Integration Involving Partial fractions:

For integrating a rational function $\frac{P(x)}{Q(x)}$, there may be three cases, when $Q(x)$ has

Case I: Non repeated linear factors.

Case II: Repeated and non-repeated linear factors

Case III: Linear and non-repeated irreducible quadratic factors or non-repeated irreducible quadratic factors.

EXERCISE 3.5

Evaluate the following integrals.

Q.1 $\int \frac{3x+1}{x^2-x-6}dx$

Solution:

$$\frac{3x+1}{x^2-x-6}$$

$$=\frac{3x+1}{x^2-3x+2x-6}$$

$$=\frac{3x+1}{x\ (x-3)+2(x-3)}$$

$$=\frac{3x+1}{(x-3)(x+2)}$$

Let $\frac{3x+1}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$ …(i)

Multiplying both side by $(x-3)(x+2)$

$3x+1 = A(x+2)+B(x-3)$ … (ii)

Put $x-3=0 \Rightarrow x=3$ in (ii)

$3(3)+1 = A(3+2)+B(3-3)$

$10=5A \Rightarrow \boxed{A=2}$

Put $x+2=0 \Rightarrow x=-2$ in (ii)

$3(-2)+1 = A(-2+2)+B(-2-3)$

$-6+1=-5B$

$-5B=-5 \Rightarrow \boxed{B=1}$

Put $A=2$ and $B=1$ in (i)

$$\frac{3x+1}{(x-3)(x+2)}=\frac{2}{x-3}+\frac{1}{x+2}$$

Integrating both sides

$$\int\frac{3x+1}{(x-3)(x+2)}dx=\int\frac{2}{x-3}dx+\int\frac{1}{x+2}dx$$

$$=2\int\frac{1}{x-3}dx+\int\frac{1}{x+2}dx$$

$$\int\frac{3x+1}{(x-3)(x+2)}dx=2\ln|x-3|+\ln|x+2|+c$$

Q.2 $\int\frac{5x+8}{(x+3)(2x-1)}dx$

Solution:

Let $\frac{5x+8}{(x+3)(2x-1)}=\frac{A}{x+3}+\frac{B}{2x-1}$ ...(i)

Multiplying both side by $(x+3)(2x-1)$

$5x+8=A(2x-1)+B(x+3)$ ...(ii)

Put $x+3=0\Rightarrow x=-3$ in (ii)

$5(-3)+8=A\{2(-3)-1\}+B(-3+3)$

$-15+8=A(-6-1)$

$-7=-7A \Rightarrow \boxed{A=1}$

Put $2x-1=0\Rightarrow x=\frac{1}{2}$ in (ii)

$5\left(\frac{1}{2}\right)+8=A\left\{2\left(\frac{1}{2}\right)-1\right\}+B\left(\frac{1}{2}+3\right)$

$\frac{5+16}{2}=0+B\left(\frac{1+6}{2}\right)$

$\frac{21}{2}=\frac{7B}{2}\Rightarrow \boxed{B=3}$

Put $A=1$ and $B=3$ in (i)

$$\frac{5x+8}{(x+3)(2x-1)}=\frac{1}{x+3}+\frac{3}{2x-1}$$

Integrating both sides

$$\int\frac{5x+8}{(x+3)(2x-1)}dx=\int\frac{1}{x+3}dx+3\int\frac{1}{2x-1}dx$$

$$\int\frac{5x+8}{(x+3)(2x-1)}dx=\int\frac{1}{x+3}dx+\frac{3}{2}\int\frac{2}{2x-1}dx$$

$$\int \frac{5x+8}{(x+3)(2x-1)}dx = \ln|x+3| + \frac{3}{2}\ln|2x-1| + c$$

Q.3 $\int \frac{x^2+3x-34}{x^2+2x-15}dx$

Solution:

By long division

$$\begin{array}{r} 1 \\ x^2+2x-15\,\overline{)\,x^2+3x-34} \\ \underline{\pm x^2 \pm 2x \mp 15} \\ x-19 \end{array}$$

So

$$\frac{x^2+3x-34}{x^2+2x-15} = 1 + \frac{x-19}{x^2+2x-15}$$

$$= 1 + \frac{x-19}{x^2+2x-15}$$

$$= 1 + \frac{x-19}{x^2+5x-3x-15}$$

$$= 1 + \frac{x-19}{x\ (x+5)-3(x+5)}$$

$$= 1 + \frac{x-19}{(x+5)(x-3)} \quad \ldots(i)$$

Let $\frac{x-19}{(x+5)(x-3)} = \frac{A}{x+5} + \frac{B}{x-3} \ldots(ii)$

Multiplying both sides by $(x+5)(x-3)$

$x-19 = A(x-3) + B(x+5) \ldots(iii)$

Put $x+5=0 \Rightarrow x=-5$ in (iii)

$-5-19 = A(-5-3) + B(-5+5)$

$-24 = A(-8) \Rightarrow \boxed{A=3}$

Put $x-3=0 \Rightarrow x=3$ in (iii)

$3-19 = A(3-3) + B(3+5)$

$-16 = 8B \Rightarrow \boxed{B=-2}$

Put $A=3$ and $B=-2$ in (ii)

$$\frac{x-19}{(x+5)(x-3)} = \frac{3}{x+5} - \frac{2}{x-3}$$

(i) becomes

$$\frac{x^2+3x-34}{x^2+2x-15}=1+\frac{3}{x+5}-\frac{2}{x-3}$$

Integrating both sides

$$\int\frac{x^2+3x-34}{x^2+2x-15}dx=\int 1dx+3\int\frac{1}{x+5}dx-2\int\frac{1}{x-3}dx$$

$$\int\frac{x^2+3x-34}{x^2+2x-15}dx=x+3\ln|x+5|-2\ln|x-3|+c$$

Q.4 $\int\frac{(a-b)x}{(x-a)(x-b)}dx,(a>b)$

Solution:

Let $\frac{(a-b)x}{(x-a)(x-b)}=\frac{A}{x-a}+\frac{B}{x-b}$ ……(i)

Multiplying both sides by $(x-a)(x-b)$

$(a-b)x=A(x-b)+B(x-a)$ ……(ii)

Put $x-a=0\Rightarrow x=a$ in (ii)

$(a-b)a=A(a-b)+0$

$(a-b)a=A(a-b)\ \Rightarrow\ \boxed{A=a}$

Put $x-b=0\Rightarrow x=b$ in (ii)

$(a-b)(b)=0+B(b-a)$

$(a-b)(b)=-B(a-b)\ \Rightarrow\ \boxed{B=-b}$

Put $A=a$ and $B=-b$ in (i)

$$\frac{(a-b)x}{(x-a)(x-b)}=\frac{a}{x-a}-\frac{b}{a-b}$$

Integrating both sides

$$\int\frac{(a-b)x}{(x-a)(x-b)}dx=\int\frac{a}{x-a}dx-\int\frac{b}{x-b}dx$$

$$=a\int\frac{1}{x-a}dx-b\int\frac{1}{x-b}dx$$

$$=a\ln|x-a|-b\ln|x-b|+c$$

Q.5 $\int\frac{3-x}{1-x-6x^2}dx$

Solution:

$$\frac{3-x}{1-x-6x^2}=\frac{-(x-3)}{-(6x^2+x-1)}$$

$$=\frac{x-3}{6x^2+x-1}=\frac{x-3}{6x^2+3x-2x-1}$$

$$=\frac{x-3}{3x(2x+1)-1(2x+1)}$$

$$= \frac{x-3}{(2x+1)(3x-1)}$$

Let $\frac{x-3}{(2x+1)(3x-1)} = \frac{A}{2x+1} + \frac{B}{3x-1}$ ...(i)

Multiplying both sides by $(2x+1)(3x-1)$

$x-3 = A(3x-1) + B(2x+1)$ ...(ii)

Put $2x+1=0 \Rightarrow x = \frac{-1}{2}$ in (ii)

$$-\frac{1}{2}-3 = A\left\{3\left(\frac{-1}{2}\right)-1\right\} + B\left\{2\left(\frac{-1}{2}\right)+1\right\}$$

$$\frac{-1-6}{2} = A\left(\frac{-3-2}{2}\right) + B(-1+1)$$

$$\frac{-7}{2} = A\left(\frac{-5}{2}\right) \Rightarrow \boxed{A = \frac{7}{5}}$$

Put $3x-1=0 \Rightarrow x = \frac{1}{3}$ in (ii)

$$\frac{1}{3}-3 = A\left\{3\left(\frac{1}{3}\right)-1\right\} + B\left\{2\left(\frac{1}{3}\right)+1\right\}$$

$$\frac{1-9}{3} = B\left(\frac{2+3}{3}\right)$$

$$\frac{-8}{3} = B\left(\frac{5}{3}\right) \Rightarrow \boxed{B = \frac{-8}{5}}$$

Put $A = \frac{7}{5}$ and $B = \frac{-8}{5}$ in eq. (i)

$$\frac{x-3}{(2x+1)(3x-1)} = \frac{\left(\frac{7}{5}\right)}{2x+1} + \frac{\left(\frac{-8}{5}\right)}{3x-1}$$

$$\frac{3-x}{1-x-6x^2} = \frac{7}{5(2x+1)} - \frac{8}{5(3x-1)}$$

Integrating both sides

$$\int \frac{3-x}{1-x-6x^2}dx = \frac{7}{5}\int \frac{1}{2x+1}dx - \frac{8}{5}\int \frac{1}{3x-1}dx$$

$$= \frac{7}{5\times 2}\int \frac{2}{2x+1}dx - \frac{8}{5\times 3}\int \frac{3}{3x-1}dx$$

$$= \frac{7}{10}\ln|2x+1| - \frac{8}{15}\ln|3x-1| + c$$

Q.6 $\int \frac{2x}{x^2-a^2}dx,\ (x>a)$

Solution:

Consider $\frac{2x}{x^2-a^2}=\frac{2x}{(x-a)(x+a)}$

Let $\frac{2x}{(x-a)(x+a)}=\frac{A}{x-a}+\frac{B}{x+a}$ ... (i)

Multiplying both sides by $(x-a)(x+a)$

$2x = A(x+a)+B(x-a)$ ... (ii)

Put $x-a=0 \Rightarrow x=a$ in (ii)

$2a = A(a+a)+B(a-a)$

$2a = 2aA \Rightarrow \boxed{A=1}$

Put $x+a=0 \Rightarrow x=-a$ in (ii)

$2(-a)=A(-a+a)+B(-a-a)$

$-2a=-2aB \Rightarrow \boxed{B=1}$

Put $A=1$ and $B=1$ in eq. (i)

$\frac{2x}{x^2-a^2}=\frac{1}{x-a}+\frac{1}{x+a}$

Integrating both sides

$\int \frac{2x}{x^2-a^2}dx=\int \frac{1}{x-a}dx+\int \frac{1}{x+a}dx$

$=\ln|x-a|+\ln|x+a|+c$

$=\ln|(x-a)(x+a)|+c$

$\int \frac{2x}{x^2-a^2}dx=\ln|x^2-a^2|+c$

Alternate Solution:

$\int \frac{2x}{x^2-a^2}dx$

$=\ln(x^2-a^2)+c \qquad \because \int \frac{f'(x)}{f(x)}dx=\ln f(x)+c$

Q.7 $\int \frac{1}{6x^2+5x-4}dx$

Solution:

$\frac{1}{6x^2+5x-4}=\frac{1}{6x^2+8x-3x-4}$

$=\frac{1}{2x(3x+4)-1(3x+4)}=\frac{1}{(3x+4)(2x-1)}$

Let $\frac{1}{(3x+4)(2x-1)}=\frac{A}{3x+4}+\frac{B}{2x-1}$ ...(i)

Multiplying both sides by $(3x+4)(2x-1)$

$1 = A(2x-1) + B(3x+4) \ldots\text{(ii)}$

Put $3x+4=0 \Rightarrow x = \frac{-4}{3}$ in (ii)

$1 = A\left\{2\left(\frac{-4}{3}\right)-1\right\} + B\left\{3\left(\frac{-4}{3}\right)+4\right\}$

$1 = A\left\{\frac{-8-3}{3}\right\}$

$1 = A\left(\frac{-11}{3}\right) \Rightarrow \boxed{A = \frac{-3}{11}}$

Put $2x-1=0 \Rightarrow x = \frac{1}{2}$ in (ii)

$1 = A\left\{2\left(\frac{1}{2}\right)-1\right\} + B\left\{3\left(\frac{1}{2}\right)+4\right\}$

$1 = B\left(\frac{3+8}{2}\right)$

$1 = B\left(\frac{11}{2}\right) \Rightarrow \boxed{B = \frac{2}{11}}$

Put $A = \frac{-3}{11}$ and $B = \frac{2}{11}$ in eq. (i)

$$\frac{1}{(3x+4)(2x-1)} = \frac{\left(\frac{-3}{11}\right)}{3x+4} + \frac{\left(\frac{2}{11}\right)}{2x-1}$$

$$\frac{1}{6x^2+5x-4} = \frac{-3}{11(3x+4)} + \frac{2}{11(2x-1)}$$

Integrating both sides

$$\int \frac{1}{6x^2+5x-4}dx = \frac{-1}{11}\int \frac{3}{3x+4}dx + \frac{1}{11}\int \frac{2}{2x-1}dx$$

$$= \frac{-1}{11}\ln|3x+4| + \frac{1}{11}\ln|2x-1| + c$$

$$= \frac{1}{11}\left(\ln|2x-1| - \ln|3x+4|\right) + c$$

$$\int \frac{1}{6x^2+5x-4}dx = \frac{1}{11}\ln\frac{|2x-1|}{|3x+4|} + c$$

Q.8 $\int \frac{2x^3-3x^2-x-7}{2x^2-3x-2}dx$

Solution:

By long division

$$\begin{array}{r|l} & \quad\quad x \\ 2x^2-3x-2 & 2x^3-3x^2-x-7 \\ & \underline{\pm 2x^3 \mp 3x^2 \mp 2x} \\ & \quad\quad x-7 \end{array}$$

$$\frac{2x^3-3x^2-x-7}{2x^2-3x-2} = x+\frac{x-7}{2x^2-3x-2}$$

$$= x+\frac{x-7}{2x^2-4x+x-2}$$

$$= x+\frac{x-7}{2x(x-2)+1(x-2)}$$

$$= x+\frac{x-7}{(x-2)(2x+1)} \text{ ...(i)}$$

Let $\frac{x-7}{(x-2)(2x+1)} = \frac{A}{x-2}+\frac{B}{2x+1}$ …(ii)

Multiplying both sides by $(x-2)(2x+1)$

$x-7 = A(2x+1)+B(x-2)$ …(iii)

Put $x-2=0 \Rightarrow x=2$ in (iii)

$2-7 = A\{2(2)+1\}+B(2-2)$

$-5 = 5A \Rightarrow \boxed{A=-1}$

Put $2x+1=0 \Rightarrow x=\frac{-1}{2}$ in (iii)

$-\frac{1}{2}-7 = A\left\{2\left(\frac{-1}{2}\right)+1\right\}+B\left(\frac{-1}{2}-2\right)$

$\frac{-15}{2} = 0+B\left(\frac{-5}{2}\right) \Rightarrow \boxed{B=3}$

Put $A=-1$ and $B=3$ in (ii)

$\frac{x-7}{(x-2)(2x+1)} = \frac{-1}{x-2}+\frac{3}{2x+1}$

(i) becomes

$\frac{2x^3-3x^2-x-7}{2x^2-3x-2} = x-\frac{1}{x-2}+\frac{3}{2x+1}$

Integrating both sides

$$\int\frac{2x^3-3x^2-x-7}{2x^2-3x-2}dx = \int xdx-\int\frac{1}{x-2}dx+\frac{3}{2}\int\frac{2}{2x+1}dx$$

$$= \frac{x^2}{2}-\ln|x-2|+\frac{3}{2}\ln|2x+1|+c$$

Q.9 $\int \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}dx$

Solution:

Let $\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)} = \frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}$ . . . (i)

Multiplying both sides by $(x-1)(x-2)(x-3)$

$3x^2-12x+11 = A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)$ …(ii)

Put $x-1=0 \Rightarrow x=1$ in (ii)

$3(1)^2-12(1)+11 = A(1-2)(1-3)$

$2 = A(-1)(-2) \Rightarrow \boxed{A=1}$

Put $x-2=0 \Rightarrow x=2$ in (ii)

$3(2)^2-12(2)+11 = B(2-1)(2-3)$

$12-24+11 = B(1)(-1)$

$-1 = B(-1) \Rightarrow \boxed{B=1}$

Put $x-3=0 \Rightarrow x=3$ in (ii)

$3(3)^2-12(3)+11 = (3-1)(3-2)$

$27-36+11 = C(2)(1)$

$2 = 2C \Rightarrow \boxed{C=1}$

Put $A=1, B=1, C=1$ in (i)

$$\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)} = \frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3}$$

Integrating both sides

$$\int \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}dx = \int \frac{1}{x-1}dx+\int \frac{1}{x-2}dx+\int \frac{1}{x-3}dx$$

$$= \ln|x-1|+\ln|x-2|+\ln|x-3|+c$$

Q.10 $\int \frac{2x-1}{x(x-1)(x-3)}dx$

Solution:

Let $\frac{2x-1}{x(x-1)(x-3)} = \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x-3}$ … (i)

Multiplying both sides by $x(x-1)(x-3)$

$2x-1 = A(x-1)(x-3)+Bx(x-3)+Cx(x-1)$ … (ii)

Put $x=0$ in (ii)

$2(0)-1 = A(0-1)(0-3)$

$-1 = A(-1)(-3) \Rightarrow \boxed{A=\frac{-1}{3}}$

Put $x-1=0 \Rightarrow x=1$ in (ii)

$2(1)-1=B(1)(1-3)$

$1=B(-2) \;\Rightarrow\; \boxed{B=\frac{-1}{2}}$

Put $x-3=0 \Rightarrow x=3$ in (ii)

