Application of Definite Integral:

Definite integral can be used to find area under the curve, volume of solids and many useful things. Here it is used to find area under curve.

Area above x-axis is $\int_a^b y\,dx$

Area below x-axis is $\int_a^b -y\,dx$

EXERCISE 3.7

Q.1 Find the area between the x-axis and the curve $y = x^2 + 1$ from $x = 1$ to $x = 2$.

Solution:

$y = x^2 + 1,\ \ a = 1,\ \ b = 2,$

Since area lies above x-axis

$$\text{Area} = \int_a^b y\,dx$$

$$= \int_1^2 \left(x^2 + 1\right)dx = \int_1^2 x^2 dx + \int_1^2 1dx$$

$$= \left[\frac{x^3}{3}\right]_1^2 + [x]_1^2$$

$$= \frac{1}{3}\left(2^3 - 1^3\right) + (2 - 1)$$

$$= \frac{1}{3}(8 - 1) + 1$$

$$= \frac{3}{7} + 1 = \frac{10}{3} \text{ Square units}$$

Q.2 Find the area above the x-axis and under the curve $y = 5 - x^2$ from $x = -1$ to $x = 2$.

Solution:

$y = 5 - x^2,\ \ a = -1,\ \ b = 2$

Since Area lies above x-axis

$$\text{Area} = \int_a^b f(x)\,dx$$

$$= \int_{-1}^2 \left(5 - x^2\right)dx$$

$$= \int_{-1}^2 5dx - \int_{-1}^2 x^2 dx$$

$$= [5x]_{-1}^2 - \left[\frac{x^3}{3}\right]_{-1}^2$$

$$= 5\left(2 - (-1)\right) - \frac{1}{3}\left(2^3 - (-1)^3\right)$$

$$= 5(3) - \frac{1}{3}(8 + 1)$$

$$= 15 - 3$$

$$= 12 \text{ Square units}$$

Q.3 Find the area below the curve $y = 3\sqrt{x}$ and above the x-axis between $x = 1$ and $x = 4$

Solution:

Let $y = 3\sqrt{x},\ \ a = 1,\ \ b = 4$

Since area lies above x-axis

$$\text{Area} = \int_a^b y\,dx$$

$$= \int_1^4 3\sqrt{x}\,dx$$

$$= 3\int_1^4 x^{1/2}dx = 3\left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^4$$

$$= 3.\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4 = 3.\frac{2}{3}\left[x^{\frac{3}{2}}\right]_1^4$$

$$=2\left((4)^{\frac{3}{2}}-(1)^{\frac{3}{2}}\right)$$

$$=2(8-1)=14 \text{ Square units}$$

Q.4 Find the area bounded by cos function from $x=-\frac{\pi}{2}$ to $x=\frac{\pi}{2}$

Solution:

Let $y=\cos x,\ a=-\frac{\pi}{2}, b=\frac{\pi}{2}$

Since area lies above x-axis

$$\text{Area}=\int_a^b y\,dx$$

$$=\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx$$

$$=[\sin x]_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$$

$$=\sin\frac{\pi}{2}-\sin\left(\frac{-\pi}{2}\right)$$

$$=1-(-1)=2 \text{ Square units}$$

Q.5 Find the area between the x-axis and the curve $y=4x-x^2$

Solution:

Let $y=4x-x^2$

To find out the limits, let $y=0$

$4x-x^2=0$

$x(4-x)=0$

$x=0, x=4$

Since area lies above x-axis

$$\text{Area}=\int_a^b y\,dx$$

$$=\int_0^4 (4x-x^2)\,dx$$

$$=\int_0^4 4x\,dx-\int_0^4 x^2\,dx$$

$$=4\int_0^4 x\,dx-\int_0^4 x^2\,dx$$

$$=4\left[\frac{x^2}{2}\right]_0^4-\left[\frac{x^3}{3}\right]_0^4$$

$$=2\left[x^2\right]_0^4-\frac{1}{3}\left[x^3\right]_0^4$$

$$=2\left(4^2-0^2\right)-\frac{1}{3}\left(4^3-0^3\right)$$

$$=2(16)-\frac{1}{3}(64)$$

$$=32-\frac{64}{3}=\frac{96-64}{3}$$

$$=\frac{32}{3} \text{ Square units}$$

Q.6 Determine the area bounded by the parabola $y=x^2+2x-3$ and the x-axis.

