UNIT 4

INTRODUCTION TO ANALYTIC GEOMETRY

Introduction.

The word geometry is derived from two Greek words **Geo** (earth) and **Metron** (measurement). It means knowledge of measurement of earth. Geometry is branch of Mathematics that deals the shape and size of things. Briefly speaking, geometry is a mathematical study of properties, relations and measurements of points, lines, angles, curves, surfaces and solids.

Around 300.B.C.**Euclid** was very first Greek mathematician who wrote **13 books** on geometry. He was founder of geometry. Geometry introduced by Euclid is known as Euclidean geometry. In his books, he wrote a number of definitions on basic concepts of point, line etc. He gave assumptions, which were actually axioms.

Note:

Euclidean geometry is divided into two parts

(i) Plane geometry **(ii)** Solid geometry

In 1637 A.D a French philosopher and mathematician **Rene-Descartes** introduced algebraic methods in geometry, named as coordinates geometry or analytic geometry. In analytic geometry, points could be represented by numbers. Lines and curves represented by equations.

Coordinate Plane:

The plane made by two mutually perpendicular lines is called coordinate plane. Let we draw two mutually perpendicular lines XX' and YY' such as O be their point of intersection. Then lines XX' and YY' are together called coordinates axes. The common point O is called origin or initial point. The horizontal line $X'OX$ is called x-axis, while the vertical line $Y'OY$ is called y-axis. The plane determined by both x-axis and y-axis is called xy-plane or Cartesian plane or Coordinate plane.

Let $P(x, y)$ be a point is coordinate plane, then first member of ordered pair (i.e. x) is called x-coordinate or abscissa of point P, and second member of ordered pair (i.e. y) is called y-coordinate or ordinate of point P.

The coordinate axes divide the coordinate plane into four equal parts, called quadrants.

Quadrant I $\{(x, y): x > 0, y > 0\}$ (i.e. x is positive and y is positive)

Quadrant II $\{(x, y): x < 0, y > 0\}$ (i.e. x is negative and y is positive)

Quadrant III $\{(x, y): x < 0, y < 0\}$ (i.e. x is negative and y is negative)

Quadrant IV $\{(x, y): x > 0, y < 0\}$ (i.e. x is positive and y is negative)

Note:

On *x*-axis ordinate is zero i.e. $y=0$, also on *y*-axis abscissa is zero i.e. $x=0$

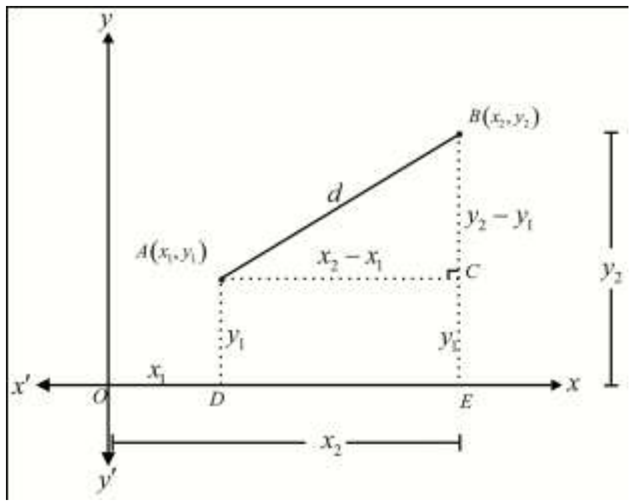
The Distance Formula:

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points in the plane. The distance between two points is given by $d=|AB|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Proof:

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points in the plane. We can find the distance $d=|\overline{AB}|$ from the right angle triangle *ABC*,

By using Pythagorean Theorem.



We have

$$d^2=|AB|^2=|AC|^2+|BC|^2 \quad \text{(i)}$$

Now $|AC|=|DE|$

$$=|OE-OD|$$

$$=|x_2-x_1|$$

And $|BC|=|BE-CE|$

$$=|BE-AD|$$

$$=|y_2-y_1|$$

Therefore, (i) takes the form

$$d^2=|AB|^2=(x_2-x_1)^2+(y_2-y_1)^2$$

$$d=|AB|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Which is the formula for the distance '*d*' the distance is always taken to be positive and it not a directed distance from *A* and *B* when

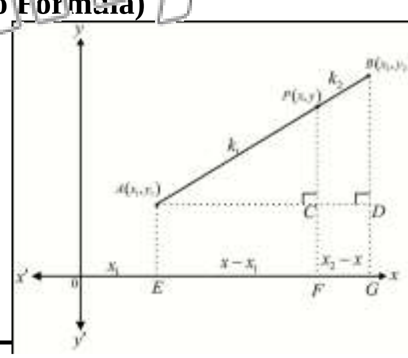
A and *B* do not lie on the same horizontal and vertical line. If *A* and *B* lie on a line parallel to one of the coordinate axes, then by the formula the distance *AB* is absolute value of the directed distance $\overrightarrow{AB}$.

Point dividing the join of two points in a given ratio: (Ratio Formula)

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points in a plane.

The coordinates of the point dividing the line segment *AB* in the ratio $k_1 : k_2$ are given by

$$\left(\frac{k_1x_2+k_2x_1}{k_1+k_2}, \frac{k_1y_2+k_2y_1}{k_1+k_2}\right)$$



Proof:

Let $P(x,y)$ be the point that divides AB in the ratio $k_1:k_2$.

From A, B and P draw perpendiculars to $x-axis$ as shown in the figure.

Also draw $AD \perp BG$.

Since PC is parallel to BD, in triangle ADB, we have

$$\frac{k_1}{k_2}=\frac{AP}{PB}=\frac{AC}{CD}=\frac{EF}{FG}=\frac{OF-OE}{OG-OF}=\frac{x-x_1}{x_2-x}$$

$$\Rightarrow \frac{k_1}{k_2}=\frac{x-x_1}{x_2-x}$$

$$k_1(x_2-x)=k_2(x-x_1)$$

$$k_1x_2-k_1x=k_2x-k_2x_1$$

$$k_1x_2+k_2x_1=k_2x+k_1x$$

$$k_1x+k_2x=k_1x_2+k_2x_1$$

$$x(k_1+k_2)=k_1x_2+k_2x_1$$

$$\Rightarrow x=\frac{k_1x_2+k_2x_1}{k_1+k_2}$$

Similarly we can show that

$$y=\frac{k_1y_2+k_2y_1}{k_1+k_2}$$

Note:

(i) If the directed distances AP and PB have the same sign, then their ratio is positive and P is said to divide AB **internally**. The coordinates of $P(x,y)$ in this case will be

$$x=\frac{k_1x_2+k_2x_1}{k_1+k_2} \text{ and } y=\frac{k_1y_2+k_2y_1}{k_1+k_2}$$

(ii) If the directed distance AP and PB have opposite signs (directions) i.e. P is beyond AB, then their ratio is negative and P is said to divide AB **externally**. The coordinates of $P(x,y)$ in this case will be

$$x=\frac{k_1x_2-k_2x_1}{k_1-k_2} \text{ and } y=\frac{k_1y_2-k_2y_1}{k_1-k_2}$$

(iii) **The mid-point formula:**

If a point $P(x,y)$ lies exactly in the middle of two points $A(x_1,y_1)$ and $B(x_2,y_2)$ i.e. $k_1:k_2=1:1$, then $x=\frac{x_1+x_2}{2}$ and $y=\frac{y_1+y_2}{2}$

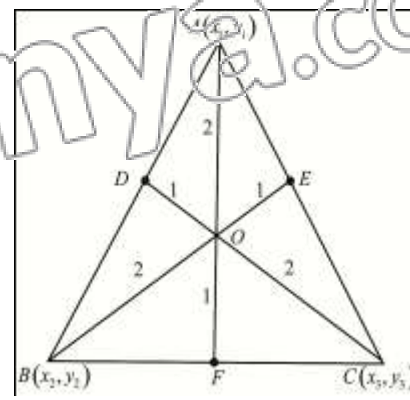
(iv) The above theorem is valid in which ever quadrant A and B lie.

Theorem:

The centroid of a ΔABC is a point that divides each median in the ratio 2:1. Using this show that medians of a triangle are concurrent.

Proof:

Let the vertices of a ΔABC have coordinates as shown in figure.



Since D is the midpoint of AB therefore its coordinates are $D = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Let $O(x, y)$ be the centroid of the triangle ABC.

Then O divides the median CD in the ratio $2:1$.

Therefore of O are

$$\left(\frac{(1)(x_3)+2\left(\frac{x_1+x_2}{2}\right)}{1+2}, \frac{(1)(y_3)+2\left(\frac{y_1+y_2}{2}\right)}{1+2}\right) = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

Similarly we can prove the same result for other two medians. Hence the medians of triangle are concurrent.

Theorem:

Bisectors of angles of a triangle are concurrent.

Proof: Let the coordinates of the vertices of a triangle ABC be as shown in the figure.

Suppose $|BC| = a, |CA| = b$ and $|AB| = c$.

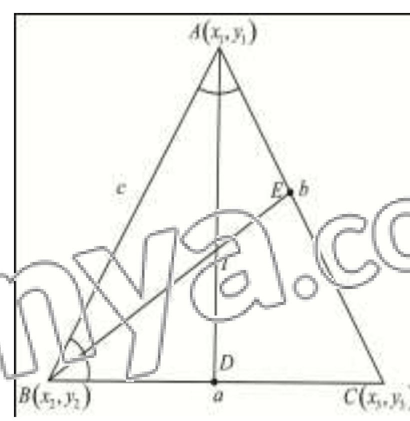
Let the bisector of $\angle A$ meets BC at D. Then D divides BC in the ratio $c:b$. Therefore coordinates of D are $\left(\frac{bx_2+cx_3}{b+c}, \frac{by_2+cy_3}{b+c}\right)$.