$2(3)-1=C(3)(3-1)$

$5=C(3)(2) \;\Rightarrow\; \boxed{C=\frac{5}{6}}$

Put $A=\frac{-1}{3}, B=\frac{-1}{2}$ and $C=\frac{5}{6}$ in (i)

$$\frac{2x-1}{x(x-1)(x-3)}=\frac{\left(\frac{-1}{3}\right)}{x}+\frac{\left(\frac{-1}{2}\right)}{x-1}+\frac{\left(\frac{5}{6}\right)}{x-3}$$

$$\frac{2x-1}{x(x-1)(x-3)}=\frac{-1}{3x}-\frac{1}{2(x-1)}+\frac{5}{6(x-3)}$$

Integrating both sides

$$\int\frac{2x-1}{x(x-1)(x-3)}dx=\frac{-1}{3}\int\frac{1}{x}dx-\int\frac{1}{2(x-1)}dx+\frac{5}{6}\int\frac{1}{x-3}dx$$

$$=\frac{-1}{3}\int\frac{1}{x}dx-\frac{1}{2}\int\frac{1}{x-1}dx+\frac{5}{6}\int\frac{1}{x-3}dx$$

$$=\frac{-1}{3}\ln|x|-\frac{1}{2}\ln|x-1|+\frac{5}{6}\ln|x-3|+c$$

Q.11 $\int\frac{5x^2+9x+6}{(x^2-1)(2x+3)}dx$

Solution:

$$\frac{5x^2+9x+6}{(x^2-1)(2x+3)}=\frac{5x^2+9x+6}{(x-1)(x+1)(2x+3)}$$

Let $\frac{5x^2+9x+6}{(x-1)(x+1)(2x+3)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{2x+3}$ … (i)

Multiplying both sides by $(x-1)(x+1)(2x+3)$

$5x^2+9x+6=A(x+1)(2x+3)+B(x-1)(2x+3)+C(x-1)(x+1)$ … (ii)

Put $x-1=0 \Rightarrow x=1$ in (ii)

$5(1)^2+9(1)+6=A(1+1)\{2(1)+3\}$

$20=A(2)(5) \;\Rightarrow\; \boxed{A=2}$

Put $x+1=0 \Rightarrow x=-1$ in (ii)

$5(-1)^2+9(-1)+6=B(-1-1)\{2(-1)+3\}$

$11-9=B(-2)(1)$

$2=-2B\Rightarrow \boxed{B=-1}$

Put $2x+3=0\Rightarrow x=\frac{-3}{2}$ in (ii)

$5\left(\frac{-3}{2}\right)^2+9\left(\frac{-3}{2}\right)+6=C\left(\frac{-3}{2}-1\right)\left(\frac{-3}{2}+1\right)$

$5\left(\frac{9}{4}\right)-\frac{27}{2}+6=C\left(\frac{-3-2}{2}\right)\left(\frac{-3+2}{2}\right)$

$\frac{45-54+24}{4}=C\left(\frac{-5}{2}\right)\left(\frac{-1}{2}\right)$

$\frac{15}{4}=C\left(\frac{5}{4}\right)\ \Rightarrow \boxed{C=3}$

Put $A=2, B=-1, C=3$ in (i)

$\frac{5x^2+9x+6}{(x^2-1)(2x+3)}=\frac{2}{x-1}-\frac{1}{x+1}+\frac{3}{2x+3}$

Integrating both sides

$$\int\frac{5x^2+9x+6}{(x^2-1)(2x+3)}dx=2\int\frac{1}{x-1}dx-\int\frac{1}{x+1}dx+3\int\frac{1}{2x+3}dx$$

$$=2\int\frac{1}{x-1}dx-\int\frac{1}{x+1}dx+\frac{3}{2}\int\frac{2}{2x+3}dx$$

$$=2\ln|x-1|-\ln|x+1|+\frac{3}{2}\ln|2x+3|+c$$

Q.12 $\int\frac{4+7x}{(1+x)^2(2x+3)}dx$

Solution:

Let $\frac{4+7x}{(1+x)^2(2+3x)}=\frac{A}{2+3x}+\frac{B}{1+x}+\frac{C}{(1+x)^2}$ … (i)

Multiplying both sides by $(1+x)^2(2+3x)$

$4+7x=A(1+x)^2+B(1+x)(2+3x)+C(2+3x)$ … (ii)

Put $2+3x=0\Rightarrow\ x=\frac{-2}{3}$ in (ii)

$4+7\left(\frac{-2}{3}\right)=A\left(1-\frac{2}{3}\right)^2$

$\frac{12-14}{3}=A\left(\frac{3-2}{3}\right)^2$

$\frac{-2}{3}=A\left(\frac{1}{9}\right)$

$A=\frac{-2}{3}\times 9 \Rightarrow \boxed{A=-6}$

Put $1+x=0 \Rightarrow x=-1$ in (ii)

$4+7(-1)=C\{2+3(-1)\}$

$4-7=C(2-3)$

$-3=C(-1) \Rightarrow \boxed{C=3}$

From (ii)

$4+7x=A(1+x^2+2x)+B(2+3x+2x+3x^2)+C(2+3x)$

$4+7x=A+Ax^2+2Ax+2B+5Bx+3Bx^2+2C+3Cx$

Comparing the coefficients of x^2

$0=A+3B$

$0=-6+3B$

$3B=6 \Rightarrow \boxed{B=2}$

Put $A=-6, B=2$ and $C=3$ in (i)

$$\frac{4+7x}{(1+x)^2(2+3x)}=\frac{-6}{2+3x}+\frac{2}{1+x}+\frac{3}{(1+x)^2}$$

Integrating both sides

$$\int\frac{4+7x}{(1+x)^2(2+3x)}dx=-2\int\frac{3}{2+3x}dx+2\int\frac{1}{1+x}dx+3\int(1+x)^{-2}dx$$

$$=-2\ln|2+3x|+2\ln|1+x|+3\frac{(1+x)^{-2+1}}{(-2+1)}+c$$

$$=-2\ln|2+3x|+2\ln|1+x|-\frac{3}{1+x}+c$$

Q.13 $\int\frac{2x^2}{(x-1)^2(x+1)}dx$

Solution:

Let $\frac{2x^2}{(x-1)^2(x+1)}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$ … (i)

Multiplying both side by $(x-1)^2(x+1)$

$2x^2=A(x-1)^2+B(x-1)(x+1)+C(x+1)$ … (ii)

Put $x+1=0 \Rightarrow x=-1$ in (ii)

$2(-1)^2=A(-1-1)^2$

$2=A(-2)^2$

$2=4A \Rightarrow \boxed{A=\frac{1}{2}}$

Put $x-1=0 \Rightarrow x=1$ in (ii)

$2(1)^2=C(1+1)$

$2=2C \Rightarrow \boxed{C=1}$

From (ii)

$2x^2 = A(x^2+1-2x)+B(x^2-1)+C(x+1)$

$2x^2 = Ax^2+A-2Ax+Bx^2-B+Cx+C$

Comparing the coefficients of $'x^2'$

$2 = A+B$

$2 = \frac{1}{2}+B$

$B = 2-\frac{1}{2} \Rightarrow \boxed{B=\frac{3}{2}}$

Put $A=\frac{1}{2}, B=\frac{3}{2}$ and $C=1$ in (i)

$$\frac{2x^2}{(x-1)^2(x+1)} = \frac{1}{2(x+1)}+\frac{3}{2(x-1)}+\frac{1}{(x-1)^2}$$

Integrating both sides

$$\int\frac{2x^2}{(x-1)^2(x+1)}dx = \frac{1}{2}\int\frac{1}{x+1}dx+\frac{3}{2}\int\frac{1}{x-1}dx+\int(x-1)^{-2}dx$$

$$= \frac{1}{2}\ln|x+1|+\frac{3}{2}\ln|x-1|+\frac{(x-1)^{-2+1}}{-2+1}+c$$

$$= \frac{1}{2}\ln|x+1|+\frac{3}{2}\ln|x-1|-(x-1)^{-1}+c$$

$$= \frac{1}{2}\ln|x+1|+\frac{3}{2}\ln|x-1|-\frac{1}{x-1}+c$$

Q.14 $\int\frac{1}{(x-1)(x+1)^2}dx$

Solution:

Let $\frac{1}{(x-1)(x+1)^2} = \frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$ … (i)

Multiplying both side by $(x-1)(x+1)^2$

$1 = A(x+1)^2+B(x-1)(x+1)+C(x-1)$ … (ii)

Put $x-1=0 \Rightarrow x=1$ in (ii)

$1 = A(1+1)^2$

$1 = 4A \Rightarrow \boxed{A=\frac{1}{4}}$

Put $x+1=0 \Rightarrow x=-1$ in (ii)

$1 = C(-1-1)$

$1 = C(-2) \Rightarrow \boxed{C=\frac{-1}{2}}$

Expanding (ii)

$1 = A(x^2+1+2x)+B(x^2-1)+C(x-1)$

$1 = Ax^2+A+2Ax+Bx^2-B+Cx-C$

Comparing the coefficients of x^2

$0 = A + B$

$0 = \frac{1}{4} + B \Rightarrow \boxed{B = -\frac{1}{4}}$

Put $A = \frac{1}{4}, B = \frac{-1}{4}$ and $C = \frac{-1}{2}$ in (i)

$$\frac{1}{(x-1)(x+1)^2} = \frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x+1)^2}$$

Integrating both sides

$$\int \frac{1}{(x-1)(x+1)^2} dx = \int \frac{1}{4(x-1)} dx - \frac{1}{4}\int \frac{1}{x+1} dx - \frac{1}{2}\int (x+1)^{-2} dx$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\frac{(x+1)^{-1}}{(-1)} + c$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| + \frac{1}{2(x+1)} + c$$

Q.15 $\int \frac{x+4}{x^3 - 3x^2 + 4} dx$

Solution:

First we factorize $x^3 - 3x^2 + 4$

Clearly $(x+1)$ is its factor

Using synthetic division

−1	1	−3	0	4
		−1	4	−4
	1	−4	4	0

Quotient $= x^2 - 4x + 4$

$$x^3 - 3x^2 + 4 = (x+1)(x^2 - 4x + 4)$$

$$= (x+1)(x-2)^2$$

$$\frac{x+4}{x^3 - 3x + 4} = \frac{x+4}{(x+1)(x-2)^2}$$

Let $\frac{x+4}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$ …(i)

