Maclaurin's Theorem:

The power series expansion of a function $f(x)$ is

$f(x)=a_{\circ}+a_1x+a_2x^2+a_3x^3+......+a_nx^n+...$ where $a_0, a_1, a_2, ...a_n, ...$ are constants and x is a variable. Now we find all constants by finding successive derivatives of the power series and evaluating them at $x=0$

$$f(x)=a_{\circ}+a_1x+a_2x^2+a_3x^3+......+a_nx^n+... \longrightarrow \textbf{(i)}$$

$$f(0)=a_{\circ} \longrightarrow \textbf{(ii)}$$

$$f'(x)=a_1+2a_2x+3a_3x^2+......+na_nx^{n-1}+...$$

$$f'(0)=a_1 \longrightarrow \textbf{(iii)}$$

$$f''(x)=2a_2+6a_3x+......+n(n-1)a_nx^{n-2}+...$$

$$f''(0)=2a_2,\quad a_2=\frac{1}{2!}f''(0) \longrightarrow \textbf{(iv)}$$

$$f'''(x)=6a_3+......+n(n-1)(n-2)a_nx^{n-3}+...$$

$$f'''(0)=6a_3,\quad a_3=\frac{1}{3!}f'''(0) \longrightarrow \textbf{(v)}$$

Putting (ii), (iii), (iv), (v) in (i)

$$\boxed{f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+......+\frac{x^n}{n!}f^n(0)+...}$$

This expansion is called **Maclaurin series** or **Maclaurin's theorem**

Note:

A function f can be expanded in the Maclaurin series if the function is defined in the interval containing 0 and its derivatives exist at $x=0$. The expansion is valid only if it is convergent.

Tailor's Series Expansions of Functions:

If f is defined in the interval containing $'a'$ and its derivatives of all orders exist at $x=a$ then we can expand $f(x)$ as

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+.....+\frac{f^n(a)}{n!}(x-a)^n+...$$

Proof:

Let

$$f(x)=a_\circ+a_1(x-a)+a_2(x-a)^2+a_3(x-a)^3+a_4(x-a)^4+.......+a_n(x-a)^n+... \longrightarrow \textbf{(i)}$$

$$f(a)=a_\circ \longrightarrow \textbf{(ii)}$$

$$f'(x)=a_1+2a_2(x-a)+3a_3(x-a)^2+4a_4(x-a)^3+....+na_n(x-a)^{n-1}+...$$

$$f'(a)=a_1 \longrightarrow \textbf{(iii)}$$

$$f''(x)=0+2a_2+6a_3(x-a)+12a_4(x-a)^2+....+n(n-1)a_n(x-a)^{n-2}+...$$

$$f''(a)=2a_2$$

$$\frac{1}{2}f''(a)=a_2 \longrightarrow \textbf{(iv)}$$

$$f'''(x)=0+6a_3+24a_4(x-a)+....+n(n-1)(n-2)a_n(x-a)^{n-3}+....$$

$$f'''(a)=6a_3$$

$$\frac{1}{6}f'''(a)=a_3$$

$$\frac{1}{3!}f'''(a)=a_3 \longrightarrow \textbf{(v)}$$

Similarly

$$\frac{1}{n!}f^n(a)=a_n \longrightarrow \textbf{(vi)}$$

Putting (ii), (iii), (iv), (v), (vi) in (i)

$$\boxed{f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+.....+\frac{f^n(a)}{n!}(x-a)^n+...}$$

This expansion of function $f(x)$ is called **Tailor series** expansion.

EXERCISE 2.8

Q.1 Apply the Maclaurin series expansion to prove that:

(i) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

Solution:

Let

$f(x) = \ln(1+x)$ and $f(0) = 0$

$f'(x) = \frac{1}{1+x}$ and $f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2}$ and $f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3}$ and $f'''(0) = 2$

$f^{iv}(x) = \frac{-6}{(1+x)^4}$ and $f^{iv}(0) = -6$

Applying Maclaurin's series expansion,

$$f(x) = f(0) + \frac{f'(0)x}{\lfloor 1} + \frac{f''(0)x^2}{\lfloor 2} + \frac{f'''(0)x^3}{\lfloor 3} + \ldots$$

$$\ln(1+x) = 0 + \frac{1}{\lfloor 1}x + \frac{-1}{\lfloor 2}x^2 + \frac{2}{\lfloor 3}x^3 + \ldots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

(ii) Prove that $\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \ldots$

Solution:

Let $f(x) = \cos x$ and $f(0) = 1$

$f'(x) = -\sin x$ and $f'(0) = 0$

$f''(x) = -\cos x$ and $f''(0) = -1$

$f'''(x) = \sin x$ and $f'''(0) = 0$

$f^{iv}(x) = \cos x$ and $f^{iv}(0) = 1$

$f^{v}(x) = -\sin x$ and $f^{v}(0) = 0$

$f^{vi}(x) = -\cos x$ and $f^{vi}(0) = -1$

Applying Maclaurin's series expansion,

$$f(x) = f(0) + \frac{f'(0)}{\lfloor 1}x + \frac{f''(0)}{\lfloor 2}x^2 + \frac{f'''(0)}{\lfloor 3}x^3 + \ldots$$

$$\cos x = 1 + \frac{0}{\lfloor 1}x + \frac{-1}{\lfloor 2}x^2 + \frac{0}{\lfloor 3}x^3 + \frac{1}{\lfloor 4}x^4 + \ldots$$

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \ldots$$

(iii) $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \ldots$

Solution:

Let

$f(x) = \sqrt{1+x}$ and $f(0) = 1$

$f'(x) = \frac{1}{2\sqrt{1+x}}$ and $f'(0) = \frac{1}{2}$

$f''(x) = \frac{-1}{4(1+x)^{\frac{3}{2}}}$ and $f''(0) = \frac{-1}{4}$

$f'''(x) = \frac{3}{8(1+x)^{\frac{5}{2}}}$ and $f'''(0) = \frac{3}{8}$

$f^{iv}(x) = \frac{-15}{16(1+x)^{\frac{7}{2}}}$ and $f^{iv}(0) = \frac{-15}{16}$

Applying Maclaurin's series expansion,

$$f(x) = f(0) + \frac{f'(0)}{\lfloor 1}x + \frac{f''(0)}{\lfloor 2}x^2 + \frac{f'''(0)}{\lfloor 3}x^3 + \ldots$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \frac{\frac{-1}{4}}{\lfloor 2}x^2 + \frac{\frac{3}{8}}{\lfloor 3}x^3 + \ldots$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \ldots$$

(iv) $e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots$

Solution:

Let

$f(x) = e^x$ and $f(0) = 1$

$f'(x) = e^x$ and $f'(0) = 1$

$f''(x) = e^x$ and $f''(0) = 1$

$f'''(x) = e^x$ and $f'''(0) = 1$

Applying Maclaurin's series expansion,

$$f(x) = f(0) + \frac{f'(0)}{\lfloor 1}x + \frac{f''(0)}{\lfloor 2}x^2 + \frac{f'''(0)}{\lfloor 3}x^3 + ...$$

$$e^x = 1 + \frac{1}{\lfloor 1}x + \frac{1}{\lfloor 2}x^2 + \frac{1}{\lfloor 3}x^3 + ...$$

$$\Rightarrow e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + ...$$

(v) $e^{2x} = 1 + 2x + \frac{4x^2}{\lfloor 2} + \frac{8x^3}{\lfloor 3} + ...$

Solution:

Let

$f(x) = e^{2x}$ and $f(0) = 1$

$f'(x) = 2e^{2x}$ and $f'(0) = 2$

$f''(x) = 4e^{2x}$ and $f''(0) = 4$

$f'''(x) = 8e^{2x}$ and $f'''(0) = 8$

$f^{iv}(x) = 16e^{2x}$ and $f^{iv}(0) = 16$

Applying Maclaurin's series expansion,

$$f(x) = f(0) + f'\frac{(0)}{\lfloor 1}x + f''\frac{(0)}{\lfloor 2}x^2 + f'''\frac{(0)}{\lfloor 3}x^3 + ...$$

$$e^{2x} = 1 + 2x + \frac{4x^2}{\lfloor 2} + \frac{8x^3}{\lfloor 3} + ...$$

Q.2 Show that $\cos(x+h) = \cos x - h\sin x - \frac{h^2}{\lfloor 2}\cos x + \frac{h^3}{\lfloor 3}\sin x + ...$

and evaluate $\cos 61°$**.**

Solution:

Let

$f(x+h) = \cos(x+h)$

$\Rightarrow f(x) = \cos x$

$f'(x) = -\sin x$ and $f''(x) = -\cos x$

$f'''(x) = \sin x$ and $f^{iv}(x) = \cos x$

Applying Tailor's Theorem, we have

$$f(x+h) = f(x) + \frac{h}{\lfloor 1}f'(x) + \frac{h^2}{\lfloor 2}f''(x) + \frac{h^3}{\lfloor 3}f'''(x) + ...$$

$$\cos(x+h) = \cos x - h\sin x + \frac{h^2}{\lfloor 2}\cos x - \frac{h^3}{\lfloor 3}\sin x + ...$$

For $x = 60°$ and $h = 1°$ or $h = 0.017455$

$$\cos(60° + 1°) = \cos 60° - (0.017455)\sin 60° - \frac{(0.017455)^2}{\lfloor 2}\cos 60° + \frac{(0.017455)^3}{\lfloor 3}\sin 60°$$

$$\cos 61^\circ = \frac{1}{2} - \frac{\sqrt{3}}{2}(0.017455) - \frac{1}{4}(0.017455)^2 + \frac{\sqrt{3}}{12}(0.017455)^3$$
$$\approx 0.5 - 0.015116 - 0.0000761 + 0.00000076$$
$$\approx 0.4848$$

Q.3 Show that $2^{x+h} = 2^x\left[1 + (\ln 2)h + \frac{(\ln 2)^2}{\lfloor 2}h^2 + \frac{(\ln 2)^3}{\lfloor 3}h^3 + \ldots\right]$

Solution:

Let $f(x+h) = 2^{x+h}$

$\Rightarrow f(x) = 2^x$

$f'(x) = 2^x(\ln 2)$ and $f''(x) = 2^x(\ln 2)^2$

$f'''(x) = 2^x(\ln 2)^3$ and $f^{iv}(x) = 2^x(\ln 2)^4$

Applying Taylor's Theorem

$$f(x+h) = f(x) + \frac{h}{\lfloor 1}f'(x) + \frac{h^2}{\lfloor 2}f''(x) + \frac{h^3}{\lfloor 3}f'''(x) + \ldots$$

$$2^{x+h} = 2^x + \frac{h}{\lfloor 1}.2^x(\ln 2) + \frac{h^2}{\lfloor 2}2^x(\ln 2)^2 + \ldots$$

$$2^{x+h} = 2^x\left[1 + (\ln 2)h + (\ln 2)^2\frac{h^2}{\lfloor 2} + (\ln 2)^3\frac{h^3}{\lfloor 3} + \ldots\right]$$