EXERCISE 4.5

Find the lines represented by each of the following and also find measure of the angle between them (problems 1-6):

Q.1 $10x^2 - 23xy - 5y^2 = 0$

Solution:

$$10x^2 - 23xy - 5y^2 = 0$$

$$10x^2 - 25xy + 2xy - 5y^2 = 0$$

$$5x(2x - 5y) + y(2x - 5y) = 0$$

$$(2x - 5y)(5x + y) = 0$$

$$2x - 5y = 0, 5x + y = 0$$

$$10x^2 - 23xy - 5y^2 = 0$$

Here, $a = 10, h = -\dfrac{23}{2}, \text{b} = -5$

As $\tan\theta = \dfrac{2\sqrt{h^2 - ab}}{a + b}$

$$\tan\theta = \frac{2\sqrt{\left(\frac{-23}{2}\right)^2 - (10)(-5)}}{10 + (-5)}$$

$$\tan\theta = \frac{2\sqrt{\frac{529}{4} + 50}}{5}$$

$$\tan\theta = \frac{2\sqrt{\frac{529 + 200}{4}}}{5}$$

$$\tan\theta = \frac{2\sqrt{\frac{729}{4}}}{5}$$

$$\tan\theta = \frac{2.\frac{27}{2}}{5}$$

$$\tan\theta = \frac{27}{5}$$

$$\theta = \tan^{-1}\left(\frac{27}{5}\right)$$

$$\boxed{\theta = 79.51^\circ}$$

Q.2 $3x^2 + 7xy + 2y^2 = 0$

Solution:

$$3x^2 + 7xy + 2y^2 = 0$$

$$3x^2 + 6xy + xy + 2y^2 = 0$$

$$3x(x + 2y) + y(x + 2y) = 0$$

$$(x + 2y)(3x + y) = 0$$

$$x + 2y = 0, 3x + y = 0$$

$$3x^2 + 7xy + 2y^2 = 0$$

Here $a = 3, \quad h = \dfrac{7}{2}, b = 2$

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$

$$\tan\theta = \frac{2\sqrt{\left(\frac{7}{2}\right)^2 - (3)(2)}}{3 + 2}$$

$$\tan\theta = \frac{2\sqrt{\frac{49}{4} - 6}}{5}$$

$$\tan\theta = \frac{2\sqrt{\frac{49 - 24}{4}}}{5}$$

$$\tan\theta = \frac{2.\sqrt{\frac{25}{4}}}{5}$$

$$\tan\theta = \frac{2.\frac{5}{2}}{5}$$

$$\tan\theta = \frac{5}{5}$$

$$\tan\theta = 1$$

$$\theta = \tan^{-1}(1)$$

$$\boxed{\theta = 45^\circ}$$

Q.3 $9x^2 + 24xy + 16y^2 = 0$

Solution:

$$9x^2 + 24xy + 16y^2 = 0$$

$$9x^2 + 12xy + 12xy + 16y^2 = 0$$

$$3x(3x + 4y) + 4y(3x + 4y) = 0$$

$$(3x + 4y)(3x + 4y) = 0$$

$$3x + 4y = 0, 3x + 4y = 0$$

As the lines are real and coincident so angle between the lines is 0º.

Q.4 $2x^2+3xy-5y^2=0$

Solution:

$$2x^2+3xy-5y^2=0$$
$$2x^2+5xy-2xy-5y^2=0$$
$$x(2x+5y)-y(2x+5y)=0$$
$$(2x+5y)(x-y)=0$$
$$2x+5y=0,\quad x-y=0$$
$$2x^2+3xy-5y^2=0$$

Here $a=2, h=\frac{3}{2}, \mathrm{b}=-5$

$$\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}$$
$$\tan\theta=\frac{2\sqrt{\left(\frac{3}{2}\right)^2-(2)(-5)}}{2+(-5)}$$
$$\tan\theta=\frac{2\sqrt{\frac{9}{4}+10}}{-3}$$
$$\tan\theta=\frac{2\sqrt{\frac{9+40}{4}}}{-3}$$
$$\tan\theta=\frac{2\sqrt{\frac{49}{4}}}{-3}$$
$$\tan\theta=\frac{2\left(\frac{7}{2}\right)}{-3}$$
$$\tan\theta=-\frac{7}{3}$$
$$\theta=\tan^{-1}\left(-\frac{7}{3}\right)$$
$$\theta=180^\circ-\tan^{-1}\left(\frac{7}{3}\right)$$
$$\boxed{\theta=113.20^\circ}$$