Multiplying both sides by $(x+1)(x-2)^2$

$x + 4 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$ … (ii)

Put $x + 1 = 0 \Rightarrow x = -1$ in (ii)

$-1 + 4 = A(-1-2)^2$

$3 = 9A \Rightarrow \boxed{A = \frac{1}{3}}$

Put $x - 2 = 0 \Rightarrow x = 2$

Put $x = 2$ in (ii)

$2 + 4 = C(2+1)$

$6 = 3C \Rightarrow \boxed{C = 2}$

Expanding (ii)

$x+4 = A(x^2+4-4x)+B(x^2-x-2)+C(x+1)$

$x+4 = Ax^2+4A-4Ax+Bx^2-Bx-2B+Cx+C$

Comparing the coefficients of x^2

$0 = A+B$

$0 = \frac{1}{3}+B \Rightarrow \boxed{B = \frac{-1}{3}}$

Put $A = \frac{1}{3}, B = \frac{-1}{3}$ and $C = 2$ in (i)

$$\frac{x+4}{(x+1)(x-2)^2} = \frac{\left(\frac{1}{3}\right)}{x+1}+\frac{\left(\frac{-1}{3}\right)}{x-2}+\frac{2}{(x-2)^2}$$

$$\frac{x+4}{x^3-3x^2+4} = \frac{1}{3(x+1)}-\frac{1}{3(x-2)}+\frac{2}{(x-2)^2}$$

Integrating both sides

$$\int\frac{x+4}{x^3-3x^2+4}dx = \frac{1}{3}\int\frac{1}{x+1}dx-\frac{1}{3}\int\frac{1}{x-2}dx+2\int(x-2)^{-2}dx$$

$$= \frac{1}{3}\ln|x+1|-\frac{1}{3}\ln|x-2|+2\frac{(x-2)^{-2+1}}{(-2+1)}+c$$

$$= \frac{1}{3}\ln|x+1|-\frac{1}{3}\ln|x-2|-\frac{2}{x-2}+c$$

Q.16 $\int\frac{x^3-6x^2+25}{(x+1)^2(x-2)^2}dx$

Solution:

Let $\frac{x^3-6x^2+25}{(x+1)^2(x-2)^2} = \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x-2}+\frac{D}{(x-2)^2}$ …(i)

Multiplying both sides by $(x+1)^2(x-2)^2$

$x^3-6x^2+25 = A(x+1)(x-2)^2+B(x-2)^2+C(x+1)^2(x-2)+D(x+1)^2$ …(ii)

Put $x+1=0 \Rightarrow x=-1$ in (ii)

$(-1)^3-6(-1)^2+25 = B(-1-2)^2$

$-1-6+25 = B(-3)^2$

$18 = 9B \Rightarrow \boxed{B = 2}$

Put $x-2=0 \Rightarrow x=2$ in (ii)

$(2)^3-6(2)^2+25 = D(2+1)^2$

$8-6(4)+25 = D(3)^2$

$9 = 9D \Rightarrow \boxed{D = 1}$

Expanding (ii)

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$$x^3-6x^2+25=A(x+1)(x^2+4-4x)+B(x^2+4-4x)+C(x^2+1+2x)(x-2)+D(x^2+1+2x)$$

$$x^3-6x^2+25=A(x^3+4x-4x^2+x^2+4-4x)+B(x^2-4x+4)+C(x^3-2x^2+x-2+2x^2-4x)$$
$$+D(x^2+1+2x)$$

$$x^3-6x^2+25=Ax^3-3Ax^2+4A+Bx^2-4Bx+4B+Cx^3-3Cx-2C+Dx^2+D+2Dx$$

Comparing the coefficients of x^3 and x^2 respectively

$1=A+C$ . . . (iii)

$-6=-3A+B+D$ . . . (iv)

Put $B=2$ and $D=1$ in (iv)

$-6=-3A+2+1$

$-6=-3A+3$

$3A=6+3$

$3A=9 \Rightarrow \boxed{A=3}$

Put $A=3$ in (iii)

$1=3+C$

$C=1-3 \Rightarrow \boxed{C=-2}$

Put $A=3, B=2, C=-2, D=1$ in (i)

$$\frac{x^3-6x^2+25}{(x+1)^2(x-2)^2}=\frac{3}{x+1}+\frac{2}{(x+1)^2}-\frac{2}{x-2}+\frac{1}{(x-2)^2}$$

Integrating both sides

$$\int\frac{x^3-6x^2+25}{(x+1)^2(x-2)^2}dx=3\int\frac{1}{x+1}dx+2\int(x+1)^{-2}dx-2\int\frac{1}{x-2}dx+\int(x-2)^{-2}dx$$

$$=3\ln|x+1|+2\frac{(x+1)^{-2+1}}{(-2+1)}-2\ln|x-2|+\frac{(x-2)^{-2+1}}{(-2+1)}+c$$

$$=3\ln|x+1|-\frac{2}{x+1}-2\ln|x-2|-\frac{1}{x-2}+c$$

Q.17 $\displaystyle\int\frac{x^3+22x^2+14x-17}{(x-3)(x+2)^3}dx$

Solution:

Let $\displaystyle\frac{x^3+22x^2+14x-17}{(x-3)(x+2)^3}=\frac{A}{x-3}+\frac{B}{x+2}+\frac{C}{(x+2)^2}+\frac{D}{(x+2)^3}$ . . . (i)

Multiplying both sides by $(x-3)(x+2)^3$

$x^3+22x^2+14x-17=A(x+2)^3+B(x-3)(x+2)^2+C(x-3)(x+2)+D(x-3)$ ...(ii)

Put $x-3=0 \Rightarrow x=3$ in (ii)

$(3)^3+22(3)^2+14(3)-17=A(3+2)^3$

$27+198+42-17=A(5)^3$

$250=125A \Rightarrow \boxed{A=2}$

Put $x+2=0 \Rightarrow x=-2$ in (ii)

$(-2)^3+22(-2)^2+14(-2)-17=D(-2-3)$

$-8+88-28-17=D(-5)$

$35=-5D \Rightarrow \boxed{D=-7}$

Expanding (ii)

$$x^3+22x^2+14x-17=A(x^3+6x^2+12x+8)+B(x-3)(x^2+4x+4)$$
$$+C(x^2+2x-3x-6)+D(x-3)$$
$$x^3+22x^2+14x-17=A(x^3+6x^2+12x+8)+B(x^3+4x^2+4x-3x^2$$
$$-12x-12)+C(x^2-x-6)+D(x-3)$$
$$x^3+22x^2+14x-17=Ax^3+6Ax^2+12Ax+8A+Bx^3+Bx^2-8Bx$$
$$-12B+Cx^2-Cx-6C+Dx-3D$$

Comparing the coefficients of x^3

$1=A+B\ldots$ (iii)

Put $A=2$

$1=2+B \Rightarrow \boxed{B=-1}$

Comparing the coefficients of x^2

$22=6A+B+C$

Put $A=2$ and $B=-1$

$22=6(2)+(-1)+C$

$22=12-1+C$

$22=11+C \Rightarrow \boxed{C=11}$

Put $A=2, B=-1, C=11, D=-7$ in (i)

$$\frac{x^3+22x^2+14x-17}{(x-3)(x+2)^2}=\frac{2}{x-3}-\frac{1}{x+2}+\frac{11}{(x+2)^2}-\frac{7}{(x+2)^3}$$

Integrating both sides

$$\int\frac{x^3+22x^2+14x-17}{(x-3)(x+2)^2}dx=2\int\frac{1}{x-3}dx-\int\frac{1}{x+2}dx+11\int(x+2)^{-2}dx-7\int(x+2)^{-3}dx$$
$$=2\ln|x-3|-\ln|x+2|+11\left(\frac{(x+2)^{-2+1}}{-2+1}\right)-7\left(\frac{(x+2)^{-3+1}}{-3+1}\right)+c$$
$$=2\ln|x-3|-\ln|x+2|-\frac{11}{x+2}+\frac{7}{2(x+2)^2}+c$$

Q.18 $\displaystyle\int\frac{x-2}{(x+1)(x^2+1)}dx$

Solution:

Let $\displaystyle\frac{x-2}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$ ... (i)

Multiplying both sides by $(x+1)(x^2+1)$

$x-2=A(x^2+1)+(Bx+C)(x+1)\ldots$ (ii)

Put $x+1=0 \Rightarrow x=-1$ in (ii)

$-1-2=A\{(-1)^2+1\}$

$-3=A(2) \Rightarrow \boxed{A=\frac{-3}{2}}$

Expanding (ii)

$x-2 = Ax^2 + A + Bx^2 + Bx + Cx + C$

Comparing the coefficients of x^2

$0 = A + B$

$0 = \frac{-3}{2} + B \Rightarrow \boxed{B = \frac{3}{2}}$

Comparing the coefficients of x

$1 = B + C$

$1 = \frac{3}{2} + C$

$C = 1 - \frac{3}{2} \Rightarrow \boxed{C = -\frac{1}{2}}$

Put $A = -\frac{3}{2}, B = \frac{3}{2}$ and $C = \frac{-1}{2}$ in (i)

$$\frac{x-2}{(x+1)(x^2+1)} = \frac{\left(\frac{-3}{2}\right)}{x+1} + \frac{\left(\frac{3x}{2} - \frac{1}{2}\right)}{x^2+1}$$

$$\frac{x-2}{(x+1)(x^2+1)} = \frac{-3}{2(x+1)} + \frac{\frac{3x-1}{2}}{x^2+1}$$

$$= \frac{-3}{2(x+1)} + \frac{3x-1}{2(x^2+1)}$$

$$\frac{x-2}{(x+1)(x^2+1)} = \frac{-3}{2(x+1)} + \frac{3x}{2(x^2+1)} - \frac{1}{2\left(x^2+1\right)}$$

