Integration by Method of Substitution:

Integration by method of substitution involves the introduction of a function that changes the integrand into an easy integral form.

Note:

This method is commonly used for irrational functions. Some useful substitutions are given in table below

Expression Involving	Suitable Substitution
$\sqrt{a^2-x^2}$	$x=a\sin\theta$
$\sqrt{x^2-a^2}$	$x=a\sec\theta$
$\sqrt{a^2+x^2}$	$x=a\tan\theta$
$\sqrt{x\pm a}$	$\sqrt{x\pm a}=t$
$\sqrt{2ax-x^2}$	$x-a=a\sin\theta$
$\sqrt{2ax+x^2}$	$x+a=a\sec\theta$

Some Important Results:

$$\int\frac{1}{a^2-x^2}dx=\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+c$$

$$\int\frac{1}{x^2-a^2}dx=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+c$$

$$\int\frac{1}{a^2+x^2}dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$$

$$\int\frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\left(\frac{x}{a}\right)+c$$

$$\int\frac{1}{\sqrt{x^2-a^2}}dx=\ln\left|x+\sqrt{x^2-a^2}\right|+c$$

$$\int\frac{1}{\sqrt{x^2+a^2}}dx=\ln\left|x+\sqrt{x^2+a^2}\right|+c$$

$$\int\frac{1}{|x|\sqrt{x^2-a^2}}dx=\frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)+c$$

$$\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$$

$$\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+c$$

$$\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right|+c$$

EXERCISE 3.3

Evaluate the following integrals

Q.1 $\int \frac{-2x}{\sqrt{4-x^2}}\,dx$

Solution:

$$\int \frac{-2x}{\sqrt{4-x^2}}\,dx$$

$$=\frac{(4-x^2)^{-\frac{1}{2}+1}}{\left(-\frac{1}{2}+1\right)}+c$$

$$=\frac{(4-x^2)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}+c$$

$$=2\sqrt{4-x^2}+c$$

Q.2 $\int \frac{dx}{x^2+4x+13}$

Solution:

$$\int \frac{dx}{x^2+4x+13}$$

$$=\int \frac{dx}{(x^2+4x+4)+9}$$

$$=\int \frac{dx}{(x+2)^2+(3)^2}$$

Using $\int \frac{1}{a^2+x^2}dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

$$=\frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right)+c$$

Q.3 $\int \frac{x^2}{4+x^2}dx$

Solution:

$\frac{x^2}{4+x^2}$ is an improper fraction

By long division

$$x^2+4\overline{\big)\ x^2\ }\quad 1$$
$$\underline{\pm x^2 \pm 4}$$
$$-4$$

$$\therefore \int \frac{x^2}{4+x^2}dx=\int\left(1-\frac{4}{4+x^2}\right)dx$$

$$=\int 1dx-4\int\frac{1}{4+x^2}dx$$

$$=x-4\int\frac{1}{2^2+x^2}dx$$

Using $\int \frac{1}{a^2+x^2}dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

$$=x-4\left(\frac{1}{2}\tan^{-1}\frac{x}{2}\right)+c$$

$$=x-2\tan^{-1}\frac{x}{2}+c$$

Q.4 $\int \frac{1}{x\ln x}dx$

Solution:

$$\int \frac{1}{x\ln x}dx$$

$$=\int\frac{\left(\frac{1}{x}\right)}{\ln x}dx$$

$$=\ln|\ln x|+c$$

Q.5 $\int \frac{e^x}{e^x+3}dx$

Solution:

$$\int \frac{e^x}{e^x+3}dx$$

$$=\ln|e^x+3|+c$$

Q.6 $\int \frac{x+b}{(x^2+2bx+c)^{\frac{1}{2}}}dx$

Solution:

$$\int \frac{x+b}{(x^2+2bx+c)^{\frac{1}{2}}}dx$$

$$=\frac{1}{2}\int\frac{2x+2b}{(x^2+2bx+c)^{\frac{1}{2}}}dx$$

$$=\frac{1}{2}\int(x^2+2bx+c)^{-\frac{1}{2}}(2x+2b)dx$$

$$=\frac{1}{2}\frac{\left(x^2+2bx+c\right)^{-\frac{1}{2}+1}}{\left(-\frac{1}{2}+1\right)}+c_1$$

