UNIT 6

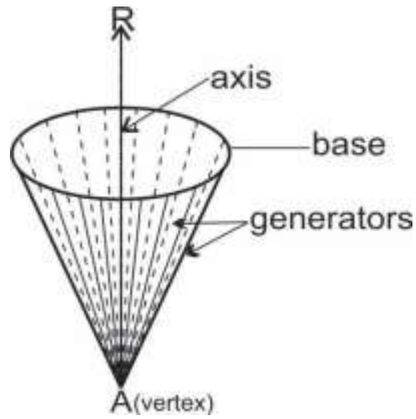
CONIC SECTION

Conic Section:

Conic sections, or simply conics, are the curves obtained by cutting a (double) right circular cone by a plane.

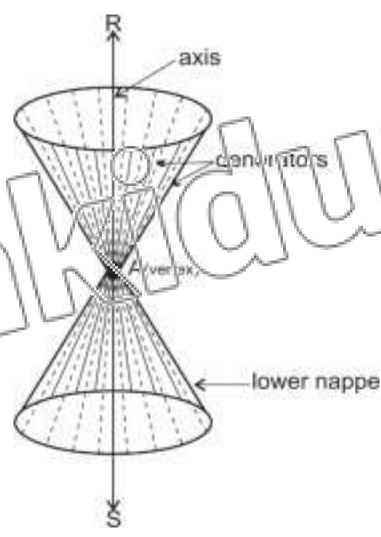
Right Circular Cone:

The surface generated by straight lines passing through fixed point and terminated on the circumference of the circle is called right circular cone. The fixed point is called vertex of cone and the straight lines are called generators of the cone.



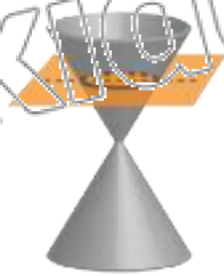
Double Right Circular Cone:

Let RS be a line through the centre "C" of a given circle and perpendicular to its plane. Let "A" be a fixed point on RS. All lines through "A" and points on the circle generate a right circular cone. The lines are called rulings or generators of the cone. The surface generated consist of two parts called nappes, meeting at the fixed point "A" called the vertex or apex of the cone and line RS is called axis of the cone as shown in figure.



Circle:

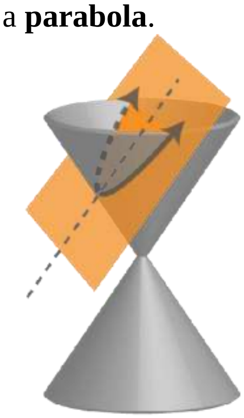
If the cone is cut by a plane perpendicular to the axis of the cone, then the section is a **circle**. If the plane passes through the vertex, perpendicular to the axis, the intersection is just a single point or a **point circle**.



Circle

Parabola:

If the intersecting plane is parallel to a generator of the cone, but intersects its one nappe only, the curve of intersection is a **parabola**.



Parabola

Ellipse:

If the cutting plane is slightly tilted and cuts only one nappe of the cone, the resulting section is an ellipse.

Ellipse

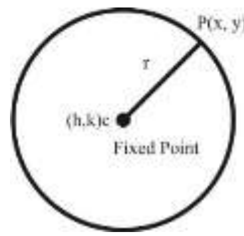
Hyperbola:

If the cutting plane is parallel to the axis of the cone and intersects both of its nappes, then the curve of intersection is a hyperbola.

Hyperbola

Circle:

The set of all points in a plane that are equidistant from a fixed point is called a **circle**. The fixed point is called the **centre** of the circle and the distance from the centre to any point on the circle is called the **radius** of the circle.



Equation of Circle in Set Notation:

If $C(h,k)$ is centre of circle, r its radius and $P(x,y)$ be ay point on the circle then in set notation.

$$S(C:r)=\{P(x,y):|\overline{CP}|=r\}$$

Standard Form of Equation of Circle:

Let $C(h,k)$ be the centre of a circle, 'r' its radius and $P(x,y)$ any point on the circle then, by definition of the circle,

$$|\overline{CP}|=r$$

$$\sqrt{(x-h)^2+(y-x)^2}=r$$

Squaring on both sides

$$(x-h)^2+(y-k)^2=r^2 \ldots \text{(i)}$$

is an equation of circle in standard form. If the centre of the circle is origin then (i) reduces to

$$x^2+y^2=r^2$$

General Form of Equation of a Circle:

The equation

$$x^2+y^2+2gx+2fy+c=0 \ldots \text{(ii)}$$

represents a circle g, f and c being constants.

By adding g^2+f^2 on both sides, the equation (ii) can be written as

$$\left(x^2+2gx+g^2\right)+\left(y^2+2fy+f^2\right)=g^2+f^2-c$$

$$(x+g)^2+(y+f)^2=g^2+f^2-c$$

$$(x-(-g))^2+(y-(-f))^2=\left(\sqrt{g^2+f^2-c}\right)^2$$

Comparing with $(x-h)^2+(y-k)^2=r^2$, we have

$h=-g,\ k=-f,\ r=\sqrt{g^2+f^2-c}$

centre$(-g,-f)$ and radius $=r=\sqrt{g^2+f^2-c}$

Note:

(i) The centre of the circle $x^2+y^2+2gx+2fy+c=0$ is $(-g,-f)$.

(ii) The radius of the circle $x^2+y^2+2gx+2fy+c=0$ is $\sqrt{g^2+f^2-c}$

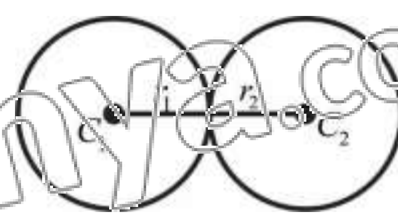
(iii) A second degree equation in which coefficients of x^2 and y^2 are equal to 1 and contains no term involving the product term xy.