Integrating both sides

$$\int \frac{x-2}{(x+1)(x^2+1)} dx = \frac{-3}{2}\int \frac{1}{x+1} dx + \frac{3}{2}\int \frac{x}{x^2+1} dx - \frac{1}{2}\int \frac{1}{x^2+1} dx$$

$$= \frac{-3}{2}\int \frac{1}{x+1} dx + \frac{3}{4}\int \frac{2x}{x^2+1} dx - \frac{1}{2}\int \frac{1}{x^2+1} dx$$

$$= \frac{-3}{2}\ln|x+1| + \frac{3}{4}\ln|x^2+1| - \frac{1}{2}\tan^{-1} x + c$$

Q.19 $\int \frac{x}{(x-1)(x^2+1)} dx$

Solution:

Let $\frac{x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$ … (i)

Multiplying both sides by $(x-1)(x^2+1)$

$x = A(x^2+1) + (Bx+C)(x-1)$ … (ii)

Put $x-1=0 \Rightarrow x=1$ in (ii)

$1 = A\left\{(1)^2+1\right\}$

$1 = A(2) \Rightarrow \boxed{A = \frac{1}{2}}$

Expanding (ii)

$x = Ax^2 + A + Bx^2 - Bx + Cx - C$

Comparing the coefficients of x^2

$0 = A + B$

$0 = \frac{1}{2} + B \Rightarrow \boxed{B = \frac{-1}{2}}$

Comparing the coefficients of "x"

$1 = -B + C$

$1 = -\left(\frac{-1}{2}\right) + C$

$C = 1 - \frac{1}{2} \Rightarrow \boxed{C = \frac{1}{2}}$

Put $A = \frac{1}{2}, B = -\frac{1}{2}$ and $C = \frac{1}{2}$ in (i)

$$\frac{x}{(x-1)(x^2+1)} = \frac{\left(\frac{1}{2}\right)}{x-1} + \frac{\left(\frac{-x}{2} + \frac{1}{2}\right)}{x^2+1}$$

$$\frac{x}{(x-1)(x^2+1)} = \frac{1}{2(x-1)} + \frac{-x+1}{2(x^2+1)}$$

$$\frac{x}{(x-1)(x^2+1)} = \frac{1}{2(x-1)} - \frac{x}{2(x^2+1)} + \frac{1}{2(x^2+1)}$$

Integrating both sides

$$\int \frac{x}{(x-1)(x^2+1)}dx = \frac{1}{2}\int \frac{1}{x-1}dx - \frac{1}{2}\int \frac{x}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx$$

$$= \frac{1}{2}\int \frac{1}{x-1}dx - \frac{1}{2\times 2}\int \frac{2x}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{4}\ln|x^2+1| + \frac{1}{2}\tan^{-1}x + c$$

Q.20 $\int \frac{9x-7}{(x+3)(x^2+1)}dx$

Solution:

Let $\frac{9x-7}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1}$ ... (i)

Multiplying both sides by $(x+3)(x^2+1)$

$9x - 7 = A(x^2+1) + (Bx+C)(x+3)$ ...(ii)

Put $x + 3 = 0 \Rightarrow x = -3$ in (ii)

$9(-3) - 7 = A\{(-3)^2 + 1\}$

$-34 = 10A \Rightarrow \boxed{A = \frac{-17}{5}}$

Expanding (ii)

$9x - 7 = Ax^2 + A + Bx^2 + 3Bx + Cx + 3C$

Comparing the coefficients of x^2

$0 = A + B$

$0 = \frac{-17}{5} + B \Rightarrow \boxed{B = \frac{17}{5}}$

Comparing the coefficients of x

$9 = 3B + C$

$9 = 3\left(\frac{17}{5}\right) + C$

$C = 9 - \frac{51}{5}$

$C = \frac{45-51}{5} \Rightarrow \boxed{C = \frac{-6}{5}}$

Put $A = \frac{-17}{5}, B = \frac{17}{5}, C = \frac{-6}{5}$ in (i)

$$\frac{9x-7}{(x+3)(x^2+1)} = \frac{\frac{-17}{5}}{x+3} + \frac{\frac{17x}{5} + \left(\frac{-6}{5}\right)}{x^2+1}$$

$$= \frac{-17}{5(x+3)} + \frac{\frac{17x-6}{5}}{x^2+1}$$

$$= \frac{-17}{5(x+3)} + \frac{17x-6}{5(x^2+1)}$$

$$\frac{9x-7}{(x+3)(x^2+1)} = \frac{-17}{5(x+3)} + \frac{17x}{5(x^2+1)} - \frac{6}{5(x^2+1)}$$

Integrating both sides

$$\int \frac{9x-7}{(x+3)(x^2+1)}dx = \frac{-17}{5}\int \frac{1}{x+3}dx + \frac{17}{5}\int \frac{x}{x^2+1}dx - \frac{6}{5}\int \frac{1}{x^2+1}dx$$

$$= \frac{-17}{5}\int \frac{1}{x+3}dx + \frac{17}{5\times 2}\int \frac{2x}{x^2+1}dx - \frac{6}{5}\int \frac{1}{x^2+1}dx$$

$$= \frac{-17}{5}\ln|x+3| + \frac{17}{10}\ln|x^2+1| - \frac{6}{5}\tan^{-1}x + c$$

Q.21 $\int \frac{1+4x}{(x-3)(x^2+4)}dx$

Solution

Let $\frac{1+4x}{(x-3)(x^2+4)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+4}$ … (i)

Multiplying both sides by $(x-3)\left(x^2+4\right)$

$1+4x = A(x^2+4)+(Bx+C)(x-3)\ \ldots\text{(ii)}$

Put $x-3=0 \Rightarrow x=3$ in (ii)

$1+4(3) = A\left\{(3)^2+4\right\}$

$13=13A \Rightarrow \boxed{A=1}$

Expanding (ii)

$1+4x = Ax^2+4A+Bx^2-3Bx+Cx-3C$

Comparing the coefficients of x^2

$0=A+B$

$0=1+B \Rightarrow \boxed{B=-1}$

Comparing the coefficients of $'x'$

$4=-3B+C$

$4=-3(-1)+C$

$4=3+C \Rightarrow \boxed{C=1}$

Put $A=1, B=-1, C=1$ in (i)

$$\frac{1+4x}{(x-3)(x^2+4)} = \frac{1}{x-3}+\frac{(-1)x+1}{x^2+4}$$

$$\frac{1+4x}{(x-3)(x^2+4)} = \frac{1}{x-3}+\frac{-x}{x^2+4}+\frac{1}{x^2+4}$$

Integrating both sides

$$\int\frac{1+4x}{(x-3)(x^2+4)}dx = \int\frac{1}{x-3}dx+\frac{-1}{2}\int\frac{2x}{x^2+4}dx+\int\frac{1}{x^2+(2)^2}dx$$

$$= \ln|x-3|-\frac{1}{2}\ln\left|x^2+4\right|+\frac{1}{2}\tan^{-1}\frac{x}{2}+c$$

Q.22 $\int\frac{12}{x^3+8}dx$

Solution:

$$\frac{12}{(x^3+8)} = \frac{12}{(x)^3+(2)^3} = \frac{12}{(x+2)(x^2-2x+4)}$$

Let $\frac{12}{(x+2)(x^2-2x+4)} = \frac{A}{x+2}+\frac{Bx+C}{x^2-2x+4}\ \ldots\text{(i)}$

Multiplying both sides by $(x+2)(x^2-2x+4)$

$12 = A(x^2-2x+4)+(Bx+C)(x+2)\ \ldots\text{(ii)}$

Put $x+2=0 \Rightarrow x=-2$ in (ii)

$12 = A\left\{(-2)^2-2(-2)+4\right\}$

$12 = A(4+4+4)$

$12=12A \Rightarrow \boxed{A=1}$

Expanding (ii)

$12 = Ax^2-2Ax+4A+Bx^2+2Bx+Cx+2C$

Comparing the coefficients of x^2

$0 = A + B$

$0 = 1 + B \Rightarrow \boxed{B = -1}$

Comparing the coefficients of $'x'$

$0 = -2A + 2B + C$

$0 = -2(1) + 2(-1) + C$

$0 = -4 + C \Rightarrow \boxed{C = 4}$

Put $A = 1, B = -1, C = 4$ in (i)

$$\frac{12}{(x+2)(x^2-2x+4)} = \frac{1}{x+2} + \frac{(-1)x+4}{x^2-2x+4}$$

$$\frac{12}{x^3+8} = \frac{1}{x+2} - \frac{x-4}{x^2-2x+4}$$

$$= \frac{1}{x+2} - \frac{x-1-3}{x^2-2x+4}$$

$$\frac{12}{x^3+8} = \frac{1}{x+2} - \frac{x-1}{x^2-2x+4} + \frac{3}{x^2-2x+4}$$

Integrating both sides

$$\int \frac{12}{x^3+8}dx = \int \frac{1}{x+2}dx - \int \frac{x-1}{x^2-2x+4}dx + 3\int \frac{1}{x^2-2x+1+3}dx$$

$$= \int \frac{1}{x+2}dx - \frac{1}{2}\int \frac{2x-2}{x^2-2x+4}dx + 3\int \frac{1}{(x-1)^2+(\sqrt{3})^2}dx$$

$$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + 3\left(\frac{1}{\sqrt{3}}\tan^{-1}\frac{x-1}{\sqrt{3}}\right) + c$$

$$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + \sqrt{3}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + c$$

