Integration By Parts:

If we want to evaluate the integral $\int f(x)g'(x)dx$, we may use integration by parts

Formula:

$$\boxed{\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx + c}$$

Results:

- $\int e^{ax}\left[af(x)+f'(x)\right]dx = e^{ax}f(x)+c$
- $\int u\,dv = uv - \int v\,du$

EXERCISE 3.4

Q.1 Evaluate the following integrals by parts.

(i) $\int x\sin x\,dx$

Solution:

$\int x\sin x\,dx$

Integrating by parts

$= x(-\cos x) - \int(-\cos x).1dx$

$= -x\cos x + \int \cos x\,dx$

$= -x\cos x + \sin x + c$

(ii) $\int \ln x dx$

Solution:

$\int \ln x.1dx$

Integrating by parts

$= (\ln x)(x) - \int x.\frac{1}{x}dx$

$= x\ln x - \int 1dx$

$= x\ln x - x + c$

(iii) $\int x\ln x dx$

Solution:

$\int \ln x.xdx$

Integrating by parts

$= \ln x\left(\frac{x^2}{2}\right) - \int\left(\frac{x^2}{2}\right).\frac{1}{x}dx$

$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$

$= \frac{x^2}{2}\ln x - \frac{1}{2}\left(\frac{x^2}{2}\right) + c$

$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$

(iv) $\int x^2\ln xdx$

Solution:

$\int \ln x.x^2dx$

Integrating by parts

$= \ln x\left(\frac{x^3}{3}\right) - \int\left(\frac{x^3}{3}\right).\frac{1}{x}dx$

$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx$

$= \frac{x^3}{3}\ln x - \frac{1}{3}\left(\frac{x^3}{3}\right) + c = \frac{x^3}{3}\ln x - \frac{x^3}{9} + c$

(v) $\int x^3\ln x\,dx$

Solution:

$\int \ln x.x^3\,dx$

Integrating by parts

$= \ln x\left(\frac{x^4}{4}\right) - \int\frac{x^4}{4}.\frac{1}{x}dx$

$= \frac{x^4}{4}(\ln x) - \frac{1}{4}\int x^3dx$

$$=\frac{x^4}{4}(\ln x)-\frac{1}{4}\left(\frac{x^4}{4}\right)+c$$

$$=\frac{x^4}{4}(\ln x)-\frac{1}{16}x^4+c$$

(vi) $\int x^4\ln x\,dx$

Solution:

$$\int \ln x.x^4\,dx$$

Integrating by parts

$$=\ln x\left(\frac{x^5}{5}\right)-\int\left(\frac{x^5}{5}\right).\left(\frac{1}{x}\right)dx$$

$$=\frac{x^5}{5}(\ln x)-\frac{1}{5}\int x^4dx$$

$$=\frac{x^5}{5}(\ln x)-\frac{1}{5}\left(\frac{x^5}{5}\right)+c=\frac{x^5}{5}\ln x-\frac{x^5}{25}+c$$

(vii) $\int \tan^{-1}x\,dx$

Solution:

$$\int \tan^{-1}x.1dx$$

Integrating by parts

$$=\tan^{-1}x(x)-\int x.\frac{1}{1+x^2}dx$$

$$=x\tan^{-1}x-\frac{1}{2}\int\frac{2x}{1+x^2}dx$$

$$=x\tan^{-1}x-\frac{1}{2}\ln\left|1+x^2\right|+c$$

(viii) $\int x^2\sin x dx$

Solution:

$$\int x^2\sin x dx$$

Integrating by parts

$$=x^2(-\cos x)-\int(-\cos x)(2x)\,dx$$

$$=-x^2\cos x+2\int x\cos x\,dx$$

Integrating by parts

$$=-x^2\cos x+2\left[x.(\sin x)-\int\sin x.1dx\right]$$

$$=-x^2\cos x+2x\sin x-2(-\cos x)+c$$

$$=-x^2\cos x+2x\sin x+2\cos x+c$$

(ix) $\int x^2\tan^{-1}x$

Solution:

$$\int \tan^{-1}x.x^2\,dx$$

Integrating by parts

$$=\tan^{-1}x\left(\frac{x^3}{3}\right)-\int\frac{x^3}{3}.\frac{1}{1+x^2}dx$$

$$=\frac{x^3}{3}\tan^{-1}x-\frac{1}{3}\int\frac{x^3}{1+x^2}dx\ \ldots\ldots\text{(i)}$$

$\frac{x^3}{1+x^2}$ is an improper fraction, therefore by long division

$$\begin{array}{r|l} & \quad x \\ x^2+1 & \quad x^3 \\ & \underline{\pm x^3\pm x} \\ & \quad -x \end{array}$$

So $\frac{x^3}{1+x^2}=x-\frac{x}{x^2+1}$

Hence (i) becomes

$$=\frac{x^3}{3}\tan^{-1}x-\frac{1}{3}\left[\int\left(x-\frac{x}{x^2+1}\right)dx\right]$$

$$=\frac{x^3}{3}\tan^{-1}x-\frac{1}{3}\left[\int x\,dx-\frac{1}{2}\int\frac{2x}{x^2+1}dx\right]$$

$$=\frac{x^3}{3}\tan^{-1}x-\frac{1}{3}\left[\frac{x^2}{2}-\frac{1}{2}\ln\left(x^2+1\right)\right]+c$$

$$=\frac{x^3}{3}\tan^{-1}x-\frac{1}{6}x^2+\frac{1}{6}\ln\left(x^2+1\right)+c$$

(x) $\int x\tan^{-1}x\,dx$

Solution:

$$\int \tan^{-1}x.x\,dx$$

Integrating by parts

$$=\tan^{-1}x\left(\frac{x^2}{2}\right)-\int\frac{x^2}{2}\left(\frac{1}{1+x^2}\right)dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\frac{x^2}{1+x^2}dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\left[\int\frac{\left(1+x^2-1\right)}{1+x^2}dx\right]$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\left[\int\frac{1+x^2}{1+x^2}dx-\int\frac{1}{1+x^2}dx\right]$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\left[\int dx-\int\frac{1}{1+x^2}dx\right]$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\left[x-\tan^{-1}x\right]+c$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}x+\frac{1}{2}\tan^{-1}x+c$$

$$=\left(\frac{1}{2}\tan^{-1}x\right)\left(x^2+1\right)-\frac{1}{2}x+c$$

(xi) $\int x^3\tan^{-1}x\,dx$

Solution:

$$\int\tan^{-1}x.x^3\,dx$$

Integrating by parts

$$=\tan^{-1}x\left(\frac{x^4}{4}\right)-\int\left(\frac{x^4}{4}.\frac{1}{1+x^2}\right)dx$$

$$=\frac{x^4}{4}\tan^{-1}x-\frac{1}{4}\int\frac{x^4}{1+x^2}dx\ldots\ldots\text{(i)}$$

