Integration:

"The inverse process of differentiation i.e the process of finding such a function whose derivative is given is called anti-derivation or integration".

For example:

An anti-derivative of $2x$ is x^2 since $\frac{d}{dx}\left(x^2\right)=2x$.

Indefinite Integral:

$\int f(x)\,dx=F(x)+c$ is called the indefinite integral of $f(x)$, where $f(x)$ is integrand, $F(x)$ is particular integral and c is the constant of integration.

Theorems on Integration:

(i) Prove that $\int\left[f(x)\right]^n f'(x)\,dx=\frac{\left[f(x)\right]^{n+1}}{n+1}+c, n\neq -1$

Proof:

Since $\frac{d}{dx}\left[f(x)\right]^{n+1}=(n+1)\left[f(x)\right]^n f'(x)$

So by definition

$$\int(n+1)\left[f(x)\right]^n f'(x)\,dx=\left[f(x)\right]^{n+1}+c_1$$

$$(n+1)\int\left[f(x)\right]^n f'(x)\,dx=\left[f(x)\right]^{n+1}+c_1$$

$$\int\left[f(x)\right]^n f'(x)\,dx=\frac{\left[f(x)\right]^{n+1}}{n+1}+\frac{c_1}{n+1}$$

$$\int\left[f(x)\right]^n f'(x)\,dx=\frac{\left[f(x)\right]^{n+1}}{n+1}+c \qquad \because c=\frac{c_1}{n+1}$$

(ii) Prove that

$$\int\left[f(x)\right]^{-1} f'(x)\,dx=\ln f(x)+c,\ f(x)>0$$

Proof:

Since $\frac{d}{dx}\ln\left[f(x)\right]=\frac{1}{f(x)}f'(x)$

By definition we have

$$\int\frac{1}{f(x)}f'(x)\,dx=\ln f(x)+c,\ f(x)>0$$

Standard Formulae for Anti-Derivatives:

General Form	Simple Form
$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)}+c, (n \neq -1)$	$\int x^n dx = \frac{x^{n+1}}{n+1}+c, (n \neq -1)$
$\int \sin(ax+b)dx = \frac{-\cos(ax+b)}{a}+c$	$\int \sin xdx = -\cos x + c$
$\int \cos(ax+b)dx = \frac{\sin(ax+b)}{a}+c$	$\int \cos xdx = \sin x + c$
$\int \sec^2(ax+b)dx = \frac{\tan(ax+b)}{a}+c$	$\int \sec^2 xdx = \tan x + c$
$\int \operatorname{cosec}^2(ax+b)dx = \frac{-\cot(ax+b)}{a}+c$	$\int \operatorname{cosec}^2 x\,dx = -\cot x + c$
$\int \sec(ax+b)\tan(ax+b)dx = \frac{\sec(ax+b)}{a}+c$	$\int \sec x \tan x\,dx = \sec x + c$
$\int \operatorname{cosec}(ax+b)\cot(ax+b)dx = \frac{-\operatorname{cosec}(ax+b)}{a}+c$	$\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$
$\int e^{\lambda x+\mu}dx = \frac{e^{\lambda x+\mu}}{\lambda}+c, (\lambda \neq 0)$	$\int e^x dx = e^x + c$
$\int a^{\lambda x+\mu}dx = \frac{a^{\lambda x+\mu}}{\lambda \ln a}+c, (a>0, a\neq 1, \lambda \neq 0)$	$\int a^x dx = \frac{a^x}{\ln a}+c\ (a>0, a\neq 1)$
$\int \frac{1}{ax+b}dx = \frac{\ln\|ax+b\|}{a}+c, (ax+b\neq 0)$	$\int \frac{1}{x}dx = \ln\|x\|+c, x\neq 0$
$\int \tan(ax+b)dx = \frac{1}{a}\ln\|\sec(ax+b)\|+c$ $= -\frac{1}{a}\ln\|\cos(ax+b)\|+c$	$\int \tan x\,dx = \ln\|\sec x\|+c$ $= -\ln\|\cos x\|+c$
$\int \cot(ax+b)dx = \frac{1}{a}\ln\|\sin(ax+b)\|+c$	$\int \cot x\,dx = \ln\|\sin x\|+c$
$\int \sec(ax+b)dx = \frac{1}{a}\ln\|\sec(ax+b)+\tan(ax+b)\|+c$	$\int \sec x\,dx = \ln\|\sec x + \tan x\|+c$
$\int \operatorname{cosec}(ax+b)dx = \frac{1}{a}\ln\|\operatorname{cosec}(ax+b)-\cot(ax+b)\|+c$	$\int \operatorname{cosec} x\,dx = \ln\|\operatorname{cosec} x - \cot x\|+c$