Equation	Description
$(x-h)^2+(y-k)^2=r^2$	Equation of circle in standard form
$x^2+y^2=r^2$	Equation of circle, when centre is at origin
$x^2+y^2=0$	Equation of circle when $r=0$ i.e point circle
$x^2+y^2+2gx+2fy+c=0$	General form of an equation of a circle.
$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$	Equation of circle with end points $A(x_1,y_1),B(x_2,y_2)$ on the diameter of the circle.
$x=r\cos\theta,\ y=r\sin\theta$	Parametric equations of circle $x^2+y^2=r^2$.
$x=h+r\cos\theta, y=k+r\sin\theta$	Parametric equations of circle $(x-h)^2+(y-k)^2=r^2$

Condition for Circles that touch each other:

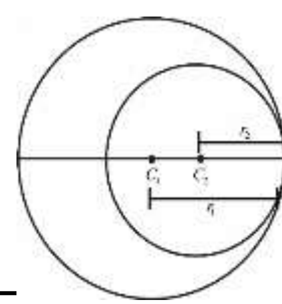
(i) Two circles touch each other externally if distance between their centres is equal to the sum of their radii.

i.e $|C_1C_2|=r_1+r_2$



(ii) Two circles touch internally if the distance between their centres is equal to the difference of their radii.

$|C_1C_2|=|r_1-r_2|$



EXERCISE 6.1

Q.1 In each of the following find an equation of the circle with

(a) Centre at $(5,-2)$ and radius 4.

Solution:

$C(h,k)=C(5,-2)$ and $r=4$

$\Rightarrow h=5,\ k=-2$

By standard form of equation of circle

$(x-h)^2+(y-k)^2=r^2$

$(x-5)^2+(y+2)^2=4^2$

$x^2+25-10x+y^2+4+4y=16$

$x^2+y^2-10x+4y+29-16=0$

$\boxed{x^2+y^2-10x+4y+13=0}$

(b) Centre at $(\sqrt{2},-3\sqrt{3})$ and radius $2\sqrt{2}$.

Solution: Here

$C(h,k)=C(\sqrt{2},-3\sqrt{3})$ and $r=2\sqrt{2}$

$\Rightarrow h=\sqrt{2},\ k=-3\sqrt{3}$

By standard form of equation of circle

$(x-h)^2+(y-k)^2=r^2$

$(x-\sqrt{2})^2+(y+3\sqrt{3})^2=(2\sqrt{2})^2$

$x^2+2-2\sqrt{2}x+y^2+27+6\sqrt{3}y=8$

$x^2+y^2-2\sqrt{2}x+6\sqrt{3}y+29-8=0$

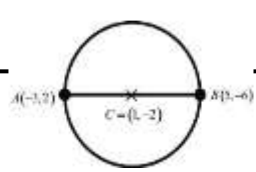
$\boxed{x^2+y^2-2\sqrt{2}x+6\sqrt{3}y+21=0}$

(c) Ends of diameter at $(-3,2)$ and $(5,-6)$

Solution: As we know that midpoint of end points of diameter is the centre of circle, so here centre of circle

$=C\left(\dfrac{-3+5}{2},\dfrac{2+(-6)}{2}\right)=C\left(\dfrac{2}{2},\dfrac{-4}{2}\right)$

$=C(1,-2)$



Now

radius of circle

$=|AC|=\sqrt{(-3-1)^2+(2+2)^2}$

$\Rightarrow r=\sqrt{16+16}$

$r=\sqrt{32}$

$r=4\sqrt{2}$

By standard form of equation of circle

$(x-h)^2+(y-k)^2=r^2$

$\Rightarrow (x-1)^2+(y+2)^2=(4\sqrt{2})^2$

$x^2+1-2x+y^2+4+4y=32$

$x^2+y^2+2x+4y+5-32=0$

$\boxed{x^2+y^2-2x+4y-27=0}$

Q.2 Find the centre and radius of the circle with the given equation.

(a) $x^2+y^2+12x-10y=0$

Solution:

$x^2+y^2+12x-10y=0$ ...(i)

Comparing equation (i) with

$x^2+y^2+2gx+2fy+c=0$, we have

$2gx=12x$	$2fy=-10y$	$c=0$
$g=6$	$f=-5$	

So, centre $(-g,-f)=C(-6,5)$ and

Radius $=r=\sqrt{g^2+f^2-c}$

$=\sqrt{36+25-0} \Rightarrow r=\sqrt{61}$

(b) $5x^2+5y^2+14x+12y-10=0$

Solution:

$5x^2+5y^2+14x+12y-10=0$

Dividing both sides by 5

$x^2+y^2+\dfrac{14}{5}x+\dfrac{12}{5}y-2=0$ ...(i)

Comparing equation (i) with

$x^2+y^2+2gx+2fy+c=0$, we have

$2gx=\frac{14}{5}x$ | $2fy=\frac{12}{5}y$

$g=\frac{7}{5}$ | $f=\frac{6}{5}$

$c=-2$

So, centre $(-g,-f)=C\left(\frac{-7}{5},\frac{-6}{5}\right)$

and

Radius $= r=\sqrt{g^2+f^2-c}$

$=\sqrt{\frac{49}{25}+\frac{36}{25}-(-2)}=\sqrt{\frac{49+36+50}{25}}$

$=\sqrt{\frac{135}{25}}\Rightarrow r=\sqrt{\frac{27}{5}}$

(c) $x^2+y^2-6x+4y+13=0$

Solution:

$x^2+y^2-6x+4y+13=0\ldots$(i)

Comparing equation (i) with

$x^2+y^2+2gx+2fy+c=0$, we have

$2gx=-6x$ | $2fy=4y$ | $c=13$

$g=-3$ | $f=2$

So, centre $(-g,-f)=C(3,-2)$

and

Radius $= r=\sqrt{g^2+f^2-c}$

$=\sqrt{(3)^2+(-2)^2-13}=\sqrt{9+4-13}$

$\Rightarrow r=0$

(d) $4x^2+4y^2-8x+12y-25=0$

Solution:

$4x^2+4y^2-8x+12y-25=0$

Dividing both sides by 4 we get

$x^2+y^2-2x+3y-\frac{25}{4}=0\ldots$ (i)

