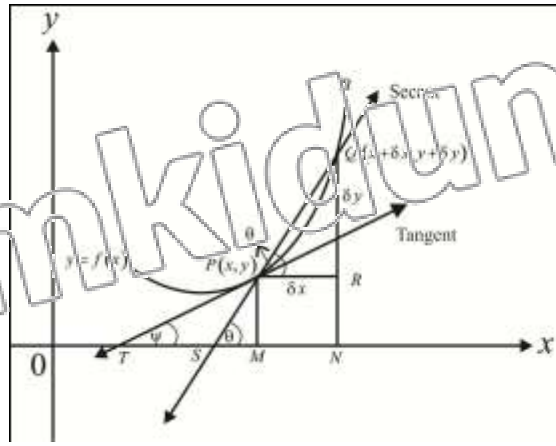
GEOMETRICAL INTERPRETATION OF A DERIVATIVE:



Let $P(x, y)$ and $Q(x+\delta x, y+\delta y)$ be two neighboring points on the graph of the function defined by the equation $y = f(x)$. The line PQ is a secant to the curve. Its inclination is θ. TP is the tangent to the curve at point P. Its inclination is ψ

In ΔPQR

$$\tan\theta = \frac{QR}{PR} = \frac{\delta y}{\delta x}$$

Applying limit $\delta x \to 0$, the secant will become the tangent at P and θ will tend to ψ.

$$\lim_{\delta x \to 0} \tan\theta = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$\tan\theta = \frac{dy}{dx}$$

The derivative w.r.t 'x' of the function defined by the equation $y = f(x)$ is equal to the slope of the tangent to the graph of the function at point $P(x, y)$.

INCREASING AND DECREASING FUNCTIONS:

Let f be defined on interval (a,b) and let $x_1, x_2 \in (a,b)$ then

(i) f is increasing on the interval (a,b) if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$

(ii) f is decreasing on the interval (a,b) if $f(x_2) < f(x_1)$ whenever $x_2 > x_1$

Note:

(i) A differentiable function f is increasing on (a,b) if tangent lines to its graph at all points $(x, f(x))$ have positive slopes i.e. $f'(x) > 0$, $\forall x \in (a,b)$.

(ii) A differentiable function f is decreasing on (a,b) if tangent lines to its graph at all points $(x, f(x))$ have negative slopes i.e. $f'(x) < 0$, $\forall x \in (a,b)$.

$f'(x) < 0 \ \forall x$ such that $a < x < b$

Stationary Point:

A point where *f* is neither increasing nor decreasing is called a stationary point, provided that $f'(x)=0$ at that point.

RELATIVE EXTREMA:

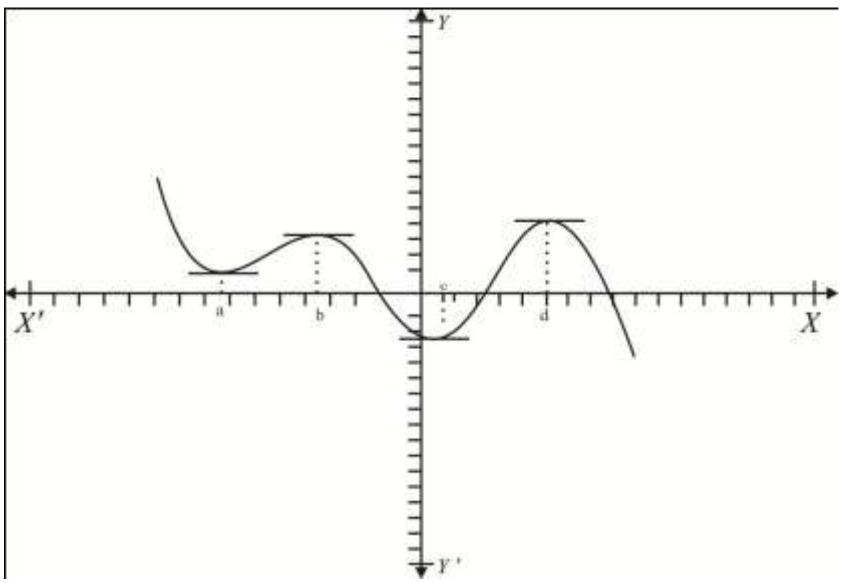
Let $(c-\delta x, c+\delta x)\subseteq D_f$ (domain of a function *f*) where δx is small positive number

If $f(c)\geq f(x)\ \forall x\in(c-\delta x, c+\delta x)$ then the function f is said to have a relative maxima at $x=c$

If $f(c)\leq f(x)\forall x\in(c-\delta x, c+\delta x)$ then the function *f* has relative minima at $x=c$.

