t "x"

$$3x^2 - 3y^2 y_1 = 0$$

$$\cancel{3}y^2 y_1 = \cancel{3}x^2$$

$$y_1 = \frac{x^2}{y^2}$$

Differentiate again w.r.t "x"

$$y_2 = \frac{y^2(2x) - x^2(2yy_1)}{(y^2)^2}$$

$$y_2 = \frac{2xy^2 - 2x^2 \cancel{2} \left(\frac{x^2}{y^2}\right)}{y^4}$$

$$y_2 = \frac{2xy^2 - 2x^4}{y^4}$$

$$y_2 = \frac{2xy^3 - 2x^4}{y^5}$$

$$y_2 = \frac{-2x(x^3 - y^3)}{y^5}$$

$$y_2 = \frac{-2a^3x}{y^5} \quad (\because x^3 - y^3 = a^3)$$

(iii) $x = a \cos \theta, \quad y = a \sin \theta$

Solution:

$$x = a \cos \theta$$

Differentiate w.r.t " θ "

$$\frac{dx}{d\theta} = -a \sin \theta \dots (i)$$

$$\frac{dy}{d\theta} = a \cos \theta \dots (ii)$$

Applying chain rule on equation (i) and (ii)

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$y_1 = \cancel{a} \cos \theta \cdot \left(\frac{-1}{\cancel{a} \sin \theta} \right)$$

$$y_1 = -\cot \theta$$

Differentiate again w.r.t " x "

$$y_2 = -(-\operatorname{cosec}^2 \theta) \cdot \frac{d\theta}{dx}$$

$$y_2 = \operatorname{cosec}^2 \theta \cdot \left(\frac{-1}{a \sin \theta} \right)$$

$$y_2 = -\frac{1}{a} \operatorname{cosec}^3 \theta$$

(iv) $x = at^2, \quad y = bt^4$

Solution:

$$x = at^2$$

Differentiate w.r.t " t "

$$\frac{dx}{dt} = 2at \dots (i)$$

$$y = bt^4$$

Differentiate w.r.t " t "

$$\frac{dy}{dt} = 4bt^3 \dots (ii)$$

Applying chain rule on equation (i) and (ii)

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$y_1 = 4bt^3 \cdot \frac{1}{2at}$$

$$y_1 = \frac{2bt^2}{a}$$

Differentiate again w.r.t " x "

$$y_2 = \frac{2b}{a} (2t) \cdot \frac{dt}{dx}$$

$$= \frac{4bt}{a} \frac{dt}{dx}$$

$$= \frac{4bt}{a} \cdot \frac{1}{2at}$$

$$y_2 = \frac{2b}{a^2}$$

(v) $x^2 + y^2 + 2gx + 2fy + c = 0$

Solution:

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Differentiate w.r.t " x "

$$2x + 2yy_1 + 2g + 2fy_1 = 0$$

$$(2y + 2f)y_1 = -2x - 2g$$

$$(y + f)y_1 = -(x + g)$$

$$y_1 = -\frac{x + g}{y + f} \dots (i)$$

Differentiate again w.r.t " x "

$$y_2 = -\frac{(y + f)(1) - (x + g)(y_1)}{(y + f)^2}$$

$$y_2 = -\frac{y + f - \left(-\frac{x + g}{y + f} \right)(x + g)}{(y + f)^2}$$

$$y_2 = -\frac{(y + f)^2 + (x + g)^2}{(y + f)^3}$$

$$y_2 = -\frac{y^2 + f^2 + 2fy + x^2 + g^2 + 2gx}{(y + f)^3}$$

$$y_2 = -\frac{x^2 + y^2 + 2gx + 2fy + g^2 + f^2}{(y+f)^3}$$

$$y_2 = \frac{c - f^2 - g^2}{(y+f)^3}$$

$$\because x^2 + y^2 + 2gx + 2fy = -c$$

Q.4 Find y_4 if

(i) $y = \sin 3x$

Solution:

$$y = \sin 3x$$

Differentiate w.r.t "x"

$$y_1 = \cos 3x \cdot 3 \Rightarrow y_1 = 3 \cos 3x$$

Differentiate again w.r.t "x"

$$y_2 = 3(-\sin 3x \cdot 3) \Rightarrow y_2 = -9 \sin 3x$$

Differentiate again w.r.t "x"

$$y_3 = -9 \cos 3x \cdot 3 \Rightarrow y_3 = -27 \cos 3x$$

Differentiate again w.r.t "x"

$$y_4 = -27(-3 \sin 3x)$$

$$y_4 = 81 \sin 3x$$

(ii) $y = \cos^3 x$

Solution:

$$y = \cos^3 x$$

Differentiate w.r.t "x"

$$y_1 = 3 \cos^2 x (-\sin x)$$

$$y_1 = -3 \sin x \cos^2 x$$

$$y_1 = -3 \sin x (1 - \sin^2 x)$$

$$y_1 = -3 \sin x + 3 \sin^3 x$$

Differentiate again w.r.t "x"

$$y_2 = -3 \cos x + 9 \sin^2 x (\cos x)$$

$$= -3 \cos x + 9 \cos x (1 - \cos^2 x)$$

$$= -3 \cos x + 9 \cos x - 9 \cos^3 x$$

$$y_2 = 6 \cos x - 9 \cos^3 x \dots (i)$$

$$y_2 = 6 \cos x - 9y_1$$

Differentiate again w.r.t "x"

$$y_3 = -6 \sin x - 9y_1$$

Differentiate y_3 w.r.t "x"

$$y_4 = -6 \cos x - 9y_2$$

$$y_4 = -6 \cos x - 9(6 \cos x - 9 \cos^3 x) \text{ using (i)}$$

$$y_4 = -6 \cos x - 54 \cos x + 81 \cos^3 x$$

$$y_4 = 81 \cos^3 x - 60 \cos x$$

(iii) $y = \ln(x^2 - 9)$

Solution:

$$y = \ln(x^2 - 9)$$

$$= \ln(x-3)(x+3)$$

$$y = \ln(x-3) + \ln(x+3)$$

Differentiate w.r.t "x"

$$y_1 = \frac{1}{x-3} + \frac{1}{x+3}$$

Differentiate again w.r.t "x"

$$y_2 = \frac{-1}{(x-3)^2} + \frac{-1}{(x+3)^2}$$

$$y_2 = -\left[\frac{1}{(x-3)^2} + \frac{1}{(x+3)^2} \right]$$

Differentiate again w.r.t "x"

$$y_3 = -\left[\frac{-2}{(x-3)^3} + \frac{-2}{(x+3)^3} \right]$$

$$y_3 = 2 \left[\frac{1}{(x-3)^3} + \frac{1}{(x+3)^3} \right]$$

Differentiate again w.r.t "x"

$$y_4 = 2 \left[\frac{-3}{(x-3)^4} + \frac{-3}{(x+3)^4} \right]$$

$$y_4 = \frac{-6}{(x-3)^4} + \frac{-6}{(x+3)^4}$$

$$y_4 = -6 \left[\frac{1}{(x-3)^4} + \frac{1}{(x+3)^4} \right]$$

Q.5 If $x = \sin \theta$, $y = \sin m\theta$ show that

$$(1-x^2)y_2 - xy_1 + m^2y = 0$$

Solution:

$$x = \sin \theta$$

$$\Rightarrow \theta = \sin^{-1} x$$

$$\text{So } y = \sin m\theta$$

$$\text{Becomes } y = \sin(m \sin^{-1} x)$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \cos(m \sin^{-1} x) \cdot \frac{d}{dx}(m \sin^{-1} x)$$

$$y_1 = \cos(m \sin^{-1} x) m \cdot \frac{1}{\sqrt{1-x^2}}$$

$$\text{Or } \sqrt{1-x^2} y_1 = m \cos(m \sin^{-1} x)$$

Differentiate again w.r.t "x"

$$\sqrt{1-x^2} y_1 + y_1 \frac{(-2x)}{2\sqrt{1-x^2}} = m \left[-\sin(m \sin^{-1} x) \right] \cdot \left[\frac{m}{\sqrt{1-x^2}} \right]$$

$$\sqrt{1-x^2} y_2 - x y_1 \cdot \frac{1}{\sqrt{1-x^2}} = -m^2 y_1 \cdot \frac{1}{\sqrt{1-x^2}} \quad \because y = \sin m \sin^{-1} x$$