$\frac{x^4}{1+x^2}$ is an improper fraction,

therefore by long division

$$\begin{array}{r|l} & x^2-1 \\ x^2+1 & x^4 \\ & \pm x^4\pm x^2 \\ \hline & -x^2 \\ & \mp x^2\mp 1 \\ \hline & 1 \end{array}$$

$$\therefore\frac{x^4}{1+x^2}=\left(x^2-1\right)+\frac{1}{x^2+1}$$

So, (i) becomes

$$=\frac{x^4}{4}\tan^{-1}x-\frac{1}{4}\left[\int\left(x^2-1+\frac{1}{1+x^2}\right)dx\right]$$

$$=\frac{x^4}{4}\tan^{-1}x-\frac{1}{4}\left[\int x^2dx-\int dx+\int\frac{1}{1+x^2}dx\right]$$

$$=\frac{x^4}{4}\tan^{-1}x-\frac{1}{4}\left[\frac{x^3}{3}-x+\tan^{-1}x\right]+c$$

$$=\frac{x^4}{4}\tan^{-1}-\frac{x^3}{12}+\frac{x}{4}-\frac{1}{4}\tan^{-1}x+c$$

$$=\frac{1}{4}\left(x^4-1\right)\tan^{-1}x-\frac{x^3}{12}+\frac{1}{4}x+c$$

(xii) $\int x^3\cos dx$

Solution:

$$\int x^3\cos x\,dx$$

Integrating by parts

$$=x^3\left(\sin x\right)-\int\sin x\left(3x^2\right)dx$$

$$=x^3\sin x-3\int x^2\sin x\,dx$$

Integrating by parts

$$=x^3\sin x-3\left[x^2\left(-\cos x\right)-\int\left(-\cos x\right)2xdx\right]$$

$$=x^3\sin x+3x^2\cos x-6\int x\cos x\,dx$$

Integrating by part

$$=x^3\sin x+3x^2\cos x-6\left[x\left(\sin x\right)-\int\left(\sin x\right).1dx\right]$$

$$=x^3\sin x+3x^2\cos x-6x\sin x+6\cos x+c$$

(xiii) $\int\sin^{-1}xdx$

Solution:

$$\int\sin^{-1}x.1dx$$

Integrating by parts

$$=\sin^{-1}x\left(x\right)-\int x.\frac{1}{\sqrt{1-x^2}}dx$$

$$=x\sin^{-1}x-\left(\frac{-1}{2}\right)\int\frac{-2x}{\sqrt{1-x^2}}dx$$

$$=x\sin^{-1}x+\frac{1}{2}\int\left(1-x^2\right)^{\frac{-1}{2}}\left(-2x\right)dx$$

$$=x\sin^{-1}x+\frac{1}{2}\frac{\left(1-x^2\right)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}+c$$

$$=x\sin^{-1}x+\sqrt{1-x^2}+c$$

(xiv) $\int x\sin^{-1}x\,dx$

Solution:

$$\int\sin^{-1}x.x\,dx$$

Integrating by parts

$$=\sin^{-1}x\left(\frac{x^2}{2}\right)-\int\frac{x^2}{2}.\frac{1}{\sqrt{1-x^2}}dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\frac{-x^2}{\sqrt{1-x^2}}dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\left[\int\frac{\left(1-x^2\right)-1}{\sqrt{1-x^2}}dx\right]$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\left[\int\frac{\left(1-x^2\right)}{\sqrt{1-x^2}}dx-\int\frac{1}{\sqrt{1-x^2}}dx\right]$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\left[\int\sqrt{1-x^2}dx-\int\frac{1}{\sqrt{1-x^2}}dx\right]\ldots\ldots\text{(i)}$$

As $\int\sqrt{a^2-x^2}dx=\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+c$

and $\int\frac{1}{\sqrt{1-x^2}}dx=\sin^{-1}x+c$

So (i) becomes

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\left[\frac{1}{2}\sin^{-1}x+\frac{x}{2}\sqrt{1-x^2}-\sin^{-1}x\right]+c$$

$$=\frac{x^2}{2}\sin^{-1}x-\frac{1}{4}\sin^{-1}x+\frac{x\sqrt{1-x^2}}{4}+c$$

(xv) $\int e^x\sin x\cos x\,dx$

Solution:

Let $I=\int e^x\left(\sin x\cos x\right)dx$

Integrating by parts

$$I=e^x\left(\frac{\sin^2 x}{2}\right)-\int\left(\frac{\sin^2 x}{2}\right).e^x dx$$

$$=\frac{e^x\sin^2 x}{2}-\frac{1}{2}\int\left(\frac{1-\cos 2x}{2}\right)e^x dx$$

$$=\frac{e^x\sin^2 x}{2}-\frac{1}{4}\int e^x dx+\frac{1}{4}\int e^x\cos 2x\,dx$$

Integrating by parts

$$=\frac{e^x\sin^2 x}{2}-\frac{1}{4}e^x+\frac{1}{4}\left[e^x\left(\frac{\sin 2x}{2}\right)-\int\frac{\sin 2x}{2}e^x dx\right]+c_1$$

$$=e^x\frac{\sin^2 x}{2}-\frac{1}{4}e^x+\frac{e^x\sin 2x}{8}-\frac{1}{4}\int\frac{2\sin x\cos x}{2}e^x dx+c_1$$

$$I=e^x\frac{\sin^2 x}{2}-\frac{1}{4}e^x+\frac{e^x\sin 2x}{8}-\frac{1}{4}\int e^x\left(\sin x\cos x\right)dx+c_1$$

$$I=e^x\frac{\sin^2 x}{2}-\frac{1}{4}e^x+\frac{e^x\sin 2x}{8}-\frac{1}{4}I+c_1$$

$$I+\frac{1}{4}I=e^x\frac{\sin^2 x}{2}-\frac{1}{4}e^x+\frac{e^x\sin 2x}{8}+c_1$$

$$\frac{5}{4}I=e^x\frac{\sin^2 x}{2}-\frac{1}{4}e^x+e^x\frac{\sin 2x}{8}+c_1$$

$$I=\frac{2}{5}e^x\sin^2 x-\frac{1}{5}e^x+\frac{1}{10}e^x\sin 2x+\frac{4}{5}c_1$$

