The Definite Integrals:

- If $\phi'(x)=f(x)$ then $\int_a^b f(x)dx$ has a definite value $\phi(b)-\phi(a)$, so it is called the definite integral of f from a to b.
- $[a,b]$ is called range of integration while a and b are known as the lower and upper limits respectively.

Properties of Definite Integrals:

(a) $\int_a^b f(x)dx$ gives area bounded by the curve $y=f(x)$ from $x=a$ to $x=b$ and $x-axis$.

(b) Fundamental theorem of calculus: If f is continuous on $[a,b]$ and $\phi'(x)=f(x)$ i.e. $\phi(x)$ is any anti derivative of f on $[a,b]$, then $\int_a^b f(x)dx=\phi(b)-\phi(a)$

(c) $\int_a^b f(x)dx=-\int_b^a f(x)dx$

(d) $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx, a<c<b$

(e) The variable of integration x in $\int_a^b f(x)dx$ can be replaced by any other variable i.e. $\int_a^b f(x)dx=\int_a^b f(t)dt$

Theorem:

(i) Prove that $\int_a^b f(x)dx=-\int_b^a f(x)dx$

Proof: If $\phi'(x)=f(x)$ that is if ϕ is an anti derivative of f then using the fundamental theorem of calculus, we get

$$\int_a^b f(x)dx=\phi(b)-\phi(a)$$

$$=-[\phi(a)-\phi(b)]$$

$$\int_a^b f(x)dx=-\int_b^a f(x)dx$$

(ii) Prove that $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$; $a < c < b$

Proof: If $\phi'(x) = f(x)$ that is if $\phi(x)$ is an anti derivative of $f(x)$, then applying the fundamental theorem of calculus we have

$$\int_a^c f(x)dx = \phi(c) - \phi(a) \ldots\ldots\text{(i)}$$

and

$$\int_c^b f(x)dx = \phi(b) - \phi(c) \ldots\ldots\text{(ii)}$$

Adding (i) and (ii)

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \phi(c) - \phi(a) + \phi(b) - \phi(c)$$

$$= \phi(b) - \phi(a)$$

$$= \int_a^b f(x)dx.$$

EXERCISE 3.6

Evaluate the following definite integrals:

Q.1 $\int_1^2 (x^2+1)dx$

Solution:

$$\int_1^2 (x^2+1)dx$$

$$= \int_1^2 x^2 dx + \int_1^2 1dx$$

$$= \left[\frac{x^3}{3}\right]_1^2 + [x]_1^2$$

$$= \frac{1}{3}\left[x^3\right]_1^2 + [x]_1^2$$

$$= \frac{1}{3}\left[2^3 - 1^3\right] + [2-1]$$

$$= \frac{1}{3}[8-1] + 1$$

$$= \frac{7}{3} + 1$$

$$= \frac{7+3}{3}$$

$$= \frac{10}{3}$$

Q.2 $\int_{-1}^1 \left(x^{\frac{1}{3}}+1\right)dx$

Solution:

$$\int_{-1}^1 \left(x^{\frac{1}{3}}+1\right)dx$$

$$= \int_{-1}^1 x^{\frac{1}{3}}dx + \int_{-1}^1 1dx$$

$$= \left[\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right]_{-1}^1 + [x]_{-1}^1$$

$$= \left[\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right]_{-1}^1 + [1-(-1)]$$

$$= \frac{3}{4}\left[x^{\frac{4}{3}}\right]_{-1}^1 + [1+1]$$

$$= \frac{3}{4}\left[1^{\frac{4}{3}} - (-1)^{\frac{4}{3}}\right] + 2$$

$$= \frac{3}{4}\left[1-\left\{(-1)^4\right\}^{\frac{1}{3}}\right]+2$$

$$= \frac{3}{4}\left[1-1^{\frac{1}{3}}\right]+2$$

$$= \frac{3}{4}[1-1]+2$$

$$= \frac{3}{4}(0)+2$$

$$= 0+2$$

$$= 2$$

Q.3 $\int_{-2}^{0} \frac{1}{(2x-1)^2}\,dx$

Solution:

$$\int_{-2}^{0} \frac{1}{(2x-1)^2}\,dx$$

$$= \int_{-2}^{0} (2x-1)^{-2}\,dx$$

$$= \frac{1}{2}\int_{-2}^{0} (2x-1)^{-2}.2dx$$

$$= \frac{1}{2}\left[\frac{(2x-1)^{-2+1}}{-2+1}\right]_{-2}^{0}$$

$$= \frac{1}{2}\left[\frac{(2x-1)^{-1}}{-1}\right]_{-2}^{0}$$

$$= -\frac{1}{2}\left[\frac{1}{2x-1}\right]_{-2}^{0}$$

$$= -\frac{1}{2}\left[\frac{1}{2(0)-1}-\frac{1}{2(-2)-1}\right]$$

$$= -\frac{1}{2}\left[\frac{1}{-1}-\frac{1}{-5}\right]$$

$$= -\frac{1}{2}\left[-1+\frac{1}{5}\right]$$

$$= -\frac{1}{2}\left[\frac{-5+1}{5}\right]$$

$$= -\frac{1}{2}\left[\frac{-4}{5}\right]$$

$$= \frac{2}{5}$$

Q.4 $\int_{-6}^{2} \sqrt{3-x}\,dx$

Solution:

$$\int_{-6}^{2} \sqrt{3-x}\,dx$$

$$= (-1)\int_{-6}^{2} \sqrt{3-x}\,(-1)\,dx$$

$$= (-1)\left[\frac{(3-x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_{-6}^{2}$$

$$= (-1)\left[\frac{(3-x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-6}^{2}$$

$$= -\frac{2}{3}\left[(3-2)^{\frac{3}{2}}-(3-(-6))^{\frac{3}{2}}\right]$$

$$= -\frac{2}{3}\left[1^{\frac{3}{2}}-9^{\frac{3}{2}}\right]$$

$$= -\frac{2}{3}\left[1-\left(3^2\right)^{\frac{3}{2}}\right]$$

$$= -\frac{2}{3}\left[1-3^3\right]$$

$$= -\frac{2}{3}[1-27]$$

$$= -\frac{2}{3}[-26]$$

$$= \frac{52}{3}$$

Q.5 $\int_{1}^{\sqrt{5}} \sqrt{(2t-1)^3}\,dt$

Solution:

$$\int_{1}^{\sqrt{5}} \sqrt{(2t-1)^3}\,dt$$

$$= \frac{1}{2}\int_{1}^{\sqrt{5}} (2t-1)^{\frac{3}{2}}.2dt$$

$$= \frac{1}{2}\left[\frac{(2t-1)^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right]_1^{\sqrt{5}}$$

$$= \frac{1}{2}\left[\frac{(2t-1)^{\frac{5}{2}}}{\frac{5}{2}}\right]_1^{\sqrt{5}}$$

$$= \frac{1}{5}\left[(2t-1)^{\frac{5}{2}}\right]_1^{\sqrt{5}}$$

$$= \frac{1}{5}\left[\left(2\sqrt{5}-1\right)^{\frac{5}{2}} - \left(2(1)-1\right)^{\frac{5}{2}}\right]$$

$$= \frac{1}{5}\left[\left(2\sqrt{5}-1\right)^{\frac{5}{2}} - 1\right]$$

Q.6 $\int_2^{\sqrt{5}} x\sqrt{x^2-1}\,dx$

Solution:

$$\int_2^{\sqrt{5}} x\sqrt{x^2-1}\,dx$$

$$= \frac{1}{2}\int_2^{\sqrt{5}} \sqrt{x^2-1}.(2x)\,dx$$

$$= \frac{1}{2}\left[\frac{(x^2-1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_2^{\sqrt{5}}$$

$$= \frac{1}{2}\left[\frac{(x^2-1)^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^{\sqrt{5}}$$

$$= \frac{1}{3}\left[(x^2-1)^{\frac{3}{2}}\right]_2^{\sqrt{5}}$$

$$= \frac{1}{3}\left[\left((\sqrt{5})^2-1\right)^{\frac{3}{2}} - \left(2^2-1\right)^{\frac{3}{2}}\right]$$

$$= \frac{1}{3}\left[(5-1)^{\frac{3}{2}} - (4-1)^{\frac{3}{2}}\right]$$

$$= \frac{1}{3}\left[4^{\frac{3}{2}} - 3^{\frac{3}{2}}\right]$$

$$= \frac{1}{3}\left[(2^2)^{\frac{3}{2}} - 3.3^{\frac{1}{2}}\right]$$

$$= \frac{1}{3}\left[2^3 - 3\sqrt{3}\right]$$

$$= \frac{1}{3}\left[8 - 3\sqrt{3}\right]$$

Q.7 $\int_1^2 \frac{x}{x^2+2}\,dx$

Solution:

$$\int_1^2 \frac{x}{x^2+2}\,dx$$

$$= \frac{1}{2}\int_1^2 \frac{2x}{x^2+2}\,dx$$

$$= \frac{1}{2}\left[\ln\left(x^2+2\right)\right]_1^2$$

$$= \frac{1}{2}\left[\ln\left(2^2+2\right) - \ln\left(1^2+2\right)\right]$$

$$= \frac{1}{2}\left[\ln 6 - \ln 3\right]$$

$$= \frac{1}{2}\left[\ln\left(\frac{6}{3}\right)\right]$$

$$= \frac{1}{2}\ln 2$$

Q.8 $\int_2^3 \left(x - \frac{1}{x}\right)^2 dx$

Solution:

$$\int_2^3 \left(x - \frac{1}{x}\right)^2 dx$$

$$= \int_2^3 \left(x^2 + \frac{1}{x^2} - 2x\frac{1}{x}\right) dx$$

$$= \int_2^3 \left(x^2 + x^{-2} - 2\right) dx$$

$$= \int_2^3 x^2 dx + \int_2^3 x^{-2} dx - 2\int_2^3 1 dx$$

$$=\left[\frac{x^{2+1}}{2+1}\right]_2^3+\left[\frac{x^{-2+1}}{-2+1}\right]_2^3-2[x]_2^3$$

$$=\left[\frac{x^3}{3}\right]_2^3+\left[\frac{x^{-1}}{-1}\right]_2^3-2[3-2]$$

$$=\frac{1}{3}\left[x^3\right]_2^3-\left[\frac{1}{x}\right]_2^3-2[1]$$

$$=\frac{1}{3}\left[3^3-2^3\right]-\left[\frac{1}{3}-\frac{1}{2}\right]-2$$

$$=\frac{1}{3}[27-8]-\left[\frac{2-3}{6}\right]-2$$

$$=\frac{19}{3}-\left(\frac{-1}{6}\right)-2$$

$$=\frac{19}{3}+\frac{1}{6}-2$$

$$=\frac{38+1-12}{6}$$

$$=\frac{27}{6}$$

$$=\frac{9}{2}$$

Q.9 $\int_{-1}^{1}\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1}\,dx$

Solution:

$$\int_{-1}^{1}\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1}\,dx$$

$$=\int_{-1}^{1}\left(\frac{2x+1}{2}\right)\sqrt{x^2+x+1}\,dx$$

$$=\frac{1}{2}\int_{-1}^{1}\left(x^2+x+1\right)^{\frac{1}{2}}(2x+1)\,dx$$

$$=\frac{1}{2}\left[\frac{\left(x^2+x+1\right)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_{-1}^{1}$$

$$=\frac{1}{2}\left[\frac{\left(x^2+x+1\right)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-1}^{1}$$

$$=\frac{1}{3}\left[\left(x^2+x+1\right)^{\frac{3}{2}}\right]_{-1}^{1}$$

$$=\frac{1}{3}\left[\left(1^2+1+1\right)^{\frac{3}{2}}-\left((-1)^2+(-1)+1\right)^{\frac{3}{2}}\right]$$

$$=\frac{1}{3}\left[3^{\frac{3}{2}}-(1-1+1)^{\frac{3}{2}}\right]$$

$$=\frac{1}{3}\left[3.3^{\frac{1}{2}}-1^{\frac{3}{2}}\right]$$

$$=\frac{1}{3}\left[3\sqrt{3}-1\right]$$

$$=\sqrt{3}-\frac{1}{3}$$

Q.10 $\int_{0}^{3}\frac{dx}{x^2+9}$

Solution:

$$\int_{0}^{3}\frac{1}{x^2+9}\,dx$$

$$=\int_{0}^{3}\frac{1}{x^2+3^2}\,dx$$

$$=\frac{1}{3}\left[\tan^{-1}\left(\frac{x}{3}\right)\right]_0^3$$

$$=\frac{1}{3}\left[\tan^{-1}\left(\frac{3}{3}\right)-\tan^{-1}\left(\frac{0}{3}\right)\right]$$

$$=\frac{1}{3}\left[\tan^{-1}(1)-\tan^{-1}(0)\right]$$

$$=\frac{1}{3}\left[\frac{\pi}{4}-0\right]$$

$$=\frac{\pi}{12}$$

Q.11 $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\cos t\,dt$

Solution:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\cos t\,dt$$

$$=\left[\sin t\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$=\sin\frac{\pi}{3}-\sin\frac{\pi}{6}$$

$$=\frac{\sqrt{3}}{2}-\frac{1}{2}$$

$$=\frac{\sqrt{3}-1}{2}$$

Q.12 $\int_1^2\left(x+\frac{1}{x}\right)^{\frac{1}{2}}\left(1-\frac{1}{x^2}\right)dx$

Solution:

$$\int_1^2\left(x+\frac{1}{x}\right)^{\frac{1}{2}}\left(1-\frac{1}{x^2}\right)dx$$

$$=\left[\frac{\left(x+\frac{1}{x}\right)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^2$$

$$=\left[\frac{\left(x+\frac{1}{x}\right)^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^2$$

$$=\frac{2}{3}\left[\left(x+\frac{1}{x}\right)^{\frac{3}{2}}\right]_1^2$$

$$=\frac{2}{3}\left[\left(2+\frac{1}{2}\right)^{\frac{3}{2}}-\left(1+\frac{1}{1}\right)^{\frac{3}{2}}\right]$$

$$=\frac{2}{3}\left[\left(\frac{4+1}{2}\right)^{\frac{3}{2}}-2^{\frac{3}{2}}\right]$$

$$=\frac{2}{3}\left[\left(\frac{5}{2}\right)^{\frac{3}{2}}-2^{\frac{3}{2}}\right]$$

$$=\frac{2}{3}\left[\frac{5^{\frac{3}{2}}}{2^{\frac{3}{2}}}-2^{\frac{3}{2}}\right]$$

$$=\frac{2}{3}\left[\frac{5^{\frac{3}{2}}-2^3}{2^{\frac{3}{2}}}\right]$$

$$=\frac{2}{3}\left[\frac{5.5^{\frac{1}{2}}-8}{2.2^{\frac{1}{2}}}\right]$$

$$=\frac{1}{3\sqrt{2}}\left[5\sqrt{5}-8\right]$$

Q.13 $\int_1^2 \ln x\,dx$

Solution:

$$\int_1^2 \ln x\,dx$$

$$=\int_1^2 \ln x.1dx$$

Integrating by parts

$$=\left[\ln x.x\right]_1^2-\int_1^2 x\frac{1}{x}dx$$

$$=2\ln 2-1.\ln(1)-\int_1^2 1dx$$

$$=2\ln 2-0-\left[x\right]_1^2$$

$$=2\ln 2-(2-1)$$

$$=2\ln 2-1$$

Q.14 $\int_0^2\left(e^{\frac{x}{2}}-e^{\frac{-x}{2}}\right)dx$

Solution:

$$\int_0^2\left(e^{\frac{x}{2}}-e^{\frac{-x}{2}}\right)dx$$

$$=\int_0^2 e^{\frac{x}{2}}dx-\int_0^2 e^{\frac{-x}{2}}dx$$

$$=2\left[\frac{e^{\frac{x}{2}}}{\frac{1}{2}}\right]_0^2-2\left[\frac{e^{\frac{-x}{2}}}{\frac{-1}{2}}\right]_0^2$$