Q.23 $\int \frac{9x+6}{x^3-8}dx$

Solution:

Consider $\frac{9x+6}{x^3-8} = \frac{9x+6}{(x)^3-(2)^3} = \frac{9x+6}{(x-2)(x^2+2x+4)}$

Let $\frac{9x+6}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}$ ... (i)

Multiplying both sides1 by $(x-2)(x^2+2x+4)$

$9x+6 = A(x^2+2x+4) + (Bx+C)(x-2)$ ... (ii)

Put $x-2=0 \Rightarrow x=2$ in (ii)

$9(2)+6 = A\{(2)^2+2(2)+4\}$

$24 = A(12) \Rightarrow \boxed{A = 2}$

Expanding (ii)

$9x+6 = Ax^2+2Ax+4A+Bx^2-2Bx+Cx-2C$

Comparing the coefficients of x^2

$0 = A + B$

$0 = 2 + B \Rightarrow \boxed{B = -2}$

Comparing the coefficients of x

$9 = 2A - 2B + C$

$9 = 2(2) - 2(-2) + C$

$9 = 4 + 4 + C \Rightarrow \boxed{C = 1}$

Put $A = 2, B = -2$ and $C = 1$ in (i)

$$\frac{9x+6}{(x-2)(x^2+2x+4)} = \frac{2}{x-2} + \frac{(-2)x+1}{x^2+2x+4}$$

$$\frac{9x+6}{x^3-8} = \frac{2}{x-2} - \frac{2x-1}{x^2+2x+4}$$

$$= \frac{2}{x-2} - \frac{2x+2-2-1}{x^2+2x+4}$$

$$\frac{9x+6}{x^3-8} = \frac{2}{x-2} - \frac{2x+2}{x^2+2x+4} + \frac{3}{x^2+2x+4}$$

Integrating both sides

$$\int \frac{9x+6}{x^3-8}dx = 2\int \frac{1}{x-2}dx - \int \frac{2x+2}{x^2+2x+4}dx + 3\int \frac{1}{x^2+2x+1+3}dx$$

$$= 2\ln|x-2| - \ln|x^2+2x+4| + 3\int \frac{1}{(x+1)^2+(\sqrt{3})^2}dx$$

$$= 2\ln|x-2| - \ln|x^2+2x+4| + 3\left(\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)\right) + c$$

$$= 2\ln|x-2| - \ln|x^2+2x+4| + \sqrt{3}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + c$$

Q.24 $\int \frac{2x^2+5x+3}{(x-1)^2(x^2+4)}dx$

Solution:

Let $\frac{2x^2+5x+3}{(x-1)^2(x^2+4)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+4}$ … (i)

Multiplying both sides by $(x-1)^2(x^2+4)$

$2x^2+5x+3 = A(x-1)(x^2+4) + B(x^2+4) + (Cx+D)(x-1)^2$ … (ii)

Put $x-1=0 \Rightarrow x=1$ in (ii)

$2(1)^2+5(1)+3 = B\{(1)^2+4\}$

$10 = 5B \Rightarrow \boxed{B = 2}$

Expanding (ii)

$2x^2+5x+3 = A(x^3+4x-x^2-4) + B(x^2+4) + (Cx+D)(x^2+1-2x)$

$2x^2+5x+3 = Ax^3 - Ax^2 + 4Ax - 4A + Bx^2 + 4B + Cx^3 + Cx - 2Cx^2 + Dx^2 + D - 2Dx$

Comparing the coefficients of x^3

$0 = A + C$ … (iii)

Comparing the coefficients of x^2

$2=-A+B-2C+D$

Put $B=2$ and $A=-C$ (from (iii))

$2=C+2-2C+D$

$0=-C+D$

$C=D\ldots$ (iv)

Comparing the coefficients of x

$5=4A+C-2D$

Put $D=C$ and $A=-C$

$5=4(-C)+C-2C$

$5=-5C \Rightarrow \boxed{C=-1}$

From (iv) $\boxed{D=-1}$

Put $C=-1$ in (iii)

$0=A+(-1) \Rightarrow \boxed{A=1}$

Put $A=1, B=2, C=-1, D=-1$ in (i)

$$\frac{2x^2+5x+3}{(x-1)^2(x^2+4)}=\frac{1}{x-1}+\frac{2}{(x-1)^2}+\frac{(-1)x+(-1)}{x^2+4}$$

$$\frac{2x^2+5x+3}{(x-1)^2(x^2+4)}=\frac{1}{x-1}+\frac{2}{(x-1)^2}-\frac{x}{x^2+4}-\frac{1}{x^2+4}$$

Integrating both sides

$$\int\frac{2x^2+5x+3}{(x-1)^2(x^2+4)}dx=\int\frac{1}{x-1}dx+2\int(x-1)^{-2}dx-\frac{1}{2}\int\frac{2x}{x^2+4}dx-\int\frac{1}{x^2+2^2}dx$$

$$=\ln|x-1|+2\left(\frac{(x-1)^{-2+1}}{-2+1}\right)-\frac{1}{2}\ln|x^2+4|-\frac{1}{2}\tan^{-1}\frac{x}{2}+c$$

$$=\ln|x-1|-\frac{2}{x-1}-\frac{1}{2}\ln|x^2+4|-\frac{1}{2}\tan^{-1}\frac{x}{2}+c$$

Q.25 $\int\frac{2x^2-x-7}{(x+2)^2(x^2+x+1)}dx$

Solution:

Let $\frac{2x^2-x-7}{(x+2)^2(x^2+x+1)}=\frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{Cx+D}{x^2+x+1}$ ... (i)

Multiplying both sides by $(x+2)^2(x^2+x+1)$

$2x^2-x-7=A(x+2)(x^2+x+1)+B(x^2+x+1)+(Cx+D)(x+2)^2\ldots$ (ii)

Put $x+2=0 \Rightarrow x=-2$ in (ii)

$2(-2)^2-(-2)-7=B\{(-2)^2+(-2)+1\}$

$2(4)+2-7=B\{4-1\}$

$3=3B \Rightarrow \boxed{B=1}$

Expanding (ii)

$2x^2-x-7=A(x^3+x^2+x+2x^2+2x+2)+B(x^2+x+1)+(Cx+D)(x^2+4x+4)$

$$2x^2 - x - 7 = Ax^3 + 3Ax^2 + 3Ax + 2A + Bx^2 + Bx + B + Cx^3 + 4Cx^2 + 4Cx + Dx^2 + 4Dx + 4D$$

Comparing the coefficients of x^3

$0 = A + C \ldots$ (iii)

Comparing the coefficients of x^2

$2 = 3A + B + 4C + D$

Put $A = -C$ (from (iii))

$2 = -3C + B + 4C + D$

$2 = B + C + D \ldots$ (iv)

Comparing the coefficients of x

$-1 = 3A + B + 4C + 4D$

$-1 = 3A + 3C + B + C + D + 3D$

$-1 = 3(A + C) + (B + C + D) + 3D$

$-1 = 3(0) + 2 + 3D \qquad (A + C = 0\,, B + C + D = 2)$

$3D = -1 - 2 \Rightarrow \boxed{D = -1}$

Put $B = 1, D = -1$ in (iv)

$2 = 1 + C + (-1) \Rightarrow \boxed{C = 2}$

Put $C = 2$ in (iii)

$0 = A + 2 \Rightarrow \boxed{A = -2}$

Put $A = -2, B = 1, C = 2, D = -1$ in (i)

$$\frac{2x^2 - x - 7}{(x+2)^2(x^2 + x + 1)} = -\frac{2}{x+2} + \frac{1}{(x+2)^2} + \frac{2x-1}{x^2 + x + 1}$$

Integrating both sides

$$\int \frac{2x^2 - x - 7}{(x+2)^2(x^2 + x + 1)}\,dx = -2\int \frac{dx}{x+2} + \int (x+2)^{-2}\,dx + \int \frac{2x-1}{x^2 + x + 1}\,dx$$

$$= -2\ln|x+2| + \frac{(x+2)^{-1}}{-1} + \int \frac{2x+1-2}{x^2 + x + 1}\,dx$$

$$= -2\ln|x+2| - \frac{1}{x+2} + \int \frac{2x+1}{x^2 + x + 1}\,dx - 2\int \frac{dx}{x^2 + x + 1}$$

$$= -2\ln|x+2| - \frac{1}{x+2} + \ln|x^2 + x + 1| - 2\int \frac{dx}{x^2 + 2.x.\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \frac{1}{4} + 1}$$

$$= -2\ln|x+2| - \frac{1}{x+2} + \ln|x^2 + x + 1| - 2\int \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= -2\ln|x+2| - \frac{1}{x+2} + \ln|x^2 + x + 1| - 2\left[\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{x + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right] + c$$

$$= -2\ln|x+2| - \frac{1}{x+2} + \ln|x^2 + x + 1| - \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$$

Q.26 $\int \frac{3x+1}{(4x^2+1)(x^2-x+1)}dx$

Solution:

Let $\frac{3x+1}{(4x^2+1)(x^2-x+1)} = \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{x^2-x+1}$ ... (i)