Comparing equation (i) with

$x^2+y^2+2gx+2fy+c=0$, we have

$2gx=-2x$ | $2fy=3y$ | $c=\frac{-25}{4}$

$g=-1$ | $f=\frac{3}{2}$

So, Centre $(-g,-f)=C\left(1,\frac{-3}{2}\right)$

and

Radius $= r=\sqrt{g^2+f^2-c}$

$=\sqrt{(-1)^2+\left(\frac{3}{2}\right)^2-\left(\frac{-25}{4}\right)}$

$=\sqrt{1+\frac{9}{4}+\frac{25}{4}}=\sqrt{\frac{4+9+25}{4}}$

$\Rightarrow r=\sqrt{\frac{19}{2}}$

Q.3 Write an equation of the circle that passes through the given points

(a) $A(4,5),\ B(-4,-3),\ C(8,-3)$

Solution: Let the equation of the required circle be

$x^2+y^2+2gx+2fy+c=0\ldots$(i)

As A,B and C lie on (i), So

Substituting $A\ (4,5)$ in (i), we have

$(4)^2+(5)^2+2g(4)+2f(5)+c=0$

$16+25+8g+10f+c=0$

$8g+10f+c=-41\ldots$(ii)

Substituting $B(-4,-3)$ in (i), we have

$(-4)^2+(-3)^2+2g(-4)+2f(-3)+c=0$

$16+9-8g-6f+c=0$

$8g+6f-c=25\ldots$(iii)

Substituting $C(8,-3)$ in (i), we have

$(8)^2+(-3)^2+2g(8)+2f(-3)+c=0$

$64+9+16g-6f+c=0$

$16g-6f+c=-73\ldots$(iv)

Adding (ii) and (iii) we get

$8g+10f+c=-41$

$\underline{8g+6f-c=25}$

$16g+16f=-16$

$\Rightarrow g+f=-1\ldots$(v)

Adding (iii) and (iv) we get

$8g+6f-c=25$
$\underline{16g-6f+c=-73}$
$24g=-48$

$\Rightarrow \boxed{g=-2}$ Putting in (v) we get

$-2+f=-1 \Rightarrow f=2-1 \Rightarrow \boxed{f=1}$

Putting $g=-2$ and $f=-1$ in (iv) we get

$16(-2)-6(1)+c=-73$
$-32-6+c=-73$
$c=38-73$
$\Rightarrow \boxed{c=-35}$

Put the value of f,g and c in (i) we have

$x^2+y^2+2(-2)x+2(1)y-35=0$

$\boxed{x^2+y^2-4x+2y-35=0}$

(b) $A(-7,7),\ B(5,-1),\ C(10,0)$

Solution: Let the equation of the required circle be

$x^2+y^2+2gx+2fy+c=0\ldots$(i)

As A,B and C lie on (i), So

Substituting $A(-7,7)$ in (i), we have

$(-7)^2+(7)^2+2g(-7)+2f(7)+c=0$
$49+49-14g+14f+c=0$
$-14g+14f+c=-98$
$\Rightarrow 14g-14f-c=98\ldots$(ii)

Substituting $B(5,-1)$ in (i), we have

$(5)^2+(-1)^2+2g(5)+2f(-1)+c=0$
$25+1+10g-2f+c=0$
$\Rightarrow 10g-2f+c=-26\ldots$(iii)

Substituting $C(10,0)$ in (i), we have

$(10)^2+(0)^2+2g(10)+2f(0)+c=0$
$100+20g+c=0$
$\Rightarrow 20g+c=-100\ldots$(iv)

Adding (ii) and (iii) we get

$14g-14f-c=98$
$\underline{10g-2f+c=-26}$
$24g-16f=72$
$\Rightarrow 8(3g-2f)=72$
$\Rightarrow 3g-2f=9\ldots$(v)

Multiply Equation (iii) with 7 and then subtracting from (ii), we get

$14g-14f-c=98$
$\underline{\underset{-}{70g}\ \underset{+}{-14f}\ \underset{-}{+7c}=\underset{+}{-182}}$
$-56g-8c=280$
$\Rightarrow 7g+c=-35\ldots$(vi)

Subtracting (vi) from (iv) we get

$20g+c=-100$
$\underline{\underset{-}{7g}\ \underset{-}{+c}=\underset{+}{-35}}$
$13g=-65$
$\Rightarrow \boxed{g=-5}$

Putting in (vi) we get

$7(-5)+c=-35$
$-35+c=-35$
$\boxed{c=0}$

Put $g=-5$ in (v) we get

$3(-5)-2f=9$
$-15-2f=9$
$-15-9=2f$
$\Rightarrow 2f=-24 \Rightarrow \boxed{f=-12}$

Put $g=-5, f=-12, c=0$ in (i) we get

$x^2+y^2+2(-5)x+2(-12)y+0=0$

$\boxed{x^2+y^2-10x-24y=0}$

(c) $A(a,0),\ B(0,b),\ C(0,0)$

Solution: Let the equation of the circle is

$x^2+y^2+2gx+2fy+c=0\ldots$(i)

As A,B and C lie on (i), So

Substituting $A(0,a)$ in (i), we have

$(a)^2+(0)^2+2g(a)+2f(0)+c=0$

$\Rightarrow a^2+2ag+c=0$ ...(ii)

Substituting $B(0,b)$ in (i), we have

$(0)^2+(b)^2+2g(0)+2f(b)+c=0$

$\Rightarrow b^2+2bf+c=0$ ...(iii)

Substituting $C(0,0)$ in (i), we have

$(0)^2+(0)^2+2g(0)+2f(0)+c=0$

$\Rightarrow \boxed{c=0}$

From (ii)

$a^2+2ag+c=0 \Rightarrow 2ag+0$

$=-a^2 \Rightarrow 2ag=-a^2$

$\Rightarrow \boxed{g=\frac{-a}{2}}$

From (iii)

$b^2+2bf+c=0 \Rightarrow 2bf+0=-b^2$

$\Rightarrow 2bf=-b^2$

$\Rightarrow \boxed{f=\frac{-b}{2}}$

Put $g=-\frac{a}{2}$, $f-\frac{b}{2}$ and $c=0$ in (i),

we get

$x^2+y^2+2\left(\frac{-a}{2}\right)x+2\left(\frac{-b}{2}\right)y+0=0$

$\boxed{x^2+y^2-ax-by=0}$

(d) $A(5,6)$, $B(-3,2)$, $C(3,-4)$

Solution: Let the equation of the circle is

$x^2+y^2+2gx+2fy+c=0$ ...(i)

As A,B and C lie on (i), So

Substituting $A(5,6)$ in (i), we have

$(5)^2+(6)^2+2g(5)+2f(6)+c=0$

$25+36+10g+12f+c=0$

$\Rightarrow 10g+12f+c=-61$ ...(ii)