$$=2\left[e^{\frac{x}{2}}\right]_0^2+2\left[e^{\frac{-x}{2}}\right]_0^2$$

$$=2\left[e^{\frac{2}{2}}-e^{\frac{0}{2}}\right]+2\left[e^{\frac{-2}{2}}-e^{\frac{0}{2}}\right]$$

$$=2\left[e^1-e^0\right]+2\left[e^{-1}-e^0\right]$$

$$=2\left[e-1\right]+2\left[\frac{1}{e}-1\right]$$

$$=2\left[e-1+\frac{1}{e}-1\right]$$

$$=2\left[e+\frac{1}{e}-2\right]$$

$$=2\left[\frac{e^2+1-2e}{e}\right]$$

$$=2\left[\frac{(e-1)^2}{e}\right]$$

$$=\frac{2}{e}(e-1)^2$$

Q.15 $\int_0^{\frac{\pi}{4}}\frac{\cos\theta+\sin\theta}{2\cos^2\theta}d\theta$

Solution:

$$\int_0^{\frac{\pi}{4}}\frac{\cos\theta+\sin\theta}{2\cos^2\theta}d\theta$$

$$=\int_0^{\frac{\pi}{4}}\left[\frac{\cos\theta}{2\cos^2\theta}+\frac{\sin\theta}{2\cos^2\theta}\right]d\theta$$

$$=\int_0^{\frac{\pi}{4}}\left[\frac{1}{2\cos\theta}+\frac{\sin\theta}{2\cos\theta\cos\theta}\right]d\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{4}}\sec\theta d\theta+\frac{1}{2}\int_0^{\frac{\pi}{4}}\sec\theta\tan\theta d\theta$$

$$=\frac{1}{2}\left[\ln\left|\sec\theta+\tan\theta\right|\right]_0^{\frac{\pi}{4}}+\frac{1}{2}\left[\sec\theta\right]_0^{\frac{\pi}{4}}$$

$$=\frac{1}{2}\left[\ln\left|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right|-\ln\left|\sec 0+\tan 0\right|\right]+\frac{1}{2}\left[\sec\frac{\pi}{4}-\sec 0\right]$$

$$=\frac{1}{2}\left[\ln\left|\sqrt{2}+1\right|-\ln\left|1+0\right|\right]+\frac{1}{2}\left[\sqrt{2}-1\right]$$

$$=\frac{1}{2}\left[\ln\left|\sqrt{2}+1\right|-0\right]+\frac{1}{2}\left[\sqrt{2}-1\right]$$

$$=\frac{1}{2}\ln\left|\sqrt{2}+1\right|+\frac{1}{2}\left[\sqrt{2}-1\right]$$

$$=\frac{1}{2}\left[\ln\left(\sqrt{2}+1\right)+\sqrt{2}-1\right]$$

Q.16 $\int_0^{\frac{\pi}{6}}\cos^3\theta d\theta$

Solution:

$$\int_0^{\frac{\pi}{6}}\cos^3\theta d\theta$$

$$=\int_0^{\frac{\pi}{6}}\cos^2\theta.\cos\theta d\theta$$

$$=\int_0^{\frac{\pi}{6}}\left(1-\sin^2\theta\right)\cos\theta d\theta$$

$$=\int_0^{\frac{\pi}{6}}\left(\cos\theta-\sin^2\theta\cos\theta\right)d\theta$$

$$=\int_0^{\frac{\pi}{6}}\cos\theta d\theta-\int_0^{\frac{\pi}{6}}\sin^2\theta\cos\theta d\theta$$

$$=\left[\sin\theta\right]_0^{\frac{\pi}{6}}-\left[\frac{\sin^3\theta}{3}\right]_0^{\frac{\pi}{6}}$$

$$=\left[\sin\frac{\pi}{6}-\sin 0\right]-\frac{1}{3}\left[\sin^3\frac{\pi}{6}-\sin^3 0\right]$$

$$=\left[\frac{1}{2}-0\right]-\frac{1}{3}\left[\left(\frac{1}{2}\right)^3-0\right]$$

$$=\frac{1}{2}-\frac{1}{3}\left(\frac{1}{8}\right)$$

$$=\frac{1}{2}-\frac{1}{24}$$

$$=\frac{12-1}{24}$$

$$=\frac{11}{24}$$

Q.17 $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos^2\theta \cot^2\theta d\theta$

Solution:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos^2\theta \cot^2\theta d\theta$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos^2\theta\left(\operatorname{cosec}^2\theta-1\right)d\theta$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos^2\theta\left(\frac{1}{\sin^2\theta}-1\right)d\theta$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\frac{\cos^2\theta}{\sin^2\theta}-\cos^2\theta\right)d\theta$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\cot^2\theta-\cos^2\theta\right)d\theta$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cot^2\theta d\theta-\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos^2\theta d\theta$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\operatorname{cosec}^2\theta-1\right)d\theta-\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec}^2\theta d\theta-\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 1d\theta-\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 1d\theta-\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos 2\theta d\theta$$