Both relative maximum and minimum are called relative extrema (in general).

The graph of a function is shown in the adjoining figure. It has relative maxima at $x=b$ and $x=d$. But at $x=a$ and $x=c$ it has relative minima.



Critical Values and Critical Points:

If $c\in D_f$ and $f'(c)=0$ or $f'(c)$ does not exist then the number $f(c)$ is called a critical value of f while the point $(c, f(c))$ on the graph of $f(x)$ is named as a critical point.

There are functions which have extrema (maxima or minima) at the points where their derivatives do not exist.

First derivative rule:

Let $f(x)$ be differentiable in neighbourhood of *c* where $f'(c)=0$

(i) If $f'(x)$ changes sign from positive to negative as *x* increases through *c* then $f(c)$ is the relative maxima of $f(x)$.

(ii) If $f'(x)$ changes sign from negative to positive as x increases through c then $f(c)$ is the relative minima of $f(x)$.

Second Derivative Rule:

Let $f(x)$ be a differentiable function in neighborhood of c where $f'(c)=0$ then

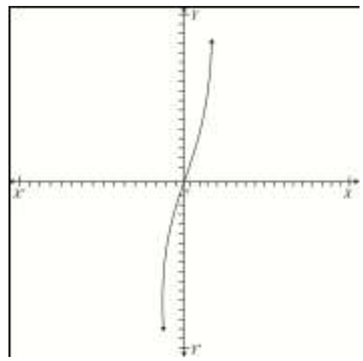
(i) $f(x)$ has relative maxima at c if $f''(c)<0$

(ii) $f(x)$ has relative minima at c if $f''(c)>0$.

Note:

(i) A stationary point is called a turning point if it is either a maximum point or a minimum point.

(ii) If $f'(x)>0$ before the point $x=a$, $f'(x)=0$ at $x=a$ and $f'(x)>0$ after $x=a$ then f does not has a relative maxima. Such a point of the function is called the point of inflection.



EXERCISE 2.9

Q.1 Determine the intervals in which f is increasing or decreasing for the domain mentioned in each case.

(i) $f(x)=\sin x,\ x\in(-\pi,\pi)$

Solution:

$f(x)=\sin x$

Differentiate w.r.t. "x"

$f'(x)=\cos x$

Put $f'(x)=0$

$\cos x=0$

$\Rightarrow x=-\frac{\pi}{2},\frac{\pi}{2}$

Intervals are $\left(-\pi,-\frac{\pi}{2}\right),\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2},\pi\right)$

For $\left(-\pi,-\frac{\pi}{2}\right)$; $f'(x)=\cos x<0$

So f is decreasing

For $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$; $f'(x)=\cos x>0$

So f is increasing.

For $\left(\frac{\pi}{2},\pi\right)$; $f'(x)=\cos x<0$

So f is decreasing

(ii) $f(x)=\cos x,\ x\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$

Solution:

$f(x)=\cos x$

Differentiate w.r.t. "x"

$f'(x)=-\sin x$

Put $f'(x)=0$

$-\sin x=0$

$\Rightarrow x=0$

Intervals are $\left(-\frac{\pi}{2},0\right)$ and $\left(0,\frac{\pi}{2}\right)$

For $\left(-\frac{\pi}{2},0\right)$; $f'(x)=-\sin x>0$

So f is increasing

For $\left(0,\frac{\pi}{2}\right)$; $f'(x)=-\sin x<0$

So f is decreasing

(iii) $f(x)=4-x^2$, $x\in(-2,2)$

Solution:

$f(x)=4-x^2$

Differentiate w.r.t. "x"

$f'(x)=-2x$

Put $f'(x)=0$

$-2x=0$

$x=0$

Intervals are $(-2,0)$ and $(0,2)$

For $(-2,0)$; $f'(x)=-2x>0$

So f is increasing

For $(0,2)$; $f'(x)=-2x<0$

So f is decreasing

(iv) $f(x)=x^2+3x+2,\ x\in(-4,1)$

Solution:

$f(x)=x^2+3x+2$

Differentiate w.r.t. "x"