Substituting $B(-3,2)$ in (i), we have

$(-3)^2+(2)^2+2g(-3)+2f(2)+c=0$

$9+4-6g+4f+c=0$

$\Rightarrow -6g+4f+c=-13$ ...(iii)

Substituting $C(3,-4)$ in (i), we have

$(3)^2+(-4)^2+2g(3)+2f(-4)+c=0$

$9+16+6g-8g+c=0$

$\Rightarrow 6g-8f+c=-25$ ... (iv)

Subtracting (iii) from (ii) we get

$10g+12f+c=-61$

$\underset{+}{-}6g\underset{-}{+}4f\underset{-}{+}c=\underset{+}{-}13$

$16g+8f=-48$

$\Rightarrow 2g+f=-6$ ...(v)

Subtracting (iv) from (iii) we get

$-6g+4f+c=-13$

$\underset{-}{+}6g\underset{+}{-}8f\underset{-}{+}c=\underset{+}{-}25$

$-12g+12f=12$

$\Rightarrow g-f=-1$ ...(vi)

Adding (v) and (vi) we get

$2g+f=-6$

$g-f=-1$

$3g=-7$

$\Rightarrow \boxed{g=\frac{-7}{3}}$

Putting in (vi), we get

$\frac{-7}{3}-f=-1$

$\frac{-7}{3}+1=f \quad \Rightarrow \boxed{f=\frac{-4}{3}}$

Putting $g=\frac{-7}{3}$ and $f=\frac{-4}{3}$ in (iii),

we get

$-6\left(\frac{-7}{3}\right)+4\left(\frac{-4}{3}\right)+c=-13$

$$\frac{42}{3}-\frac{16}{3}+c=-13$$

$$\frac{26}{3}+c=-13$$

$$c=-13-\frac{26}{3}$$

$$c=\frac{-39-26}{3}$$

$$\Rightarrow \boxed{c=-\frac{65}{3}}$$

Putting $g=\frac{-7}{3}, f=\frac{-4}{3}$ and $c=-\frac{65}{3}$ in (i), we get

$$x^2+y^2+2\left(\frac{-7}{3}\right)x+2\left(\frac{-4}{3}\right)y-\frac{65}{3}=0$$

$$\Rightarrow \boxed{3\left(x^2+y^2\right)-14x-8y-65=0}$$

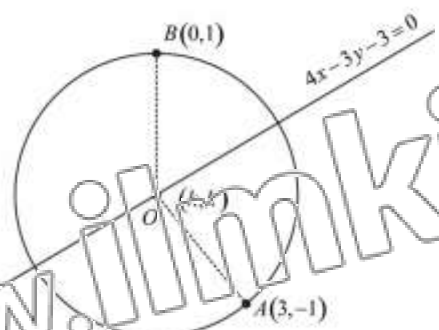
Q.4 In each of the following, find an equation of the circle passing through.

(a) $A(3,-1), B(0,1)$ **and having centre at** $4x-3y-3=0$

Solution:

Consider the required circle is

$$(x-h)^2+(y-k)^2=r^2 \quad \ldots (i)$$



$$\left|\overline{OA}\right|=\left|\overline{OB}\right|$$

$$(h-3)^2+(k+1)^2=(h-0)^2+(k-1)^2$$

$$h^2+k^2-6h+2k+10=h^2+k^2-2k+1$$

$$-6h+4k=-9 \ldots (ii)$$

Centre (h,k) lies on the line

$4x-3y-3=0$

$\therefore \ 4h-3k=3 \ldots (iii)$

Multiply (ii) by 3 and (iii) by 4 and Adding

$$-18h+12k+16h-12k=-27+12$$

$$-2h=-15$$

$$\boxed{h=\frac{15}{2}}$$

Put in (iii) $4\left(\frac{15}{2}\right)-3k=3$

$$30-3k=3$$

$$-3k=-27$$

$$\boxed{k=9}$$

$\therefore$ centre is $(h,k)=\left(\frac{15}{2},9\right)$

Radius

$$r=|OB|=\sqrt{\left(\frac{15}{2}-0\right)^2+(9-1)^2}$$

$$=\sqrt{\frac{225}{4}+64}$$

$$=\sqrt{\frac{225+256}{4}}=\frac{\sqrt{481}}{2}$$

Putting in (i)

$$\left(x-\frac{15}{2}\right)^2+(y-9)^2=\left(\frac{\sqrt{481}}{2}\right)^2$$

$$x^2+y^2-15x-18y+\frac{225}{4}+81=\frac{481}{4}$$

$$x^2+y^2-15x-18y+\frac{225+324-481}{4}=0$$

$$\boxed{x^2+y^2-15x-18y+17=0}$$

(b) A(–3,1) with radius 2 and centre at $2x-3y+3=0$

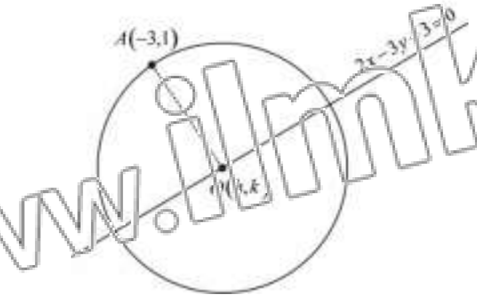
Solution:

Consider the required circle is

$$(x-h)^2+(y-k)^2=r^2$$

$$(x-h)^2+(y-k)^2=4\ \ldots(A)$$

$$\because\ r=2$$



The point $A(-3,1)$ lies on the circle

$$\therefore\ (h+3)^2+(k-1)^2=4\ \ldots(i)$$

Since $C(h,k)$ lies on

$$2x-3y+3=0$$

$$2h-3k+3=0$$

$$\Rightarrow \boxed{k-1=\frac{2h}{3}}\ \ldots(ii)$$

Putting in (i), we get

$$(h+3)^2+\left(\frac{2h}{3}\right)^2=4$$

$$\Rightarrow 9\left(h^2+6h+9\right)+4h^2=36$$

$$13h^2+54h+45=0$$

$$\Rightarrow h=\frac{-54\pm\sqrt{(54)^2-4(13)(45)}}{2(13)}$$

$$\Rightarrow h=\frac{-54\pm 24}{26}\quad \Rightarrow \boxed{h=\frac{-15}{13},-3}$$