$$=\left[-\cot\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}-\left[\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}-\frac{1}{2}\left[\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}-\frac{1}{2}\left[\frac{\sin 2\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$=-\left[\cot\frac{\pi}{4}-\cot\frac{\pi}{6}\right]-\left[\frac{\pi}{4}-\frac{\pi}{6}\right]-\frac{1}{2}\left[\frac{\pi}{4}-\frac{\pi}{6}\right]$$
$$-\frac{1}{4}\left[\sin\left(\frac{2\pi}{4}\right)-\sin\left(\frac{2\pi}{6}\right)\right]$$

$$=-\left[1-\sqrt{3}\right]-\left[\frac{3\pi-2\pi}{12}\right]-\frac{1}{2}\left[\frac{3\pi-2\pi}{12}\right]$$
$$-\frac{1}{4}\left[\sin\frac{\pi}{2}-\sin\frac{\pi}{3}\right]$$

$$=-1+\sqrt{3}-\frac{\pi}{12}-\frac{1}{2}\left(\frac{\pi}{12}\right)-\frac{1}{4}\left[1-\frac{\sqrt{3}}{2}\right]$$

$$=-1+\sqrt{3}-\frac{\pi}{12}-\frac{\pi}{24}-\frac{1}{4}+\frac{\sqrt{3}}{8}$$

$$=-1+\sqrt{3}-\frac{1}{4}+\frac{\sqrt{3}}{8}-\left[\frac{\pi}{12}+\frac{\pi}{24}\right]$$

$$=\frac{-8+8\sqrt{3}-2+\sqrt{3}}{8}-\left[\frac{2\pi+\pi}{24}\right]$$

$$=\frac{-10+9\sqrt{3}}{8}-\frac{3\pi}{24}$$

$$=\frac{-10+9\sqrt{3}}{8}-\frac{\pi}{8}$$

$$=\frac{9\sqrt{3}-10-\pi}{8}$$

Q.18 $\displaystyle\int_{0}^{\frac{\pi}{4}} \cos^4 t\, dt$

Solution:

$$\int_{0}^{\frac{\pi}{4}} \left(\cos^2 t\right)^2 dt$$

$$=\int_{0}^{\frac{\pi}{4}} \left(\frac{1+\cos 2t}{2}\right)^2 dt$$

$$=\frac{1}{4}\int_{0}^{\frac{\pi}{4}} \left[1+\cos^2 2t+2\cos 2t\right]dt$$

$$=\frac{1}{4}\int_{0}^{\frac{\pi}{4}} 1dt+\frac{1}{4}\int_{0}^{\frac{\pi}{4}} \cos^2 2t\, dt+\frac{2}{4}\int_{0}^{\frac{\pi}{4}} \cos 2t\, dt$$

$$=\frac{1}{4}\left[t\right]_0^{\frac{\pi}{4}}+\frac{1}{4}\int_{0}^{\frac{\pi}{4}} \frac{1+\cos 4t}{2}dt+\frac{1}{2}\left[\frac{\sin 2t}{2}\right]_0^{\frac{\pi}{4}}$$

$$=\frac{1}{4}\left[\frac{\pi}{4}-0\right]+\frac{1}{8}\int_0^{\frac{\pi}{4}}(1+\cos 4t)\,dt+\frac{1}{4}[\sin 2t]_0^{\frac{\pi}{4}}$$

$$=\frac{\pi}{16}+\frac{1}{8}\left[t+\frac{\sin 4t}{4}\right]_0^{\frac{\pi}{4}}+\frac{1}{4}\left[\sin 2\left(\frac{\pi}{4}\right)-\sin 2(0)\right]$$

$$=\frac{\pi}{16}+\frac{1}{8}\left[\frac{\pi}{4}+\frac{\sin 4\left(\frac{\pi}{4}\right)}{4}\right]-\frac{1}{8}\left[0+\frac{\sin 4(0)}{4}\right]+\frac{1}{4}\left[\sin\frac{\pi}{2}-0\right]$$

$$=\frac{\pi}{16}+\frac{1}{8}\left[\frac{\pi}{4}+0\right]-\frac{1}{8}[0+0]+\frac{1}{4}[1-0]$$

$$=\frac{\pi}{16}+\frac{\pi}{32}+\frac{1}{4}$$

$$=\frac{2\pi+\pi+8}{32}=\frac{3\pi+8}{32}$$

Q.19 $\int_0^{\frac{\pi}{3}}\cos^2\theta\sin\theta\,d\theta$

Solution:

$$\int_0^{\frac{\pi}{3}}\cos^2\theta\sin\theta\,d\theta$$

$$=-\int_0^{\frac{\pi}{3}}\cos^2\theta(-\sin\theta)\,d\theta$$

$$=-\left[\frac{\cos^3\theta}{3}\right]_0^{\frac{\pi}{3}}$$

$$=-\frac{1}{3}\left[\cos^3\theta\right]_0^{\frac{\pi}{3}}$$

$$=-\frac{1}{3}\left[\cos^3\frac{\pi}{3}-\cos^3 0\right]$$

$$=-\frac{1}{3}\left[\left(\frac{1}{2}\right)^3-1^3\right]$$

$$=-\frac{1}{3}\left[\frac{1}{8}-1\right]$$

$$=-\frac{1}{3}\left[\frac{1-8}{8}\right]$$

$$=-\frac{1}{3}\left[\frac{-7}{8}\right]$$

$$=\frac{7}{24}$$

Q.20 $\int_0^{\frac{\pi}{4}}\left(1+\cos^2\theta\right)\tan^2\theta\,d\theta$

Solution:

$$\int_0^{\frac{\pi}{4}}\left(1+\cos^2\theta\right)\tan^2\theta\,d\theta$$

$$=\int_0^{\frac{\pi}{4}}\left(\tan^2\theta+\cos^2\theta\tan^2\theta\right)d\theta$$

$$=\int_0^{\frac{\pi}{4}}\left(\tan^2\theta+\sin^2\theta\right)d\theta$$

$$=\int_0^{\frac{\pi}{4}}\tan^2\theta\,d\theta+\int_0^{\frac{\pi}{4}}\sin^2\theta\,d\theta$$

$$=\int_0^{\frac{\pi}{4}}\left(\sec^2\theta-1\right)d\theta+\int_0^{\frac{\pi}{4}}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$=\int_0^{\frac{\pi}{4}}\sec^2\theta\,d\theta-\int_0^{\frac{\pi}{4}}1d\theta+\frac{1}{2}\int_0^{\frac{\pi}{4}}1d\theta-\frac{1}{2}\int_0^{\frac{\pi}{4}}\cos 2\theta\,d\theta$$

$$=\left[\tan\theta\right]_0^{\frac{\pi}{4}}-\left[\theta\right]_0^{\frac{\pi}{4}}+\frac{1}{2}\left[\theta\right]_0^{\frac{\pi}{4}}-\frac{1}{2}\left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}$$

$$=\left[\tan\frac{\pi}{4}-\tan 0\right]-\left[\frac{\pi}{4}-0\right]+\frac{1}{2}\left[\frac{\pi}{4}-0\right]$$

$$-\frac{1}{4}\left[\sin 2\left(\frac{\pi}{4}\right)-\sin 2(0)\right]$$

$$=[1-0]-\frac{\pi}{4}+\frac{\pi}{8}-\frac{1}{4}\left[\sin\frac{\pi}{2}-\sin 0\right]$$

$$=1-\frac{\pi}{4}+\frac{\pi}{8}-\frac{1}{4}[1-0]$$

$$=1-\frac{\pi}{4}+\frac{\pi}{8}-\frac{1}{4}$$

$$=\frac{8-2\pi+\pi-2}{8}$$

$$=\frac{6-\pi}{8}$$

Q.21 $\int_0^{\frac{\pi}{4}} \frac{\sec\theta}{\sin\theta+\cos\theta} d\theta$

Solution:

$$\int_0^{\frac{\pi}{4}} \frac{\sec\theta}{\sin\theta+\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec\theta d\theta}{\cos\theta\left(\frac{\sin\theta}{\cos\theta}+1\right)}$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec\theta.\sec\theta d\theta}{\tan\theta+1}$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta d\theta}{\tan\theta+1}$$

Using $\int \frac{f'(x)}{f(x)} dx = \ln\left|f(x)\right| + c$

$$= \left[\ln(\tan\theta+1)\right]_0^{\frac{\pi}{4}}$$

$$= \ln\left(\tan\frac{\pi}{4}+1\right) - \ln(\tan 0+1)$$

$$= \ln(1+1) - \ln(0+1)$$

$$= \ln 2 - \ln 1$$

$$= \ln 2 \qquad \because \ln 1 = 0$$

Q.22 $\int_{-1}^{5} |x-3| dx$

Solution:

$$\int_{-1}^{5} |x-3| dx$$

Using $|x-3| = \begin{cases} x-3 & \text{if } x \geq 3 \\ -(x-3) & \text{if } x < 3 \end{cases}$

$$= -\int_{-1}^{3} (x-3) dx + \int_{3}^{5} (x-3) dx$$

$$= -\left[\frac{x^2}{2} - 3x\right]_{-1}^{3} + \left[\frac{x^2}{2} - 3x\right]_{3}^{5}$$

$$= -\left\{\left(\frac{9}{2}-9\right) - \left(\frac{1}{2}+3\right)\right\} + \left\{\left(\frac{25}{2}-15\right) - \left(\frac{9}{2}-9\right)\right\}$$

$$= -\left\{\frac{-9}{2} - \frac{7}{2}\right\} + \left\{\frac{-5}{2} + \frac{9}{2}\right\}$$

$$= -\left\{\frac{-16}{2}\right\} + \left\{\frac{4}{2}\right\}$$

$$= 8+2$$

$$= 10$$

Q.23 $\int_{\frac{1}{8}}^{1} \frac{\left(x^{\frac{1}{3}}+2\right)^2}{x^{\frac{2}{3}}} dx$

Solution:

$$\int_{\frac{1}{8}}^{1} \frac{\left(x^{\frac{1}{3}}+2\right)^2}{x^{\frac{2}{3}}} dx$$

$$= 3\int_{\frac{1}{8}}^{1} \left(x^{\frac{1}{3}}+2\right)^2 \left(\frac{1}{3} x^{\frac{-2}{3}}\right) dx$$