The bisector of $\angle B$ meets AD at I and I divides AD in the ratio $c:|BD|$

Now $\left|\frac{BD}{DC}\right| = \frac{c}{b}$ or $\left|\frac{DC}{BD}\right| = \frac{b}{c}$

or $\frac{|DC|+|BD|}{|BD|} = \frac{b+c}{c}$

or $\frac{a}{|BD|} = \frac{b+c}{c}$ or $|BD| = \frac{ac}{b+c}$



Thus I divides AD in the ratio $c:\frac{ac}{b+c}$ or in the ratio $b+c:a$

So, coordinates of I are

$$\left(\frac{(b+c)\frac{bx_2+cx_3}{b+c}+ax_1}{a+b+c}, \frac{(b+c)\frac{by_2+cy_3}{b+c}+ay_1}{a+b+c}\right)$$

i.e. $\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$

The symmetry of these co-ordinates shows that the bisector of $\angle C$ will also pass through this point. Thus the angle bisectors of a triangle are concurrent.

EXERCISE 4.1

Q.1 Describe the location in the plane of the point $P(x,y)$ for which

(i) $x>0$

Ans: The right half plane

(ii) $x>0$ and $y>0$

Ans: First quadrant

(iii) $x=0$

Ans: y-axis

(iv) $y=0$

Ans: x-axis

(v) $x<0$ and $y\geq 0$

Ans: Second quadrant and negative x-axis.

(vi) $x=y$

Ans: The set of points in the 1st and 3rd quadrants having equal abscissa and ordinates.

(vii) $|x|=|y|$

Ans: The set of points in 1st and 3rd quadrants having both the coordinates equal and the set of points in 2nd and 4th quadrants having both the coordinates equal but opposite in signs.

(viii) $|x|\geq 3$

Ans: Points on the x-axis having abscissa less than or equal to '–3' or greater than or equal to '3'.

(ix) $x>2$ and $y=2$

Ans: Points in the 1st quadrant with ordinate 2 and abscissa greater than 2.

(x) x and y have opposite signs.

Ans: Set of points in 2nd and 4th quadrants.

Q.2 Find in each of the following:

(i) The distance between the two given points

(ii) Midpoint of the line segment joining the two point

Solution:

(a) $A(3,1);\ B(-2,-4)$

$$\left|\overline{AB}\right|=\sqrt{(-2-3)^2+(-4-1)^2}$$

$$=\sqrt{(-5)^2+(-5)^2}$$

$$=\sqrt{25+25}$$

$$\left|\overline{AB}\right|=\sqrt{2\times 25}=5\sqrt{2}$$

Mid-point of

$$\overline{AB}=\left(\frac{3+(-2)}{2},\frac{1+(-4)}{2}\right)$$

$$=\left(\frac{1}{2},-\frac{3}{2}\right)$$

(b) $A(-8,3);B(2,-1)$

Solution:

$$\left|\overline{AB}\right|=\sqrt{(2-(-8))^2+(-1-3)^2}$$

$$=\sqrt{(2+8)^2+(-4)^2}$$

$$=\sqrt{100+16}$$

$$=\sqrt{116}$$

$$=\sqrt{4\times 29}$$

$$\left|\overline{AB}\right|=2\sqrt{29}$$

Mid-point of $\overline{AB}=\left(\frac{-8+2}{2},\frac{3+(-1)}{2}\right)$

$$=\left(\frac{-6}{2},\frac{2}{2}\right)$$

$$=(-3,1)$$

(c) $A\left(-\sqrt{5},-\frac{1}{3}\right);B\left(-3\sqrt{5},5\right)$

Solution:

$$\left|\overline{AB}\right|=\sqrt{\left(-3\sqrt{5}+\sqrt{5}\right)^2+\left(5+\frac{1}{3}\right)^2}$$

$$=\sqrt{\left(-2\sqrt{5}\right)^2+\left(\frac{15+1}{3}\right)^2}$$

$$=\sqrt{20+\frac{256}{9}}$$

$$=\sqrt{\frac{180+256}{9}}$$

$$=\sqrt{\frac{436}{9}}$$

$$=\sqrt{\frac{4\times 109}{9}}$$

$$\left|\overline{AB}\right|=\frac{2}{3}\sqrt{109}$$

Mid-point of

$$\overline{AB}=\left(\frac{-\sqrt{5}-3\sqrt{5}}{2},\frac{-\frac{1}{3}+5}{2}\right)$$

$$=\left(\frac{-4\sqrt{5}}{2},\frac{\frac{-1+15}{3}}{2}\right)$$

$$=\left(-2\sqrt{5},\frac{14}{6}\right)$$

$$=\left(-2\sqrt{5},\frac{7}{3}\right)$$

Q.3 Which of the following points are at a distance of 15 units from the origin?

(a) $\left(\sqrt{176},7\right)$

Solution:

Let $A\left(\sqrt{176},7\right)$, $O(0,0)$

$$|OA|=\sqrt{\left(\sqrt{176}-0\right)^2+(7-0)^2}$$

$$=\sqrt{176+49}$$

$$=\sqrt{225}$$

$$|OA|=15\text{ units}$$

Point $\left(\sqrt{176},7\right)$ is at distance of 15 units from origin.

(b) $(10,-10)$

Solution:

Let $A(10,-10)$, $O(0,0)$

$$|OA|=\sqrt{(10-0)^2+(-10-0)^2}$$

$$=\sqrt{100+100}$$

$$=\sqrt{200}$$

$$=\sqrt{100\times 2}$$

$$|OA|=10\sqrt{2}\neq 15$$

Point $(10,-10)$ is not at distance of 15 units from origin.

(c) $(1,15)$

Solution:

Let $A(1,15)$, $O(0,0)$

$$|OA|=\sqrt{(1-0)^2+(15-0)^2}$$

$$=\sqrt{1+225}$$

$$|OA|=\sqrt{226}\neq 15$$

Point $(1,15)$ is not at distance of 15 units from origin.

(d) $\left(\frac{15}{2},\frac{15}{2}\right)$

Solution:

Let $A\left(\frac{15}{2},\frac{15}{2}\right)$, $O(0,0)$

$$|OA|=\sqrt{\left(\frac{15}{2}-0\right)^2+\left(\frac{15}{2}-0\right)^2}$$

$$=\sqrt{\frac{225}{4}+\frac{225}{4}}$$

$$=\sqrt{\frac{450}{4}}$$

$$=\sqrt{\frac{225}{2}}$$

$$|OA|=\frac{15}{\sqrt{2}}\neq 15$$

Point $\left(\frac{15}{2},\frac{15}{2}\right)$ is not at distance of 15 units from origin.

Q.4 Show that

(i) The point $A(0,2), B(\sqrt{3},-1)$ and $C(0,-2)$ are vertices of a right triangle.

Solution:

$|\overline{AB}| = \sqrt{(\sqrt{3}-0)^2 + (-1-2)^2}$

$= \sqrt{3+9}$

$|\overline{AB}| = \sqrt{12}$

$|\overline{BC}| = \sqrt{(0-\sqrt{3})^2 + (-2+1)^2}$

$= \sqrt{3+1}$

$= \sqrt{4}$

$|\overline{BC}| = 2$

$|\overline{AC}| = \sqrt{(0-0)^2 + (-2-2)^2}$

$= \sqrt{0+(-4)^2}$

$= \sqrt{16}$

$|\overline{AC}| = 4$

Using Pythagoras theorem

$|\overline{AC}|^2 = |\overline{AB}|^2 + |\overline{BC}|^2$

$(4)^2 = (\sqrt{12})^2 + (2)^2$

$16 = 12 + 4$

$16 = 16$

Which is true

So, given points form a right triangle.

(ii) The points $A(3,1), B(-2,-3)$ and $C(2,2)$ are vertices of an isosceles triangle.

Solution:

$|\overline{AB}| = \sqrt{(-2-3)^2 + (-3-1)^2}$

$= \sqrt{(-5)^2 + (-4)^2}$

$= \sqrt{25+16}$

$|\overline{AB}| = \sqrt{41}$ …(i)

$|\overline{AC}| = \sqrt{(2-3)^2 + (2-1)^2}$

$= \sqrt{1+1}$

$|\overline{AC}| = \sqrt{2}$ …(ii)

$|\overline{BC}| = \sqrt{(2+2)^2 + (2+3)^2}$

$= \sqrt{(4)^2 + (5)^2}$

$= \sqrt{16+25}$

$|\overline{BC}| = \sqrt{41}$ …(iii)

By comparing (i) and (iii)

$|\overline{AB}| = |\overline{BC}|$

Since two sides of triangle are equal so, the triangle is isosceles.

(iii) The points $A(5,2), B(-2,3), C(-3,-4)$ and $D(4,-5)$ are vertices of a parallelogram. Is the parallelogram a square?

Solution:

$|\overline{AB}| = \sqrt{(-2-5)^2 + (3-2)^2}$

$= \sqrt{49+1}$

$|\overline{AB}| = \sqrt{50}$ …(i)

$|\overline{BC}| = \sqrt{(-3+2)^2 + (-4-3)^2}$

$= \sqrt{1+49}$

$|\overline{BC}| = \sqrt{50}$ …(ii)

$|\overline{CD}| = \sqrt{(4+3)^2 + (-5+4)^2}$

$= \sqrt{49+1}$

$|\overline{CD}| = \sqrt{50}$ …(iii)

$|\overline{AD}| = \sqrt{(4-5)^2 + (-5-2)^2}$

$= \sqrt{1+49}$

$\left|\overline{AD}\right| = \sqrt{50}$ …(iv)

By comparing (i), (ii), (iii) and (iv)

$\left|\overline{AB}\right| = \left|\overline{BC}\right| = \left|\overline{CD}\right| = \left|\overline{AD}\right|$

Since all four sides are equal so A, B, C and D are vertices of a parallelogram.

