UNIT 3

INTEGRATION

Introduction:

The technique or method to find such a function whose derivative is given involves the inverse process of differentiation, called anti-derivation or integration.

Differentials of Variables:

Let *f* be a differentiable function in the interval $a<x<b$, defined as $y=f\left(x\right),$ then

the differential of independent variable *x* is denoted by *dx* and is defined to be the increment δx

Thus $\delta x = dx$

The differential of dependent variable *y* is denoted by dy and is written as

$$dy = f'\left(x\right)dx \text{ and } \delta y \approx dy$$

EXERCISE 3.1

Q.1 Find δy and dy in the following cases

(i) $y=x^2-1,$ **when** x **changes from 3 to 3.02**

Solution:

$x=3$ and $dx=\delta x=3.02-3=0.02$

$y=x^2-1$

$y+\delta y=\left(x+\delta x\right)^2-1$

$\delta y=\left(x+\delta x\right)^2-1-\left(x^2-1\right)$

$=x^2+\left(\delta x\right)^2+2x\delta x-1-x^2+1$

$=\delta x\left(\delta x+2x\right)$

$=\left(0.02\right)\left(0.02+2\left(3\right)\right)$

$=\left(0.02\right)\left(6.02\right)$

$\boxed{\delta y=0.1204}$

$y=f\left(x\right)=x^2-1$

$f'\left(x\right)=2x$

$dy=f'\left(x\right)dx$

$dy=2xdx \qquad \because f'\left(x\right)=2x$

$dy=2\left(3\right)\left(0.02\right)$

$dy=\left(6\right)\left(0.02\right)$

$\boxed{dy=0.12}$

(ii) $y=x^2+2x,$ **when** x **changes from 2 to 1.8**

Solution:

$x=2$ and $dx=\delta x=1.8-2=-0.2$

$y=x^2+2x$

$y+\delta y=\left(x+\delta x\right)^2+2\left(x+\delta x\right)$

$\delta y=\left(x+\delta x\right)^2+2\left(x+\delta x\right)-\left(x^2+2x\right)$

$=x^2+\left(\delta x\right)^2+2x\delta x+2x+2\delta x-x^2-2x$

$=\left(\delta x\right)^2+2x\delta x+2\delta x$

$=\left(-0.2\right)^2+2\left(2\right)\left(-0.2\right)+2\left(-0.2\right)$

$\delta y = 0.04 - 0.8 - 0.4$

$\boxed{\delta y = -1.16}$

$y = f(x) = x^2 + 2x$

$f'(x) = 2x + 2$

$dy = f'(x)dx$

$dy = (2x+2)dx$

$dy = [2(2)+2](-0.2)$

$dy = (6)(-0.2)$

$\boxed{dy = -1.2}$

(iii) $y = \sqrt{x}$ **when** x **changes from 4 to 4.1**

Solution:

$x = 4$ and $\delta x = dx = 4.41 - 4 = 0.41$

$y = \sqrt{x}$

Then $y + \delta y = \sqrt{x + \delta x}$

$\delta y = (x+\delta x)^{\frac{1}{2}} - x^{\frac{1}{2}}$

$\delta y = (4+0.4)^{\frac{1}{2}} - (4)^{\frac{1}{2}}$

$\delta y = \sqrt{4.41} - \sqrt{4}$

$\boxed{\delta y = 0.1}$

$y = f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2\sqrt{x}}$

$dy = f'(x)dx$

$dy = \frac{1}{2\sqrt{x}}dx$

$dy = \frac{1}{2\sqrt{4}}(0.41)$

$dy = \frac{0.41}{4}$

$\boxed{dy = 0.1025}$

Q.2 Using differentials, find $\frac{dy}{dx}$ and $\frac{dx}{dy}$ in the following equations

(i) $xy + x = 4$

Solution:

$xy + x = 4$

Taking differentials on both sides

$d(xy + x) = d(4)$

$d(xy) + d(x) = 0$

$xdy + ydx + dx = 0$

$xdy = -(y+1)dx \ldots$(i)

$dy = -\left[\frac{y+1}{x}\right]dx$

$\boxed{\frac{dy}{dx} = -\frac{y+1}{x}}$

From (i)