$f'(x)=2x+3$

Put $f'(x)=0$

$2x+3=0$

$x=-\frac{3}{2}$

Intervals are $\left(-4,\frac{-3}{2}\right)$ and $\left(\frac{-3}{2},1\right)$

For $\left(-4,-\frac{3}{2}\right)$; $f'(x)=2x+3<0$

So f is decreasing

For $\left(-\frac{3}{2},1\right)$; $f'(x)=2x+3>0$

So f is increasing

Q.2 Find the extreme values for the following functions defined as,

(i) $f(x)=1-x^3$

Solution:

$f(x)=1-x^3$

Differentiate w.r.t. "x"

$f'(x)=-3x^2$

Put $f'(x)=0$

$-3x^2=0\Rightarrow x=0$

Now $f''(x)=-6x$

Putting $x=0,$

$f''(0)=0$

So the second derivative is not helpful in determining the extreme values.

Now we use first derivative test.

Let $x=0-\varepsilon$

and $f'(0-\varepsilon)=-3(0-\varepsilon)^2$

$=-3\varepsilon^2<0$

and $f'(0+\varepsilon)=-3(0+\varepsilon)^2$

$=-3\varepsilon^2<0$

So the first derivative does not change sign at $x=0$ and $f(x)=1$,

$\therefore(0,1)$ is the point of inflexion.

(ii) $f(x)=x^2-x-2$

Solution:

$f(x)=x^2-x-2$

Differentiate w.r.t. "x"

$f'(x)=2x-1$

Put $f'(x)=0\Rightarrow 2x-1=0$

we get $x=\frac{1}{2}$

take $f'(x)=2x-1$

Differentiate again w.r.t. "x"

$f''(x)=2$

at $x=\frac{1}{2}$

$f''(x)=2>0$

so f has relative minima at $x=\frac{1}{2}$

and $f\left(\frac{1}{2}\right)=\frac{-9}{4}$

(iii) $f(x)=5x^2-6x+2$

Solution:

$f(x)=5x^2-6x+2$

Differentiate w.r.t. "x"

$f'(x)=10x-6$

Put $f'(x)=0 \Rightarrow 10x-6=0$

we get $x=\frac{3}{5}$

take $f'(x)=10x-6$

Differentiate again w.r.t. "x"

$f''(x)=10$

at $x=\frac{3}{5}$

$f''(x)=10>0$

So f has relative minima at $x=\frac{3}{5}$

$f\left(\frac{3}{5}\right)=5\left(\frac{3}{5}\right)^2-6\left(\frac{3}{5}\right)+2$

$=\frac{9}{5}-\frac{18}{5}+2$

$=\frac{9-18+10}{5}$

$f\left(\frac{3}{5}\right)=\frac{1}{5}$

(iv) $f(x)=3x^2$

Solution:

$f(x)=3x^2$

Differentiate w.r.t. "x"

$f'(x)=6x$

Put $f'(x)=0 \Rightarrow 6x=0$

We get $x=0$

take $f'(x)=6x$

Differentiate again w.r.t. "x"

$f''(x)=6$

at $x=0$

$f''(x)=6>0$

So $f(x)$ has minimum value

and $f(0)=0$

(v) $f(x)=2x^3-2x^2-36x+3$

Solution:

$f(x)=2x^3-2x^2-36x+3$

Differentiate w.r.t. "x"

$f'(x)=6x^2-4x-36$

Put $f'(x)=0$

$6x^2-4x-36=0$

$\Rightarrow \quad x=\frac{1\pm\sqrt{55}}{3}$

Take $f'(x)=6x^2-4x-36$

Differentiate again w.r.t. "x"

Now $f''(x)=12x-4$

At $x=\frac{1+\sqrt{55}}{3}$

$f''\left(\frac{1+\sqrt{55}}{3}\right)=12\left(\frac{1+\sqrt{55}}{3}\right)-4$

$=4+4\sqrt{55}-4=4\sqrt{55}>0$

Thus f has relative minima

$f\left(\frac{1+\sqrt{55}}{3}\right)=2\left(\frac{1+\sqrt{55}}{3}\right)^3-2\left(\frac{1+\sqrt{55}}{3}\right)^2-36\left(\frac{1+\sqrt{55}}{3}\right)+3$