Now we find diagonals $\left|\overline{AC}\right|$ and $\left|\overline{BD}\right|$

$\left|\overline{AC}\right| = \sqrt{(-3-5)^2 + (-4-2)^2}$

$= \sqrt{64+36}$

$= \sqrt{100}$

$\left|\overline{AC}\right| = 10$ …(v)

$\left|\overline{BD}\right| = \sqrt{(4+2)^2 + (-5-3)^2}$

$= \sqrt{36+64}$

$= \sqrt{100}$

$\left|\overline{BD}\right| = 10$ …(vi)

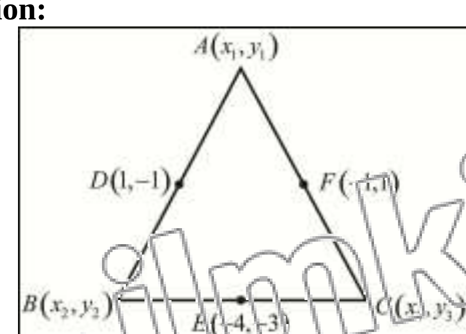
By comparing (v) and (vi)

$\left|\overline{AC}\right| = \left|\overline{BD}\right|$

As diagonals are also equal so the parallelogram is a square.

Q.5 The midpoints of the sides of a triangle are $(1,-1),(-4,-3)$ and $(-1,1)$ Find coordinates of the vertices of the triangle.

Solution:



Let $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of triangle.

Let $D(1,-1), E(-1,1)$ and $F(-4,-3)$ be the mid-points of sides $\left|\overline{AB}\right|, \left|\overline{BC}\right|$ and $\left|\overline{AC}\right|$ respectively.

As $D(1,-1)$ is the mid-point of $\overline{AB}$, So

$1 = \dfrac{x_1 + x_2}{2}$ $\quad -1 = \dfrac{y_1 + y_2}{2}$

$2 = x_1 + x_2$ $\quad -2 = y_1 + y_2$

$x_1 + x_2 = 2$ …(i) $\quad y_1 + y_2 = -2$ …(ii)

As $E(-4,-3)$ is the mid-point of $\overline{BC}$, So

$-4 = \dfrac{x_2 + x_3}{2}$ $\quad -3 = \dfrac{y_2 + y_3}{2}$

$-8 = x_2 + x_3$ $\quad -6 = y_2 + y_3$

$x_2 + x_3 = -8$ …(iii) $y_2 + y_3 = -6$ …(iv)

Now, as $F(-1,1)$ is the mid-point of $\overline{CA}$, So

$-1 = \dfrac{x_3 + x_1}{2}$ $\quad 1 = \dfrac{y_3 + y_1}{2}$

$-2 = x_3 + x_1$ $\quad 2 = y_3 + y_1$

$x_3 + x_1 = -2$ …(v) $\quad y_3 + y_1 = 2$ …(vi)

Adding (i), (iii) and (v)

$2x_1 + 2x_2 + 2x_3 = -8$

$x_1 + x_2 + x_3 = -4$ …(vii)

Adding (ii), (iv) and (vi)

$2y_1 + 2y_2 + 2y_3 = -6$

$y_1 + y_2 + y_3 = -3$ …(viii)

Putting $x_2 + x_3 = -8$ in equation (vii) and $y_2 + y_3 = -6$ in equation (viii), we get

$x_1 - 8 = -4$ $\quad y_1 - 6 = -3$

$x_1 = -4 + 8$ $\quad y_1 = -3 + 6$

$x_1 = 4$ $\quad y_1 = 3$

$\Rightarrow A(x_1, y_1) = A(4,3)$

Now putting $x_3 + x_1 = -2$ in equation (vii) and $y_3 + y_1 = 2$ in equation (viii), we get

$x_2 - 2 = -4$ $\qquad$ $y_2 + 2 = -3$

$x_2 = -4 + 2$ $\qquad$ $y_2 = -3 - 2$

$x_2 = -2$ $\qquad$ $y_2 = -5$

$\Rightarrow B(x_2, y_2) = B(-2, -5)$

Similarly, putting $x_1 + x_2 = 2$ in equation (vii) and $y_1 + y_2 = -2$ in equation (viii), we get

$x_3 + 2 = -4$ $\qquad$ $y_3 - 2 = -3$

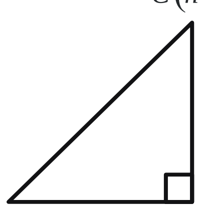
$x_3 = -4 - 2$ $\qquad$ $y_3 = -3 + 2$

$x_3 = -6$ $\qquad$ $y_3 = -1$

$\Rightarrow C(x_3, y_3) = C(-6, -1)$

Q.6 Find 'h' such that the points $A(\sqrt{3}, -1), B(0, 2)$ and $C(h, -2)$ are vertices of a right triangle with right angle at vertex A.

Solution:



$\left|\overline{AB}\right| = \sqrt{\left(\sqrt{3} - 0\right)^2 + (-1-2)^2}$

$= \sqrt{3 + 9}$

$\left|\overline{AB}\right| = \sqrt{12}$

$\left|\overline{AC}\right| = \sqrt{\left(h - \sqrt{3}\right)^2 + (-2+1)^2}$

$= \sqrt{h^2 + 3 - 2h\sqrt{3} + 1}$

$\left|\overline{AC}\right| = \sqrt{h^2 - 2\sqrt{3}h + 4}$

$\left|\overline{BC}\right| = \sqrt{(h-0)^2 + (-2-2)^2}$

$\left|\overline{BC}\right| = \sqrt{h^2 + 16}$

Using Pythagorean Theorem

$\left|\overline{BC}\right|^2 = \left|\overline{AB}\right|^2 + \left|\overline{AC}\right|^2$

$h^2 + 16 = 12 + h^2 - 2\sqrt{3}h + 4$

$2\sqrt{3}h = 16 - 16$

$2\sqrt{3}h = 0 \Rightarrow h = 0$

Q.7 Find 'h' such that $A(-1, h), B(3, 2)$ and $C(7, 3)$ are collinear.

Solution:

Since the points are collinear so

$$\begin{vmatrix} -1 & h & 1 \\ 3 & 2 & 1 \\ 7 & 3 & 1 \end{vmatrix} = 0$$

Expanding by 'R_1'

$$-1\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} - h\begin{vmatrix} 3 & 1 \\ 7 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & 2 \\ 7 & 3 \end{vmatrix} = 0$$

$-1(2-3) - h(3-7) + 1(9-14) = 0$

$-1(-1) - h(-4) + 1(-5) = 0$

$1 + 4h - 5 = 0$

$4h - 4 = 0$

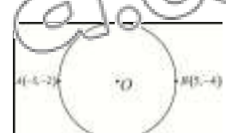
$4h = 4$

$h = 1$

Q.8 The points $A(-5, -2)$ and $B(5, -4)$ are ends of a diameter of a circle. Find the centre and radius of the circle.

Solution:

As centre O of circle is the mid-point of end points of diameter so applying mid-point formula on $A(-5, -2)$ and $B(5, -4)$



$\text{Centre} = O\left(\dfrac{5 + (-5)}{2}, \dfrac{-4 + (-2)}{2}\right)$

$= O\left(0, \dfrac{-6}{2}\right)$

$\text{Centre} = O(0, -3)$

Radius of circle is the distance from centre to any of the end point

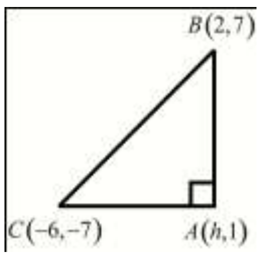
Radius $= |\overline{OA}| = \sqrt{(0+5)^2 + (-3+2)^2}$

$= \sqrt{25+(-1)^2}$

Radius $= \sqrt{26}$

Q.9 Find 'h' such that the points $A(h,1), B(2,7)$ and $C(-6,-7)$ are vertices of a right triangle with right angle at the vertex A.

Solution:



$|\overline{AB}| = \sqrt{(2-h)^2 + (7-1)^2}$

$= \sqrt{4+h^2-4h+36}$

$|\overline{AB}| = \sqrt{h^2-4h+40}$

$|\overline{AC}| = \sqrt{(-6-h)^2 + (-7-1)^2}$

$= \sqrt{36+h^2+12h+64}$

$|\overline{AC}| = \sqrt{h^2+12h+100}$

$|\overline{BC}| = \sqrt{(-6-2)^2 + (-7-7)^2}$

$= \sqrt{64+196}$

$|\overline{BC}| = \sqrt{260}$

Using Pythagorean Theorem

$|\overline{BC}|^2 = |\overline{AB}|^2 + |\overline{AC}|^2$

$260 = h^2 - 4h + 40 + h^2 + 12h + 100$

$2h^2 + 8h + 140 - 260 = 0$

$2h^2 + 8h - 120 = 0$

$h^2 + 4h - 60 = 0$ (Dividing by 2)

$h^2 + 10h - 6h - 60 = 0$

$h(h+10) - 6(h+10) = 0$

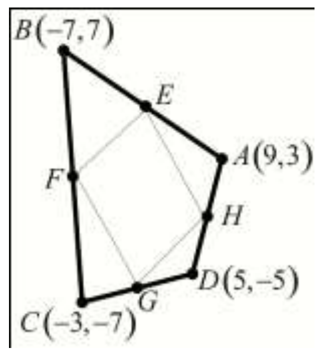
$(h+10)(h-6) = 0$

Either $h+10=0$ or $h-6=0$

$h=-10$ or $h=6$

Q.10 A quadrilateral has the points $A(9,3), B(-7,7), C(-3,-7)$ and $D(5,-5)$ as its vertices. Find the mid-points of its sides. Show that the figure formed by joining the mid-points consecutively is a parallelogram.

Solution:



Let E, F, G and H be the mid-points of the sides $\overline{AB}, \overline{BC}, \overline{CD}$ and $\overline{AD}$ respectively.