$(y+1)dx = -xdy$

$\boxed{\frac{dx}{dy} = -\frac{x}{y+1}}$

(ii) $x^2 + 2y^2 = 16$

Solution:

$x^2 + 2y^2 = 16$

Taking differentials on both sides

$d(x^2 + 2y^2) = d(16)$

$2xdx + 2(2ydy) = 0$

$2xdx + 4ydy = 0$

$2ydy = -xdx \ldots$(i)

$dy = \left(\frac{-x}{2y}\right)dx$

$\boxed{\frac{dy}{dx} = \frac{-x}{2y}}$

From (i)

$xdx = -2ydy$

$dx = \left(\frac{-2y}{x}\right)dy$

$\boxed{\frac{dx}{dy} = \frac{-2y}{x}}$

(iii) $x^4 + y^2 = xy^2$

Solution:

$x^4 + y^2 = xy^2$

Taking differentials on both sides

$d\left(x^4 + y^2\right) = d\left(xy^2\right)$

$4x^3dx + 2y\,dy = y^2dx + x\left(2ydy\right)$

$2y\,dy - 2xy\,dy = y^2dx - 4x^3\,dx$

$2y\left(1-x\right)dy = \left(y^2 - 4x^3\right)dx \ldots$(i)

$dy = \left[\dfrac{y^2 - 4x^3}{2y\left(1-x\right)}\right]dx$

$$\boxed{\dfrac{dy}{dx} = \dfrac{y^2 - 4x^3}{2y\left(1-x\right)}}$$

From (i)

$dx = \left(\dfrac{2y\left(1-x\right)}{y^2 - 4x^3}\right)dy$

$$\boxed{\dfrac{dx}{dy} = \dfrac{2y\left(1-x\right)}{y^2 - 4x^3}}$$

(iv) $xy - \ln x = c$

Solution:

$xy - \ln x = c$

Taking differentials on both sides

$d\left(xy - \ln x\right) = d\left(c\right)$

$d\left(xy\right) - d\left(\ln x\right) = 0$

$xdy + ydx - \dfrac{1}{x}dx = 0$

$xdy = \dfrac{1}{x}dx - ydx$

$xdy = \left(\dfrac{1}{x} - y\right)dx$

$dy = \dfrac{1}{x}\left(\dfrac{1-xy}{x}\right)dx$

$dy = \left(\dfrac{1-xy}{x^2}\right)dx \ldots$(i)

$$\boxed{\dfrac{dy}{dx} = \dfrac{1-xy}{x^2}}$$

From (i),

$\dfrac{1}{x}dx - ydx = xdy$

$dx\left(\dfrac{1-xy}{x}\right) = xdy$

$$\boxed{\dfrac{dx}{dy} = \dfrac{x^2}{1-xy}}$$

Q.3 Use differentials to approximate the values of

(i) $\sqrt[4]{17}$

Solution:

Let $f\left(x\right) = \sqrt[4]{x} = \left(x\right)^{\frac{1}{4}}$

Then $f\left(x+\delta x\right) = \left(x+\delta x\right)^{\frac{1}{4}}$

$f\left(x+\delta x\right) = \left(x+dx\right)^{\frac{1}{4}} \left(\because \delta x = dx\right)$

As the nearest perfect fourth root to 17 is 16 so, we take $x = 16$ and $\delta x = dx = 1$

$f\left(16\right) = \left(16\right)^{\frac{1}{4}} = 2$

As $f\left(x\right) = x^{\frac{1}{4}}$

Differentiate w.r.t x

$f'\left(x\right) = \dfrac{1}{4}x^{\frac{1}{4}-1} = \dfrac{1}{4}x^{-\frac{3}{4}}$

$f'\left(x\right) = \dfrac{1}{4x^{\frac{3}{4}}}$

put $x = 16$ in

$f'\left(16\right) = \dfrac{1}{4\left(16\right)^{\frac{3}{4}}} = \dfrac{1}{4\left[\left(16\right)^{\frac{1}{4}}\right]^3}$