$$=\frac{1}{2}\frac{\left(x^2+2bx+c\right)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}+c_1$$

$$=\sqrt{x^2+2bx+c}+c_1$$

Q.7 $\int\frac{\sec^2 x}{\sqrt{\tan x}}dx$

Solution:

$$\int\frac{\sec^2 x}{\sqrt{\tan x}}dx$$

$$=\int\left(\tan x\right)^{-\frac{1}{2}}\sec^2 xdx$$

$$=\frac{\left(\tan x\right)^{-\frac{1}{2}+1}}{\left(-\frac{1}{2}+1\right)}+c$$

$$=\frac{\left(\tan x\right)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}+c$$

$$=2\left(\tan x\right)^{\frac{1}{2}}$$

$$=2\sqrt{\tan x}+c$$

Q.8 (a) Show that

$$\int\frac{dx}{\sqrt{x^2-a^2}}dx=\ln\left(x+\sqrt{x^2-a^2}\right)+c$$

Solution:

$$\int\frac{dx}{\sqrt{x^2-a^2}}$$

Let $x=a\sec\theta\Rightarrow\sec\theta=\frac{x}{a}$

Then $dx=a\sec\theta\tan\theta d\theta$

$$\therefore\int\frac{dx}{\sqrt{x^2-a^2}}$$

$$=\int\frac{a\sec\theta\tan\theta d\theta}{\sqrt{a^2\sec^2\theta-a^2}}$$

$$=\int\frac{a\sec\theta\tan\theta d\theta}{\sqrt{a^2\left(\sec^2\theta-1\right)}}$$

$$=\frac{a}{a}\int\frac{\sec\theta\tan\theta}{\sqrt{\tan^2\theta}}d\theta$$

$$=\int\sec\theta d\theta$$

$$=\ln\left|\sec\theta+\tan\theta\right|+c_1\quad\ldots(\text{i})$$

$$\because\sec\theta=\frac{x}{a}$$

$$\tan\theta=\sqrt{\sec^2\theta-1}$$

$$=\sqrt{\frac{x^2}{a^2}-1}$$

$$\Rightarrow\tan\theta=\frac{\sqrt{x^2-a^2}}{a}$$

(i) becomes

$$=\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+c_1$$

$$=\ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|+c_1$$

$$=\ln\left|x+\sqrt{x^2-a^2}\right|-\ln a+c_1$$

$$=\ln\left|x+\sqrt{x^2-a^2}\right|+c\quad\because-\ln a+c_1=c$$

(b) Show that

$$\int\sqrt{a^2-x^2}dx=\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+c$$

Solution:

$$\int\sqrt{a^2-x^2}dx$$

Let $x=a\sin\theta\Rightarrow\theta=\sin^{-1}\frac{x}{a}$

$\Rightarrow dx=a\cos\theta d\theta$

$$\therefore\int\sqrt{a^2-x^2}dx$$

$$=\int\sqrt{a^2-a^2\sin^2\theta}\,a\cos\theta d\theta$$

$$=\int a\sqrt{1-\sin^2\theta}.a\cos\theta d\theta$$

$$=a^2\int\sqrt{\cos^2\theta}.\cos\theta d\theta$$

$$= a^2\int \cos^2\theta d\theta$$

$$= a^2\int\left(\frac{1+\cos 2\theta}{2}\right)d\theta \quad \because \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= \frac{a^2}{2}\left[\int 1d\theta + \int \cos 2\theta d\theta\right]$$

$$= \frac{a^2}{2}\left[\theta + \left(\frac{\sin 2\theta}{2}\right)\right] + c$$

$$= \frac{a^2}{2}\left[\theta + \frac{2\sin\theta\cos\theta}{2}\right] + c$$

As $x = a\sin\theta$

$$\Rightarrow \sin\theta = \frac{x}{a} \quad \Rightarrow \theta = \sin^{-1}\frac{x}{a}$$

$$\Rightarrow \cos\theta = \sqrt{1-\sin^2\theta}$$

$$\cos\theta = \sqrt{1-\frac{x^2}{a^2}} = \frac{\sqrt{a^2-x^2}}{a}$$

$$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \frac{x}{a}\frac{\sqrt{a^2-x^2}}{a}\right] + c$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + c$$