Put $\frac{4}{5}c_1=c$

$$I=\frac{1}{5}e^x\left[2\sin^2 x-1+\frac{1}{2}\sin 2x\right]+c$$

(xvi) $\int x\sin x\cos x\,dx$

Solution:

$$\int x\left(\sin x\cos x dx\right)$$

Integrating by parts

$$=x\left(\frac{\sin^2 x}{2}\right)-\int\left(\frac{\sin^2 x}{2}\right).(1)dx$$

$$=\frac{x}{2}\sin^2 x-\frac{1}{4}\int\left(2\sin^2 x\right)dx$$

$$=\frac{x}{2}\sin^2 x-\frac{1}{4}\int\left(1-\cos 2x\right)dx$$

$$=\frac{x}{2}\sin^2 x-\frac{1}{4}\int 1dx+\frac{1}{4}\int\cos 2x\,dx$$

$$=\frac{x}{2}\sin^2 x-\frac{1}{4}x+\frac{1}{4}\frac{\sin 2x}{2}+c$$

$$=\frac{x}{2}\sin^2 x-\frac{1}{4}x+\frac{1}{8}\sin 2x+c$$

(xvii) $\int x\cos^2 x\,dx$

Solution:

$$\int x\cos^2 x\,dx$$

$$=\int x\frac{\left(1+\cos 2x\right)}{2}dx \because \cos^2 x=\frac{1+\cos 2x}{2}$$

$$=\frac{1}{2}\int x\,dx+\frac{1}{2}\int x\cos 2x\,dx$$

Integrating by parts

$$=\frac{1}{2}\left(\frac{x^2}{2}\right)+\frac{1}{2}\left[x\frac{\sin 2x}{2}-\int\frac{\sin 2x}{2}(1)\,dx\right]$$

$$=\frac{1}{4}x^2+\frac{x\sin 2x}{4}-\frac{1}{4}\int\sin 2x\,dx$$

$$=\frac{1}{4}x^2+\frac{x\sin 2x}{4}-\frac{1}{4}\left(\frac{-\cos 2x}{2}\right)+c$$

$$=\frac{1}{4}x^2+\frac{x\sin 2x}{4}+\frac{\cos 2x}{8}+c$$

$$=\frac{1}{4}\left[x^2+x\sin 2x+\frac{1}{2}\cos 2x\right]+c$$

(xviii) $\int x\sin^2 x\,dx$

Solution:

$$\int x\sin^2 x\,dx$$

$$=\int x\left(\frac{1-\cos 2x}{2}\right)dx \because \sin^2 x=\frac{1-\cos 2x}{2}$$

$$=\frac{1}{2}\int x\,dx-\frac{1}{2}\int x\cos 2x\,dx$$

$$=\frac{1}{2}\left(\frac{x^2}{2}\right)-\frac{1}{2}\int x\cos 2x\,dx$$

Integrating by parts

$$=\frac{x^2}{4}-\frac{1}{2}\left[x\frac{\sin 2x}{2}-\int\frac{\sin 2x}{2}(1)\,dx\right]$$

$$=\frac{x^2}{4}-\frac{1}{4}x\sin 2x+\frac{1}{4}\int\sin 2x\,dx$$

$$=\frac{1}{4}\left[x^2-x\sin 2x+\left(\frac{-\cos 2x}{2}\right)\right]+c$$

$$=\frac{1}{4}\left[x^2-x\sin 2x-\frac{1}{2}\cos 2x\right]+c$$

(xix) $\int(\ln x)^2\,dx$

Solution:

$$\int(\ln x)^2(1).dx$$

Integrating by parts

$$=(\ln x)^2(x)-\int x.2(\ln x).\frac{1}{x}dx$$

$$=x(\ln x)^2-2\int\ln x.1dx$$

Integrating by parts

$$=x(\ln x)^2-2\left[(\ln x).x-\int x.\frac{1}{x}dx\right]$$

$$=x(\ln x)^2-2x\ln x+2\int 1dx$$

$$=x(\ln x)^2-2x\ln x+2x+c$$

$$=x\ln x[\ln x-2]+2x+c$$

(xx) $\int\ln(\tan x)\sec^2 x\,dx$

Solution:

$$\int\ln(\tan x)\sec^2 x\,dx$$

Integrating by parts

$$=\ln(\tan x)(\tan x)-\int(\tan x).\frac{1}{\tan x}\sec^2 x\,dx$$

$$=\tan x\ln(\tan x)-\int\sec^2 x\,dx$$

$$=\tan x\ln(\tan x)-\tan x+c$$

(xxi) $\int\frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$

Solution:

$$\int\frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$$

$$=\int\sin^{-1}x\frac{x}{\sqrt{1-x^2}}dx$$

$$=-\frac{1}{2}\left[\int\sin^{-1}x\left\{\left(1-x^2\right)^{\frac{-1}{2}}(-2x)\right\}dx\right]$$

Integrating by parts

$$=\frac{-1}{2}\left[\sin^{-1}x.\frac{\sqrt{1-x^2}}{\left(\frac{1}{2}\right)}-\int\frac{\sqrt{1-x^2}}{\left(\frac{1}{2}\right)}.\frac{1}{\sqrt{1-x^2}}dx\right]$$

$$=-\sqrt{1-x^2}\sin^{-1}x+\int 1dx$$

$$=-\sqrt{1-x^2}\sin^{-1}x+x+c$$

Q.2 Evaluate the following Integrals.