Multiply both sides by $\sqrt{1-x^2}$

$$\text{We get } (1-x^2) y_2 - x y_1 = -m^2 y$$

$$\text{Or } \boxed{(1-x^2) y_2 - x y_1 + m^2 y = 0}$$

Q.6 If $y = e^x \sin x$, show that

$$\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0$$

Solution:

$$y = e^x \sin x$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = e^x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(e^x)$$

$$\frac{dy}{dx} = e^x \cos x + e^x \sin x$$

$$\frac{dy}{dx} = e^x \cos x + y \dots \text{(i)}$$

Differentiate again w.r.t "x"

$$\frac{d^2 y}{dx^2} = e^x ((-\sin x) + \cos x) + \frac{dy}{dx}$$

$$\frac{d^2 y}{dx^2} = -e^x \sin x + e^x \cos x + \frac{dy}{dx}$$

$$\frac{d^2 y}{dx^2} = -y + e^x \cos x + \frac{dy}{dx}$$

Adding and subtracting " $e^x \sin x$ "

$$\frac{d^2 y}{dx^2} = -y + e^x \sin x + e^x \cos x + \frac{dy}{dx} - e^x \sin x$$

$$\frac{d^2 y}{dx^2} = -y + \frac{dy}{dx} - y + \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = e^x \cos x + e^x \sin x$$

$$\frac{d^2 y}{dx^2} = 2 \frac{dy}{dx} - 2y$$

$$\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0$$

Hence proved

Q.7 If $y = e^{ax} \sin bx$, show that

$$\frac{d^2 y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2)y = 0$$

Solution:

$$y = e^{ax} \sin bx \dots \text{(i)}$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = e^{ax} \cdot \cos bx \cdot b + \sin bx (e^{ax} \cdot a)$$

$$\frac{dy}{dx} = a e^{ax} \sin bx + b e^{ax} \cos bx$$

$$\frac{dy}{dx} = a y + b e^{ax} \cos bx \dots \text{(ii)} \quad (\because e^{ax} \sin bx = y)$$

Differentiate again w.r.t "x"

$$\frac{d^2 y}{dx^2} = a \frac{dy}{dx} + b [e^{ax} (-b \sin bx) + \cos bx \cdot e^{ax} \cdot a]$$

$$= a \frac{dy}{dx} - e^{ax} b^2 \sin bx + a b e^{ax} \cos bx$$

$$\frac{d^2 y}{dx^2} = a \frac{dy}{dx} - b^2 e^{ax} \sin bx + a b e^{ax} \cos bx$$

Adding and subtracting ' $a^2 y$ '

$$\frac{d^2 y}{dx^2} = a \frac{dy}{dx} - b^2 y + a b e^{ax} \cos bx + a^2 y - a^2 y$$

$$\frac{d^2 y}{dx^2} = a \frac{dy}{dx} - b^2 y + a (b e^{ax} \cos bx + a y) - a^2 y$$

$$\frac{d^2 y}{dx^2} = a \frac{dy}{dx} - b^2 y + a \left(\frac{dy}{dx} \right) - a^2 y$$

$$\frac{d^2 y}{dx^2} = a \frac{dy}{dx} - b^2 y + a \frac{dy}{dx} - a^2 y$$

$$\frac{d^2 y}{dx^2} = 2a \frac{dy}{dx} - (a^2 + b^2) y$$

$$\Rightarrow \boxed{\frac{d^2 y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2) y = 0}$$

Q.8 If $y = (\cos^{-1}x)^2$, show that $(1-x^2)y_2 - xy_1 - 2 = 0$

Solution:

$$y = (\cos^{-1}x)^2$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = 2\cos^{-1}x \left(\frac{-1}{\sqrt{1-x^2}} \right)$$

$$\sqrt{1-x^2} y_1 = -2\cos^{-1}x$$

Differentiate again w.r.t "x"

$$\sqrt{1-x^2} \frac{d}{dx}(y_1) + y_1 \frac{d}{dx}\sqrt{1-x^2} = -2 \left(\frac{-1}{\sqrt{1-x^2}} \right)$$

$$\sqrt{1-x^2} y_2 - y_1 \cdot \frac{1(-2x)}{2\sqrt{1-x^2}} = -2 \left(\frac{-1}{\sqrt{1-x^2}} \right)$$

$$\frac{(1-x^2)y_2 - xy_1}{\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}$$

$$\Rightarrow \boxed{(1-x^2)y_2 - xy_1 - 2 = 0}$$

Q.9 If $y = a \cos(\ln x) + b \sin(\ln x)$, show that $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$

Solution:

$$y = a \cos(\ln x) + b \sin(\ln x)$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{-a \sin(\ln x)}{x} + \frac{b \cos(\ln x)}{x}$$

$$x \frac{dy}{dx} = -a \sin(\ln x) + b \cos(\ln x)$$

Differentiate again w.r.t "x"

$$x \frac{d^2y}{dx^2} + \frac{dy}{dx}(1) = \frac{-a \cos(\ln x)}{x} - \frac{b \sin(\ln x)}{x}$$

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} = -[a \cos(\ln x) + b \sin(\ln x)]$$

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} = -y$$

$$\Rightarrow \boxed{x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0}$$