Using $\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c$

$$= 3\left[\frac{\left(x^{\frac{1}{3}}+2\right)^3}{3}\right]_{\frac{1}{8}}^{1}$$

$$= \frac{3}{3}\left[\left\{(1)^{\frac{1}{3}}+2\right\}^3 - \left\{\left(\frac{1}{8}\right)^{\frac{1}{3}}+2\right\}^3\right]$$

$$= (1+2)^3 - \left(\frac{1}{2}+2\right)^3$$

$$= 27 - \frac{125}{8}$$

$$= \frac{216-125}{8}$$

$$= \frac{91}{8}$$

Q.24 $\int_1^3 \frac{x^2-2}{x+1}dx$

Solution:

$$\int_1^3 \frac{x^2-2}{x+1}dx$$

By long division

$$\begin{array}{r} x-1 \\ x+1\overline{)\,x^2-2} \\ \pm x^2 \pm x \\ \hline -x-2 \\ \mp x \mp 1 \\ \hline -1 \end{array}$$

$$=\int_1^3\left(x-1-\frac{1}{x+1}\right)dx$$

$$=\int_1^3 xdx-\int_1^3 1dx-\int_1^3\frac{1}{x+1}dx$$

$$=\left[\frac{x^2}{2}\right]_1^3-\left[x\right]_1^3-\left[\ln\left(x+1\right)\right]_1^3$$

$$=\left[\frac{9}{2}-\frac{1}{2}\right]-(3-1)-\left[\ln(3+1)-\ln(1+1)\right]$$

$$=4-2-\ln 4+\ln 2$$

$$=2-\ln\left(\frac{4}{2}\right)=2-\ln\left(2\right)$$

Q.25 $\int_2^3 \frac{3x^2-2x+1}{(x-1)(x^2+1)}dx$

Solution:

$$\int_2^3 \frac{3x^2-2x+1}{(x-1)(x^2+1)}dx$$

$$=\int_2^3 \frac{3x^2-2x+1}{x^3-x^2+x-1}dx$$

$$=\int_2^3 \frac{\frac{d}{dx}(x^3-x^2+x-1)}{x^3-x^2+x-1}dx$$

$$=\left[\ln\left(x^3-x^2+x-1\right)\right]_2^3$$

$$=\ln\left(3^3-3^2+3-1\right)-\ln\left(2^3-2^2+2-1\right)$$

$$=\ln\left(27-9+2\right)-\ln\left(8-4+1\right)$$

$$=\ln 20-\ln 5$$

$$=\ln\frac{20}{5}=\ln 4$$

Q.26 $\int_0^{\frac{\pi}{4}} \frac{\sin x-1}{\cos^2 x}dx$

Solution:

$$\int_0^{\frac{\pi}{4}} \frac{\sin x-1}{\cos^2 x}dx$$

$$=\int_0^{\frac{\pi}{4}}\left(\frac{\sin x}{\cos^2 x}-\frac{1}{\cos^2 x}\right)dx$$

$$=\int_0^{\frac{\pi}{4}}\sec x\tan xdx-\int_0^{\frac{\pi}{4}}\sec^2 xdx$$

$$=\left[\sec x\right]_0^{\frac{\pi}{4}}-\left[\tan x\right]_0^{\frac{\pi}{4}}$$

$$=\sec\left(\frac{\pi}{4}\right)-\sec(0)-\left(\tan\frac{\pi}{4}-\tan 0\right)$$

$$=\sqrt{2}-1-1-0$$

$$=\sqrt{2}-2$$

Q.27 $\int_0^{\frac{\pi}{4}} \frac{1}{1+\sin x}dx$

Solution:

$$\int_0^{\frac{\pi}{4}} \frac{1}{1+\sin x}dx$$

$$=\int_0^{\frac{\pi}{4}}\frac{1}{1+\sin x}\times\frac{1-\sin x}{1-\sin x}dx$$

$$=\int_0^{\frac{\pi}{4}}\frac{1-\sin x}{1-\sin^2 x}dx=\int_0^{\frac{\pi}{4}}\frac{1}{\cos^2 x}dx-\int_0^{\frac{\pi}{4}}\frac{\sin x}{\cos^2 x}dx$$

$$=\int_0^{\frac{\pi}{4}}\sec^2 xdx-\int_0^{\frac{\pi}{4}}\sec x\tan xdx$$

$$=\left[\tan x\right]_0^{\frac{\pi}{4}}-\left[\sec x\right]_0^{\frac{\pi}{4}}$$

$$= \tan\frac{\pi}{4} - \tan 0 - \left(\sec\frac{\pi}{4} - \sec 0\right)$$

$$= 1 - 0 - \sqrt{2} + 1 = 2 - \sqrt{2}$$

Q.28 $\int_0^1 \frac{3x}{\sqrt{4-3x}}dx$

Solution:

$$\int_0^1 \frac{3x}{\sqrt{4-3x}}dx$$

$$= -\int_0^1 \frac{-3x}{\sqrt{4-3x}}dx$$

$$= -\int_0^1 \frac{4-3x-4}{\sqrt{4-3x}}dx$$

$$= -\int_0^1 \left(\frac{4-3x}{\sqrt{4-3x}} - \frac{4}{\sqrt{4-3x}}\right)dx$$

$$= -\int_0^1 \sqrt{4-3x}\,dx + 4\int_0^1 \frac{1}{\sqrt{4-3x}}dx$$

$$= -\int_0^1 (4-3x)^{\frac{1}{2}}dx + 4\int_0^1 (4-3x)^{\frac{-1}{2}}dx$$

$$= \frac{1}{3}\int_0^1 (4-3x)^{\frac{1}{2}}(-3)dx - \frac{4}{3}\int_0^1 (4-3x)^{\frac{-1}{2}}(-3)dx$$

$$= \frac{1}{3}\left[\frac{(4-3x)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^1 - \frac{4}{3}\left[\frac{(4-3x)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}\right]_0^1$$