(i) $\int\tan^4 x\,dx$

Solution:

$$\int\tan^4 x\,dx$$

$$=\int\tan^2 x.\tan^2 x\,dx$$

$$=\int\tan^2 x\left(\sec^2 x-1\right)dx$$

$$=\int\left(\tan^2 x\sec^2 x-\tan^2 x\right)dx$$

$$=\int\left(\tan x\right)^2\sec^2 x\,dx-\int\tan^2 x\,dx$$

$$=\frac{\tan^3 x}{3}-\int\left(\sec^2 x-1\right)dx$$

$$=\frac{\tan^3 x}{3}-\int\sec^2 x\,dx+\int 1dx$$

$$=\frac{\tan^3 x}{3}-\tan x+x+c$$

(ii) $\int\sec^4 x\,dx$

Solution:

$$\int\sec^4 x\,dx$$

$$=\int\sec^2 x.\sec^2 x\,dx$$

$$=\int\left(1+\tan^2 x\right)\sec^2 x\,dx$$

$$=\int\sec^2 x\,dx+\int\tan^2 x\sec^2 x\,dx$$

$$=\tan x+\frac{\tan^3 x}{3}+c$$

(iii) $\int e^x\sin 2x\cos x\,dx$

Solution:

Let $I=\int e^x\sin 2x\cos dx$

$$I=\frac{1}{2}\int\left[e^x\,2\sin 2x\cos x\right]dx$$

Using $2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta)$

$$I=\frac{1}{2}\int e^x\left[\sin\left(2x+x\right)+\sin\left(2x-x\right)\right]dx$$

$$I=\frac{1}{2}\int e^x\left(\sin 3x+\sin x\right)dx$$

$$I=\frac{1}{2}\int e^x\sin 3x\,dx+\frac{1}{2}\int e^x\sin x\,dx$$

$$=\frac{1}{2}I_1+\frac{1}{2}I_2\ ......(\text{i})$$

Now $I_1=\int e^x\sin 3x\,dx$

Integrating by parts

$$I_1=e^x\left(\frac{-\cos 3x}{3}\right)-\int\left(\frac{-\cos 3x}{3}\right).e^x\,dx$$

$$I_1=\frac{-\cos 3x}{3}e^x+\frac{1}{3}\int e^x\cos 3x\,dx$$

Integrating by parts

$$I_1=\frac{-\cos 3xe^x}{3}+\frac{1}{3}\left[e^x\frac{\sin 3x}{3}-\int\frac{\sin 3x}{3}.e^x dx\right]$$

$$I_1=\frac{-\cos 3x}{3}e^x+\frac{1}{9}e^x\sin 3x-\frac{1}{9}\int e^x\sin 3x\,dx$$

$$I_1=\frac{-\cos 3x}{3}e^x+\frac{1}{9}e^x\sin 3x-\frac{1}{9}I_1+c_1$$

$$I_1+\frac{1}{9}I_1=\frac{-\cos 3x}{3}e^x+\frac{1}{9}e^x\sin 3x+c_1$$

$$\frac{10}{9}I_1=\frac{-\cos 3x}{3}e^x+\frac{1}{9}e^x\sin 3x+c_1$$

$$I_1=\frac{9}{10}\left[\frac{-\cos 3x}{3}e^x+\frac{\sin 3x}{9}e^x\right]+\frac{9}{10}c_1$$

$$I_1=\frac{e^x}{10}\left[-3\cos 3x+\sin 3x\right]+\frac{9}{10}c_1$$

Now

$$I_2=\int e^x\sin x\,dx$$

Integrating by parts

$$I_2=e^x\left(-\cos x\right)-\int\left(-\cos x\right).e^x dx$$

$$I_2=-e^x\cos x+\int e^x\cos xdx$$

Integrating by parts

$$I_2=-e^x\cos x+\left[e^x\sin x-\int\sin xe^x dx\right]$$

$$I_2=-e^x\cos x+e^x\sin x-I_2+c_2$$

$$2I_2=-e^x\cos x+e^x\sin x+c_2$$

$$I_2=\frac{1}{2}e^x\left[\sin x-\cos x\right]+\frac{1}{2}c_2$$

By putting the values of I_1 and I_2 in

(i) $I=\frac{1}{2}\left[\frac{e^x}{10}\left(\sin 3x-3\cos 3x\right)+\frac{9}{10}c_1\right]$

$$+\frac{1}{2}\left[\frac{1}{2}e^{x}\left(\sin x-\cos x\right)+\frac{1}{2}c_2\right]$$

$$I=\frac{1}{2}e^{x}\left[\frac{1}{10}\sin 3x-\frac{3}{10}\cos 3x+\frac{1}{2}\sin x\right.$$

$$\left.-\frac{1}{2}\cos x\right]+\frac{9}{20}c_1+\frac{1}{4}c_2$$

$$I=\frac{1}{4}e^{x}\left[\frac{1}{5}\sin 3x-\frac{3}{5}\cos 3x+\sin x-\cos x\right]+c$$

Where $c=\frac{9}{20}c_1+\frac{1}{4}c_2$

(iv) $\int \tan^3 x\sec x\,dx$

Solution:

Let $I=\int \tan^3 x\sec x\,dx$

$$I=\int \tan^2 x\left(\tan x\sec x\right)dx$$

Integrating by parts

$$I=\tan^2 x\left(\sec x\right)-\int \sec x\left(2\tan x\sec^2 x\right)dx$$

$$I=\sec x\tan^2 x-2\int \sec x\tan x\left(1+\tan^2 x\right)dx$$

$$I=\sec x\tan^2 x-2\int \sec x\tan x\,dx-2\int \tan^3 x\sec x\,dx$$

$$I=\sec x\tan^2 x-2\sec x-2I+c_1$$

$$3I=\sec x\tan^2 x-2\sec x+c$$

$$I=\frac{1}{3}\left[\sec x\tan^2 x-2\sec x\right]+\frac{c_1}{3}$$

$$I=\frac{1}{3}\left[\sec x\tan^2 x-2\sec x\right]+c$$

Where $c=\frac{c_1}{3}$

(v) $\int x^3e^{5x}dx$

Solution:

$$\int x^3e^{5x}dx$$

Integrating by parts

$$=x^3\left(\frac{e^{5x}}{5}\right)-\int \frac{e^{5x}}{5}\left(3x^2\right)dx$$

Integrating by parts

$$=\frac{x^3e^{5x}}{5}-\frac{3}{5}\left[x^2\left(\frac{e^{5x}}{5}\right)-\int \frac{e^{5x}}{5}\left(2x\right)dx\right]$$

$$=\frac{x^3e^{5x}}{5}-\frac{3x^2e^{5x}}{25}+\frac{6}{25}\int xe^{5x}dx$$

Integrating by parts

$$=\frac{x^3e^{5x}}{5}-\frac{3x^2e^{5x}}{25}+\frac{6}{25}\left[x.\frac{e^{5x}}{5}-\int \frac{e^{5x}}{5}.1dx\right]$$

$$=\frac{x^3e^{5x}}{5}-\frac{3x^2e^{5x}}{25}+\frac{6xe^{5x}}{125}-\frac{6}{125}\int e^{5x}dx$$

$$=\frac{x^3e^{5x}}{5}-\frac{3x^2e^{5x}}{25}+\frac{6xe^{5x}}{125}-\frac{6}{125}\left(\frac{e^{5x}}{5}\right)+c$$

$$=\frac{x^3e^{5x}}{5}-\frac{3x^2e^{5x}}{25}+\frac{6xe^{5x}}{125}-\frac{6e^{5x}}{625}+c$$

$$=\frac{1}{5}e^{5x}\left[x^3-\frac{3x^2}{5}+\frac{6}{25}x-\frac{6}{125}\right]+c$$