Q.5 $6x^2-19xy+15y^2=0$

Solution:

$$6x^2-19xy+15y^2=0$$
$$6x^2-10xy-9xy+15y^2=0$$
$$2x(3x-5y)-3y(3x-5y)=0$$
$$(3x-5y)(2x-3y)=0$$
$$3x-5y=0,\quad 2x-3y=0$$
$$6x^2-19xy+15y^2=0$$

Here $a=6, \mathrm{h}=\frac{-19}{2}, \mathrm{b}=15$

$$\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}$$
$$\tan\theta=\frac{2\sqrt{\left(\frac{-19}{2}\right)^2-(6)(15)}}{6+15}$$
$$\tan\theta=\frac{2\sqrt{\frac{361}{4}-90}}{21}$$
$$\tan\theta=\frac{2\sqrt{\frac{361-360}{4}}}{21}$$
$$\tan\theta=\frac{2\sqrt{\frac{1}{4}}}{21}$$
$$\tan\theta=\frac{2.\frac{1}{2}}{21}$$
$$\theta=\tan^{-1}\left(\frac{1}{21}\right)$$
$$\boxed{\theta=2.73^\circ}$$

Q.6 $x^2+2xy\sec\alpha+y^2=0$

Solution:

$$x^2+2xy\sec\alpha+y^2=0$$

Dividing both sides by x^2

$$1+2\sec\alpha\left(\frac{y}{x}\right)+\frac{y^2}{x^2}=0$$

Let $\frac{y}{x}=t\ \ldots$(i)

$$1+(2\sec\alpha)t+t^2=0$$

$$\Rightarrow \quad t^2+(2\sec\alpha)t+1=0$$

Here $a=1, \quad b=2\sec\alpha, \quad c=1$

$$t=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$t=\frac{-2\sec\alpha\pm\sqrt{(2\sec\alpha)^2-4(1)(1)}}{2(1)}$$

$$t=\frac{-2\sec\alpha\pm\sqrt{4\sec^2\alpha-4}}{2(1)}$$

$$t=\frac{-2\sec\alpha\pm\sqrt{4(\sec^2\alpha-1)}}{2}$$

$$t=\frac{-2\sec\alpha\pm\sqrt{4\tan^2\alpha}}{2}$$

$$t=\frac{-2\sec\alpha\pm2\tan\alpha}{2}$$

$$t=\frac{-2(-\sec\alpha\pm\tan\alpha)}{2}$$

$$t=-(\sec\alpha\pm\tan\alpha)$$

$$t=-(\sec\alpha+\tan\alpha), t=-(\sec\alpha-\tan\alpha)$$

From equation (i)

$$\frac{y}{x}=-(\sec\alpha+\tan\alpha),\ \frac{y}{x}=-(\sec\alpha-\tan\alpha)$$

$$y=-(\sec\alpha+\tan\alpha)x,\ y=-(\sec\alpha-\tan\alpha)x$$

$$(\sec\alpha+\tan\alpha)x+y=0,\ (\sec\alpha-\tan\alpha)x+y=0$$

$$x^2+2xy\sec\alpha+y^2=0$$

Here $a=1$, $h=\sec\alpha$, $b=1$

As $\tan\theta=\dfrac{2\sqrt{h^2-ab}}{a+b}$

$$\tan\theta=\frac{2\sqrt{\sec^2\alpha-(1)(1)}}{1+1}$$

$$\tan\theta=\frac{\not{2}\sqrt{\sec^2\alpha-1}}{\not{2}}$$

$$\tan\theta=\tan\alpha$$

$$\boxed{\theta=\alpha}$$

Q.7 Find a joint equation of the lines through the origin and perpendicular to the lines:

$$x^2-2xy\tan\alpha-y^2=0$$

Solution:

$$x^2-2xy\tan\alpha-y^2=0\text{—}(i)$$

let $y=m_1x$, $y=m_2x$ be the lines represented by equation (i)