EXERCISE 3.2

Q.1 Evaluate the following indefinite integrals

(i) $\int(3x^2-2x+1)dx$

Solution:

$\int(3x^2-2x+1)dx$

$=\int 3x^2dx-\int 2xdx+\int 1dx$

$=3\int x^2dx-2\int xdx+1\int dx$

Using $\int x^n dx=\dfrac{x^{n+1}}{n+1}+c$

$=3\left(\dfrac{x^3}{3}\right)-2\left(\dfrac{x^2}{2}\right)+x+c$

$=x^3-x^2+x+c$

(ii) $\int\left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)dx,(x>0)$

Solution:

$\int\left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)dx$

$=\int x^{\frac{1}{2}}dx+\int x^{\frac{-1}{2}}dx$

$=\dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}+\dfrac{x^{\frac{-1}{2}+1}}{\frac{-1}{2}+1}+c$

$=\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}}+\dfrac{x^{\frac{1}{2}}}{\frac{1}{2}}+c$

$=\dfrac{2}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}+c$

(iii) $\int x\left(\sqrt{x}+1\right)dx,\left(x>0\right)$

Solution:

$\int x\left(\sqrt{x}+1\right)dx$

$=\int x\sqrt{x}dx+\int xdx$

$=\int x^{\frac{3}{2}}dx+\int xdx$

$=\dfrac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}+\dfrac{x^2}{2}+c$

$=\dfrac{x^{\frac{5}{2}}}{\frac{5}{2}}+\dfrac{x^2}{2}+c$

$=\dfrac{2}{5}x^{\frac{5}{2}}+\dfrac{1}{2}x^2+c$

(iv) $\int(2x+3)^{\frac{1}{2}}dx$

Solution:

$\int(2x+3)^{\frac{1}{2}}dx$

Multiplying and dividing by 2

$=\dfrac{1}{2}\int(2x+3)^{\frac{1}{2}}(2)dx$

$=\dfrac{1}{2}\dfrac{(2x+3)^{\frac{1}{2}+1}}{\frac{1}{2}+1}+c$

$=\dfrac{1}{2}\dfrac{(2x+3)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}+c$

$=\dfrac{1}{2}\times\dfrac{2}{3}(2x+3)^{\frac{3}{2}}+c$

$=\dfrac{1}{3}(2x+3)^{\frac{3}{2}}+c$

(v) $\int\left(\sqrt{x}+1\right)^2dx,(x>0)$

Solution:

$\int\left(\sqrt{x}+1\right)^2dx$

$=\int\left(x+2\sqrt{x}+1\right)dx$

$=\int xdx+2\int x^{\frac{1}{2}}dx+\int 1dx$

$=\dfrac{x^2}{2}+\dfrac{2x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}+x+c$

$=\dfrac{x^2}{2}+\dfrac{4}{3}x^{\frac{3}{2}}+x+c$

(vi) $\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx,(x>0)$

Solution:

$$\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx$$

$$=\int\left(x+\frac{1}{x}-2\right)dx$$

$$=\int x\,dx+\int x^{-1}dx-2\int dx$$

$$=\frac{x^2}{2}+\ln|x|-2x+c$$

$$\because \int\frac{1}{x}dx=\ln|x|+c$$

(vii) $\int\frac{3x+2}{\sqrt{x}}dx,(x>0)$

Solution:

$$\int\frac{3x+2}{\sqrt{x}}dx$$

$$=\int\left(\frac{3x}{\sqrt{x}}+\frac{2}{\sqrt{x}}\right)dx$$

$$=\int 3x^{\frac{1}{2}}dx+\int 2x^{\frac{-1}{2}}dx$$

$$=3\int x^{\frac{1}{2}}dx+2\int x^{\frac{-1}{2}}dx$$

$$=3\frac{x^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)}+2\frac{x^{\frac{-1}{2}+1}}{\left(\frac{-1}{2}+1\right)}+c$$