For $h=-3$

equation (ii)

$$k-1=\frac{2(-3)}{3}$$

$$k-1=-2$$

$$k=-1$$

centre $(h,k)=(-3,1)$

equation (A)

$$(x-h)^2+(y-k)^2=4$$

$$\Rightarrow (x+3)^2+(y-1)^2=4$$

$$\Rightarrow x^2+y^2+6x-2y+9+1=4$$

$$\Rightarrow \boxed{x^2+y^2+6x+2y+6=0}$$

For $h=-\frac{15}{13}$

equation (ii)

$$k-1=\frac{2\left(-\frac{15}{13}\right)}{3}$$

$$k-1=-\frac{30}{39}$$

$$k=\frac{3}{13}$$

centre $(h,k)=\left(\frac{-15}{13},\frac{3}{13}\right)$

equation (A)

$$(x-h)^2+(y-k)^2=4$$

$$\Rightarrow \left(x+\frac{15}{13}\right)^2+\left(y-\frac{3}{13}\right)^2=4$$

$$\Rightarrow x^2+y^2+2\left(\frac{15}{13}\right)x$$

$$-2\left(\frac{3}{13}\right)y+\left(\frac{15}{13}\right)^2+\left(\frac{3}{13}\right)^2=4$$

$$\Rightarrow \boxed{13x^2+13y^2+30x-6y-34=0}$$

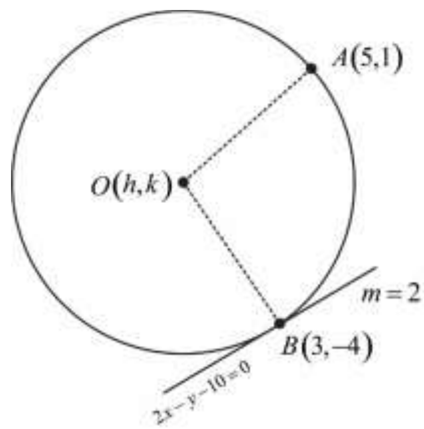
(c) A (5,1) **and tangent to the line** $2x-y-10=0$ **at** $B(3,-4)$

Solution:

Consider the required circle is

$$(x-h)^2+(y-k)^2=r^2\ \ldots\ (i)$$

$$\left|\overline{OA}\right|=\left|\overline{OB}\right|$$



$$(h-5)^2+(k-1)^2=(h-3)^2+(k+4)^2$$

$$h^2+k^2-10h-2k+26=h^2+k^2-6h+8k+25$$

$$-4h-10k+1=0$$

$$4h+10k-1=0\ \ldots(ii)$$

Slope of $OB=\frac{k+4}{h-3}$

Slope of tangent line = 2

$$\frac{k+4}{h-3}\times 2=-1\quad (\because m_1m_2=-1)$$

$$2k+8=-h+3$$

$$h+2k+5=0\ \ldots(iii)$$

Multiply (iii) by 4 then subtract from (ii)

$$4h+8k+20=0$$
$$4h+10k-1=0$$
$$-\quad -\quad\quad +$$
$$-2k+21=0$$
$$\boxed{k=\frac{21}{2}}$$

Put in (iii)

$$h+2\left(\frac{21}{2}\right)+5=0$$
$$\boxed{h=-26}$$

centre $(h,k)=\left(-26,\frac{21}{2}\right)$

to find radius

$$\left|\overline{OA}\right|=\sqrt{(-26-5)^2+\left(\frac{21}{2}-1\right)^2}$$
$$r=\sqrt{961+\left(-\frac{19}{2}\right)^2}$$
$$=\sqrt{\frac{3844+361}{4}}$$
$$r^2=\frac{4025}{4}$$

Put in (i)

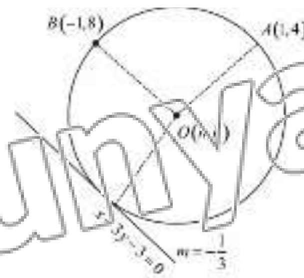
$$(x+26)^2+\left(y-\frac{21}{2}\right)^2=\frac{4025}{4}$$
$$x^2+y^2+52x-21y+676+\frac{441}{4}-\frac{4025}{4}=0$$
$$x^2+y^2+52x-21y+676-\frac{3764}{4}=0$$
$$\boxed{x^2+y^2+52x-21y-265=0}$$

(d) $A(1,4), B(-1,8)$ **and tangent to the line** $x+3y-3=0$

Solution:

Consider the required circle is

$$(x-h)^2+(y-k)^2=r^2 \quad \ldots \text{(i)}$$



$$\left|\overline{OA}\right|=\left|\overline{OB}\right|$$
$$(h-1)^2+(k-4)^2=(h+1)^2+(k-8)^2$$
$$-2h+1-8k+16=2h+1-16k-64$$
$$4h-8k+48=0$$
$$h-2k+12=0$$
$$\boxed{h=2k-12} \quad \ldots \text{(ii)}$$

Also the distance from the centre $C(h,k)$ to the line $x+3y-3=0$ is equal to the radius to the circle.

So, we have

$$\frac{|h+3k-3|}{\sqrt{1^2+3^2}}=|AC|=\sqrt{(h-1)^2+(k-4)^2}$$
$$\Rightarrow \frac{(h+3k-3)^2}{10}=(h-1)^2+(k-4)^2$$