So by using mid-point formula

$E\left(\frac{9+(-7)}{2}, \frac{7+3}{2}\right) = E\left(\frac{2}{2}, \frac{10}{2}\right)$

$= E(1,5)$

$F\left(\frac{-7+(-3)}{2}, \frac{7+(-7)}{2}\right) = F\left(-\frac{10}{2}, \frac{0}{2}\right)$

$= F(-5,0)$

$G\left(\frac{-3+5}{2}, \frac{-7+(-5)}{2}\right) = G\left(\frac{2}{2}, -\frac{12}{2}\right)$

$= G(1,-6)$

$H\left(\frac{9+5}{2}, \frac{3+(-5)}{2}\right) = H\left(\frac{14}{2}, -\frac{2}{2}\right)$

$= H(7,-1)$

Now by distance formula

$|\overline{EF}| = \sqrt{(-5-1)^2 + (0-5)^2}$

$= \sqrt{36+25}$

$|\overline{EF}| = \sqrt{61}$ ...(i)

$|\overline{FG}| = \sqrt{(1+5)^2 + (-6-0)^2}$

$= \sqrt{36+36}$

$= \sqrt{2\times 36}$

$|\overline{FG}| = 6\sqrt{2}$ ...(ii)

$|\overline{GH}| = \sqrt{(7-1)^2 + (-1+6)^2}$

$= \sqrt{36+25}$

$|\overline{GH}| = \sqrt{61}$ ...(iii)

$|\overline{EH}| = \sqrt{(7-1)^2 + (-1-5)^2}$

$= \sqrt{36+36}$

$= \sqrt{2\times 36}$

$|\overline{EH}| = 6\sqrt{2}$ ...(iv)

By comparing (i) and (iii)

$|\overline{EF}| = |\overline{GH}|$

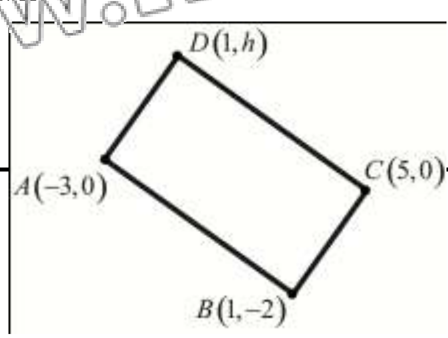
By comparing (ii) and (iv)

$|\overline{FG}| = |\overline{EH}|$

As the figure formed by joining the mid-points consecutively have opposite sides equal so it is a parallelogram.

Q.11 Find '*h*' such that the quadrilateral with vertices $A(-3,0), B(1,-2), C(5,0)$ and $D(1,h)$ is a parallelogram. Is it a square?

Solution:



As quadrilateral is a parallelogram so

$|\overline{AB}| = |\overline{CD}|$

$\sqrt{(1+3)^2 + (-2-0)^2} = \sqrt{(1-5)^2 + (h-0)^2}$

$\sqrt{16+4} = \sqrt{16+h^2}$

$\sqrt{20} = \sqrt{16+h^2}$

$20 = 16 + h^2$

$h^2 = 4$

$h = \pm 2$

$h = -2$ will make $B(1,-2)$ and $D(1,h)$ identical so $h \neq -2$

Hence $h = 2$

Now we find diagonals

$|\overline{AC}|$ and $|\overline{BD}|$

$|\overline{AC}| = \sqrt{(5+3)^2 + (0-0)^2} = \sqrt{64} = 8$

$|\overline{BD}| = \sqrt{(1-1)^2 + (2+2)^2} = \sqrt{16} = 4$

As $|\overline{AC}| \neq |\overline{BD}|$

so $ABCD$ is not a square.

Q.12 If two vertices of an equilateral triangle are $A(-3,0)$ and $B(3,0)$, find the third vertex. How many of these triangles are possible?

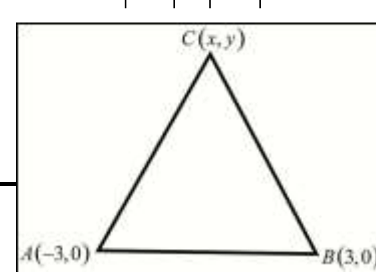
Solution:

Let third vertex be $C(x,y)$.

As ABC is an equilateral triangle so,

$|\overline{AB}| = |\overline{BC}| = |\overline{CA}|$

$\Rightarrow |\overline{AB}| = |\overline{BC}|$



$$\sqrt{(3+3)^2+(0-0)^2}=\sqrt{(x-3)^2+(y-0)^2}$$
$$\sqrt{36+0}=\sqrt{x^2-6x+9+y^2}$$
$$36=x^2+y^2-6x+9$$
$$x^2+y^2-6x=27\ldots\text{(i)}$$
$$|\overline{AB}|=|\overline{CA}|$$
$$\sqrt{(3+3)^2+(0-0)^2}=\sqrt{(x+3)^2+(y-0)^2}$$
$$\sqrt{36+0}=\sqrt{x^2+6x+9+y^2}$$
$$36=x^2+y^2+6x+9$$
$$x^2+y^2+6x=27\ldots\text{(ii)}$$

Subtracting (i) and (ii)

$$x^2+y^2-6x=27$$
$$\pm x^2\pm y^2\pm 6x=\pm 27$$
$$12x=0$$
$$x=0$$

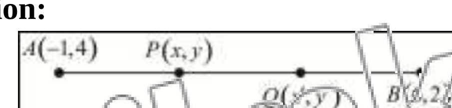
Putting $x=0$ in equation (i)

$$0+y^2-0=27$$
$$y^2=27$$
$$y=\pm 3\sqrt{3}$$

So, vertex C can be $\left(0,3\sqrt{3}\right)$ or $\left(0,-3\sqrt{3}\right)$ and hence two triangles are possible.

Q.13 Find the points trisecting the join of $A(-1,4)$ and $B(6,2)$.

Solution:



Let $P(x,y)$ and $Q(x',y')$ be the points which trisect the join of $A(-1,4)$ and $B(6,2)$.

As $P(x,y)$ divides the join of $A(-1,4)$ and $B(6,2)$ in the ratio $k_1:k_2=1:2$ so,

By using ratio formula

$$x=\frac{k_1x_2+k_2x_1}{k_1+k_2}\quad,\quad y=\frac{k_1y_2+k_2y_1}{k_1+k_2}$$
$$=\frac{1.6+2(-1)}{1+2},\qquad =\frac{1.2+2.4}{1+2}$$
$$x=\frac{4}{3}\quad,\quad y=\frac{10}{3}$$

So $P(x,y)=P\left(\frac{4}{3},\frac{10}{3}\right)$

As $Q(x',y')$ is the mid-point of $P(x,y)$ and $B(6,2)$ so,

$$x'=\frac{x+x_2}{2}\quad,\quad y'=\frac{y+y_2}{2}$$
$$=\frac{\frac{4}{3}+6}{2}\quad,\quad =\frac{\frac{10}{3}+2}{2}$$
$$=\frac{\frac{4+18}{3}}{2}\quad,\quad =\frac{\frac{10+6}{3}}{2}$$
$$=\frac{22}{6}\quad,\quad =\frac{16}{6}$$
$$x'=\frac{11}{3}\quad,\quad y'=\frac{8}{3}$$

So $Q(x,y)=Q\left(\frac{11}{3},\frac{8}{3}\right)$

Q.14 Find the point three fifth of the way along the line segment from $A(-5,8)$ to $B(5,3)$

Solution:

Let $P(x,y)$ be the required point the point $P(x,y)$ divides the line segment from $A(-5,8)$ to $B(5,3)$ in ratio 3:2.

So, by using ratio formula

$$x = \frac{k_1 x_2 + k_2 x_1}{k_1 + k_2} \quad , \quad y = \frac{k_1 y_2 + k_2 y_1}{k_1 + k_2}$$

$$= \frac{3.5 + 2(-5)}{3+2} \quad , \quad = \frac{3.3 + 2.8}{3+2}$$

$$= \frac{15 - 10}{5} \quad , \quad = \frac{9+16}{5}$$

$$x = \frac{5}{5} = 1 \quad , \quad y = \frac{25}{5} = 5$$

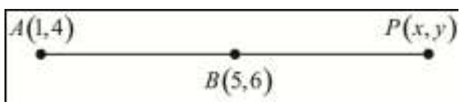
So, $P(x,y) = P(1,5)$

Q.15 Find the point P on the join of $A(1,4)$ and $B(5,6)$ that is twice as far from A as B is from A and lies

(i) On the same side of A as B does

Solution:

Let $P(x,y)$ be the required point



As P is twice as far from A as B is from A so 'B' is mid-point of A and P.

$$\frac{1+x}{2} = 5 \quad , \quad \frac{4+y}{2} = 6$$

$$1 + x = 10 \quad , \quad 4 + y = 12$$

$$x = 9 \quad , \quad y = 8$$

So, $P(x,y) = P(9,8)$

(ii) On the opposite side of A as B does.

Solution:

As point $P(x,y)$ is twice as far from A as B is from A so $\overline{AP}$ is double of $\overline{AB}$, so externally

P divides $\overline{AB}$ externally in the ratio $AP : PB = 2 : 3$

By using ratio formula

$$x = \frac{k_1 x_2 - k_2 x_1}{k_1 - k_2} \quad , \quad y = \frac{k_1 y_2 - k_2 y_1}{k_1 - k_2}$$

$$= \frac{2.5 - 3.1}{2-3} \quad , \quad = \frac{2.6 - 3.4}{2-3}$$

$$= \frac{10-3}{-1} \quad , \quad = \frac{12-12}{-1}$$

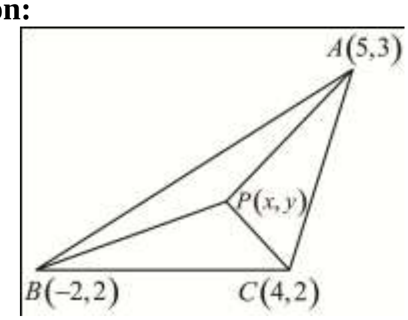
$$= \frac{7}{-1} \quad , \quad = \frac{0}{-1}$$

$$= -7 \quad , \quad = 0$$

So, $P(x,y) = P(-7,0)$

Q.16 Find the point which is equidistant from the points $A(5,3)$, $B(-2,2)$ and $C(4,2)$. What is the radius of the circum-circle of ΔABC?