$f\left(\frac{1+\sqrt{55}}{3}\right)=\frac{-1}{27}\left(247+220\sqrt{55}\right)$

at $x=\frac{1-\sqrt{55}}{3}$

$f''\left(\frac{1-\sqrt{55}}{3}\right)=-4\sqrt{55}<0$

So f has relative maxima

$$f\left(\frac{1-\sqrt{55}}{3}\right)=2\left(\frac{1-\sqrt{55}}{3}\right)^3-2\left(\frac{1-\sqrt{55}}{3}\right)^2-36\left(\frac{1-\sqrt{55}}{3}\right)+3$$

$$f\left(\frac{1-\sqrt{55}}{3}\right)=\frac{-1}{27}\left(247-220\sqrt{55}\right)$$

(vi) $f(x)=x^4-4x^2$

Solution:

$f(x)=x^4-4x^2$

Differentiate w.r.t. "x"

$f'(x)=4x^3-8x$

Put $f'(x)=0$

$4x^3-8x=0$

$x=0,\quad x=\sqrt{2},\quad x=-\sqrt{2}$

Take $f'(x)=4x^3-8x$

Differentiate again w.r.t. "x"

$f''(x)=12x^2-8$

at $x=-\sqrt{2}$

$f''(-\sqrt{2})=12(-\sqrt{2})^2-8$

$f''(-\sqrt{2})=24-8$

$f''(-\sqrt{2})=16>0$

So f has relative minima

at $x=0,\ f''(0)=-8<0$

so f has relative maxima at $x=0$

at $x=\sqrt{2}$, $f''(\sqrt{2})=11>0$

so f has relative minima at $x=\sqrt{2}$

and $f(\sqrt{2})=f(-\sqrt{2})=-4$

also $f(0)=0$

(vii) $f(x)=(x-2)^2(x-1)$

Solution:

$f(x)=(x-2)^2(x-1)$

$f(x)=(x^2-4x+4)(x-1)$

$f(x)=x^3-5x^2+8x-4$

Differentiate w.r.t. "x"

$f'(x)=3x^2-10x+8$

Put $f'(x)=0$

$3x^2-10x+8=0$

$\Rightarrow x=2,\qquad x=\frac{4}{3}$

Take $f'(x)=3x^2-10x+8$

Differentiate again w.r.t. "x"

$f''(x)=6x-10$

at $x=2$,

$f''(2)=2>0$

So f has relative minima

and $f(2)=0$

at $x=\frac{4}{3}$

$f''\left(\frac{4}{3}\right)=-2<0$

So f has relative minima

And

$$f\left(\frac{4}{3}\right)=\left(\frac{4}{3}\right)^3-5\left(\frac{4}{3}\right)^2+8\left(\frac{4}{3}\right)-4$$

$$=\frac{64}{27}-\frac{80}{9}+\frac{32}{3}-4$$

$$f\left(\frac{4}{3}\right)=\frac{4}{27}$$

(viii) $f(x)=5+3x-x^3$

Solution:

$f(x)=5+3x-x^3$

Differentiate w.r.t. "x"

$f'(x)=3-3x^2$

Put $f'(x)=0$

$3-3x^2=0$

$\Rightarrow\qquad x=\pm1$

Take $f'(x)=3-3x^2$

Differentiate again w.r.t. "x"

$f''(x)=-6x$

at $x=-1,\ f''(-1)=6>0$

So f has relative minima

And $f(-1)=3$

At $x=1$, $f''(1)=-6<0$

So f has relative maxima

And $f(1)=7$

Q.3 Find the maximum and minimum values of the function defined by the following equation occurring in the interval $[0,2\pi]$

$f(x)=\sin x+\cos x$

Solution:

$f(x)=\sin x+\cos x$

Differentiate w.r.t. "x"

$f'(x)=\cos x-\sin x$

Put $f'(x)=0$

$\cos x-\sin x=0$

$\sin x=\cos x \quad \Rightarrow \tan x=1$

$x=\frac{\pi}{4}$ & $x=\pi+\frac{\pi}{4}=\frac{5\pi}{4}$

Take $f'(x)=\cos x-\sin x$

Differentiate again w.r.t. "x"

$f''(x)=-\sin x-\cos x$

At $x=\frac{\pi}{4}$

$f''\left(\frac{\pi}{4}\right)=-\sin\frac{\pi}{4}-\cos\frac{\pi}{4}$

$=\frac{-1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=-\frac{2}{\sqrt{2}}$

$=-\sqrt{2}<0$

So f has maximum value

And

$f\left(\frac{\pi}{4}\right)=\sin\frac{\pi}{4}+\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\frac{2}{\sqrt{2}}$