Using (ii)

$$\Rightarrow \frac{(2k-12+3k-3)^2}{10}=(2k-12-1)^2+(k-4)^2$$
$$\Rightarrow \frac{(5k-15)^2}{10}=(2k-13)^2+(k-4)^2$$
$$\Rightarrow \frac{25(k-3)^2}{10}=5k^2-60k+185$$
$$\frac{5(k-3)^2}{2}=5\left(k^2-12k+37\right)$$
$$k^2-6k+9=2k^2-24k+74$$
$$k^2-18k+65=0$$
$$\Rightarrow k=\frac{18\pm\sqrt{324-260}}{2}$$

$$\Rightarrow k = \frac{18 \pm 8}{2}$$

$$\Rightarrow \boxed{k = 13, 5}$$

Putting $k = 13$ in (ii), we get $h = 14$

Putting $k = 5$ in (ii), we get $h = -2$

For	For
centre $(h,k) = (14,13)$	centre $(h,k) = (-2,5)$
and radius	and radius
$r = \lvert AC \rvert$	$r = \lvert AC \rvert$
$= \sqrt{(14-1)^2 + (13-4)^2}$	$= \sqrt{(-2-1)^2 + (5-4)^2}$
$\Rightarrow r = \sqrt{250}$	$\Rightarrow r = \sqrt{10}$
the equation of the circle is	the equation of the circle is
$(x-h)^2 + (y-k)^2 = r^2$	$(x-h)^2 + (y-k)^2 = r^2$
$\Rightarrow (x-14)^2 + (y-13)^2 = 250$	$\Rightarrow (x+2)^2 + (y-5)^2 = 10$
Or	Or
$x^2 + y^2 - 28x - 26y + 115 = 0$	$x^2 + y^2 + 4x - 10y + 19 = 0$

Q.5 Find an equation of circle of radius 'a' and lying in the second quadrant such that it is tangent to both the axes.

Solution: As circle is tangent to both axes and lie in 2nd quadrant.

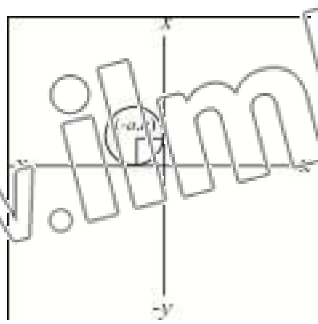
So that

Centre of the circle is $(-a, a)$

And

radius of the circle is "a"

Now, Equation of the circle is



$$(x-h)^2 + (y-k)^2 = r^2$$

$$(x+a)^2 + (y-a)^2 = a^2$$

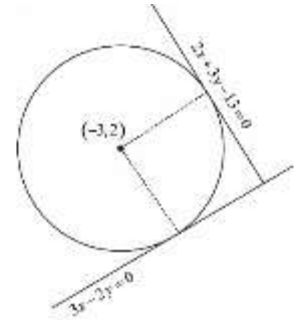
$$x^2 + a^2 + 2ax + y^2 + a^2 - 2ay = a^2$$

$$\boxed{x^2 + y^2 + 2ax - 2ay + a^2 = 0}$$

Q.6 Show that the lines $3x - 2y = 0$ and $2x + 3y - 13 = 0$ are tangent to the circle $x^2 + y^2 + 6x - 4y = 0$

Solution:

We show distance from centre of the circle to the given lines is equal to radius of circle.



$x^2 + y^2 + 6x - 4y = 0 \cdots$(i)

Comparing (i) with

$x^2 + y^2 + 2gx + 2fy + c = 0$

$2gx = 6x$	$2fy = -4y$	$c = 0$
$g = 3$	$f = -2$	

So, Centre $(-g, -f) = C(-3, 2)$ and

Radius $= r = \sqrt{g^2 + f^2 - c}$

$= \sqrt{(3)^2 + (-2)^2 - 0} = \sqrt{9+4}$

$\Rightarrow \boxed{r = \sqrt{13}}$

Now, we find distance from centre to the line $3x - 2y = 0$ which is

$$d = \frac{\lvert 3(-3) - 2(2) \rvert}{\sqrt{9+4}} = \frac{\lvert -9-4 \rvert}{\sqrt{13}} = \frac{13}{\sqrt{13}}$$

$= \sqrt{13} = r$

Similarly we find distance from centre to the line $2x + 3y - 13 = 0$ which is

$$d = \frac{|2(-g)+3(-f)-3|}{\sqrt{(2)^2+(3)^2}}$$

$$= \frac{|2(-3)+3(2)-13|}{\sqrt{4+9}} = \frac{|-6+6-13|}{\sqrt{13}}$$

$$= \frac{|-13|}{\sqrt{13}}$$

$$= \frac{13}{\sqrt{13}} = \sqrt{13} = r$$

Hence distance of both given lines from centre is equal to radius of circle so both lines are tangent to circle.

Q.7 Show that the circles

$x^2+y^2+2x-2y-7=0$ **and**

$x^2+y^2-6x+4y+9=0$ **touch externally.**

Solution:

$x^2+y^2+2x-2y-7=0$

$2gx=2x \qquad 2fy=-2y \qquad c=-7$

$g=1 \qquad f=-1$

Centre $(-g,-f)=C_1(-1,1)$

and radius is

$r_1=\sqrt{g^2+f^2-c}$

$=\sqrt{(-1)^2+(1)^2-(-7)}$

$=3$

$x^2+y^2-6x+4y+9=0$

$2gx=-6x \quad 2fy=4y \quad c=9$

$g=-3 \qquad f=2$

Now, centre $(-g,-f)=C_2(3,-2)$

and radius is

$r_2=\sqrt{g^2+f^2-c}=\sqrt{(-3)^2+(2)^2-9}$

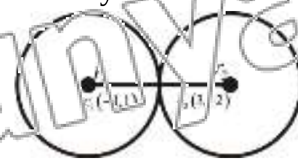
$=\sqrt{9+4-9}=\sqrt{4}=2$

Now distance between centres of circles is

$|C_1C_2|=\sqrt{(3+1)^2+(-2-1)^2}=\sqrt{16+9}=\sqrt{25}=5$

And $r_1+r_2=3+2=5$

Since $|C_1C_2|=r_1+r_2$ so circles touch externally.