(vi) $\int e^{-x}\sin 2x\,dx$

Solution:

Let $I=\int e^{-x}\sin 2x\,dx$

Integrating by parts

$$I=e^{-x}\left(\frac{-\cos 2x}{2}\right)-\int \left(\frac{-\cos 2x}{2}\right)\left(-1\right)e^{-x}dx$$

$$I=\frac{-e^{-x}}{2}\cos 2x-\frac{1}{2}\int \cos 2x.e^{-x}dx$$

Integrating by parts

$$I=\frac{-e^{-x}}{2}\cos 2x-\frac{1}{2}e^{-x}\left[\left(\frac{\sin 2x}{2}\right)-\int \frac{\sin 2x}{2}\left(e^{-x}\right)\left(-1\right)dx\right]$$

$$I=\frac{-e^{-x}}{2}\cos 2x-\frac{1}{4}e^{-x}\sin 2x-\frac{1}{4}\int e^{-x}\sin 2x\,dx+c_1$$

$$I=\frac{-e^{-x}}{2}\cos 2x-\frac{1}{4}e^{-x}\sin 2x-\frac{1}{4}I+c_1$$

$$I+\frac{1}{4}I=\frac{-e^{-x}\cos 2x}{2}-\frac{1}{4}e^{-x}\sin 2x+c_1$$

$$\left(\frac{5}{4}\right)I=\frac{-e^{-x}}{2}\cos 2x-\frac{1}{4}e^{-x}\sin 2x+c_1$$

$$I=\frac{4}{5}\left(\frac{-e^{-x}}{2}\cos 2x-\frac{1}{4}e^{-x}\sin 2x\right)+\frac{4}{5}c_1$$

$$I=\frac{-2}{5}e^{-x}\left[\cos 2x+\frac{1}{2}\sin 2x\right]+c$$

Where $\frac{4}{5}c_1=c$

(vii) $\int e^{2x}\cos 3x\,dx$

Solution:

Let $I = \int e^{2x}\cos 3x\,dx$

Integrating by parts

$$I = e^{2x}\frac{\sin 3x}{3} - \int \frac{\sin 3x}{3}e^{2x}\times 2dx$$

$$I = \frac{\sin 3x}{3}e^{2x} - \frac{2}{3}\int \sin 3x e^{2x}dx$$

Integrating by parts

$$I = \frac{\sin 3x}{3}e^{2x} - \frac{2}{3}\left[e^{2x}\left(\frac{-\cos 3x}{3}\right) - \int \frac{-\cos 3x}{3}e^{2x}(2)dx + c_1\right]$$

$$I = \frac{\sin 3x}{3}e^{2x} + \frac{2}{9}e^{2x}\cos 3x - \frac{4}{9}\int e^{2x}\cos 3x\,dx + c_1$$

$$I = \frac{\sin 3x}{3}e^{2x} + \frac{2}{9}e^{2x}\cos 3x - \frac{4}{9}I + c_1$$

$$I + \frac{4}{9}I = \frac{\sin 3x}{3}e^{2x} + \frac{2}{9}e^{2x}\cos 3x + c_1$$

$$\frac{13}{9}I = \frac{\sin 3x}{3}e^{2x} + \frac{2}{9}e^{2x}\cos 3x + c_1$$

$$I = \frac{9}{13}\left[\frac{\sin 3x}{3}e^{2x} + \frac{2}{9}e^{2x}\cos 3x\right] + \frac{9}{13}c_1$$

$$I = \frac{e^{2x}}{13}\left[3\sin 3x + 2\cos 3x\right] + c$$

Where $c = \frac{9}{13}c_1$

(viii) $\int \text{cosec}^3 x\,dx$

Solution:

Let $I = \int \text{cosec}^3 x\,dx$

$$I = \int \text{cosec}\,x\left(\text{cosec}^2 x\right)dx$$

Integrating by parts

$$I = \text{cosec}\,x(-\cot x) - \int(-\cot x)(-\text{cosec}\,x\cot x)dx$$

$$I = -\cot x\,\text{cosec}\,x - \int \cot^2 x\,\text{cosec}\,x\,dx$$

$$I = -\cot x\,\text{cosec}\,x - \int\left(\text{cosec}^2 x - 1\right)\text{cosec}\,x\,dx$$

$$I = -\cot x\,\text{cosec}\,x - \int \text{cosec}^3 x\,dx + \int \text{cosec}\,x\,dx$$

$$I = -\cot x\,\text{cosec}\,x - I + \ln\left|\text{cosec}\,x - \cot x\right| + c_1$$

$$2I = -\cot x\,\text{cosec}\,x + \ln\left|\text{cosec}\,x - \cot x\right| + c_1$$

$$I = -\frac{1}{2}\left[\cot x\,\text{cosec}\,x - \ln\left|\text{cosec}\,x - \cot x\right|\right] + \frac{1}{2}c_1$$

$$I = \frac{-1}{2}\left[\cot x\,\text{cosec}\,x - \ln\left|\text{cosec}\,x - \cot x\right|\right] + c$$

Where $c = \frac{1}{2}c_1$

Q.3 Show that

$$\int e^{ax}\sin bx\,dx = \frac{1}{\sqrt{a^2+b^2}}e^{ax}\sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c$$

Solution:

Let $I = \int e^{ax}\sin bx\,dx$

Integrating by parts

$$I = e^{ax}\left(-\frac{\cos bx}{b}\right) - \int\left(-\frac{\cos bx}{b}\right)\left(e^{ax}\right)(a)dx$$

$$I = \frac{-e^{ax}}{b}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx$$

Integrating by parts

$$I = \frac{-e^{ax}}{b}\cos bx + \frac{a}{b}\left[e^{ax}\left(\frac{\sin bx}{b}\right) - \int\frac{\sin bx}{b}\left(e^{ax}\right)(a)dx\right]$$

$$I = \frac{-e^{ax}}{b}\cos bx + \frac{a}{b}\left[\frac{e^{ax}}{b}\sin bx - \frac{a}{b}\int e^{ax}\sin bx\,dx\right]$$

$$I = \frac{-e^{ax}}{b}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}\int e^{ax}\sin bx\,dx + c_1$$

$$I = \frac{-e^{ax}}{b}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}I + c_1$$

$$I + \frac{a^2}{b^2}I = \frac{-e^{ax}}{b}\cos bx + \frac{a}{b^2}e^{ax}\sin bx + c_1$$

$$\left(\frac{a^2+b^2}{b^2}\right)I = e^{ax}\left[\frac{-b\cos bx + a\sin bx}{b^2}\right] + c$$

$$I = \left(\frac{b^2}{a^2+b^2}\right)\frac{e^{ax}}{b^2}\left[-b\cos bx + a\sin bx\right] + c_1\left(\frac{b^2}{a^2+b^2}\right)$$

$$I = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right] + c$$

Where $c = c_1\left(\frac{b^2}{a^2+b^2}\right)$

Put $a = r\cos\theta, b = r\sin\theta$

$$a^2 + b^2 = r^2\left(\cos^2\theta + \sin^2\theta\right) \text{ and } \tan\theta = \frac{b}{a}$$

$$\Rightarrow r = \sqrt{a^2+b^2} \quad \text{and} \quad \theta = \tan^{-1}\frac{b}{a}$$

$$I=\frac{1}{a^2+b^2}e^{ax}\left[r\cos\theta\sin bx-r\sin\theta\cos bx\right]+c$$

$$I=\frac{r}{a^2+b^2}e^{ax}\left[\cos\theta\sin bx-\sin\theta\cos bx\right]+c$$

Using $\sin\alpha\cos\beta-\cos\alpha\sin\beta=\sin(\alpha-\beta)$

$$I=\frac{\sqrt{a^2+b^2}}{a^2+b^2}e^{ax}\sin(bx-\theta)+c$$

$$I=\frac{1}{\sqrt{a^2+b^2}}e^{ax}\sin\left(bx-\tan^{-1}\left(\frac{b}{a}\right)\right)+c$$

Q.1 Evaluate the following indefinite integrals.

(i) $\int\sqrt{a^2-x^2dx}$

Solution:

Let $I=\int\sqrt{a^2-x^2}dx$

$$I=\int\sqrt{a^2-x^2}\,(1)dx$$

Integrating by parts

$$I=\sqrt{a^2-x^2}\,(x)-\int x.\frac{1}{2}\left(a^2-x^2\right)^{\frac{-1}{2}}(-2x)dx$$

$$I=x\sqrt{a^2-x^2}+\int\frac{x^2}{\sqrt{a^2-x^2}}dx$$

$$I=x\sqrt{a^2-x^2}+\int\frac{a^2-a^2+x^2}{\sqrt{a^2-x^2}}dx$$

$$I=x\sqrt{a^2-x^2}+\int\frac{a^2-\left(a^2-x^2\right)}{\sqrt{a^2-x^2}}dx$$

$$I=x\sqrt{a^2-x^2}+\int\frac{a^2}{\sqrt{a^2-x^2}}dx-\int\frac{a^2-x^2}{\sqrt{a^2-x^2}}dx$$

$$I=x\sqrt{a^2-x^2}+a^2\int\frac{dx}{\sqrt{a^2-x^2}}-\int\sqrt{a^2-x^2}dx$$

$$I=x\sqrt{a^2-x^2}+a^2\sin^{-1}\left(\frac{x}{a}\right)-I+c_1$$

$$2I=x\sqrt{a^2-x^2}+a^2\sin^{-1}\left(\frac{x}{a}\right)+c_1$$

$$I=\frac{1}{2}\left[x\sqrt{a^2-x^2}+a^2\sin^{-1}\left(\frac{x}{a}\right)\right]+\frac{c_1}{2}$$

$$I=\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$$

Where $\frac{c_1}{2}=c$

(ii) $\int\sqrt{x^2-a^2}dx$

Solution:

Let $I=\int\sqrt{x^2-a^2}dx$

$$I=\int\sqrt{x^2-a^2}\,(1)dx$$

Integrating by parts

$$I=\sqrt{x^2-a^2}\,(x)-\int x\frac{1}{2}\left(x^2-a^2\right)^{\frac{-1}{2}}(2x)dx$$

$$I=x\sqrt{x^2-a^2}-\int\frac{x^2}{\sqrt{x^2-a^2}}dx$$

$$I=x\sqrt{x^2-a^2}-\int\frac{\left(x^2-a^2\right)+a^2}{\sqrt{x^2-a^2}}dx$$

$$I=x\sqrt{x^2-a^2}-\int\frac{x^2-a^2}{\sqrt{x^2-a^2}}dx-\int\frac{a^2}{\sqrt{x^2-a^2}}dx$$

$$I=x\sqrt{x^2-a^2}-\int\sqrt{x^2-a^2}dx-a^2\int\frac{1}{\sqrt{x^2-a^2}}dx$$

Using $\int\frac{1}{\sqrt{x^2-a^2}}dx=\ln\left|x+\sqrt{x^2-a^2}\right|+c$

$$I=x\sqrt{x^2-a^2}-I-a^2\ln\left|x+\sqrt{x^2-a^2}\right|+c_1$$

$$2I=x\sqrt{x^2-a^2}-a^2\ln\left|x+\sqrt{x^2-a^2}\right|+c_1$$

$$I=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+\frac{c_1}{2}$$

Where $c=\frac{c_1}{2}$

$$I=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+c$$

(iii) $\int\sqrt{4-5x^2}dx$

Solution:

Let $I=\int\sqrt{4-5x^2}\,(1)dx$

Integrating by parts

$$I=\sqrt{4-5x^2}\,(x)-\int x.\frac{1}{2}\left(4-5x^2\right)^{\frac{-1}{2}}(-10x)dx$$

$$I=x\sqrt{4-5x^2}-\int\frac{-5x^2}{\sqrt{4-5x^2}}dx$$

$$I = x\sqrt{4-5x^2} - \int \frac{\left(4-5x^2\right)-4}{\sqrt{4-5x^2}}dx$$

$$I = x\sqrt{4-5x^2} - \int \frac{\left(4-5x^2\right)}{\sqrt{4-5x^2}}dx + 4\int \frac{1}{\sqrt{4-5x^2}}dx$$

$$I = x\sqrt{4-5x^2} - \int \sqrt{4-5x^2}dx + 4\int \frac{dx}{\sqrt{(2)^2-\left(\sqrt{5}x\right)^2}}$$

$$I = x\sqrt{4-5x^2} - I + 4\left(\frac{\sin^{-1}\left(\frac{\sqrt{5}x}{2}\right)}{\sqrt{5}}\right) + c_1$$

$$2I = x\sqrt{4-5x^2} + \frac{4}{\sqrt{5}}\sin^{-1}\frac{\sqrt{5}x}{2} + c_1$$

$$I = \frac{x}{2}\sqrt{4-5x^2} + \frac{2}{\sqrt{5}}\sin^{-1}\frac{\sqrt{5}x}{2} + c$$

Where $c = \frac{c_1}{2}$

(iv) $\int \sqrt{3-4x^2}dx$

Solution:

Let $I = \int \sqrt{3-4x^2}.(1)dx$

Integrating by parts

$$I = \sqrt{3-4x^2}(x) - \int x.\frac{1}{2}\left(3-4x^2\right)^{\frac{-1}{2}}(-8x)dx$$

$$I = x\sqrt{3-4x^2} - \int \frac{-4x^2}{\sqrt{3-4x^2}}dx$$

$$I = x\sqrt{3-4x^2} - \int \frac{\left(3-4x^2\right)-3}{\sqrt{3-4x^2}}dx$$

$$I = x\sqrt{3-4x^2} - \int \frac{3-4x^2}{\sqrt{3-4x^2}}dx + 3\int \frac{1}{\sqrt{3-4x^2}}dx$$

$$I = x\sqrt{3-4x^2} - \int \sqrt{3-4x^2}dx + 3\int \frac{1}{\sqrt{\left(\sqrt{3}\right)^2-(2x)^2}}dx$$

$$I = x\sqrt{3-4x^2} - I + 3\left(\frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right)\right) + c_1$$

$$2I = x\sqrt{3-4x^2} + \frac{3}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + c_1$$

$$I = \frac{x}{2}\sqrt{3-4x^2} + \frac{3}{4}\sin^{-1}\frac{2x}{\sqrt{3}} + c$$

Where $c = \frac{c_1}{2}$

(v) $\int \sqrt{x^2+4}dx$

Solution:

Let $I = \int \sqrt{x^2+4}.(1)dx$

Integrating by parts

$$I = x\sqrt{x^2+4} - \int x.\frac{1}{2}\left(x^2+4\right)^{\frac{-1}{2}}(2x)dx$$

$$I = x\sqrt{x^2+4} - \int \frac{x^2}{\sqrt{x^2+4}}dx$$

$$I = x\sqrt{x^2+4} - \int \frac{\left(x^2+4\right)-4}{\sqrt{x^2+4}}dx$$

$$I = x\sqrt{x^2+4} - \int \frac{x^2+4}{\sqrt{x^2+4}}dx + \int \frac{4}{\sqrt{x^2+4}}dx$$

$$I = x\sqrt{x^2+4} - \int \sqrt{x^2+4}dx + 4\int \frac{1}{\sqrt{x^2+4}}dx$$

$$I = x\sqrt{x^2+4} - \int \sqrt{x^2+4}dx + 4\int \frac{1}{\sqrt{x^2+2^2}}dx$$

$$\because \int \frac{1}{\sqrt{x^2+2^2}}dx = \ln\left|x+\sqrt{x^2+4}\right| + c_1$$

$$I = x\sqrt{x^2+4} - I + 4\ln\left|x+\sqrt{x^2+4}\right| + c_1$$

$$2I = x\sqrt{x^2+4} + 4\ln\left|x+\sqrt{x^2+4}\right| + c_1$$

$$I = \frac{x}{2}\sqrt{x^2+4} + 2\ln\left|x+\sqrt{x^2+4}\right| + c$$

Where $c = \frac{c_1}{2}$

(vi) $\int x^2e^{ax}dx$

Solution:

$$\int x^2e^{\alpha x}dx$$

Integrating by parts

$$= x^2\left(\frac{e^{ax}}{a}\right) - \int \left(\frac{e^{ax}}{a}\right)(2x)dx$$

$$=\frac{x^2}{a}e^{\alpha x}-\frac{2}{a}\int xe^{ax}dx$$

Integrating by parts

$$=\frac{x^2}{a}e^{ax}-\frac{2}{a}\left[x\left(\frac{e^{ax}}{a}\right)-\int\left(\frac{e^{ax}}{a}\right).1dx\right]$$

$$=\frac{x^2}{a}e^{ax}-\frac{2}{a}\left[\frac{x}{a}e^{ax}+\frac{1}{a}\int e^{ax}dx\right]$$

$$=\frac{x^2}{a}e^{ax}-\frac{2}{a}\left[\frac{x}{a}e^{ax}+\frac{1}{a}\left(\frac{e^{ax}}{a}\right)\right]+c$$

$$=\frac{x^2}{a}e^{ax}-\frac{2}{a^2}xe^{ax}+\frac{2}{a^3}e^{ax}+c$$

$$=\frac{1}{a}e^{ax}\left[x^2-\frac{2}{a}x+\frac{2}{a^2}\right]+c$$

Q.5 Evaluate the following indefinite integrals

(i) $\int e^x\left(\frac{1}{x}+\ln x\right)dx$

Solution:

$$\int e^x\left(\frac{1}{x}+\ln x\right)dx$$

$$=\int e^x\left(\ln x+\frac{1}{x}\right)dx$$

Using $\int e^{ax}\left(a\,f(x)+f'(x)\right)dx=e^{ax}f(x)+c$

$$=e^x\ln x+c$$

Alternate method:

$$\int e^x\left(\frac{1}{x}+\ln x\right)dx$$

$$=\int e^x.\frac{1}{x}dx+\int e^x.\ln xdx$$

Integrating by parts

$$=e^x.\ln x-\int \ln x.e^x dx+\int e^x.\ln xdx$$

$$=e^x\ln x+c$$

(ii) $\int e^x\left(\cos x+\sin x\right)dx$

Solution:

$$\int e^x\left(\cos x+\sin x\right)dx$$

$$=\int e^x\left(\sin x+\cos x\right)dx$$

Using $\int e^{ax}\left(a\,f(x)+f'(x)\right)dx=e^{ax}f(x)+c$

$$=e^x\sin x+c$$

(iii) $\int e^{ax}\left[a\sec^{-1}x+\frac{1}{x\sqrt{x^2-1}}\right]dx$

Solution:

$$\int e^{ax}\left[a\sec^{-1}x+\frac{1}{x\sqrt{x^2-1}}\right]dx$$

Using $\int e^{ax}\left(a\,f(x)+f'(x)\right)dx=e^{ax}f(x)+c$

$$=e^{ax}\sec^{-1}x+c$$

(iv) $\int e^{3x}\left(\frac{3\sin x-\cos x}{\sin^2 x}\right)dx$

Solution:

$$\int e^{3x}\left(\frac{3\sin x-\cos x}{\sin^2 x}\right)dx$$

$$=\int e^{3x}\left(\frac{3\sin x}{\sin^2 x}-\frac{\cos x}{\sin^2 x}\right)dx$$

$$=\int e^{3x}\left[3\operatorname{cosec}x-\operatorname{cosec}x\cot x\right]dx$$

$$=\int e^{3x}\left[3\operatorname{cosec}x+\left(-\operatorname{cosec}x\cot x\right)\right]dx$$

Using $\int e^{ax}\left(a\,f(x)+f'(x)\right)dx=e^{ax}f(x)+c$

$$=e^{3x}\operatorname{cosec}x+c$$

(v) $\int e^{2x}\left(-\sin x+2\cos x\right)dx$

Solution:

$$\int e^{2x}\left(-\sin x+2\cos x\right)dx$$

$$=\int e^{2x}\left(2\cos x+\left(-\sin x\right)\right)dx$$

Using $\int e^{ax}\left(a\,f(x)+f'(x)\right)dx=e^{ax}f(x)+c$

$$=e^{2x}\cos x+c$$

(vi) $\int \frac{xe^x}{(1+x)^2}dx$

Solution:

$$\int \frac{xe^x}{(1+x)^2}$$

$$=\int \frac{(1+x-1)e^x}{(1+x)^2}dx$$

$$=\int e^x\left(\frac{(1+x)}{(1+x)^2}-\frac{1}{(1+x)^2}\right)dx$$

$$=\int e^x\left(\frac{1}{(1+x)}+\left(\frac{-1}{(1+x)^2}\right)\right)dx$$

Using $\int e^{ax}\left(a\,f(x)+f'(x)\right)dx=e^{ax}f(x)+c$

$$=e^x\left(\frac{1}{1+x}\right)+c$$

(vii) $\int e^{-x}\left[\cos x-\sin x\right]dx$

Solution:

$$\int e^{-x}\left[\cos x-\sin x\right]dx$$

$$=\int e^{-x}\left[-\sin x+\cos x\right]dx$$

Using $\int e^{ax}\left(a\,f(x)+f'(x)\right)dx=e^{ax}f(x)+c$

$$=e^{-x}\sin x+c$$

(viii) $\int \frac{e^{m\tan^{-1}x}}{1+x^2}dx$

Solution:

$$\int \frac{e^{m\tan^{-1}x}}{1+x^2}dx$$

$$=\int e^{m\tan^{-1}x}\frac{1}{1+x^2}dx$$

Let $\tan^{-1}x=t$ then $\left(\frac{1}{1+x^2}\right)dx=dt$

$$=\int e^{mt}dt$$

$$=\frac{e^{mt}}{m}+c$$

$$=\frac{1}{m}e^{m\tan^{-1}x}+c$$

(ix) $\int \frac{2x}{1-\sin x}dx$

Solution:

$$\int \frac{2x}{1-\sin x}dx$$

Using $\cos\left(\frac{\pi}{2}-x\right)=\sin x$

$$=\int \frac{2x}{1-\cos\left(\frac{\pi}{2}-x\right)}dx$$

$$\because 1-\cos\left(\frac{\pi}{2}-x\right)=2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)$$

$$=\int \frac{2x}{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}dx$$

$$=\int x\,\text{cosec}^2\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$$

Integrating by parts

$$=x(-2)\left(-\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)\right)-\int(-2)\left(-\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)\right).1dx$$

$$=2x\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)-2\int\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$$

$$=2x\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)-2(-2)\ln\left|\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\right|+c$$

$$=2x\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)+4\ln\left|\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\right|+c$$

(x) $\int \frac{e^x(1+x)}{(2+x)^2}dx$

Solution:

$$\int \frac{e^x(1+x)}{(2+x)^2}dx$$

$$=\int e^x\frac{[2+x-1]}{(2+x)^2}dx$$

$$=\int e^x\left[\frac{2+x}{(2+x)^2}-\frac{1}{(2+x)^2}\right]dx$$

$$=\int e^x\left[\frac{1}{(2+x)}+\frac{-1}{(2+x)^2}\right]dx$$

Using $\int e^{ax}\left(a\,f(x)+f'(x)\right)dx=e^{ax}f(x)+c$

$$=e^{x}\left(\frac{1}{2+x}\right)+c$$

(xi) $\int\left(\frac{1-\sin x}{1-\cos x}\right)e^{x}dx$

Solution:

$$\int\left(\frac{1-\sin x}{1-\cos x}\right)e^{x}dx$$

$$=\int\left(\frac{1-\sin x}{2\sin^{2}\frac{x}{2}}\right)e^{x}dx\ \because 1-\cos x=2\sin^{2}\frac{x}{2}$$

$$=\int e^{x}\left[\frac{1}{2\sin^{2}\frac{x}{2}}-\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^{2}\frac{x}{2}}\right]dx \qquad \because \sin x=2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$=\int e^{x}\left[\frac{1}{2}\text{cosec}^{2}\frac{x}{2}-\cot\frac{x}{2}\right]dx$$

$$=\int e^{x}\left[\left(-\cot\frac{x}{2}\right)+\left(\frac{1}{2}\text{cosec}^{2}\frac{x}{2}\right)\right]dx$$

Using $\int e^{ax}\left(a\,f(x)+f'(x)\right)dx=e^{ax}f(x)+c$

$$=e^{x}\left(-\cot\frac{x}{2}\right)+c \quad \because f(x)=-\cot\frac{x}{2}$$

$$=-e^{x}\cot\frac{x}{2}+c$$