$$=3\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}+2\frac{x^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}+c$$

$$=2x^{\frac{3}{2}}+4x^{\frac{1}{2}}+c$$

(viii) $\int\frac{\sqrt{y}(y+1)}{y}dy,(y>0)$

Solution:

$$\int\frac{\sqrt{y}(y+1)}{y}dy$$

$$=\int\left(\frac{y+1}{\sqrt{y}}\right)dy$$

$$=\int\frac{y}{\sqrt{y}}dy+\int\frac{1}{\sqrt{y}}dy$$

$$=\int y^{\frac{1}{2}}dy+\int y^{\frac{-1}{2}}dy$$

$$=\frac{y^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)}+\frac{y^{\frac{-1}{2}+1}}{\left(\frac{-1}{2}+1\right)}+c$$

$$=\frac{y^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}+\frac{y^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}+c$$

$$=\frac{2}{3}y^{\frac{3}{2}}+2y^{\frac{1}{2}}+c$$

(ix) $\int\frac{\left(\sqrt{\theta}-1\right)^2}{\sqrt{\theta}}d\theta,(\theta>0)$

Solution:

$$\int\frac{\left(\sqrt{\theta}-1\right)^2}{\sqrt{\theta}}d\theta$$

$$=\int\frac{\left(\theta+1-2\sqrt{\theta}\right)}{\sqrt{\theta}}d\theta$$

$$=\int\left(\frac{\theta}{\sqrt{\theta}}+\frac{1}{\sqrt{\theta}}-\frac{2\sqrt{\theta}}{\sqrt{\theta}}\right)d\theta$$

$$=\int\theta^{\frac{1}{2}}d\theta+\int\theta^{\frac{-1}{2}}d\theta-2\int d\theta$$

$$=\frac{\theta^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)}+\frac{\theta^{\frac{-1}{2}+1}}{\left(\frac{-1}{2}+1\right)}-2\theta+c$$

$$=\frac{\theta^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}+\frac{\theta^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}-2\theta+c$$

$$= \frac{2}{3}\theta^{\frac{3}{2}} + 2\theta^{\frac{1}{2}} - 2\theta + c$$

(x) $\int \frac{\left(1-\sqrt{x}\right)^2}{\sqrt{x}}\, dx, (x>0)$

Solution:

$$\int \frac{\left(1-\sqrt{x}\right)^2}{\sqrt{x}}\, dx$$

$$= \int \frac{\left(1+x-2\sqrt{x}\right)}{\sqrt{x}}\, dx$$

$$= \int \left(\frac{1}{\sqrt{x}} + \frac{x}{\sqrt{x}} - \frac{2\sqrt{x}}{\sqrt{x}} \right) dx$$

$$= \int x^{\frac{-1}{2}} dx + \int x^{\frac{1}{2}} dx - 2\int dx$$

$$= \frac{x^{\frac{-1}{2}+1}}{\left(\frac{-1}{2}+1\right)} + \frac{x^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)} - 2x + c$$

$$= \frac{x^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + \frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} - 2x + c$$

$$= 2x^{\frac{1}{2}} + \frac{2}{3}x^{\frac{3}{2}} - 2x + c$$

(xi) $\int \frac{e^{2x}+e^x}{e^x}\, dx$

Solution:

$$\int \frac{e^{2x}+e^x}{e^x}\, dx$$

$$= \int \frac{e^{2x}}{e^x}\, dx + \int \frac{e^x}{e^x}\, dx$$

$$= \int e^x dx + \int dx$$

$$= e^x + x + c$$

Q.2 Evaluate

(i) $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}, \begin{pmatrix} x+a>0 \\ x+b>0 \end{pmatrix}$