Q.8 Show that the circles

$x^2+y^2+2x-8=0$ **and**

$x^2+y^2-6x+6y-46=0$ **touch internally**

Solution:

$x^2+y^2+2x-8=0$ ...(i)

$2gx=2x \quad 2fy=0y \quad c=-8$

$g=1 \qquad f=0$

centre $(-g,-f)=C_1(-1,0)$

and radius is

$r_1=\sqrt{g^2+f^2-c}$

$=\sqrt{(1)^2+(0)^2-(-8)}$

$=3$

$x^2+y^2-6x+6y-46=0$

$2gx=-6x \quad 2fy=6y \quad c=-46$

$g=-3 \qquad f=3$

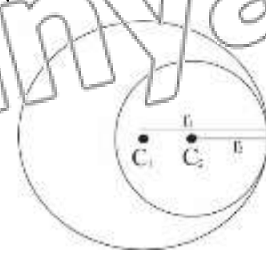
Now, centre $(-g,-f)=C_2(3,-3)$

And radius is

$r_2=\sqrt{g^2+f^2-c}$

$=\sqrt{(3)^2+(-3)^2-(-46)}$

$=8$



Here $|C_1C_2|=\sqrt{(3+1)^2+(-3-0)^2}$

$=\sqrt{(4)^2+(-3)^2}$

$= \sqrt{16+9} = \sqrt{25} = 5$

$r_1 - r_2 = 8 - 3 = 5$

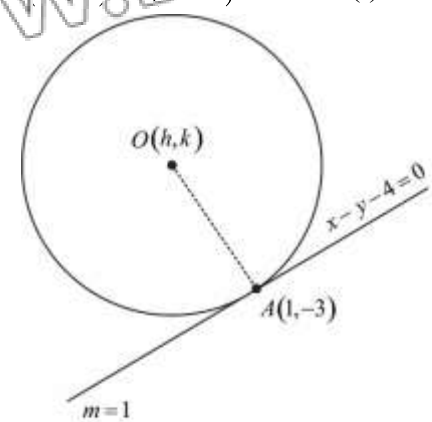
As $|C_1C_2| = r_1 - r_2$, so circles touch internally.

Q.9 Find equations of the circle of radius 2 and tangent to the line $x - y - 4 = 0$ at $A(1,-3)$

Solution:

Consider the required circle is

$(x-h)^2 + (y-k)^2 = r^2$ … (i)



$(x-h)^2 + (y-k)^2 = 4 \quad \because \; r = 2$

It passes through $A(1,-3)$ so

$(1-h)^2 + (-3-k)^2 = 4$

$h^2 + k^2 - 2h + 6k + 1 + 9 - 4 = 0$

$h^2 + k^2 + 6k - 2h + 6 = 0$ …(ii)

Slope of $OA = \dfrac{k+3}{h-1}$

Slope of tangent line = 1

$\dfrac{k+3}{h-1} \times 1 = -1 \quad (\because \; m_1m_2 = -1)$

$k + 3 = -h + 1$

$h = -k - 2$ …(iii)

Put in (ii)

$(-k-2)^2 + k^2 + 6k - 2(-k-2) + 6 = 0$

$k^2 + 4k + 4 + k^2 + 6k + 2k + 4 + 6 = 0$

$2k^2 + 12k + 14 = 0$

$k^2 + 6k + 7 = 0$

Using quadratic formula

$k = \dfrac{-6 \pm \sqrt{36-28}}{2}$

$= \dfrac{-6 \pm \sqrt{8}}{2}$

$= \dfrac{-6 \pm 2\sqrt{2}}{2} = \dfrac{2(-3 \pm \sqrt{2})}{2}$

$k = -3 + \sqrt{2}$ and $k = -3 - \sqrt{2}$

Putting $k = -3 + \sqrt{2}$ in (ii), we get

$h = -(-3 + \sqrt{2} + 2) = 1 - \sqrt{2}$

<u>For</u>

Centre: $C(h,k) = (\sqrt{2}+1, -3-\sqrt{2})$

and radius $r = 2$,

The equation of the circle is

$(x-h)^2 + (y-k)^2 = r^2$

Putting $k = -3 - \sqrt{2}$ in (ii), we get

$h = -(-3 - \sqrt{2} + 2) = 1 + \sqrt{2}$

<u>For</u>

Center: $C(h,k) = (1-\sqrt{2}, -3+\sqrt{2})$

and radius $r = 2$,

The equation of the circle is

$(x-h)^2 + (y-k)^2 = r^2$

$$\Rightarrow \left(x+\left(\sqrt{2}-1\right)\right)^2+\left(y+\left(3-\sqrt{2}\right)\right)^2=4$$

Or

$$x^2+y^2+2\left(\sqrt{2}-1\right)x+2\left(3-\sqrt{2}\right)y+10-8\sqrt{2}=0$$

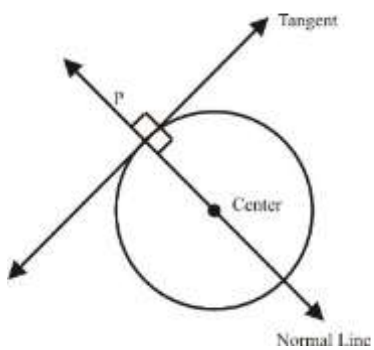
$$\Rightarrow \left(x-\left(\sqrt{2}+1\right)\right)^2+\left(y+\left(\sqrt{3}+1\right)\right)^2=4$$

Or

$$x^2+y^2-2\left(\sqrt{2}+1\right)x+2\left(3+\sqrt{2}\right)y+10+8\sqrt{2}=0$$

Tangent and Normals

A **tangent** to a curve is a line that touches the curve without cutting through it.
A **normal** to the curve at P is the line through P perpendicular to the tangent to the curve at P.



EQUATION OF TANGENT AND NORMAL LINE TO THE CIRCLE:

Equation of tangent and normal to the circle $x^2+y^2=a^2$ at $P(x_1,y_1)$

Let equation of circle is

$x^2+y^2=a^2$ (I)

Let $P(x_1,y_1)$ be any point on (I)

Then $x_1^2+y_1^2=a^2$ (1)

Differentiating eq. (I) w.r.t "x"

$$2x+2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-2x}{2y}=\frac{-x}{y}$$

Let "m" be the slope of tangent line at $P(x_1,y_1)$ then $m=\frac{-x_1}{y_1}$

By point slope form equation of tangent line to (I) is

$$y-y_1=m(x-x_1)$$

$$y-y_1=\frac{-x_1}{y_1}(x-x_1)$$

$$yy_1-y_1^2=-xx_1+x_1^2$$

$$xx_1+yy_1=x_1^2+y_1^2$$

$$\boxed{xx_1 + yy_1 = a^2}$$

using eq.(1)

Let m_1 be the slope of normal line then

$mm_1 = -1$

$\left(\dfrac{-x_1}{y_1}\right)m_1 = -1 \quad \Rightarrow m_1 = \dfrac{y_1}{x_1}$