Multiplying both sides by $(4x^2+1)(x^2-x+1)$

$3x+1 = (Ax+B)(x^2-x+1) + (Cx+D)(4x^2+1)$

$3x+1 = Ax^3 - Ax^2 + Ax + Bx^2 - Bx + B + 4Cx^3 + Cx + 4Dx^2 + D$

Comparing coefficients of like powers of x, we have

(for x^3) $A+4C=0$ ... (ii)

(for x^2) $-A+B+4D=0$ ... (iii)

(for x) $A-B+C=3$ ...(iv)

(Constant) $B+D=1$

$B=1-D$ ... (v)

Adding (iii) and (iv)

$4D+C=3$

$C=3-4D$ ...(vi)

From (ii) $A=-4C$ ... (vii)

So $A=-4(3-4D)$

$A=-12+16D$ ... (viii)

Put values of A, B and C in (iv)

$-12+16D-(1-D)+3-4D=3$

$-12+16D-1+D+3-4D=3$

$13D-10=3$

$13D=13 \Rightarrow \boxed{D=1}$

From (v) $B=1-1 \Rightarrow \boxed{B=0}$

From (vi) $C=3-4(1) \Rightarrow \boxed{C=-1}$

From (viii) $A=-12+16(1) \Rightarrow \boxed{A=4}$

Putting values of A, B, C and D in (i)

$$\frac{3x+1}{(4x^2+1)(x^2-x+1)} = \frac{4x+0}{4x^2+1} + \frac{-x+1}{x^2-x+1}$$

$$= \frac{4x}{4x^2+1} - \frac{x-1}{x^2-x+1}$$

Integrating both sides

$$\int \frac{3x+1}{(4x^2+1)(x^2-x+1)}dx = \frac{1}{2}\int \frac{8x}{4x^2+1}dx - \frac{1}{2}\int \frac{2x-2}{x^2-x+1}dx$$

$$= \frac{1}{2}\ln(4x^2+1) - \frac{1}{2}\int \frac{2x-1-1}{x^2-x+1}dx$$

$$= \frac{1}{2}\ln(4x^2+1) - \frac{1}{2}\int \frac{2x-1}{x^2-x+1}dx + \frac{1}{2}\int \frac{1}{x^2-x+1}dx$$

$$=\frac{1}{2}\ln\left(4x^2+1\right)-\frac{1}{2}\ln\left(x^2-x+1\right)+\frac{1}{2}\int\frac{1}{x^2-x+\left(\frac{1}{2}\right)^2-\frac{1}{4}+1}dx$$

$$=\frac{1}{2}\ln\left(4x^2+1\right)-\frac{1}{2}\ln\left(x^2-x+1\right)+\frac{1}{2}\int\frac{1}{x^2-x+\left(\frac{1}{2}\right)^2+\frac{3}{4}}dx$$

$$=\frac{1}{2}\ln\left(4x^2+1\right)-\frac{1}{2}\ln\left(x^2-x+1\right)+\frac{1}{2}\int\frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx$$

$$=\frac{1}{2}\ln\left(4x^2+1\right)-\frac{1}{2}\ln\left(x^2-x+1\right)+\frac{1}{2}\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c$$

$$=\frac{1}{2}\ln\left(4x^2+1\right)-\frac{1}{2}\ln\left(x^2-x+1\right)+\frac{1}{2}\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\frac{2x-1}{2}}{\frac{\sqrt{3}}{2}}\right)+c$$

$$=\frac{1}{2}\ln\left(4x^2+1\right)-\frac{1}{2}\ln\left(x^2-x+1\right)+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+c$$

Q.27 $\int\frac{4x+1}{(x^2+4)(x^2+4x+5)}dx$

Solution:

Let $\frac{4x+1}{(x^2+4)(x^2+4x+5)}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{x^2+4x+5}$ ... (i)

Multiplying both sides by $(x^2+4)(x^2+4x+5)$

$4x+1=(Ax+B)(x^2+4x+5)+(Cx+D)(x^2+4)$

$4x+1=Ax^3+4Ax^2+5Ax+Bx^2+4Bx+5B+Cx^3+4Cx+Dx^2+4D$

Comparing coefficients of like powers of *x*.

(For x^3) $A+C=0$ … (ii)

(For x^2) $4A+B+D=0$ … (iii)

(For *x*) $5A+4B+4C=4$ … (iv)

(For constant) $5B+4D=1$ … (v)

From (ii) $A=-C$ … (vi)

Put $A=-C$ in (iv)

$-5C+4B+4C=4$

$4B-C=4$ ...(vii)

From (v) $D=\frac{1-5B}{4}$ … (viii)

Putting values *A* and D in (iii)

$$4(-C)+B+\frac{1-5B}{4}=0$$

$$\frac{-16C+4B+1-5B}{4}=0$$

$$-16C-B+1=0$$

$$B+16C=1 \;.....(ix)$$

By Eq (vii) -4Eq (ix)

$$4B-C=4$$

$$\underline{\pm 4B \pm 64C = \pm 4}$$

$$-65C=0$$

$$\boxed{C=0}$$

From (vii) $4B-0=4 \Rightarrow \boxed{B=1}$

From (vi) $\boxed{A=0}$

From (viii) $D=\frac{1-5}{4} \Rightarrow \boxed{D=-1}$

Putting values of A, B, C and D in (i)

$$\frac{4x+1}{(x^2+4)(x^2+4x+5)}=\frac{1}{x^2+4}-\frac{1}{x^2+4x+5}$$

Integrating both sides

$$\int \frac{4x+1}{(x^2+4)(x^2+4x+1)}dx=\int \frac{1}{x^2+4}dx-\int \frac{1}{x^2+4x+5}dx$$

$$=\int \frac{1}{(x)^2+(2)^2}dx-\int \frac{1}{(x+2)^2+(1)^2}dx$$

$$=\frac{1}{2}\tan^{-1}\frac{x}{2}-\tan^{-1}(x+2)+c$$

Q.28 $\int \frac{6a^2}{(x^2+a^2)(x^2+4a^2)}dx$

Solution:

Let $\frac{6a^2}{(x^2+a^2)(x^2+4a^2)}=\frac{Ax+B}{x^2+a^2}+\frac{Cx+D}{x^2+4a^2}$ … (i)

Multiplying both sides by $(x^2+a^2)(x^2+4a^2)$

$$6a^2=(Ax+B)(x^2+4a^2)+(Cx+D)(x^2+a^2)$$

$$6a^2=Ax^3+4a^2Ax+Bx^2+4a^2B+Cx^3+a^2Cx+Dx^2+a^2D$$

Comparing coefficients of like powers of x, we have

(For x^3) $A+C=0$ … (ii)

(For x^2) $B+D=0$ … (iii)

(For x) $4a^2A+a^2C=0$ … (iv)

(For constant) $4a^2B+a^2D=6a^2$ … (v)

From (ii) $A=-C$ … (vi)

From (iv) $4a^2A=-a^2C \Rightarrow A=-\frac{1}{4}C$ … (vii)

From (vi) and (vii)

$-C=-\frac{1}{4}C \Rightarrow \frac{1}{4}C-C=0$

$-\frac{3}{4}C=0 \Rightarrow \boxed{C=0}$

From (ii) $\boxed{A=0}$

From (v) $4a^2B+a^2D=6a^2$

$4B+D=6$ … (viii)

By Eq (viii) – Eq (iii)

$$\begin{array}{r} 4B+D=6 \\ \underline{\pm B\pm D=0} \\ 3B=6 \end{array}$$

$\Rightarrow \boxed{B=2}$

From (iii) $2+D=0 \Rightarrow \boxed{D=-2}$

Putting values of $A,B,C,$ and D in eq. (i)

$$\frac{6a^2}{(x^2+a^2)(x^2+4a^2)}=\frac{2}{x^2+a^2}-\frac{2}{x^2+4a^2}$$

Integrating both sides

$$\int\frac{6a^2}{(x^2+a^2)(x^2+4a^2)}dx=2\int\frac{1}{x^2+a^2}dx-2\int\frac{1}{x^2+(2a)^2}dx$$

$$=2\left(\frac{1}{a}\right)\tan^{-1}\frac{x}{a}-2\left(\frac{1}{2a}\right)\tan^{-1}\frac{x}{2a}+c$$

$$=\frac{2}{a}\tan^{-1}\frac{x}{a}-\frac{1}{a}\tan^{-1}\frac{x}{2a}+c$$

Q.29 $\int\frac{2x^2-2}{x^4+x^2+1}dx$

Solution:

$$\frac{2x^2-2}{x^4+x^2+1}=\frac{2x^2-2}{x^4+1+2x^2-2x^2+x^2}$$

$$=\frac{2x^2-2}{(x^2+1)^2-(x)^2}=\frac{2x^2-2}{(x^2+1+x)(x^2+1-x)}$$

Let $\frac{2x^2-2}{(x^2+x+1)(x^2-x+1)}=\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{x^2-x+1}$ … (i)