Solution:



Let $P(x,y)$ be the point which is equidistant from $A(5,3)$, $B(-2,2)$ and $C(4,2)$, then

$$|\overline{PA}| = |\overline{PB}| = |\overline{PC}|$$

$$\Rightarrow |\overline{PA}| = |\overline{PB}|$$

$$\sqrt{(x-5)^2 + (y-3)^2} = \sqrt{(x+2)^2 + (y-2)^2}$$

$$\sqrt{x^2 - 10x + 25 + y^2 - 6y + 9} = \sqrt{x^2 + 4x + 4 + y^2 - 4y + 4}$$

$$\sqrt{x^2 + y^2 - 10x - 6y + 34} = \sqrt{x^2 + y^2 + 4x - 4y + 8}$$

$$x^2 + y^2 - 10x - 6y + 34 = x^2 + y^2 + 4x - 4y + 8$$

$$-10x - 4x - 6y + 4y = 8 - 34$$

$$-14x - 2y = -26$$

$$14x + 2y = 26$$

$$7x + y = 13 \ldots \text{(i)}$$

$$\Rightarrow \left|\overline{PA}\right| = \left|\overline{PC}\right|$$

$$\sqrt{(x-5)^2+(y-3)^2} = \sqrt{(x-4)^2+(y-2)^2}$$

$$\sqrt{x^2-10x+25+y^2-6y+9} =$$

$$\sqrt{x^2-8x+16+y^2-4y+4}$$

$$\sqrt{x^2+y^2-10x-6y+34} =$$

$$\sqrt{x^2+y^2-8x-4y+20}$$

$$x^2+y^2-10x-6y+34 = x^2+y^2-8x-4y+20$$

$$-10x+8x-6y+4y = 20-34$$

$$-2x-2y = -14$$

$$x+y = 7 \quad \ldots\text{(ii)}$$

Subtracting (i) and (ii)

$$7x+y-x-y = 13-7$$

$$6x = 6$$

$$x = 1$$

Putting in equation (ii)

$$7(1)+y = 13$$

$$y = 13-7$$

$$y = 6$$

So, $P(x,y) = P(1,6)$

Radius of circum-circle is

$$\left|\overline{PA}\right| = \left|\overline{PB}\right| = \left|\overline{PC}\right|$$

So,

Radius$=\left|\overline{PA}\right| = \sqrt{(1-5)^2+(6-3)^2}$

$$= \sqrt{16+9}$$

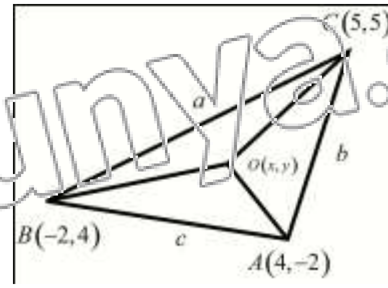
$$= \sqrt{25}$$

$$= 5$$

So, radius of circum-circle is '5'.

Q.17 The points $(4,-2),(-2,4)$ and $(5,5)$ are vertices of a triangle. Find in-centre of the triangle.

Solution:



Consider ΔABC with $A(4,-2), B(-2,4)$ and $C(5,5)$ as its vertices, and 'O' its in-centre.

We find $\left|\overline{AB}\right| = c, \left|\overline{BC}\right| = a, \left|\overline{AC}\right| = b$ by using distance formula.

$$a = \left|\overline{BC}\right| = \sqrt{(5+2)^2+(5-4)^2}$$

$$= \sqrt{49+1}$$

$$= \sqrt{50}$$

$$a = 5\sqrt{2}$$

$$b = \left|\overline{AC}\right| = \sqrt{(5-4)^2+(5+2)^2}$$

$$= \sqrt{1+49}$$

$$= \sqrt{50}$$

$$b = 5\sqrt{2}$$

$$c = \left|\overline{AB}\right| = \sqrt{(-2-4)^2+(4+2)^2}$$

$$= \sqrt{36+36} = \sqrt{2\times 36}$$

$$c = 6\sqrt{2}$$

Using in-centre formula

$$x = \frac{ax_1+bx_2+cx_3}{a+b+c}$$

$$= \frac{5\sqrt{2}(4)+5\sqrt{2}(-2)+6\sqrt{2}(5)}{5\sqrt{2}+5\sqrt{2}+6\sqrt{2}}$$

$$x = \frac{\sqrt{2}(20-10+30)}{\sqrt{2}(16)} = \frac{40}{16} = \frac{5}{2}$$

$$y = \frac{ay_1+by_2+cy_3}{a+b+c}$$

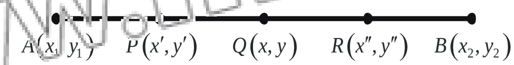
$$= \frac{5\sqrt{2}(-2)+5\sqrt{2}(4)+6\sqrt{2}(5)}{5\sqrt{2}+5\sqrt{2}+6\sqrt{2}}$$

$$y = \frac{\sqrt{2}(-10+20+30)}{\sqrt{2}(16)} = \frac{40}{16} = \frac{5}{2}$$

So in-centre is $\left(\frac{5}{2}, \frac{5}{2}\right)$

Q.18 Find the points that divide the line segments joining $A(x_1, y_1)$ and $B(x_2, y_2)$ into four equal parts.

Solution:



Let $P(x', y'), Q(x, y)$ and $R(x'', y'')$ be the points which divide line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ into four equal parts.

As $Q(x, y)$ is the mid-point of $A(x_1, y_1)$ and $B(x_2, y_2)$ so

$$Q(x, y) = Q\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

From the figure, $P(x', y')$ is the mid-point of $A(x_1, y_1)$ and $Q\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$$x' = \frac{x_1 + \frac{x_1+x_2}{2}}{2}, \quad y' = \frac{y_1 + \frac{y_1+y_2}{2}}{2}$$

$$= \frac{\frac{2x_1+x_1+x_2}{2}}{2}, \quad = \frac{\frac{2y_1+y_1+y_2}{2}}{2}$$

$$= \frac{3x_1+x_2}{4}, \quad = \frac{3y_1+y_2}{4}$$

So $P(x', y') = P\left(\frac{3x_1+x_2}{4}, \frac{3y_1+y_2}{4}\right)$

From the figure $R(x'', y'')$ is the mid-point of $Q\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ and $B(x_2, y_2)$

$$x'' = \frac{\frac{x_1+x_2}{2} + x_2}{2}, \quad y'' = \frac{\frac{y_1+y_2}{2} + y_2}{2}$$

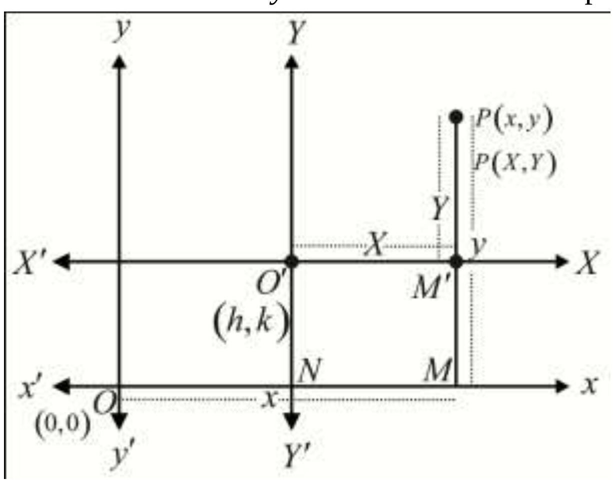
$$= \frac{\frac{x_1+x_2+2x_2}{2}}{2}, \quad = \frac{\frac{y_1+y_2+2y_2}{2}}{2}$$

$$= \frac{x_1 + 3x_2}{4} \quad , \quad = \frac{y_1 + 3y_2}{4}$$

So $R(x'', y'') = R\left(\frac{x_1 + 3x_2}{4}, \frac{y_1 + 3y_2}{4}\right)$

Translation of axes:

In translation of axes, origin is shifted to another point in the plane but the axes remain parallel to the old axes. Let P be the point with co-ordinates (x, y) referred to $xy-$ coordinates system and the axes be translated through the point $O'(h,k)$ and $O'X$ and $O'Y$ be the new axes. If P has coordinates (X, Y) referred to the new axes, then we need to find X and Y in terms of x and y. Draw PM and $O'N$ perpendiculars to Ox.



From the figure

$OM = x$, $MP = y$, $ON = h$, $NO' = k = MM'$

Now $X = O'M' = NM = OM - ON = x - h$

$\Rightarrow X = x - h$

Similarly

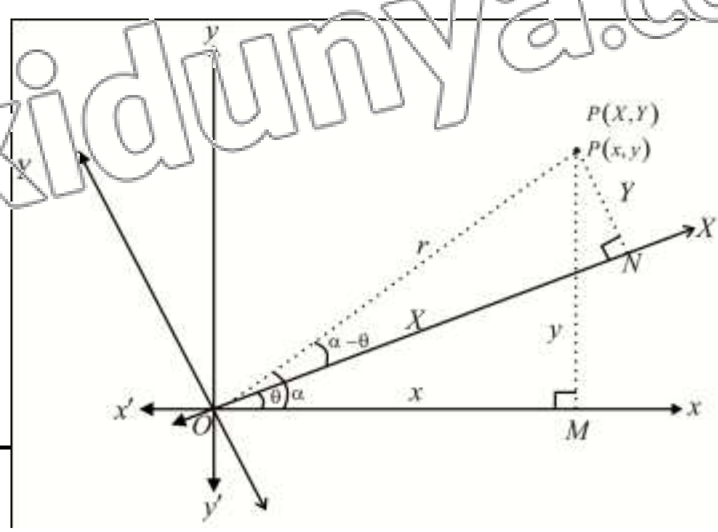
$Y = M'P = MP - MM' = y - k$

$\Rightarrow Y = y - k$

Moreover $x = X + h$, $y = Y + k$

Rotation of axes:

Let a point P have coordinates (x, y) referred to the xy-system coordinates (X, Y) referred to the xy-coordinate system. We have to find X, Y in terms of the given coordinates x, y. Let α be measure of the angle that OP makes with Ox.



From P, draw PM perpendicular to Ox and PM' perpendicular to OX.

Let $|OP| = r$.

