Differential Equations:

An equation containing at least one derivative of a dependent variable with respect to an independent variable is called a differential equation.

For example: $y\dfrac{dy}{dx}+2x=0, x\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}-2x=0$ etc.

Order of Differential Equation:

The order of differential equation is the order of the highest derivative in the equation.
For example:

$y\dfrac{dy}{dx}+2x=0$ is first order differential equation.

$x\dfrac{d^2y}{dx^2}-2x=0$ is second order differential equation.

Solution of differential Equation:

A solution of differential equation is a relation between the variables (not involving derivatives) which satisfies the differential equation.

Initial value conditions:

The arbitrary constants involving in the solution of a differential equation can be determined by the given conditions. Such conditions are called initial value conditions.

Note:

(i) A differential equation containing only derivative of first order can be written in terms of differentials.

(ii) General solution of differential equation of order n contains n arbitrary constants which can be determined by n initial value conditions.\

EXERCISE 3.8

Q.1 Check that each of the following equations written against the differential equation is its solution.

(i) $x\dfrac{dy}{dx}=1+y \quad , \quad y=cx-1$

Solution:

$y=cx-1$

Differentiate w.r.t. "x"

$\dfrac{dy}{dx}=\dfrac{d}{dx}(cx-1)$

$\dfrac{dy}{dx}=c$

Also $x\dfrac{dy}{dx}=1+y$ …(i)

$\Rightarrow xc=1+cx-1$

$cx=cx$

Hence $y=cx-1$ is the solution of (i)

Alternate solution:

$x\dfrac{dy}{dx}=1+y$

$xdy=(1+y)dx$

$\dfrac{1}{1+y}dy=\dfrac{1}{x}.dx$

Integrating both sides

$\int\dfrac{1}{1+y}dy=\int\dfrac{1}{x}.dx$

$\ln(1+y)=\ln x+\ln c$

$\ln(1+y)=\ln cx$

$1+y=cx$

$y=cx-1$

(ii) $x^2(2y+1)\dfrac{dy}{dx}-1=0 \quad , \quad y^2+y=c-\dfrac{1}{x}$

Solution:

$y^2+y=c-\dfrac{1}{x}$

Differentiate w.r.t. "x"

$2y\dfrac{dy}{dx}+\dfrac{dy}{dx}=0-\left(\dfrac{-1}{x^2}\right)$

$(2y+1)\dfrac{dy}{dx}=\dfrac{1}{x^2}$

Also $x^2(2y+1)\dfrac{dy}{dx}-1=0$ … (i)

$$x^2\left(\frac{1}{x^2}\right)-1=0$$

$$1-1=0$$

Hence $y^2+y=c-\frac{1}{x}$ is the solution of (i)

(iii) $y\frac{dy}{dx}-e^{2x}=1$, $y^2=e^{2x}+2x+c$

Solution:

$$y^2=e^{2x}+2x+c$$

Differentiate w.r.t. "x"

$$2y\frac{dy}{dx}=2e^{2x}+2$$

Dividing both sides by 2

$$y\frac{dy}{dx}=e^{2x}+1$$

Also $y\frac{dy}{dx}-e^{2x}=1$ … (i)

(i) becomes

$$e^{2x}+1-e^{2x}=1$$

Hence $y^2=e^{2x}+2x+c$ is the solution of (i)

(iv) $\frac{1}{x}\frac{dy}{dx}-2y=0$, $y=ce^{x^2}$

Solution:

$$y=ce^{x^2}$$

Differentiate w.r.t. "x"

$$\frac{dy}{dx}=ce^{x^2}(2x)$$

$$\frac{dy}{dx}=2cxe^{x^2}$$

Also $\frac{1}{x}\frac{dy}{dx}-2y=0$ … (i)

(i) becomes

$$\frac{1}{x}\left(2cxe^{x^2}\right)-2\left(ce^{x^2}\right)=0$$

$$2ce^{x^2}-2ce^{x^2}=0$$

Hence $y=ce^{x^2}$ is the solution of (i)