$= \dfrac{1}{4\left(2\right)^3}$

$f'\left(16\right) = \dfrac{1}{32}$

By using

$f\left(x+\delta x\right) \approx f\left(x\right) + f'\left(x\right)dx$

$f\left(16+1\right) \approx f\left(16\right) + f'\left(16\right)\left(1\right)$

$f\left(17\right) \approx 2 + \dfrac{1}{32}$

$\sqrt[4]{17} \approx 2+0.03125$

$\boxed{\sqrt[4]{17} \approx 2.03125}$

(ii) $(31)^{\frac{1}{5}}$

Solution:

Let $f(x)=x^{\frac{1}{5}}$

As the nearest perfect fifth root to 31 is 32, so we take $x=32$ and $\delta x=dx=-1$

$f(32)=(32)^{\frac{1}{5}}=2$

$f(x)=x^{\frac{1}{5}}$

Differentiate w.r.t x

$f'(x)=\frac{1}{5}x^{\frac{1}{5}-1}=\frac{1}{5}x^{-\frac{4}{5}}$

$f'(x)=\frac{1}{5x^{\frac{4}{5}}}$

Put $x=32$ in

$f'(32)=\frac{1}{5(32)^{\frac{4}{5}}}=\frac{1}{5\left[(32)^{\frac{1}{5}}\right]^4}$

$=\frac{1}{5(2)^4}$

$f'(32)=\frac{1}{80}$

By using

$f(x+\delta x)\approx f(x)+f'(x)dx$

$f(32+(-1))\approx f(32)+f'(32)(-1)$

$f(31)\approx 2-\frac{1}{80}$

$\sqrt[5]{31}\approx 2-0.0125$

$\boxed{\sqrt[5]{31}\approx 1.9875}$

(iii) $\cos 29°$

Solution:

Let $f(x)=\cos x$

We take

$x=30°$ and

$\delta x=dx=-1°=\frac{-\pi}{180}=-0.01745$ radians

$f(30°)=\cos 30°=0.8660$

$f(x)=\cos x$

Differentiate w.r.t x

$f'(x)=-\sin x$

$f'(30°)=-\sin 30°=-\frac{1}{2}$

$f(x+\delta x)\approx f(x)+f'(x)dx$

$f(30°-1°)\approx f(30°)+f'(30°)(-0.01745)$

$f(29°)\approx 0.8660+\left(-\frac{1}{2}\right)(-0.01745)$

$\cos 29°\approx 0.8660+0.0087$

$\boxed{\cos 29° = 0.8747}$

(iv) $\sin 61°$

Solution:

Let $f(x)=\sin x$

We take

$x=60°$ and

$\delta x=dx=1°=\frac{\pi}{180}=0.01745$ radians

$f(60°)=\sin 60°=0.8660$

$f(x)=\sin x$

Differentiate w.r.t x

$f'(x)=\cos x$

$f'(60°)=\cos 60°=\frac{1}{2}$

Using

$f(x+\delta x)\approx f(x)+f'(x)dx$

$f(60°+1°)\approx f(60°)+f'(60°)(0.01745)$

$f(61°)\approx(0.8660)+\frac{1}{2}(0.01745)$

$\sin 61°\approx 0.08660+(0.5)(0.01745)$

$\boxed{\sin 61° \approx 0.8747}$

Q.4 Find the approximate increase in the volume of a cube of if the length of its each edge changes from 5 to 5.02

Solution:

Let x be the length of each edge of cube and V be the volume of cube, then

$V = x^3$ ...(i)

Here

$x = 5, dx = 5.02 - 5 = 0.02$

Taking differentials on both sides of (i)

$dV = 3x^2 dx$

$dV = 3(5)^2(0.02)$

$dV = 75(0.02)$

$\boxed{dV = 1.5}$

Thus the approximate increase in volume of the cube is 1.5 cubic units

Q.5 Find the approximate increase in the area of circular disc if its diameter is increased from 44 cm to 44.4 cm

Solution:

Let r be the radius of the circle

then r changes from 22 cm to 22.2 cm

So $r = 22, dr = 22.2 - 22 = 0.2$

$A = \pi r^2$

Taking differentials of both sides

$dA = \pi(2rdr)$

$dA = 2\pi(22)(0.2)$

$dA = 8.8\pi$

$\boxed{dA = 27.646}$

Thus the approximate increase in area of circular disc is $27.646\, cm^2$