In ΔOPM

$\cos\alpha = \frac{x}{r}$, $\sin\alpha = \frac{y}{r}$

$\Rightarrow x = r\cos\alpha$ (i) , $y = r\sin\alpha$ (ii)

In ΔOPN

$\cos(\alpha-\theta) = \frac{X}{r}$ Also, $\sin(\alpha-\theta) = \frac{Y}{r}$

$\Rightarrow X = r\cos(\alpha-\theta)$ $\Rightarrow Y = r\sin(\alpha-\theta)$

$X = r[\cos\alpha\cos\theta + \sin\alpha\sin\theta]$ $Y = r[\sin\alpha\cos\theta - \cos\alpha\sin\theta]$

$= (r\cos\alpha)\cos\theta + (r\sin\alpha)\sin\theta$ $Y = (r\sin\alpha)\cos\theta - (r\cos\alpha)\sin\theta$

$\Rightarrow X = x\cos\theta + y\sin\theta$ (Using (i) and (ii) $Y = y\cos\theta - x\sin\theta$

$X = x\cos\theta + y\sin\theta$ (1)

$Y = y\cos\theta - x\sin\theta$ (2)

Multiplying equation (1) with $\cos\theta$ and (2) with $\sin\theta$ and subtracting

$$X\cos\theta = x\cos^2\theta + y\sin\theta\cos\theta$$
$$\pm Y\sin\theta = \pm x\sin^2\theta \pm y\sin\theta\cos\theta$$
$$x\left(\cos^2\theta + \sin^2\theta\right) = X\cos\theta - Y\sin\theta$$
$$x = X\cos\theta - Y\sin\theta$$

Multiplying equation (1) with $\sin\theta$ and equation (2) with $\cos\theta$ and adding

$$X\sin\theta = x\sin\theta\cos\theta + y\sin^2\theta$$
$$\pm Y\cos\theta = \pm x\sin\theta\cos\theta \pm y\cos^2\theta$$
$$y\left(\sin^2\theta + \cos^2\theta\right) = X\sin\theta + Y\cos\theta$$
$$y = X\sin\theta + Y\cos\theta$$

So the transformation equations are

$X = x\cos\theta + y\sin\theta \qquad x = X\cos\theta - Y\sin\theta$

$Y = y\cos\theta - x\sin\theta \qquad y = X\sin\theta + Y\cos\theta$

EXERCISE 4.2

Q.1 The two points P and O' are given in xy-coordinate system. Find the XY-coordinates of P referred to the translated axes $O'X$ and $O'Y$.

(i) $P(3,2); O'(1,3)$

Solution:

$P(x,y) = P(3,2), O'(h,k) = O'(1,3)$

$\Rightarrow x = 3\ ,\ y = 2\ ,\ \Rightarrow h = 1\ ,\ k = 3$

Coordinates of P referred to the new axes are (X,Y) given by

$X = x - h \quad , \quad Y = y - k$

$= 3 - 1 \quad , \quad = 2 - 3$

$X = 2 \quad , \quad Y = -1$

So, $P(X,Y) = P(2,-1)$

(ii) $P(-2,6); O'(-3,2)$

Solution:

$P(x,y) = P(-2,6), O'(h,k) = O'(-3,2)$

$\Rightarrow x = -2\ ,\ y = 6, \Rightarrow h = -3\ ,\ k = 2$

Coordinates of P referred to the new axes are (X,Y) given by

$X = x - h \quad , \quad Y = y - k$

$= -2 - (-3) \quad , \quad = 6 - 2$

$X = 1 \quad , \quad Y = 4$

So, $P(X,Y) = P(1,4)$

(iii) $P(-6,-8); O'(-4,-6)$

Solution:

$P(x,y) = P(-6,-8)$,

$O'(h,k) = O'(-4,-6)$

$\Rightarrow x = -6\ ,\ y = -8$,

$\Rightarrow h = -4\ ,\ k = -6$

Coordinates of P referred to the new axes are (X,Y) given by

$X = x - h \quad , \quad Y = y - k$

$= -6 - (-4) \quad , \quad = -8 - (-6)$

$X = -2 \quad , \quad Y = -2$

So, $P(X,Y) = P(-2,-2)$

(iv) $P\left(\frac{3}{2},\frac{5}{2}\right); O'\left(-\frac{1}{2},\frac{7}{2}\right)$

Solution:

$P(x,y) = P\left(\frac{3}{2},\frac{5}{2}\right)$,

$O'(h,k) = O'\left(-\frac{1}{2},\frac{7}{2}\right)$

$\Rightarrow x = \frac{3}{2}\ ,\ y = \frac{5}{2}, \Rightarrow h = -\frac{1}{2}\ ,\ k = \frac{7}{2}$

Coordinates of P referred to the new axes are (X,Y) given by

$X = x - h \quad , \quad Y = y - k$

$= \frac{3}{2} - \left(\frac{-1}{2}\right) \quad , \quad = \frac{5}{2} - \frac{7}{2}$

$= \frac{3+1}{2} = \frac{4}{2} \quad , \quad = \frac{5-7}{2} = \frac{-2}{2}$

$X = 2 \quad , \quad Y = -1$

So, $P(X,Y) = P(2,-1)$

Q.2 The xy-coordinate axes are translated through the point O' whose coordinates are given in xy-coordinate system. The coordinates of P are given in the XY-coordinate system. Find the

coordinates of P in xy-coordinate system.

(i) $P(8,10);O'(3,4)$

Solution:

$P(X,Y)=P(8,10)$,

$O'(h,k)=O'(3,4)$

$\Rightarrow X=8\ ,\ Y=10,\Rightarrow h=3\ ,\ k=4$

We have

$x=X+h \qquad , \qquad y=Y+k$

$\quad =8+3 \qquad , \qquad =10+4$

$x=11 \qquad , \qquad y=14$

So, $P(x,y)=P(11,14)$

(ii) $P(-5,-3);O'(-2,-6)$

Solution:

$P(X,Y)=P(-5,-3)$,

$O'(h,k)=O'(-2,-6)$

$\Rightarrow X=-5\ ,\ Y=-3$,

$\Rightarrow h=-2\ ,\ k=-6$

We have

$x=X+h \qquad , \qquad y=Y+k$

$\quad =-5+(-2) \qquad , \qquad =-3+(-6)$

$x=-7 \qquad , \qquad y=-9$

So, $P(x,y)=P(-7,-9)$

(iii) $P\left(-\frac{3}{4},-\frac{7}{6}\right);O'\left(\frac{1}{4},-\frac{1}{6}\right)$

Solution:

$P(X,Y)=P\left(-\frac{3}{4},-\frac{7}{6}\right)$

$\Rightarrow X=-\frac{3}{4}\ ,\ Y=-\frac{7}{6}$

And $O'(h,k)=O'\left(\frac{1}{4},-\frac{1}{6}\right)$

$\Rightarrow h=\frac{1}{4}\ ,\ k=-\frac{1}{6}$

We have

$x=X+h \qquad , \qquad y=Y+k$

$\quad =\frac{-3}{4}+\frac{1}{4} \qquad , \qquad =\frac{-7}{6}+\left(\frac{-1}{6}\right)$

$\quad =\frac{-2}{4} \qquad , \qquad =\frac{-7-1}{6}$

$x=\frac{-1}{2} \qquad , \qquad y=\frac{-8}{6}=-\frac{4}{3}$

So, $P(x,y)=P\left(\frac{-1}{2},\frac{-4}{3}\right)$

(iv) $P(4,-3);O'(-2,3)$

Solution:

$P(X,Y)=P(4,-3),O'(h,k)=O'(-2,3)$

$\Rightarrow X=4\ ,\ Y=-3,\Rightarrow h=-2\ ,\ k=3$

We have

$x=X+h \qquad , \qquad y=Y+k$

$\quad =4+(-2) \qquad , \qquad =-3+3$

$x=2 \qquad , \qquad y=0$

So, $P(x,y)=P(2,0)$

Q.3 The xy-coordinate axes are rotated about the origin through the indicated angle. The new axes are OX and OY. Find the XY-coordinates of the point P with the given xy-coordinates.

(i) $P(5,3);\theta=45^\circ$

Solution:

Let (X,Y) be the coordinates of P referred to the XY-axes.

$P(x,y)=P(-5,-3),\ \theta=45^\circ$

$\Rightarrow x=-5\ ,\ y=-3$

We have

$X=x\cos\theta+y\sin\theta \quad ,$

$\quad =5\cos 45^\circ+3\sin 45^\circ$

$$= 5\left(\frac{1}{\sqrt{2}}\right)+3\left(\frac{1}{\sqrt{2}}\right)$$

$$X = \frac{5+3}{\sqrt{2}} = \frac{8}{\sqrt{2}} = \frac{4\times 2}{\sqrt{2}} = 4\sqrt{2}$$

$$Y = -x\sin\theta + y\cos\theta$$

$$= -5\sin 45^\circ + 3\cos 45^\circ$$

$$= -5\left(\frac{1}{\sqrt{2}}\right)+3\left(\frac{1}{\sqrt{2}}\right)$$

$$Y = \frac{-5+3}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$$

So, $P(X,Y) = P\left(4\sqrt{2}, -\sqrt{2}\right)$

(ii) $P(3,-7), \theta = 30^\circ$

Solution:

Let (X,Y) be the coordinates of P referred to the XY-axes.

$P(x,y) = P(3,-7),\ \theta = 30^\circ$

$\Rightarrow x = 3,\ y = -7$

We have

$$X = x\cos\theta + y\sin\theta$$

$$= 3\cos 30^\circ - 7\sin 30^\circ$$

$$X = 3\left(\frac{\sqrt{3}}{2}\right) - 7\left(\frac{1}{2}\right) = \frac{3\sqrt{3}-7}{2}$$

$$Y = -x\sin\theta + y\cos\theta$$

$$= -3\sin 30^\circ - 7\cos 30^\circ$$

$$Y = -3\left(\frac{1}{2}\right) - 7\left(\frac{\sqrt{3}}{2}\right) = \frac{-3-7\sqrt{3}}{2}$$

So

$$P(X,Y) = P\left(\frac{3\sqrt{3}-7}{2}, \frac{-3-7\sqrt{3}}{2}\right)$$

(iii) $P(11,-15); \theta = 60^\circ$

Solution:

Let (X,Y) be the coordinates of P referred to the XY-axes.