(v) $\frac{dy}{dx}=\frac{y^2+1}{e^{-x}}$, $y=\tan\left(e^x+c\right)$

Solution:

$$y=\tan\left(e^x+c\right)$$

Differentiate w.r.t. "x"

$$\frac{dy}{dx}=\sec^2\left(e^x+c\right)e^x$$

Also $\frac{dy}{dx}=\frac{y^2+1}{e^{-x}}$ … (i)

(i) becomes

$$e^x\sec^2\left(e^x+c\right)=\frac{\tan^2\left(e^x+c\right)+1}{e^{-x}}$$

$$e^x\sec^2\left(e^x+c\right)=e^x\sec^2\left(e^x+c\right)$$

Hence $y=\tan\left(e^x+c\right)$ is the solution of (i)

Solve the following differential equations:

Q.2 $\frac{dy}{dx}=-y$

Solution:

$$\frac{dy}{dx}=-y$$

By separation of variables

$$\frac{dy}{y}=-dx$$

Integrating both sides

$$\int\frac{dy}{y}=-\int 1dx$$

$$\ln y=-x+\ln c$$

$$\ln y=\ln e^{-x}+\ln c$$

$$\ln y=\ln ce^{-x}$$

$$y=ce^{-x}$$

Q.3 $ydx+xdy=0$

Solution:

$$ydx+xdy=0$$

By separation of variables

$$xdy=-ydx$$

$$\frac{dy}{y}=-\frac{dx}{x}$$

Integrating both sides

$$\int\frac{dy}{y}=-\int\frac{dx}{x}$$

$$\ln y=-\ln x+\ln c$$

$$\ln x+\ln y=\ln c$$

$$\ln xy=\ln c$$

$$xy=c$$

Q.4 $\frac{dy}{dx}=\frac{1-x}{y}$

Solution:

$$\frac{dy}{dx}=\frac{1-x}{y}$$

By separation of variables

$$ydy=(1-x)dx$$

Integrating both sides

$$\int y\,dy=\int(1-x)dx$$

$$\frac{y^2}{2}=x-\frac{x^2}{2}+c_1$$

$$y^2=2x-x^2+2c_1$$

$$y^2=2x-x^2+c \qquad \because 2c_1=c$$

$$y^2=x(2-x)+c$$

Q.5 $\frac{dy}{dx}=\frac{y}{x^2},(y>0)$

Solution:

$$\frac{dy}{dx}=\frac{y}{x^2}$$

By separation of variables

$$\frac{dy}{y}=\frac{dx}{x^2}$$

Integrating both sides

$$\int\frac{dy}{y}=\int x^{-2}dx$$

$$\ln y=\frac{x^{-1}}{-1}+\ln c$$

$$\ln y=\frac{-1}{x}+\ln c$$

$$\ln y=\ln e^{\frac{-1}{x}}+\ln c$$

$$\ln y=\ln ce^{\frac{-1}{x}}$$

$$y=ce^{\frac{-1}{x}}$$

Q.6 $\sin y\operatorname{cosec} x\frac{dy}{dx}=1$

Solution:

$$\sin y\operatorname{cosec} x\frac{dy}{dx}=1$$

By separation of variables

$$\sin ydy=\frac{dx}{\operatorname{cosec} x}$$

Integrating both sides

$$\int\sin ydy=\int\sin xdx$$

$$-\cos y=-\cos x+c_1$$

$$\cos y=\cos x+c \qquad \because -c_1=c$$

Q.7 $xdy+y(x-1)dx=0$

Solution:

$$xdy+y(x-1)dx=0$$

By separation of variables

$$xdy=-y(x-1)dx$$

$$\frac{dy}{y}=-\left(\frac{x-1}{x}\right)dx$$

$$\frac{dy}{y}=\left(\frac{-x}{x}+\frac{1}{x}\right)dx$$

$$\frac{dy}{y}=\left(-1+\frac{1}{x}\right)dx$$

Integrating both sides

$$\int\frac{dy}{y}=\int-1dx+\int\frac{1}{x}dx$$

$$\ln y=-x+\ln x+\ln c$$

$$\ln y-\ln x-\ln c=-x$$

$$\ln\left(\frac{y}{xc}\right)=-x$$

$$\frac{y}{xc}=e^{-x}$$

$$y=cxe^{-x}$$

Q.8 $\frac{x^2+1}{y+1}=\frac{x}{y}\frac{dy}{dx}$

Solution:

$$\frac{x^2+1}{y+1}=\frac{x}{y}\frac{dy}{dx}$$

By separation of variables

$$\frac{x}{y}.\frac{dy}{dx}=\frac{x^2+1}{y+1}$$

$$\left(\frac{y+1}{y}\right)dy=\left(\frac{x^2+1}{x}\right)dx$$

$$\left(\frac{y}{y}+\frac{1}{y}\right)dy=\left(\frac{x^2}{x}+\frac{1}{x}\right)dx$$

$$\left(1+\frac{1}{y}\right)dy=\left(x+\frac{1}{x}\right)dx$$

Integrating both sides

$$\int\left(1+\frac{1}{y}\right)dy=\int\left(x+\frac{1}{x}\right)dx$$

$$y+\ln y=\frac{x^2}{2}+\ln x+\ln c$$

$$\ln y-\ln x-\ln c=\frac{x^2}{2}-y$$

$$\ln\left(\frac{y}{xc}\right)=\frac{x^2}{2}-y$$

$$\frac{y}{xc}=e^{\frac{x^2}{2}-y}$$

$$\frac{y}{xc}=e^{\frac{x^2}{2}}e^{-y}$$

$$\frac{y}{xc}=\frac{e^{\frac{x^2}{2}}}{e^y}$$

$$ye^y=cxe^{\frac{x^2}{2}}$$

Q.9 $\frac{1}{x}\frac{dy}{dx}=\frac{1}{2}\left(1+y^2\right)$

Solution:

$$\frac{1}{x}\frac{dy}{dx}=\frac{1}{2}\left(1+y^2\right)$$

$$\frac{dy}{1+y^2}=\frac{1}{2}xdx$$

Integrating both sides

$$\int\frac{dy}{1+y^2}=\frac{1}{2}\int xdx$$

$$\tan^{-1}y=\frac{1}{2}\frac{x^2}{2}+c$$

$$\tan^{-1}y=\frac{x^2}{4}+c$$

Q.10 $2x^2y\frac{dy}{dx}=x^2-1$

Solution:

$$2x^2y\frac{dy}{dx}=x^2-1$$

By separation of variables

$$2ydy=\frac{x^2-1}{x^2}dx$$

$$2ydy=\left(1-\frac{1}{x^2}\right)dx$$

Integrating both sides

$$\int 2ydy=\int\left(1-x^{-2}\right)dx$$

$$\frac{2y^2}{2}=x+\frac{1}{x}+c$$

$$y^2=x+\frac{1}{x}+c$$

Q.11 $\frac{dy}{dx}+\frac{2xy}{2y+1}=x$

Solution:

$$\frac{dy}{dx}+\frac{2xy}{2y+1}=x$$

$$\frac{dy}{dx}=x-\frac{2xy}{2y+1}$$

$$=\frac{2xy+x-2xy}{2y+1}$$

$$\frac{dy}{dx}=\frac{x}{2y+1}$$

$$\left(2y+1\right)dy=xdx$$

Integrating both sides

$$\int\left(2y+1\right)dy=\int xdx$$

$$\frac{2y^2}{2}+y=\frac{x^2}{2}+c$$

$$y^2+y=\frac{x^2}{2}+c$$

$$y\left(y+1\right)=\frac{x^2}{2}+c$$

Q.12 $\left(x^2-yx^2\right)\frac{dy}{dx}+y^2+xy^2=0$

Solution:

$$\left(x^2-yx^2\right)\frac{dy}{dx}+y^2+xy^2=0$$

By separation of variables

$$x^2\left(1-y\right)\frac{dy}{dx}+y^2\left(1+x\right)=0$$

$$x^2\left(1-y\right)\frac{dy}{dx}=-y^2\left(1+x\right)$$

$$\left(\frac{1-y}{y^2}\right)dy=-\frac{\left(1+x\right)}{x^2}dx$$

$$\left(\frac{1}{y^2}-\frac{y}{y^2}\right)dy=-\left(\frac{1}{x^2}+\frac{x}{x^2}\right)dx$$

$$\left(\frac{1}{y^2}-\frac{1}{y}\right)dy=-\left(\frac{1}{x^2}+\frac{1}{x}\right)dx$$