Multiplying both sides by $(x^2+x+1)(x^2-x+1)$

$2x^2-2=(Ax+B)(x^2-x+1)+(Cx+D)(x^2+x+1)$

$2x^2-2=Ax^3-Ax^2+Ax+Bx^2-Bx+B+Cx^3+Cx^2+Cx+Dx^2+Dx+D$

Comparing coefficients of like powers of x,

(For x^3) $A+C=0\ldots$ (ii)
(For x^2) $-A+B+C+D=2\ldots$ (iii)
(For x) $A-B+C+D=0\ldots$ (iv)
(For constant) $B+D=-2\ldots$ (v)
Put $A=-C$ (From (ii)) in (iv)
$-C-B+C+D=0$
$-B+D=0\ldots$ (vi)
By Eq (vi) + Eq (v)

$$\begin{array}{r} -B+D=0 \\ B+D=-2 \\ \hline 2D=-2 \end{array}$$

$\boxed{D=-1}$

From (v) $B-1=-2\Rightarrow \boxed{B=-1}$
Put values of B and D in (iii)
$-A-1+C-1=2$
$-A+C=4$ ... (vii)
By Eq (ii) + Eq (vii)

$$\begin{array}{r} A+C=0 \\ -A+C=4 \\ \hline 2C=4 \end{array}$$

$\boxed{C=2}$

From (ii) $\boxed{A=-2}$
Putting values of A,B,C and D in (i)

$$\frac{2x^2-2}{\left(x^2+x+1\right)\left(x^2-x+1\right)}=\frac{-2x-1}{x^2+x+1}+\frac{2x-1}{x^2-x+1}$$

$$\frac{2x^2-2}{x^4+x^2+1}=-\frac{2x+1}{x^2+x+1}+\frac{2x-1}{x^2-x+1}$$

Integrating both sides

$$\int\frac{2x^2-2}{x^4+x^2+1}dx=-\int\frac{2x+1}{x^2+x+1}dx+\int\frac{2x-1}{x^2-x+1}dx$$

$$=-\ln\left(x^2+x+1\right)+\ln\left(x^2-x+1\right)+c$$

$$=\ln\left(\frac{x^2-x+1}{x^2+x+1}\right)+c$$

Q.30 $\displaystyle\int\frac{3x-8}{\left(x^2-x+2\right)\left(x^2+x+2\right)}dx$

Solution:

Let $\displaystyle\frac{3x-8}{(x^2-x+2)(x^2+x+2)}=\frac{Ax+B}{x^2-x+2}+\frac{Cx+D}{x^2+x+2}\ldots$ (i)

Multiplying both sides by $\left(x^2-x+2\right)\left(x^2+x+2\right)$

$$3x-8=(Ax+B)(x^2+x+2)+(Cx+D)(x^2-x+2)$$

$$3x-8=Ax^3+Ax^2+2Ax+Bx^2+Bx+2B+Cx^3-Cx^2+2Cx+Dx^2-Dx+2D$$

Comparing coefficients of like powers of x

(For x^3) $A+C=0\ldots$ (ii)

(For x^2) $A+B-C+D=0\ldots$ (iii)

(For x) $2A+B+2C-D=3\ldots$ (iv)

(For constant) $2B+2D=-8 \Rightarrow B+D=-4\ldots$ (v)

By Eq (iii) − Eq (v)

$A-C=4\ldots$(vi)

By Eq (ii) + Eq (vi)

$$\begin{array}{r} A+C=0 \\ A-C=4 \\ \hline 2A=4 \end{array}$$

$\boxed{A=2}$

From (ii) $\boxed{C=-2}$

From (ii) and (iv) $B-D=3\ldots$(vii)

By Eq (v) + Eq (vii)

$$\begin{array}{r} B+D=-4 \\ B-D=3 \\ \hline 2B=-1 \end{array}$$

$\boxed{B=-\frac{1}{2}}$

From (v) $\frac{-1}{2}+D=-4$

$D=-4+\frac{1}{2} \Rightarrow \boxed{D=-\frac{7}{2}}$

Putting values of A,B,C and D in (i)

$$\frac{3x-8}{(x^2-x+2)(x^2+x+2)}=\frac{2x-\frac{1}{2}}{x^2-x+2}+\frac{-2x-\frac{7}{2}}{x^2+x+2}$$

Integrating both sides

$$\int\frac{3x-8}{(x^2-x+2)(x^2+x+2)}=\int\frac{2x-1+1-\frac{1}{2}}{x^2-x+2}dx-\int\frac{2x+1-1+\frac{7}{2}}{x^2+x+2}dx$$

$$=\int\frac{2x-1}{x^2-x+2}dx+\frac{1}{2}\int\frac{1}{x^2-x+2}dx-\int\frac{2x+1}{x^2+x+2}dx-\frac{5}{2}\int\frac{1}{x^2+x+2}dx$$

$$=\ln(x^2-x+2)+\frac{1}{2}\int\frac{1}{x^2-x+\left(\frac{1}{2}\right)^2-\frac{1}{4}+2}dx-\ln(x^2+x+2)-\frac{5}{2}\int\frac{1}{x^2+x+\frac{1}{4}+2-\frac{1}{4}}dx$$

$$=\ln(x^2-x+2)+\frac{1}{2}\int\frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{7}{4}}dx-\ln(x^2+x+2)-\frac{5}{2}\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{7}{4}}dx$$

$$=\ln(x^2-x+2)+\frac{1}{2}\int\frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}dx-\ln(x^2+x+2)-\frac{5}{2}\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}dx$$

$$=\ln(x^2-x+2)+\frac{1}{2}\frac{1}{\left(\frac{\sqrt{7}}{2}\right)}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{7}}{2}}\right)-\ln(x^2+x+2)-\frac{5}{2}\frac{1}{\left(\frac{\sqrt{7}}{2}\right)}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{7}}{2}}\right)+c$$

$$=\ln(x^2-x+2)+\frac{1}{2}\times\frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{\frac{2x-1}{2}}{\frac{\sqrt{7}}{2}}\right)-\ln(x^2+x+2)-\frac{5}{2}\times\frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{\frac{2x+1}{2}}{\frac{\sqrt{7}}{2}}\right)+c$$

$$=\ln(x^2-x+2)+\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{2x-1}{\sqrt{7}}\right)-\ln(x^2+x+2)-\frac{5}{\sqrt{7}}\tan^{-1}\left(\frac{2x+1}{\sqrt{7}}\right)+c$$

Q.31 $\int\frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)}dx$

Solution:

Let $\frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)}=\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{x^2+2x+3}$ … (i)

Multiplying both sides by $(x^2+x+1)(x^2+2x+3)$

$3x^3+4x^2+9x+5=(Ax+B)(x^2+2x+3)+(Cx+D)(x^2+x+1)$

$3x^3+4x^2+9x+5=Ax^3+2Ax^2+3Ax+Bx^2+2Bx+3B+Cx^3+Cx^2+Cx+Dx^2+Dx+D$

Comparing coefficients of like powers of x,

(For x^3) $A+C=3$ … (ii)

(For x^2) $2A+B+C+D=4$ … (iii)

(For x) $3A+2B+C+D=9$ … (iv)

(For constant) $3B+D=5 \Rightarrow D=5-3B$ … (v)

From (ii) $A=3-C$

Put values of A and D in (iii)

$$6-2C+B+C+5-3B=4$$

$$-C-2B=-7 \Rightarrow C+2B=7 \ldots \text{(vi)}$$

Put values of A and D in (iv)

$$9-3C+2B+C+5-3B=9$$

$$-2C-B=-5 \Rightarrow 2C+B=5 \ldots \text{(vii)}$$

By Eq (vi) − 2.Eq (vii)

$$\begin{array}{r} C+2B=7 \\ \pm 4C \pm 2B = \pm 10 \\ \hline -3C=-3 \end{array}$$

$$\boxed{C=1}$$

From (vii) $2+B=5 \Rightarrow \boxed{B=3}$

From (v) $D=5-3(3) \Rightarrow \boxed{D=-4}$

Put C = 1 in (ii)

$$A+1=3 \Rightarrow \boxed{A=2}$$

Putting values of A,B,C and D in (i)

$$\frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)}=\frac{2x+3}{x^2+x+1}+\frac{x-4}{x^2+2x+3}$$

Integrating both sides

$$\int\frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)}dx=\int\frac{2x+3}{x^2+x+1}dx+\int\frac{x-4}{x^2+2x+3}dx$$

$$=\int\frac{2x+1+2}{x^2+x+1}dx+\frac{1}{2}\int\frac{2x+2-10}{x^2+2x+3}dx$$

$$=\int\frac{2x+1}{x^2+x+1}dx+2\int\frac{1}{x^2+x+1}dx+\frac{1}{2}\int\frac{2x+2}{x^2+2x+3}dx-\frac{10}{2}\int\frac{1}{x^2+2x+3}dx$$

$$=\ln(x^2+x+1)+2\int\frac{1}{x^2+x+\left(\frac{1}{2}\right)^2+1-\frac{1}{4}}dx+\frac{1}{2}\ln(x^2+2x+3)-5\int\frac{1}{x^2+2x+1+2}dx$$

$$=\ln(x^2+x+1)+2\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx+\frac{1}{2}\ln(x^2+2x+3)-5\int\frac{1}{(x+1)^2+\left(\sqrt{2}\right)^2}dx$$

$$= \ln\left(x^2+x+1\right)+2\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+\frac{1}{2}\ln\left(x^2+2x+3\right)-5\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+c$$

$$= \ln\left(x^2+x+1\right)+\frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+\frac{1}{2}\ln\left(x^2+2x+1\right)-\frac{5}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+c$$

$$= \ln\left(x^2+x+1\right)+\ln\left(\sqrt{x^2+2x+1}\right)+\frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)-\frac{5}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+c$$

$$= \ln\left|\left(x^2+x+1\right)\sqrt{x^2+2x+3}\right|+\frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)-\frac{5}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+c$$