Evaluate the following integrals:

Q.9 $\int \frac{dx}{\left(1+x^2\right)^{\frac{3}{2}}}$

Solution:

$$\int \frac{dx}{\left(1+x^2\right)^{\frac{3}{2}}}$$

Let $x = \tan\theta$

$$\Rightarrow dx = \sec^2\theta d\theta$$

$$\therefore \int \frac{dx}{\left(1+x^2\right)^{\frac{3}{2}}}$$

$$= \int \frac{\sec^2\theta d\theta}{\left(1+\tan^2\theta\right)^{\frac{3}{2}}}$$

$$= \int \frac{\sec^2\theta d\theta}{\left(\sec^2\theta\right)^{\frac{3}{2}}}$$

$$= \int \frac{\sec^2\theta}{\sec^3\theta}d\theta$$

$$= \int \frac{1}{\sec\theta}d\theta$$

$$= \int \cos\theta d\theta$$

$$= \sin\theta + c \ \ldots\text{(i)}$$

$$\because \tan\theta = x \Rightarrow \cot\theta = \frac{1}{x}$$

$$\sin\theta = \frac{1}{\sqrt{1+\cot^2\theta}} = \frac{1}{\sqrt{1+\frac{1}{x^2}}} = \frac{x}{\sqrt{x^2+1}}$$

(i) Becomes

$$= \frac{x}{\sqrt{1+x^2}} + c$$

Q.10 $\int \frac{1}{\left(1+x^2\right)\tan^{-1}x}dx$

Solution:

$$\int \frac{1}{\left(1+x^2\right)\tan^{-1}x}dx$$

$$= \int \frac{\frac{1}{1+x^2}}{\tan^{-1}x}dx$$

$$= \ln\left|\tan^{-1}x\right| + c$$

Q.11 $\int \sqrt{\frac{1+x}{1-x}}dx$

Solution:

$$\int \sqrt{\frac{1+x}{1-x}}dx$$

$$= \int \sqrt{\frac{1+x}{1-x}\times\frac{1+x}{1+x}}dx$$

$$= \int \sqrt{\frac{(1+x)^2}{1-x^2}}dx = \int \frac{1+x}{\sqrt{1-x^2}}dx$$

$$= \int \frac{1}{\sqrt{1-x^2}}dx + \int \frac{x}{\sqrt{1-x^2}}dx$$

$$= \sin^{-1}x - \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}dx$$

$$=\sin^{-1}x-\frac{1}{2}\int\left(1-x^2\right)^{\frac{-1}{2}}(-2x)\,dx$$

$$=\sin^{-1}x-\frac{1}{2}\frac{\left(1-x^2\right)^{-\frac{1}{2}+1}}{\left(\frac{-1}{2}+1\right)}+c$$

$$=\sin^{-1}x-\frac{1}{2}\frac{\left(1-x^2\right)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}+c$$

$$=\sin^{-1}x-\sqrt{1-x^2}+c$$

Q.12 $\int\frac{\sin\theta}{1+\cos^2\theta}d\theta$

Solution:

$$\int\frac{\sin\theta}{1+\cos^2\theta}d\theta$$

Let $\cos\theta=x\Rightarrow-\sin\theta\,d\theta=dx$

$$\Rightarrow\sin\theta\,d\theta=-dx$$

$$\therefore\int\frac{\sin\theta\,d\theta}{1+\cos^2\theta}$$

$$=\int\frac{-dx}{1+x^2}$$

$$=-\int\frac{1}{1+x^2}dx$$

$$=-\tan^{-1}x+c$$

$$=-\tan^{-1}(\cos\theta)+c \qquad \because\cos\theta=x$$

Q.13 $\int\frac{ax}{\sqrt{a^2-x^4}}dx$

Solution:

$$\int\frac{ax}{\sqrt{a^2-x^4}}dx$$

$$=\int\frac{ax}{\sqrt{a^2-\left(x^2\right)^2}}dx$$

Let

$$x^2=a\sin\theta\Rightarrow\frac{x^2}{a}=\sin\theta\Rightarrow\theta=\sin^{-1}\frac{x^2}{\alpha}$$

$$2xdx=a\cos\theta\,d\theta$$

$$\Rightarrow xdx=\frac{a}{2}\cos\theta\,d\theta$$

$$\therefore\int\frac{ax}{\sqrt{a^2-x^4}}dx$$

$$=\int\frac{a.\frac{a}{2}\cos\theta}{\sqrt{a^2-a^2\sin^2\theta}}d\theta$$

$$=\frac{a^2}{2}\int\frac{\cos\theta.d\theta}{a\sqrt{1-\sin^2\theta}}$$

$$=\frac{a}{2}\int\frac{\cos\theta.d\theta}{\cos\theta}$$

$$=\frac{a}{2}\int d\theta$$

$$=\frac{a}{2}(\theta)+c$$

$$=\frac{a}{2}\sin^{-1}\frac{x^2}{a}+c$$

Q.14 $\int\frac{dx}{\sqrt{7-6x-x^2}}$

Solution:

$$\int\frac{dx}{\sqrt{7-6x-x^2}}$$

$$=\int\frac{dx}{\sqrt{7-6x-x^2-9+9}}$$

$$=\int\frac{dx}{\sqrt{16-\left(x^2+6x+9\right)}}$$

$$=\int\frac{dx}{\sqrt{(4)^2-(x+3)^2}}$$

$$\because\int\frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\frac{x}{a}+c$$

$$=\sin^{-1}\left(\frac{x+3}{4}\right)+c$$

Q.15 $\int\frac{\cos x}{\sin x\ln\sin x}dx$

Solution:

$$\int\frac{\cos x}{\sin x\ln\sin x}dx$$

Let $\ln\sin x=t$

$$\Rightarrow \frac{1}{\sin x}(\cos x)\,dx = dt$$

$$\therefore \int \frac{\cos x\,dx}{\sin x \ln \sin x}$$

$$= \int \frac{dt}{t}$$

$$= \ln|t| + c$$

$$= \ln(\ln \sin x) + c$$

Q.16 $\int \cos x\left(\frac{\ln \sin x}{\sin x}\right)dx$

Solution:

$$\int \cos x\left(\frac{\ln \sin x}{\sin x}\right)dx$$

Let $\ln \sin x = t$

$$\Rightarrow \frac{1}{\sin x}(\cos x)\,dx = dt$$

$$\therefore \int \cos\left(\frac{\ln \sin x}{\sin x}\right)dx$$

$$= \int t\,dt$$

$$= \frac{t^2}{2} + c$$

$$= \frac{(\ln \sin x)^2}{2} + c$$

Q.17 $\int \frac{x\,dx}{4+2x+x^2}$

Solution:

$$\int \frac{x\,dx}{4+2x+x^2}$$

$$= \frac{1}{2}\int \frac{2x}{x^2+2x+4}\,dx$$

$$= \frac{1}{2}\int \frac{2x+2-2}{x^2+2x+4}\,dx$$

$$= \frac{1}{2}\int \frac{2x+2}{x^2+2x+4}\,dx - \frac{1}{2}\int \frac{2}{x^2+2x+4}\,dx$$

$$= \frac{1}{2}\ln\left|x^2+2x+4\right| - \int \frac{1}{x^2+2x+1+3}\,dx$$

$$= \frac{1}{2}\ln\left|x^2+2x+4\right| - \int \frac{1}{(x+1)^2+\left(\sqrt{3}\right)^2}\,dx$$

$$= \frac{1}{2}\ln\left|x^2+2x+4\right| - \frac{1}{\sqrt{3}}\tan^{-1}\frac{x+1}{\sqrt{3}} + c$$