Integrating both sides

$$\int\left(y^{-2}-\frac{1}{y}\right)dy=-\int\left(x^{-2}+\frac{1}{x}\right)dx$$

$$\frac{y^{-1}}{-1}-\ln y=-\frac{1}{(-x)}-\ln x-c$$

$$-\frac{1}{y}-\ln y=\frac{1}{x}-\ln x-c$$

$$\ln y+\frac{1}{y}=\ln x-\frac{1}{x}+c$$

Q.13 $\sec^2 x\tan y dx+\sec^2 y\tan x dy=0$

Solution:

$$\sec^2 x\tan y dx+\sec^2 y\tan x dy=0$$

By separation of variables

$$\sec^2 y\tan x dy=-\sec^2 x\tan y dx$$

$$\frac{\sec^2 y dy}{\tan y}=-\frac{\sec^2 x}{\tan x}dx$$

Integrating both sides

$$\int\frac{\sec^2 y dy}{\tan y}=-\int\frac{\sec^2 x}{\tan x}dx$$

$$\ln\tan y=-\ln\tan x+\ln c$$

$$\ln\tan y+\ln\tan x=\ln c$$

$$\ln\tan x\tan y=\ln c$$

$$\tan x\tan y=c$$

Q.14 $y-x\frac{dy}{dx}=2\left(y^2+\frac{dy}{dx}\right)$

Solution:

$$y-x\frac{dy}{dx}=2\left(y^2+\frac{dy}{dx}\right)$$

By separation of variables

$$y-x\frac{dy}{dx}=2y^2+2\frac{dy}{dx}$$

$$-x\frac{dy}{dx}-2\frac{dy}{dx}=2y^2-y$$

$$-(x+2)\frac{dy}{dx}=2y^2-y$$

$$\frac{dy}{2y^2-y}=\frac{-dx}{x+2}$$

$$\frac{dy}{y(2y-1)}=-\frac{dx}{x+2}\quad\ldots(i)$$

Let $\frac{1}{y(2y-1)}=\frac{A}{y}+\frac{B}{2y-1}\quad\ldots(ii)$

Multiplying both sides by $y(2y-1)$

$$1=A(2y-1)+B(y)\quad\ldots(iii)$$

Put $y=0$ in eq. (iii)

$$1=A(-1)\Rightarrow \boxed{A=-1}$$

Put $2y-1=0\Rightarrow y=\frac{1}{2}$ in eq. (iii)

$$1=A(0)+B\left(\frac{1}{2}\right)\Rightarrow \boxed{B=2}$$

Put values of A and B in (ii)

$$\frac{1}{y(2y-1)}=\frac{-1}{y}+\frac{2}{2y-1}$$

(i) becomes

$$\left(\frac{-1}{y}+\frac{2}{2y-1}\right)dy=-\frac{dx}{x+2}$$

Integrating both sides

$$\int\left(\frac{-1}{y}+\frac{2}{2y-1}\right)dy=-\int\frac{dx}{x+2}$$

$$-\ln y+\ln(2y-1)=-\ln(x+2)-\ln c$$

$$\ln y-\ln(2y-1)=\ln(x+2)+\ln c$$

$$\ln\frac{y}{2y-1}=\ln(c(x+2))$$

$$\frac{y}{2y-1}=c(x+2)$$

Q.15 $1+\cos x\tan y\frac{dy}{dx}=0$

Solution:

$$1+\cos x\tan y\frac{dy}{dx}=0$$

By separation of variables

$$\cos x\tan y\frac{dy}{dx}=-1$$

$$\tan y dy=\frac{-dx}{\cos x}$$

Integrating both sides

$$\int\tan y dy=-\int\sec x dx$$

$$-\ln\cos y=-\ln(\sec x+\tan x)-\ln c$$

$$\ln\cos y=\ln(\sec x+\tan x)+\ln c$$

$$\ln\cos y=\ln(c(\sec x+\tan x))$$

$$\cos y=c(\sec x+\tan x)$$

Q.16 $y-x\frac{dy}{dx}=3\left(1+x\frac{dy}{dx}\right)$

Solution:

$$y-x\frac{dy}{dx}=3\left(1+x\frac{dy}{dx}\right)$$

By separation of variables

$$y-x\frac{dy}{dx}=3+3x\frac{dy}{dx}$$

$$y-3=4x\frac{dy}{dx}$$
$$4x\frac{dy}{dx}=y-3$$
$$\frac{dy}{y-3}=\frac{dx}{4x}$$
Integrating both sides
$$\int\frac{dy}{y-3}=\frac{1}{4}\int\frac{dx}{x}$$
$$\ln(y-3)=\frac{1}{4}\ln x+\ln c$$
$$\ln(y-3)=\ln x^{\frac{1}{4}}+\ln c$$
$$\ln(y-3)=\ln\left(cx^{\frac{1}{4}}\right)$$
$$y-3=cx^{\frac{1}{4}}$$
$$y=3+cx^{\frac{1}{4}}$$

Q.17 $\sec x+\tan y\frac{dy}{dx}=0$

Solution:
$$\sec x+\tan y\frac{dy}{dx}=0$$
By separation of variables
$$\tan y\frac{dy}{dx}=-\sec x$$
$$\tan y dy=-\sec x dx$$
Integrating both sides
$$\int\tan y dy=-\int\sec x dx$$
$$-\ln\cos y=-\ln(\sec x+\tan x)-\ln c$$
$$\ln\cos y=\ln(\sec x+\tan x)+\ln c$$
$$\ln(\cos y)=\ln(c(\sec x+\tan x))$$
$$\cos y=c(\sec x+\tan x)$$

Q.18 $\left(e^x+e^{-x}\right)\frac{dy}{dx}=e^x-e^{-x}$

Solution:
$$\left(e^x+e^{-x}\right)\frac{dy}{dx}=e^x-e^{-x}$$
$$\frac{dy}{dx}=\frac{e^x-e^{-x}}{e^x+e^{-x}}$$
$$dy=\frac{e^x-e^{-x}}{e^x+e^{-x}}dx$$
Integrating both sides
$$\int dy=\int\frac{e^x-e^{-x}}{e^x+e^{-x}}dx$$
$$\because\int\frac{f'(x)}{f(x)}dx=\ln|f(x)|+c$$
$$y=\ln|e^x+e^{-x}|+c$$

Q.19 **Find the general solution of the equation $\frac{dy}{dx}-x=xy^2$. Also find the particular solution if $y=1$ when $x=0$**

Solution:
$$\frac{dy}{dx}-x=xy^2$$
By separation of variables
$$\frac{dy}{dx}=x+xy^2$$
$$\frac{dy}{dx}=x\left(1+y^2\right)$$
$$\frac{dy}{1+y^2}=xdx$$
Integrating both sides
$$\int\frac{dy}{1+y^2}=\int xdx$$
$$\tan^{-1}y=\frac{x^2}{2}+c\ \dots\ \text{(i)}$$
is the general solution

put $y=1, x=0$ in (i)
$$\tan^{-1}(1)=c$$
$$\frac{\pi}{4}=c$$
Put value of c in (i)
$$\tan^{-1}y=\frac{x^2}{2}+\frac{\pi}{4}$$

Q.20 **Solve the differential equation $\frac{dx}{dt}=2x$, given that $x=4$ when $t=0$.**

Solution:
$$\frac{dx}{dt}=2x$$
By separation of variables
$$\frac{dx}{x}=2dt$$
Integrating both sides

$$\int \frac{dx}{x} = 2\int dt$$
$$\ln x = 2t + \ln c$$
$$\ln x = \ln e^{2t} + \ln c$$
$$\ln x = \ln ce^{2t}$$
$$x = ce^{2t} \quad \ldots(i)$$ is the general solution

Put $x = 4, \quad t = 0$ in (i)

$$\ln 4 = c$$

Put value of c in (i)

$$\ln x = 2t + \ln 4$$
$$\ln x - \ln 4 = 2t$$
$$\ln\left(\frac{x}{4}\right) = 2t$$
$$\frac{x}{4} = e^{2t}$$
$$x = 4e^{2t}$$

Q.21 Solve the differential equation $\frac{ds}{dt} + 2st = 0$. Also find the particular solution if $s = 4e$, when $t = 0$.