$$= \frac{2}{9}\left[(4-3)^{\frac{3}{2}} - (4-0)^{\frac{3}{2}}\right] - \frac{8}{3}\left[(4-3)^{\frac{1}{2}} - (4-0)^{\frac{1}{2}}\right]$$

$$= \frac{2}{9}\left[(1)^{\frac{3}{2}} - \left(2^2\right)^{\frac{3}{2}}\right] - \frac{8}{3}\left[(1)^{\frac{1}{2}} - \left(2^2\right)^{\frac{1}{2}}\right]$$

$$= \frac{2}{9}[1-8] - \frac{8}{3}[1-2]$$

$$= \frac{2}{9}(-7) - \frac{8}{3}(-1)$$

$$= \frac{-14}{9} + \frac{8}{3} = \frac{-14+24}{9} = \frac{10}{9}$$

Q.29 $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x(2+\sin x)}dx$

Solution:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x(2+\sin x)}dx$$

Put $\sin x = t \Rightarrow \cos x dx = dt$

when $x \to \frac{\pi}{6}$, $t \to \frac{1}{2}$

when $x \to \frac{\pi}{2}$, $t \to 1$

$$= \int_{\frac{1}{2}}^{1} \frac{dt}{t(2+t)}$$

Let $\frac{1}{t(2+t)} = \frac{A}{t} + \frac{B}{2+t}$ … (i)

Multiplying both sides by $t(2+t)$

$1 = A(2+t) + B(t)$ … (ii)

Put $t = 0$ in eq. (ii)

$1 = A(2) \Rightarrow \boxed{A = \frac{1}{2}}$

Put $t+2=0 \Rightarrow t=-2$ in eq. (ii)

$1 = A(0) + B(-2) \Rightarrow \boxed{B = \frac{-1}{2}}$

Put values of A and B in eq. (i)

$\frac{1}{t(2+t)} = \frac{1}{2t} - \frac{1}{2(2+t)}$ … (iii)

$$\int_{\frac{1}{2}}^{1} \frac{dt}{t(2+t)}$$

$$= \frac{1}{2}\int_{\frac{1}{2}}^{1} \frac{1}{t}dt - \frac{1}{2}\int_{\frac{1}{2}}^{1} \frac{1}{2+t}dt \quad \text{(by (iii))}$$

$$= \frac{1}{2}[\ln t]_{\frac{1}{2}}^{1} - \frac{1}{2}\left[\frac{1}{2}\ln(2+t)\right]_{\frac{1}{2}}^{1}$$

$$=\frac{1}{2}\left(\ln 1-\ln\frac{1}{2}\right)-\frac{1}{2}\left(\ln(2+1)-\ln\left(2+\frac{1}{2}\right)\right)$$

$$=\frac{1}{2}\left(0-\ln\frac{1}{2}\right)-\frac{1}{2}\left(\ln 3-\ln\frac{5}{2}\right)\quad \because \ln 1=0$$

$$=\frac{-1}{2}\ln\frac{1}{2}-\frac{1}{2}\ln 3+\frac{1}{2}\ln\frac{5}{2}$$

$$=\frac{1}{2}\left(\ln\frac{5}{2}-\ln\frac{1}{2}-\ln 3\right)$$

$$=\frac{1}{2}\ln\left[\frac{\left(\frac{5}{2}\right)}{\left(\frac{1}{2}\right)(3)}\right]$$

$$=\frac{1}{2}\ln\left(\frac{5}{3}\right)$$

Q.30 $\int_0^{\frac{\pi}{2}}\frac{\sin x}{(1+\cos x)(2+\cos x)}dx$

Solution:

$$\int_0^{\frac{\pi}{2}}\frac{\sin x}{(1+\cos x)(2+\cos x)}dx$$

Put $\cos x=t \Rightarrow -\sin x dx=dt$

$\Rightarrow \sin x dx=-dt$

when $x\to 0,\quad t\to 1$

when $x\to\frac{\pi}{2},\quad t\to 0$

$$=\int_1^0\frac{-dt}{(1+t)(2+t)}$$

$$=\int_0^1\frac{dt}{(1+t)(2+t)}$$

Let

$$\frac{1}{(1+t)(2+t)}=\frac{A}{1+t}+\frac{B}{2+t}\ \dots\text{(i)}$$

Multiplying both side by $(1+t)(2+t)$

$1=A(2+t)+B(1+t)\ \dots$ (ii)

Put $1+t=0\Rightarrow t=-1$ in eq. (ii)

$1=A(2-1)+B(0)$

$\boxed{A=1}$

Put $2+t=0\Rightarrow t=-2$ in eq. (ii)

$1=A(2-2)+B(1-2)$

$1=B(-1)$

$\boxed{B=-1}$

Put values of A and B in eq. (i)

$$\frac{1}{(1+t)(2+t)}=\frac{1}{1+t}-\frac{1}{2+t}\ \dots\text{(iii)}$$

Consider

$$\int_0^1\frac{1}{(1+t)(2+t)}dt$$

$$=\int_0^1\frac{1}{1+t}dt-\int_0^1\frac{1}{2+t}dt\quad\text{(by (iii))}$$

$$=\left[\ln(1+t)\right]_0^1-\left[\ln(2+t)\right]_0^1$$

$$=\ln(1+1)-\ln(1+0)-\left[\ln(2+1)-\ln(2)\right]$$

$$=\ln 2-\ln 1-\ln 3+\ln 2$$

$$=\ln\left(\frac{2\times 2}{3}\right)$$

$$=\ln\frac{4}{3}$$