Solution:

Let $y=x^2+2x-3$

To find out the limits, let $y=0$

$x^2+2x-3=0$

$x^2+3x-x-3=0$

$x(x+3)-1(x+3)=0$

$(x-1)(x+3)=0$

$x=1, x=-3$

Since area lies below x-axis

$$\text{Area}=-\int_a^b y\,dx$$

$$=-\int_{-3}^{1}\left(x^2+2x-3\right)dx$$

$$=-\left[\frac{x^3}{3}+x^2-3x\right]_{-3}^{1}$$

$$=-\left(\frac{1}{3}+1-3\right)+\left(\frac{-27}{3}+9+9\right)$$

$$=-\left(\frac{1}{3}-2\right)+9$$

$$=\frac{5}{3}+9$$

$$=\frac{5+27}{3}$$

$$=\frac{32}{3} \text{ Square units}$$

Q.7 Find the area bounded by the curve $y = x^3 + 1$, the x-axis and the line $x = 2$.

Solution:

$y = x^3 + 1, x = 2$

To find out the limits, let $y = 0$

$x^3 + 1 = 0 \Rightarrow (x+1)(x^2 - x + 1) = 0$

Either $x + 1 = 0 \Rightarrow x = -1$

Or $x^2 - x + 1 = 0$ (neglected because it gives complex roots)

Since area lies above $x - axis$

$$\text{Area} = \int_a^b f(x)dx$$

$$= \int_{-1}^{2} (x^3 + 1)dx = \int_{-1}^{2} x^3 dx + \int_{-1}^{2} 1dx$$

$$= \left[\frac{x^4}{4}\right]_{-1}^{2} + [x]_{-1}^{2}$$

$$= \frac{1}{4}\left(2^4 - (-1)^4\right) + \left(2 - (-1)\right)$$

$$= \frac{1}{4}(16-1) + 3$$

$$= \frac{15}{4} + 3$$

$$= \frac{15+12}{4}$$

$$= \frac{27}{4} \text{ Square units}$$

Q.8 Find the area bounded by the curve $y = x^3 - 4x$ and the x-axis.

Solution:

$y = x^3 - 4x$

To find out the limits, let $y = 0$

$x^3 - 4x = 0$

$x(x^2 - 4) = 0$

$x = 0, x = \pm 2$

Since area lies above $x - axis$ for $-2 < x < 0$

$$A_1 = \int_a^b ydx$$

$$= \int_{-2}^{0} (x^3 - 4x)dx$$

$$= \left[\frac{x^4}{4}\right]_{-2}^{0} - \left[\frac{4x^2}{2}\right]_{-2}^{0}$$

$$= \frac{1}{4}\left(0 - (-2)^4\right) - 2\left(0 - (-2)^2\right)$$

$$A_1 = \frac{1}{4}(-16) - 2(-4) = -4 + 8 = 4$$

Since area lies below $x - axis$ for $0 < x < 2$

$$A_2 = -\int_a^b ydx$$

$$= -\int_0^2 (x^3 - 4x)dx$$

$$= -\int_0^2 x^3 dx + 4\int_0^2 xdx$$

$$= -\left[\frac{x^4}{4}\right]_0^2 + 4\left[\frac{x^2}{2}\right]_0^2$$

$$= \frac{-1}{4}(16-0) + 2(4-0)$$

$$A_2 = -4 + 8 = 4$$

Total area $= A_1 + A_2$

$A = 4 + 4 = 8$ Square units

Q.9 Find the area between the curve $y = x(x-1)(x+1)$ and the x-axis.

Solution:

$y = x(x-1)(x+1)$

$= x^3 - x$

To find out the limits, let $y = 0$

$x(x-1)(x+1) = 0$

$x = 0, x - 1 = 0, x + 1 = 0$

$x = 0, x = 1, x = -1$

Since area lies above $x - axis$ for $-1 < x < 0$

$$A_1 = \int_a^b ydx$$

$$= \int_{-1}^{0} \left(x^3 - x\right) dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0}$$

$$= 0 - 0 - \left(\frac{1}{4} - \frac{1}{2}\right)$$

$$A_1 = -\frac{1}{4} + \frac{1}{2} = \frac{-1+2}{4} = \frac{1}{4}$$

Since area lies below $x - axis$ for $0 < x < 1$

$$A_2 = -\int_a^b y\, dx$$

$$= -\int_{0}^{1} \left(x^3 - x\right) dx$$

$$= \int_{0}^{1} \left(-x^3 + x\right) dx$$

$$= \left[-\frac{x^4}{4} + \frac{x^2}{2}\right]_{0}^{1}$$

$$= -\frac{1}{4} + \frac{1}{2} - \left(0 + 0\right)$$

$$A_2 = \frac{-1+2}{4} = \frac{1}{4}$$

Total area $= A_1 + A_2$

$$A = \frac{1}{4} + \frac{1}{4} = \frac{1+1}{4} = \frac{2}{4} = \frac{1}{2} \text{ Square units}$$

Q.10 Find the area above the *x*-axis, bounded by the curve $y^2 = 3 - x$ from $x = -1$ to $x = 2$

Solution:

$y^2 = 3 - x$

$\Rightarrow y = \pm\sqrt{3 - x}$

$y = \sqrt{3 - x}$, $a = -1$, $b = 2$

Since area lies above *x*-axis

$$\text{Area} = \int_a^b y dx$$

$$= \int_{-1}^{2} \sqrt{3 - x}\, dx$$

$$= -\int_{-1}^{2} (3 - x)^{\frac{1}{2}} (-1) dx$$

$$= -\left[\frac{(3-x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-1}^{2} = -\frac{2}{3}\left[(3-x)^{\frac{3}{2}}\right]_{-1}^{2}$$

$$= \frac{-2}{3}\left[(3-2)^{\frac{3}{2}} - (3+1)^{\frac{3}{2}}\right]$$

$$= \frac{-2}{3}\left[(1)^{\frac{3}{2}} - (4)^{\frac{3}{2}}\right]$$

$$= \frac{-2}{3}(1 - 8)$$

$$= \frac{-2}{3}(-7)$$

$$= \frac{14}{3} \text{ Square units}$$

Q.11 Find the area between the *x*-axis and the curve $y = \cos\frac{1}{2}x$ from $x = -\pi$ to π.

Solution:

$y = \cos\frac{x}{2}$, $a = -\pi$, $b = \pi$

Since area lies above *x*-axis

$$\text{Area} = \int_a^b y dx$$

$$= \int_{-\pi}^{\pi} \cos\frac{x}{2} dx$$

$$= \left[\frac{\sin\frac{x}{2}}{\frac{1}{2}}\right]_{-\pi}^{\pi}$$

$$= 2\left[\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right]$$

$$= 2\left[1 - (-1)\right] = 2(2)$$

$= 4$ Square units

Q.12 Find the area between the *x*-axis and the curve $y = \sin 2x$ from $x = 0$ to $x = \frac{\pi}{3}$

Solution:

$y = \sin 2x$ $a = 0$, $b = \frac{\pi}{3}$

Since area lies above *x*-axis

$$\text{Area} = \int_a^b f(x)\,dx$$
$$= \int_0^{\frac{\pi}{3}} \sin 2x\,dx$$
$$= \left[\frac{-\cos 2x}{2}\right]_0^{\frac{\pi}{3}}$$
$$= -\frac{1}{2}\left[\cos\frac{2\pi}{3} - \cos 0\right]$$
$$= \frac{-1}{2}\left[-\frac{1}{2} - 1\right]$$
$$= -\frac{1}{2}\left(\frac{-3}{2}\right)$$
$$= \frac{3}{4} \text{ Square units}$$

Q.13 Find the area between the *x*-axis and the curve $y = \sqrt{2ax - x^2}$ when $a > 0$

Solution:

$$y = \sqrt{2ax - x^2}$$

To find out the limits, let $y = 0$

$$2ax - x^2 = 0$$
$$x(2a - x) = 0$$
$$x = 0,\ 2a - x = 0$$
$$x = 0,\ x = 2a$$

Since area lies above *x*-axis

$$\text{Area} = \int_a^b y\,dx$$
$$= \int_0^{2a} \sqrt{2ax - x^2}\,dx$$
$$= \int_0^{2a} \sqrt{a^2 + 2ax - x^2 - a^2}\,dx$$
$$= \int_0^{2a} \sqrt{a^2 - (x^2 - 2ax + a^2)}\,dx$$
$$= \int_0^{2a} \sqrt{a^2 - (x - a)^2}\,dx$$

Put $x - a = a\sin\theta \Rightarrow dx = a\cos\theta\,d\theta$

when $x \to 0, \quad \theta \to \frac{-\pi}{2}$

when $x \to 2a, \quad \theta \to \frac{\pi}{2}$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2\theta}\; a\cos\theta\,d\theta$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{a^2(1 - \sin^2\theta)}\; a\cos\theta\,d\theta$$
$$= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - \sin^2\theta}\,\cos\theta\,d\theta$$
$$= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$
$$= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1 + \cos 2\theta}{2}\right) d\theta$$
$$= \frac{a^2}{2}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 1\,d\theta + \frac{a^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2\theta\,d\theta$$
$$= \frac{a^2}{2}[\theta]_{\frac{-\pi}{2}}^{\frac{\pi}{2}} + \frac{a^2}{2}\left[\frac{\sin 2\theta}{2}\right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$$
$$= \frac{a^2}{2}\left(\frac{\pi}{2} - \left(\frac{-\pi}{2}\right)\right) + \frac{a^2}{4}(\sin\pi - \sin(-\pi))$$
$$= \frac{a^2}{2}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) + \frac{a^2}{4}(0 - 0)$$
$$= \frac{a^2}{2}(\pi)$$
$$= \frac{1}{2}\pi a^2 \text{ Square units}$$