Equations of lines passing through origin and perpendicular to the lines represented by equation (i) are

$$y=\frac{-1}{m_1}x \quad , \quad y=\frac{-1}{m_2}x$$

$$m_1y=-x \quad , \quad m_2y=-x$$

$$x+m_1y=0 \quad , \quad x+m_2y=0$$

Now, joint equation of these lines is

$$(x+m_1y)(x+m_2y)=0$$

$$x^2+m_1xy+m_2xy+m_1m_2y^2=0$$

$$x^2+(m_1+m_2)xy+m_1m_2y^2=0\text{—}(ii)$$

From equation (i)

$$x^2-2xy\tan\alpha-y^2=0$$

Here $a=1, h=-\tan\alpha, b=-1$

$$m_1+m_2=\frac{-2h}{b}=\frac{-2(-\tan\alpha)}{-1}=-2\tan\alpha$$

$$m_1m_2 = \frac{a}{b} = \frac{1}{-1} = -1$$

Putting in equation (ii)

$$x^2 + (-2\tan\alpha)xy + (-1)y^2 = 0$$

$$\boxed{x^2 - 2\tan\alpha\, xy - y^2 = 0}$$

Q.8 Find a joint equation of the lines through the origin and perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$

Solution:

$$ax^2 + 2hxy + by^2 = 0 \ldots \text{(i)}$$

Let $y = m_1x$, $y = m_2x$ be the lines represented by equation (i)

Equations of lines through origin and perpendicular to the lines represented by equation (i) are

$$y = \frac{-1}{m_1}x \quad , \quad y = \frac{-1}{m_2}x$$

$$m_1y = -x \quad , \quad m_2y = -x$$

$$x + m_1y = 0 \quad , \quad x + m_2y = 0$$

Now, joint equation of these lines is

$$(x + m_1y)(x + m_2y) = 0$$

$$x^2 + m_1xy + m_2xy + m_1m_2y^2 = 0$$

$$x^2 + (m_1 + m_2)xy + m_1m_2 = 0 \ldots \text{(ii)}$$

From equation (i)

$$ax^2 + 2hxy + by^2 = 0$$

$$m_1 + m_2 = \frac{-2h}{b}$$

$$m_1m_2 = \frac{a}{b}$$

Putting in equation (ii)

$$x^2 + \left(\frac{-2h}{b}\right)xy + \frac{a}{b}y^2 = 0$$

Multiplying by b on both sides.

$$\boxed{bx^2 - 2hxy + ay^2 = 0}$$

Q.9 Find the area of the region bounded by $10x^2 - xy - 21y^2 = 0$ **and** $x + y + 1 = 0$

Solution:

$$10x^2 - xy - 21y^2 = 0$$

$$10x^2 - 15xy + 14xy - 21y^2 = 0$$

$$5x(2x - 3y) + 7y(2x - 3y) = 0$$

$$(2x - 3y)(5x + 7y) = 0$$

Either $2x - 3y = 0$ or $5x + 7y = 0$

$$2x - 3y = 0 \ldots \text{(i)}$$

$$5x + 7y = 0 \ldots \text{(ii)}$$

$$x + y + 1 = 0 \ldots \text{(iii)}$$

Solving (i) and (iii)

By Eq (i) -2 Eq (iii)

$$2x - 3y = 0$$
$$\pm 2x \pm 2y \pm 2 = 0$$
$$\overline{-5y - 2 = 0}$$

$$-5y = 2$$

$$y = \frac{-2}{5}$$

Equation (iii)

$$x + y + 1 = 0$$

$$x + \left(\frac{-2}{5}\right) + 1 = 0$$

$x=-\frac{3}{5}$

So $A\left(\frac{-3}{5},\frac{-2}{5}\right)$ is the point of intersection of (i) and (iii)

By eq (ii) -5 eq (iii)

$$\begin{array}{r} 5x+7y=0 \\ -\ \pm5x\pm5y\pm5=0 \\ \hline 2y-5=0 \end{array}$$

$2y=5$

$y=\frac{5}{2}$

Equation (iii)

$x+y+1=0$

$x+\frac{5}{2}+1=0$

$x=\frac{-7}{2}$

So $B\left(\frac{-7}{2},\frac{5}{2}\right)$ is the point of intersection of (ii) and (iii)

Clearly, the point of intersection of (i) and (ii) is origin, $O(0,0)$

Then area of $\Delta OAB=\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ \frac{-3}{5} & \frac{-2}{5} & 1 \\ \frac{-7}{2} & \frac{5}{2} & 1 \end{vmatrix}$

Expanding by R_1

$$=\frac{1}{2}\left[0-0+1\begin{vmatrix} \frac{-3}{5} & \frac{-2}{5} \\ \frac{-7}{2} & \frac{5}{2} \end{vmatrix}\right]$$

$$=\frac{1}{2}\left[-\frac{15}{10}-\frac{14}{10}\right]$$

$$=\frac{1}{2}\left[\frac{-15-14}{10}\right]$$

$$=\frac{-29}{20}$$

As area can never be negative, so

Area of $\Delta OAB=\frac{29}{20}$ square units