$P(x,y) = P(11,-15),\ \theta = 60^\circ$

$\Rightarrow x = 11,\ y = -15$

We have

$$X = x\cos\theta + y\sin\theta$$

$$= 11\cos 60^\circ + (-15)\sin 60^\circ$$

$$X = 11\left(\frac{1}{2}\right) - 15\left(\frac{\sqrt{3}}{2}\right) = \frac{11-15\sqrt{3}}{2}$$

$$Y = -x\sin\theta + y\cos\theta$$

$$= -11\sin 60^\circ + (-15)\cos 60^\circ$$

$$Y = -11\left(\frac{\sqrt{3}}{2}\right) - 15\left(\frac{1}{2}\right) = \frac{-11\sqrt{3}-15}{2}$$

So

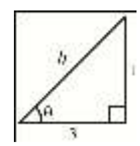
$$P(X,Y) = P\left(\frac{11-15\sqrt{3}}{2}, \frac{-11\sqrt{3}-15}{2}\right)$$

(iv) $P(15,10); \theta = \arctan\frac{1}{3}$

Solution:

$P(x,y) = P(15,10),\ \theta = \arctan\frac{1}{3}$

$\Rightarrow x = 15,\ y = 10$



Also $\tan\theta = \frac{1}{3}$

By Pythagorean Theorem

$$(h)^2 = (1)^2 + (3)^2$$

$$h^2 = 1 + 9$$

$$h^2 = 10$$

$$h = \sqrt{10}$$

$$\therefore \sin\theta = \frac{1}{\sqrt{10}},\ \cos\theta = \frac{3}{\sqrt{10}}$$

We have

$$X = x\cos\theta + y\sin\theta$$

$$= 15\left(\frac{3}{\sqrt{10}}\right) + 10\left(\frac{1}{\sqrt{10}}\right)$$

$$X = \frac{45+10}{\sqrt{10}} = \frac{55}{\sqrt{10}}$$

$$Y = -x\sin\theta + y\cos\theta$$

$$= -15\left(\frac{1}{\sqrt{10}}\right) + 10\left(\frac{3}{\sqrt{10}}\right)$$

$$Y = \frac{-15+30}{\sqrt{10}} = \frac{15}{\sqrt{10}}$$

So, $P(X,Y)=P\left(\dfrac{55}{\sqrt{10}},\dfrac{15}{\sqrt{10}}\right)$

Q.4 The *xy*-coordinate axes are rotated about the origin through the indicated angle and the new axes are *OX* and *OY*. Find the *xy*-coordinates of *P* with the given *XY*-coordinates.

(i) $P(-5,3);\theta=30^\circ$

Solution:

$P(X,Y)=P(-5,3),\ \theta=30^\circ$

$\Rightarrow X=-5,\ Y=3$

We have

$x=X\cos\theta-Y\sin\theta$

$=-5\cos30^\circ-3\sin30^\circ$

$=-5\left(\dfrac{\sqrt{3}}{2}\right)-3\left(\dfrac{1}{2}\right)=\dfrac{-5\sqrt{3}-3}{2}$

$y=X\sin\theta+Y\cos\theta$

$=-5\sin30^\circ+3\cos30^\circ$

$=-5\left(\dfrac{1}{2}\right)+3\left(\dfrac{\sqrt{3}}{2}\right)=\dfrac{-5+3\sqrt{3}}{2}$

So, $P(x,y)=P\left(\dfrac{-5\sqrt{3}-3}{2},\dfrac{3\sqrt{3}-5}{2}\right)$

(ii) $P\left(-7\sqrt{2},5\sqrt{2}\right);\theta=45^\circ$

Solution:

$P(X,Y)=P\left(-7\sqrt{2},5\sqrt{2}\right),\ \theta=45^\circ$

$\Rightarrow X=-7\sqrt{2},\ Y=5\sqrt{2}$

We have

$x=X\cos\theta-Y\sin\theta\ =-7\sqrt{2}\cos45^\circ-5\sqrt{2}\sin45^\circ$

$=-7\sqrt{2}\left(\dfrac{1}{\sqrt{2}}\right)-5\sqrt{2}\left(\dfrac{1}{\sqrt{2}}\right)$

$=-7-5=-12$

$y=X\sin\theta+Y\cos\theta$

$=-7\sqrt{2}\sin45^\circ+5\sqrt{2}\cos45^\circ$

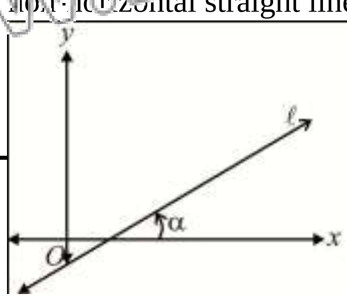
$=-7\sqrt{2}\left(\dfrac{1}{\sqrt{2}}\right)+5\sqrt{2}\left(\dfrac{1}{\sqrt{2}}\right)$

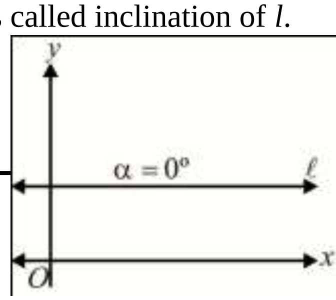
$=-7+5=-2$

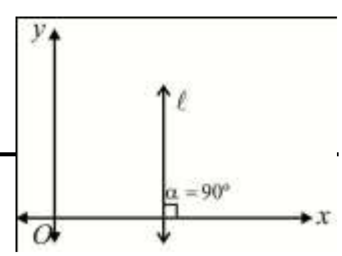
So, $P(x,y)=P(-12,-2)$

<u>Inclination of a line:</u>

The angle $\alpha\,(0^\circ<\alpha<180^\circ)$ measured anti-clockwise from positive *x*-axis to a non-horizontal straight line l is called inclination of l.







Note:

(i) If ℓ is parallel to x-axis then $\alpha = 0°$.

(ii) If ℓ is parallel to y-axis then $\alpha = 90°$.

Slope or gradient of a line:

The slope or gradient of a non-vertical line ℓ with α as its inclination is defined by $m = \tan\alpha$

Note:

(i) If ℓ is horizontal then $m = 0$.

(ii) If ℓ is vertical then m is undefined.

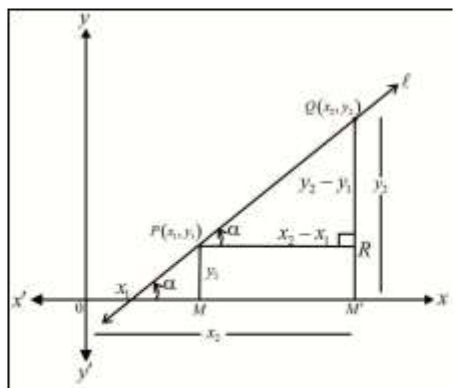
(iii) If $0 < \alpha < 90°$ then m is positive.

(iv) If $90° < \alpha < 180°$ then m is negative.

Slope or Gradient of a straight line joining two points:

If a non-vertical line ℓ with inclination α passes through two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, then slope m is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Proof:



Let m be the slope of the line ℓ. Draw perpendiculars PM and QM' on $x-axis$ and a perpendicular PR on QM'. Then $m\angle RPQ = \alpha$

$m\overline{PR} = m\overline{MM'} = m\overline{OM'} - m\overline{OM} = x_2 - x_1$ and

$m\overline{RQ} = m\overline{M'Q} - m\overline{M'R} = y_2 - m\overline{MP} = y_2 - y_1$

In ΔPQR

$$\tan\alpha = \frac{QR}{PR}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

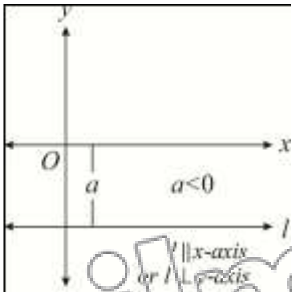
Parallel lines:

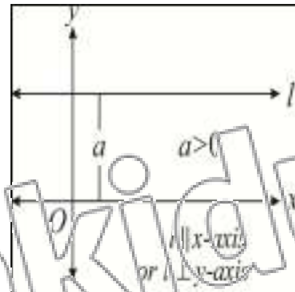
Two non-vertical lines ℓ_1 and ℓ_2 having slopes m_1 and m_2 respectively are parallel iff $m_1 = m_2$

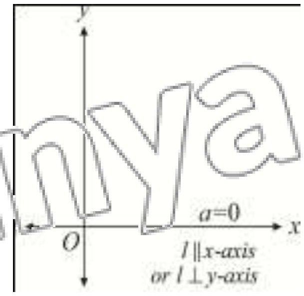
Perpendicular lines:

Two non-vertical lines ℓ_1 and ℓ_2 having slopes m_1 and m_2 will be perpendicular if $m_1 m_2 = -1$

Equation of Straight Line Parallel to x-axis: (or perpendicular to y-axis)







All the points on the line l parallel to x-axis remain at a constant distance (say a) from x-axis. Therefore, each point on the line has its distance from x-axis equal to a. which is its y-coordinate (ordinate). So, all the points on this line satisfy the equation: $y = a$

Note:

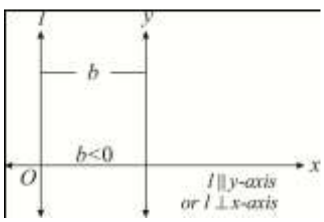
(i) If $a > 0$, then the value l is above the x-axis.

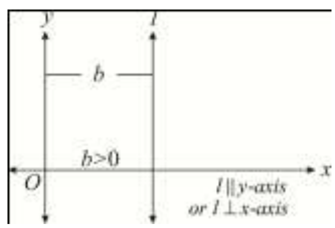
(ii) If $a < 0$, then then line l is below the x-axis.