Solution:

$$\frac{ds}{dt} + 2st = 0$$

By separation of variables

$$\frac{ds}{dt} = -2st$$
$$\frac{ds}{s} = -2tdt$$

Integrating both sides

$$\int \frac{ds}{s} = -2\int tdt$$
$$\ln s = -\frac{2t^2}{2} + \ln c$$
$$\ln s = \ln e^{-t^2} + \ln c$$
$$\ln s = \ln ce^{-t^2}$$
$$s = ce^{-t^2} \quad \ldots(i)$$

is the general solution

Put $s = 4e, t = 0$ in (i)

$$4e = c$$

Put value of c in eq. (i)

$$s = 4ee^{-t^2}$$
$$s = 4e^{1-t^2}$$

Q.22 In a culture, bacteria increases at the rate proportional to the number of bacteria present. If bacteria are 200 initially and are doubled in 2 hours, find the number of bacteria present four hours later.

Solution:

Let p be the number of bacteria present, then

$$\frac{dp}{dt} \propto p$$
$$\frac{dp}{dt} = kp$$

By separation of variables

$$\frac{dp}{p} = kdt$$

Integrating both sides

$$\int \frac{dp}{p} = \int kdt$$
$$\ln p = kt + \ln c$$
$$= \ln e^{kt} + \ln c$$
$$\ln p = \ln ce^{kt}$$
$$p = ce^{kt} \quad \ldots(i)$$

Put $p = 200, t = 0$ in (i)

$$200 = c$$

Hence (i) becomes

$$p = 200e^{kt} \quad \ldots(ii)$$

Put $t = 2, p = 400$ in (ii)

$$400 = 200\, e^{2k}$$
$$2 = e^{2k}$$
$$\ln 2 = 2k \Rightarrow k = \frac{1}{2}\ln 2$$

Hence

$$p = 200e^{\left(\frac{1}{2}\ln 2\right)t}$$

If $t = 4$, then

$$p = 200e^{\left(\frac{1}{2}\ln 2\right)(4)}$$
$$p = 200e^{2\ln 2}$$
$$p = 200e^{\ln 4}$$
$$p = 200(4)$$
$$p = 800$$

which is required number of bacteria.

Q.23 A ball is thrown vertically upward with a velocity of 2450 cm/sec.

Neglecting air resistance, find

(i) Velocity of ball at any time 't'

(ii) Distance traveled in any time 't'

(iii) Maximum height attained by the ball.

Solution:

Let v be the velocity of ball at any time t, then

(i) $\frac{dv}{dt} = -g$

By separation of variables

$dv = -gdt$

Integrating both sides

$\int dv = -g\int dt$

$v = -gt + c_1 \ldots$ (i)

put $v = 2450,\ t = 0$ in (i)

$2450 = c_1$

(i) becomes

$v = -gt + 2450$

$v = 2450 - 980t \ldots$ (ii)

Where $g = 980cm/\sec^2$

(ii) Let 'h' be the height of ball at any time t, then

$\frac{dh}{dt} = 2450 - 980t$

By separation of variables

$dh = (2450 - 980t)dt$

Integrating both sides

$\int dh = \int (2450 - 980t)dt$

$h = 2450t - \frac{980t^2}{2} + c_2$

$h = 2450t - 490t^2 + c_2 \ldots$(iii)

put $h = 0,\ t = 0$ in (iii)

$0 = 0 - 0 + c_2 \quad \Rightarrow c_2 = 0$

Then (iii) becomes

$h = 2450t - 490t^2 \ldots$(iv)

(iii) Maximum height is attained by the ball when $v = 0$

i.e. $2450 - 980t = 0$

$2450 = 980t \Rightarrow t = \frac{2450}{980}$

$t = 2.5$ sec

From (iv) the maximum height attained in cm is

$= 2450 \times (2.5) - 490(2.5)^2$

$= 6125 - 3062.5$

$= 3062.5cm$