Q.18 $\int \frac{x}{x^4+2x^2+5}\,dx$

Solution:

$$\int \frac{x}{x^4+2x^2+5}\,dx$$

$$= \int \frac{x\,dx}{x^4+2x^2+1+4}$$

$$= \int \frac{x\,dx}{\left(x^2+1\right)^2+(2)^2}$$

$$= \frac{1}{2}\int \frac{2x}{\left(x^2+1\right)^2+(2)^2}\,dx$$

$$= \frac{1}{2}\left(\frac{1}{2}\tan^{-1}\left(\frac{x^2+1}{2}\right)\right) + c$$

$$= \frac{1}{4}\tan^{-1}\left(\frac{x^2+1}{2}\right) + c$$

Q.19 $\int \cos\left(\sqrt{x} - \frac{x}{2}\right)\times\left(\frac{1}{\sqrt{x}} - 1\right)dx$

Solution:

$$\int \cos\left(\sqrt{x} - \frac{x}{2}\right)\times\left(\frac{1}{\sqrt{x}} - 1\right)dx$$

Put $\sqrt{x} - \frac{x}{2} = t$

$$\left(\frac{1}{2\sqrt{x}} - \frac{1}{2}\right)dx = dt$$

$$\frac{1}{2}\left(\frac{1}{\sqrt{x}} - 1\right)dx = dt$$

$$\left(\frac{1}{\sqrt{x}} - 1\right)dx = 2dt$$

$$= \int \cos t\,2dt$$

$$= 2\int \cos t\,dt$$

$$= 2\sin t + c$$

$$= 2\sin\left(\sqrt{x} - \frac{x}{2}\right) + c$$

Q.20 $\int \frac{x+2}{\sqrt{x+3}}dx$

Solution:

$$\int \frac{x+2}{\sqrt{x+3}}dx$$

$$= \int \frac{x+2+1-1}{\sqrt{x+3}}dx$$

$$= \int \frac{x+3-1}{\sqrt{x+3}}dx$$

$$= \int \left(\frac{x+3}{\sqrt{x+3}} - \frac{1}{\sqrt{x+3}}\right)dx$$

$$= \int \sqrt{x+3}\,dx - \int \frac{1}{\sqrt{x+3}}dx$$

$$= \int (x+3)^{\frac{1}{2}}dx - \int (x+3)^{-\frac{1}{2}}dx$$

$$= \frac{(x+3)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} - \frac{(x+3)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + c$$

$$= \frac{2}{3}(x+3)^{\frac{3}{2}} - 2(x+3)^{\frac{1}{2}} + c$$

Q.21 $\int \frac{\sqrt{2}}{\sin x + \cos x}dx$

Solution:

$$\int \frac{\sqrt{2}}{\sin x + \cos x}dx$$

$$= \int \frac{1}{\frac{1}{\sqrt{2}}(\sin x + \cos x)}dx$$

$$= \int \frac{1}{\sin x \frac{1}{\sqrt{2}} + \cos x \frac{1}{\sqrt{2}}}dx$$

$$= \int \frac{1}{\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}}dx$$

Using $\sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha+\beta)$

$$= \int \frac{1}{\sin\left(x+\frac{\pi}{4}\right)}dx$$

$$= \int \operatorname{cosec}\left(x+\frac{\pi}{4}\right)dx$$

$$= \ln\left|\operatorname{cosec}\left(x+\frac{\pi}{4}\right) - \cot\left(x+\frac{\pi}{4}\right)\right| + c$$

Q.22 $\int \frac{dx}{\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x}$

Solution:

$$\int \frac{dx}{\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x}$$

$$= \int \frac{1}{\sin\frac{\pi}{6}\sin x + \cos\frac{\pi}{6}\cos x}$$

$$= \int \frac{dx}{\cos x\cos\frac{\pi}{6} + \sin x\sin\frac{\pi}{6}}$$

Using $\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta)$

$$= \int \frac{dx}{\cos\left(x-\frac{\pi}{6}\right)}$$

$$= \int \sec\left(x-\frac{\pi}{6}\right)dx$$

$$= \ln\left|\sec\left(x-\frac{\pi}{6}\right) + \tan\left(x-\frac{\pi}{6}\right)\right| + c$$