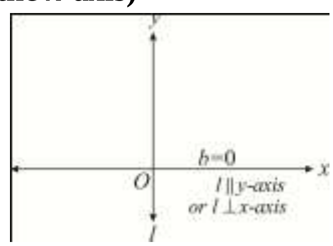
(iii) If $a = 0$, then the line l becomes the x-axis.

Thus the equation of x-axis is $y = 0$

Equation of Straight Line Parallel to *y*-axis: (or parallel to the *x*-axis)







All points on the line l parallel to y-axis remain at a constant distance (say b) from the y-axis. Each point on the line has its distance from the y-axis equal to b which is its x-coordinate (abscissa). So, all the points on this line satisfy the equation: $x = b$ which is an equation of the line l parallel to the y-axis (or perpendicular to the x-axis).

Note:

(i) If $b > 0$, then the line is on the right of the y-axis.

(ii) If $b < 0$, then the line is on the left of the y-axis.

(iii) If $b = 0$, then the line becomes the y-axis.

Thus the equation of y-axis is $x = 0$.

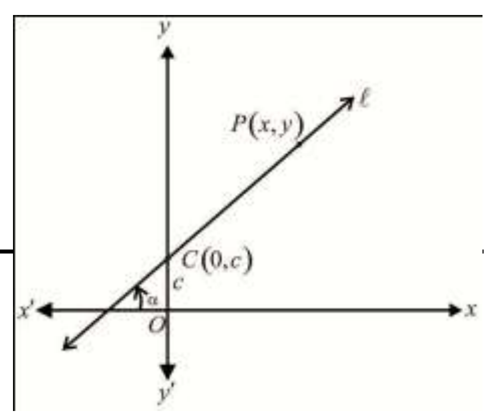
Standard Forms of Equations of Straight Lines:

(i) Slope-intercept form

Equation of a non-vertical line l with slope m and y-intercept c is $y = mx + c$

Proof:

Let $P(x, y)$ be any point of the straight line ℓ with slope m and $y-$intercept c. As $C(0, c)$ and $P(x, y)$ lie on the line, so slope of the line is



$$m=\frac{y_2-y_1}{x_2-x_1}$$
$$m=\frac{y-c}{x-0}$$
$$y-c=mx$$
$$y=mx+c$$

Note: If $c=0$ then the equation becomes $y=mx$ and the line passes through origin.

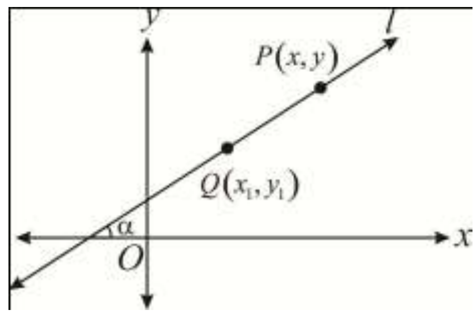
(ii) Point-Slope Form

Equation of a non-vertical line ℓ with slope m and passing through a point $Q(x_1,y_1)$ is

$$y-y_1=m(x-x_1)$$

Proof:

Let $P(x,y)$ be any point of the straight line with slope m and passing through $Q(x_1,y_1)$. As $Q(x_1,y_1)$ and $P(x,y)$ lie on the line, so slope of the line is



$$m=\frac{y_2-y_1}{x_2-x_1}$$
$$m=\frac{y-y_1}{x-x_1}$$
$$y-y_1=m(x-x_1)$$

(iii) Symmetric Form of Equation of a Straight Line

We have $m=\dfrac{y-y_1}{x-x_1}=\tan\alpha$, where α is the inclination of the line.

Or $\dfrac{x-x_1}{\cos\alpha}=\dfrac{y-y_1}{\sin\alpha}=r(\text{say})$

Proof:

By points slope form

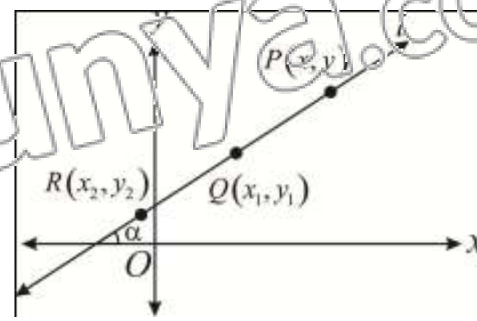
$$y-y_1=m(x-x_1)$$
$$y-y_1=\tan\alpha(x-x_1)$$
$$y-y_1=\frac{\sin\alpha}{\cos\alpha}(x-x_1)$$
$$\frac{y-y_1}{\sin\alpha}=\frac{x-x_1}{\cos\alpha}$$
$$\frac{x-x_1}{\cos\alpha}=\frac{y-y_1}{\sin\alpha}=r(\text{say})$$

This is called symmetric form of equation of the line.

(iv) Two Points Form

Equation of a non-vertical line ℓ passing through two points $Q(x_1,y_1)$ and $R(x_2,y_2)$ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$



Proof:

Let $P(x, y)$ be any point of the line, then

Slope of $PQ = \dfrac{y - y_1}{x - x_1}$

Slope of $QR = \dfrac{y_2 - y_1}{x_2 - x_1}$

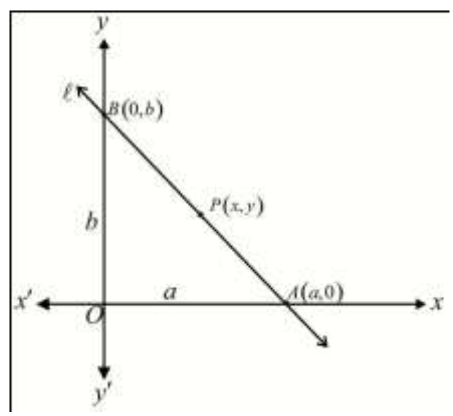
Since P, Q and R are collinear therefore

Slope of $PQ =$ slope of QR

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

(v) Intercept Form



Equation of line ℓ having non-zero x-intercept $= a$ and y-intercept $= b$ is $\dfrac{x}{a} + \dfrac{y}{b} = 1$

Proof:

Let $P(x, y)$ be any point of the line. Clearly $A(a, 0)$ and $B(0, b)$ lie on the required line,

So, by two points slope form

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - 0 = \frac{b - 0}{0 - a}(x - a)$$

$$y = \frac{-b}{a}(x - a)$$

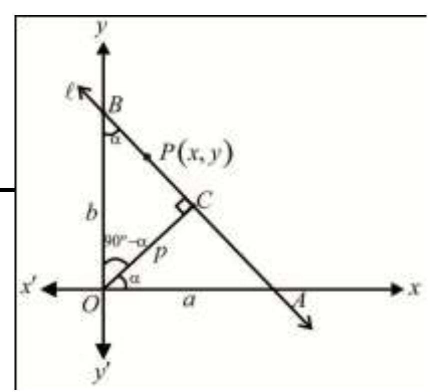
$$ay = -bx + ab$$

Dividing by ab

$$\frac{y}{b} = \frac{-x}{a} + 1$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1$$

(vi) Normal Form



An equation of non-vertical straight line ℓ, such that length of perpendicular from origin to ℓ is p and α is the inclination of this perpendicular, is $x\cos\alpha + y\sin\alpha = p$

Proof:

Let the line meet the $x-axis$ and $y-axis$ at the points A and B respectively. Let $P(x, y)$ be any point of the line and OC be perpendicular to the line ℓ with $|\overline{OC}| = p$.

In ΔAOC `

$$\cos\alpha = \frac{p}{a}$$

$$a = \frac{p}{\cos\alpha}$$

In ΔBOC

$$\sin\alpha = \frac{p}{b} \Rightarrow b = \frac{p}{\sin\alpha}$$

By two intercepts form of equation of straight lines

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\frac{x}{\frac{p}{\cos\alpha}} + \frac{y}{\frac{p}{\sin\alpha}} = 1$$

$$\frac{x\cos\alpha}{p} + \frac{y\sin\alpha}{p} = 1$$

$$\Rightarrow x\cos\alpha + y\sin\alpha = p$$

<u>A Linear Equation in Two Variables Represents a Straight Line</u>:

The linear equation in two variables represents a straight line. A linear equation in two variable *x* and *y* is $ax + by + c = 0$

Where *a*, *b* and *c* are constants and *a* and *b* are not simultaneously zero.

<u>To Transform the General Linear Equation in Standard Forms</u>:

To transform the equation $ax + by + c = 0$ in standard form:

(i) <u>Slope Intercept Form</u>:

$$y = -\frac{a}{b}x - \frac{c}{b}$$

(ii) <u>Point Slope Form</u>:

$$y = -\frac{a}{b}\left(x + \frac{c}{a}\right)$$

(iii) <u>Symmetric Form</u>:

$$\frac{x-\left(-\frac{c}{a}\right)}{\frac{b}{\pm\sqrt{a^2+b^2}}}=\frac{y-0}{\frac{a}{\pm\sqrt{a^2+b^2}}}=r\,(\text{say})$$

(iv) **Two Points Slope Form:**

$$y=-\frac{a}{b}\left(x+\frac{c}{a}\right)$$

(v) **Intercepts Form**

$$\frac{x}{-\frac{c}{a}}+\frac{y}{-\frac{c}{b}}=1$$

(vi) **Normal Form:**

$$\frac{ax+by}{\pm\sqrt{a^2+b^2}}=\frac{-c}{\pm\sqrt{a^2+b^2}}$$

Position of a Point With Respect to a Line:

Let $P(x_1, y_1)$ be a point in the plane not lying on the line ℓ

$\ell : ax+by+c=0$ (i)

with $b>0$ then P lies

(a) Above the line (i) if $ax_1+by_1+c>0$

(b) Below the line (i) if $ax_1+by_1+c<0$

Distance of a Point from a Line:

The distance d from a point $P(x_1, y_1)$ to the line of $l: ax+by+c=0$ is

$$d=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$$

Area of triangle:

Area of triangle with vertices $P(x_1, y_1), Q(x_2, y_2), R(x_3, y_3)$ is given by

$$\Delta=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Note:

(i) If the points P, Q and R are collinear, then $\Delta=0$

(ii) In numerical problems, if sign of area is negative, then it is to be omitted.