$f\left(\frac{\pi}{4}\right)=\sqrt{2}$

At $x=\frac{5\pi}{4}$

$f''\left(\frac{5\pi}{4}\right)=-\sin\frac{5\pi}{4}-\cos\frac{5\pi}{4}$

$=\sqrt{2}>0$

So f has minimum value

and $f\left(\frac{5\pi}{4}\right)=\sin\frac{5\pi}{4}+\cos\frac{5\pi}{4}$

$f\left(\frac{5\pi}{4}\right)=-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}$

$f\left(\frac{5\pi}{4}\right)=\frac{-1-1}{\sqrt{2}}=\frac{-2}{\sqrt{2}}$

$f\left(\frac{5\pi}{4}\right)=-\sqrt{2}$

So maximum and minimum values of $f(x)$ are $\sqrt{2}$ and $-\sqrt{2}$ respectively.

Q.4 Show that $y=\frac{\ln x}{x}$ has maximum value at $x=e$

Solution:

$y=\frac{\ln x}{x}$

Differentiate w.r.t "x"

$\frac{dy}{dx}=\frac{x\left(\frac{1}{x}\right)-\ln x(1)}{x^2}$

$\frac{dy}{dx}=\frac{1-\ln x}{x^2}$

Put $\frac{dy}{dx}=0$

$\frac{1-\ln x}{x^2}=0$

$1-\ln x=0$

$\ln x=1\Rightarrow \ln x=\ln e$

$\Rightarrow x=e$

Take $\frac{dy}{dx}=\frac{1-\ln x}{x^2}$

Differentiate again w.r.t. "x"

$\frac{d^2y}{dx^2}=\frac{x^2\left(-\frac{1}{x}\right)-(1-\ln x)(2x)}{x^4}$

$$\frac{d^2y}{dx^2} = \frac{-x-2x(1-\ln x)}{x^4}$$

$$\frac{d^2y}{dx^2} = \frac{-3x+2x\ln x}{x^4}$$

$$\frac{d^2y}{dx^2} = \frac{-3+2\ln x}{x^3}$$

at $x = e$

$$\left.\frac{d^2y}{dx^2}\right|_{x=e} = \frac{-3+2\ln e}{e^3}$$

$$= \frac{-1}{e^3} < 0$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=e} < 0$$

so *f* has maximum value at $x = e$.

Q.5 Show that $y = x^x$ has minimum value at $x = \frac{1}{e}$

Solution:

$y = x^x$

Taking natural log on both sides

$\ln y = \ln x^x$

$\ln y = x\ln x$

Differentiate w.r.t. "x"

$$\frac{1}{y}\frac{dy}{dx} = x.\frac{1}{x} + \ln x(1)$$

$$\frac{1}{y}\frac{dy}{dx} = 1+\ln x$$

$$\frac{dy}{dx} = y(1+\ln x) \Rightarrow \frac{dy}{dx} = x^x(1+\ln x)$$

Put $\frac{dy}{dx} = 0$

$\Rightarrow x^x(1+\ln x) = 0$

$\Rightarrow 1+\ln x = \quad \therefore x^x \neq 0$

$\ln x = -1$

$\ln x = -\ln e$

$(\because \ln e = 1)$

$\ln x = \ln e^{-1}$

$\ln x = \ln\left(\frac{1}{e}\right)$

$\Rightarrow x = \frac{1}{e}$

Take $\frac{dy}{dx} = x^x(1+\ln x)$

Differentiate again w.r.t. "x"

$$\frac{d^2y}{dx^2} = x^x\left(\frac{1}{x}\right) + (1+\ln x)x^x(1+\ln x)$$

$$\frac{d^2y}{dx^2} = \frac{x^x}{x} + x^x(1+\ln x)^2$$

$$\frac{d^2y}{dx^2} = x^x\left(\frac{1}{x} + (1+\ln x)^2\right)$$

at $x = \frac{1}{e}$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{e}} = \left(\frac{1}{e}\right)^{\frac{1}{e}}\left[\frac{1}{1/e} + \left(1+\ln\frac{1}{e}\right)^2\right]$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{e}} = \left(\frac{1}{e}\right)^{\frac{1}{e}}\left[e + (1+\ln 1-\ln e)^2\right]$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{e}} = e\left(\frac{1}{e}\right)^{\frac{1}{e}} > 0$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{e}} > 0$$

$\therefore$ $y = x^x$ has minimum value at

$x = \frac{1}{e}$