Solution:

$$\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$$

By rationalizing the denominator

$$= \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}} \times \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}\, dx$$

$$= \int \frac{\sqrt{x+a}-\sqrt{x+b}}{\left(\sqrt{x+a}\right)^2 - \left(\sqrt{x+b}\right)^2}\, dx$$

$$= \int \frac{\sqrt{x+a}-\sqrt{x+b}}{x+a-x-b}\, dx$$

$$= \frac{1}{a-b}\left(\int (x+a)^{\frac{1}{2}} dx - \int (x+b)^{\frac{1}{2}} dx \right)$$

$$= \frac{1}{a-b}\left(\frac{(x+a)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)} - \frac{(x+b)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)} \right) + c$$

$$= \frac{1}{(a-b)}\left(\frac{2}{3}(x+a)^{\frac{3}{2}} - \frac{2}{3}(x+b)^{\frac{3}{2}} \right) + c$$

$$= \frac{2}{3(a-b)}\left((x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}} \right) + c$$

(ii) $\int \frac{1-x^2}{1+x^2}\, dx$

Solution:

$\frac{1-x^2}{1+x^2}$ is an improper fraction therefor

$$x^2+1 \overline{\smash{)}\, -x^2+1} \quad \text{quotient } -1$$

$$\underline{\mp x^2 \mp 1}$$

$$2$$

So, $\int \frac{1-x^2}{1+x^2} dx$

$$= \int \left(-1 + \frac{2}{1+x^2} \right) dx$$

$$= -\int 1 dx + 2\int \frac{1}{1+x^2}\, dx$$

$$= -x + 2\tan^{-1}x + c \because \int \frac{1}{1+x^2}dx = \tan^{-1}x$$

$$= 2\tan^{-1}x - x + c$$

(iii) $\int \frac{dx}{\sqrt{x+a}+\sqrt{x}}, (x>0, a>0)$

Solution:

$$\int \frac{dx}{\sqrt{x+a}+\sqrt{x}}$$

By rationalizing the denominator

$$= \int \frac{1}{\sqrt{x+a}+\sqrt{x}} \times \frac{\sqrt{x+a}-\sqrt{x}}{\sqrt{x+a}-\sqrt{x}}dx$$

$$= \int \frac{\sqrt{x+a}-\sqrt{x}}{\left(\sqrt{x+a}\right)^2 - \left(\sqrt{x}\right)^2}dx$$

$$= \int \frac{\sqrt{x+a}-\sqrt{x}}{x+a-x}dx$$

$$= \frac{1}{a}\left(\int x+a)^{\frac{1}{2}}dx - \int x^{\frac{1}{2}}dx\right)$$

$$= \frac{1}{a}\left(\frac{(x+a)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} - \frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right) + c$$

$$= \frac{1}{a}\left(\frac{2}{3}(x+a)^{\frac{3}{2}} - \frac{2}{3}x^{\frac{3}{2}}\right) + c$$

$$= \frac{2}{3a}\left((x+a)^{\frac{3}{2}} - x^{\frac{3}{2}}\right) + c$$

(iv) $\int (a-2x)^{\frac{3}{2}}dx$

Solution:

$$\int (a-2x)^{\frac{3}{2}}dx$$

$$= -\frac{1}{2}\int (a-2x)^{\frac{3}{2}}(-2)dx$$

$$= -\frac{1}{2}\left(\frac{(a-2x)^{\frac{3}{2}+1}}{\left(\frac{3}{2}+1\right)}\right) + c$$

$$= -\frac{1}{2}\left(\frac{(a-2x)^{\frac{5}{2}}}{\left(\frac{5}{2}\right)}\right) + c$$

$$= -\frac{1}{5}(a-2x)^{\frac{5}{2}} + c$$

(v) $\int \frac{(1+e^x)^3}{e^x}dx$

Solution:

$$\int \frac{(1+e^x)^3}{e^x}dx$$

Using $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$

$$= \int \left(\frac{(1+e^{3x} + 3(1)(e^x)(1+e^x))}{e^x}\right)dx$$

$$= \int \left(\frac{(1+e^{3x} + 3e^x + 3e^{2x})}{e^x}\right)dx$$

$$= \int \left(\frac{1}{e^x} + \frac{e^{3x}}{e^x} + \frac{3e^x}{e^x} + \frac{3e^{2x}}{e^x}\right)dx$$

$$= \int e^{-x}dx + \int e^{2x}dx + 3\int dx + 3\int e^x dx$$

$$= \frac{e^{-x}}{-1} + \frac{e^{2x}}{2} + 3x + 3e^x + c$$

$$= -\frac{1}{e^x} + \frac{e^{2x}}{2} + 3x + 3e^x + c$$

(vi) $\int \sin(a+b)x\,dx$

Solution:

$$\int \sin(a+b)x\,dx$$

$$= \frac{1}{a+b}\left(-\cos(a+b)x\right) + c$$

$$= -\frac{1}{(a+b)}\cos(a+b)x + c$$

(vii) $\int \sqrt{1-\cos 2x}\,dx, \ (1-\cos 2x > 0)$

Solution:

$$\int \sqrt{1-\cos 2x}\,dx$$

$$= \int \sqrt{2\sin^2 x}dx \ \because 1-\cos 2x = 2\sin^2 x$$

$$= \sqrt{2}\int \sin x dx$$

$$= \sqrt{2}(-\cos x) + c$$

(viii) $\int (\ln x) \times \frac{1}{x} dx, \ (x > 0)$

Solution:

$$\int (\ln x) \times \frac{1}{x} dx$$

$$= \frac{(\ln x)^2}{2} + c$$

(ix) $\int \sin^2 x dx$

Solution:

$$\int \sin^2 x dx$$

$$\because \sin^2 x = \left(\frac{1-\cos 2x}{2}\right)$$

$$= \int \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{2}\left[\int (1-\cos 2x) dx\right]$$

$$= \frac{1}{2}\left[\int dx - \int \cos 2x \, dx\right]$$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + c$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

(x) $\int \frac{1}{1+\cos x} dx, \left(\frac{-\pi}{2} < x < \frac{\pi}{2}\right)$

Solution:

$$\int \frac{1}{1+\cos x} dx$$

$$= \int \frac{1}{2\cos^2 \frac{x}{2}} dx \quad \because 1+\cos x = 2\cos^2 \frac{x}{2}$$

$$= \frac{1}{2}\left(\int \sec^2 \frac{x}{2}\right) dx$$

$$= -\sqrt{2}\cos x + c$$

$$= \frac{1}{2}\left(\frac{\tan \frac{x}{2}}{\left(\frac{1}{2}\right)}\right) + c$$

$$= \tan \frac{x}{2} + c$$

(xi) $\int \frac{ax+b}{ax^2+2bx+c} dx$

Solution:

$$\int \frac{ax+b}{ax^2+2bx+c} dx$$

$$= \frac{1}{2}\int \left(\frac{2ax+2b}{ax^2+2bx+c}\right) dx$$

$$= \frac{1}{2}\ln\left|ax^2+2bx+c\right| + c_1$$

(xii) $\int \cos 3x \sin 2x dx$

Solution:

$$\int \cos 3x \sin 2x dx$$

$$= \frac{1}{2}\int (2\cos 3x \sin 2x) dx$$

Using

$$2\cos\alpha \sin\alpha = \sin(\alpha+\beta) - \sin(\alpha-\beta)$$

$$= \frac{1}{2}\int \left[\sin(3x+2x) - \sin(3x-2x)\right] dx$$

$$= \frac{1}{2}\left(\int \sin 5x dx - \int \sin x dx\right)$$

$$=\frac{1}{2}\left[\frac{-\cos 5x}{5}-(-\cos x)\right]+c$$

$$=-\frac{1}{2}\left(\frac{\cos 5x}{5}-\cos x\right)+c$$

(xiii) $\int \frac{\cos 2x-1}{1+\cos 2x}dx,\ (1+\cos 2x\neq 0)$

Solution:

$$\int \frac{\cos 2x-1}{1+\cos 2x}dx$$

$$=-\int\left(\frac{1-\cos 2x}{1+\cos 2x}\right)dx$$

As

$$1+\cos 2x=2\cos^2 x$$

$$1-\cos 2x=2\sin^2 x$$

$$=-\int \frac{2\sin^2 x}{2\cos^2 x}dx$$

$$=-\int \tan^2 x dx$$

$$=-\int\left(\sec^2 x-1\right)dx$$

$$=\int 1dx-\int \sec^2 x dx$$

$$=x-\tan x+c$$

(xiv) $\int \tan^2 x\, dx$

Solution:

$$\int \tan^2 x\, dx$$

$$=\int\left(\sec^2 x-1\right)dx$$

$$=\int \sec^2 x dx-\int dx$$

$$=\tan x-x+c$$