By point slope form equation of normal line is

$y - y_1 = m_1(x - x_1)$

$y - y_1 = \dfrac{y_1}{x_1}(x - x_1)$

$x_1y - x_1y_1 = xy_1 - x_1y_1$

$$\boxed{xy_1 - x_1y = 0}$$

Equation of tangent and normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ **at** $P(x_1, y_1)$

$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad$ (I)

Let $P(x_1, y_1)$ be any point the circle then

$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad$ (1)

Differentiate equation (i) w.r.t 'x'

$2x + 2y\dfrac{dy}{dx} + 2g + 2f\dfrac{dy}{dx} + 0 = 0$

$2\dfrac{dy}{dx}(y + f) = -2x - 2g$

$\dfrac{dy}{dx} = \dfrac{-2(x+g)}{2(y+f)}$

$\dfrac{dy}{dx} = -\dfrac{x+g}{y+f}$

$\left.\dfrac{dy}{dx}\right|_{(x_1,y_1)} = -\dfrac{x_1+g}{y_1+f} =$ slope of tangent at $P(x_1, y_1)$

Equation of tangent at $P(x_1, y_1)$ is given by

$y - y_1 = -\dfrac{x_1+g}{y_1+f}(x - x_1)$

$(y - y_1)(y_1 + f) = -(x_1 + g)(x - x_1)$

$yy_1 + yf - y_1^2 - y_1f = -\left[xx_1 - x_1^2 + gx - gx_1\right]$

$yy_1 + yf - y_1^2 - y_1f = -xx_1 + x_1^2 - gx + gx_1 \quad xx_1 + yy_1 + fy + g = x_1^2 + y_1^2 + gx_1 + fy_1$ Adding gx_1, fy_1 and c on both sides

$xx_1 + yy_1 + gx + gx_1 + fy + fy_1 + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

From equation (ii)

$$\boxed{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0}$$

Equation of normal line at $P(x_1, y_1)$

Slope of normal line $= \frac{y_1 + f}{x_1 + g}$

$$y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$$

$$\boxed{(y - y_1)(x_1 + g) = (x - x_1)(y_1 + f)}$$

Circle	Tangent and normal line
Equation of tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at $P(x_1, y_1)$	$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$
Equation of normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at $P(x_1, y_1)$	$(y - y_1)(x_1 + g) = (x - x_1)(y_1 + f)$
Equation of tangent to the circle $x^2 + y^2 = a^2$ **at** $P(x_1, y_1)$	$xx_1 + yy_1 = a^2$
Equation of normal to the circle $x^2 + y^2 = a^2$ **at** $P(x_1, y_1)$	$xy_1 - x_1y = 0$

Theorem:

The line $y = mx + c$ intersects the circle $x^2 + y^2 = a^2$ in at the most two points.

Proof:

We solve the simultaneous equations of line and the circle to get their point of intersection.

$y = mx + c$ …(i)

$x^2 + y^2 = a^2$ …(ii)

From equation (i) and equation (ii)

$$x^2 + (mx + c)^2 = a^2$$

$$x^2 + m^2x^2 + c^2 + 2mcx = a^2$$

$$x^2(1 + m^2) + 2mcx + c^2 - a^2 = 0 \ \ldots(iii)$$

This being quadratic in x gives two values of x say x_1 and x_2.

Thus the line intersect the circle at most two point. For nature of the points we examine the discriminant of equation (iii).

Discriminant $= b^2 - 4ac$

$$= (2mc)^2 - 4(1 + m^2)(c^2 - a^2)$$

$$= 4m^2c^2 - 4\left[c^2 - a^2 + m^2c^2 - a^2m^2\right]$$

$$= 4m^2c^2 - 4c^2 + 4a^2 - 4m^2c^2 + 4a^2m^2$$

$$= 4\left[-c^2 + a^2 + a^2m^2\right]$$
$$= 4\left[-c^2 + a^2\left(1+m^2\right)\right]$$

These points are

(i) Real and distinct, if $a^2\left(1+m^2\right) - c^2 > 0$

(ii) Real and coincident if $a^2\left(1+m^2\right) - c^2 = 0$

(iii) Imaginary if $a^2\left(1+m^2\right) - c^2 < 0$

Deduction:

(iv) The line $y = mx + c$ will be tangent to the circle if $c^2 = a^2\left(1+m^2\right)$ *or* $c = \pm a\sqrt{1+m^2}$

Theorem:

The point $P\left(x_1, y_1\right)$ lies outside, on or inside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

according as $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \begin{matrix} > \\ = \\ < \end{matrix} 0$

Proof:

Since $\left(-g, -f\right)$ and $r = \sqrt{f^2 + g^2 - c}$ are the centre and radius of circle respectively.

The point $P\left(x_1, y_1\right)$ lies outside on or inside the circle according as

$$m\left|\overline{CP}\right| \begin{matrix} > \\ = \\ < \end{matrix} r$$

$$\sqrt{\left(x_1 + g\right)^2 + \left(y_1 + f\right)^2} \begin{matrix} > \\ = \\ < \end{matrix} \sqrt{f^2 + g^2 - c}$$

Taking square on both sides

$$x^2{}_1 + 2gx_1 + g^2 + y^2{}_1 + 2fy_1 + f^2 \begin{matrix} > \\ = \\ < \end{matrix} g^2 + f^2 - c$$

$$x^2{}_1 + y^2{}_1 + 2gx_1 + 2fy_1 + c \begin{matrix} > \\ = \\ < \end{matrix} 0$$

Theorem:

Two tangents can be drawn to a circle from any point $P\left(x_1, y_1\right)$. The tangents are real and distinct, coincident or imaginary according as the point lies outside, on or inside the circle.

Proof:

Let an equation of the circle

$x^2 + y^2 = a^2$ ...(i)

As equation of tangent line of circle having slope *m* is.

$$y = mx + a\sqrt{1+m^2}$$

If this tangent passes through passes thought $P\left(x_1, y_1\right)$

$$y_1 = mx_1 + a\sqrt{1+m^2}$$
$$y_1 - mx_1 = a\sqrt{1+m^2}$$

Taking square on both sides

$$(y_1 - mx_1)^2 = a^2(1+m^2)$$

$$y^2{}_1 + m^2x^2{}_1 - 2mx_1y_1 = a^2 + a^2m^2$$

$$m^2x^2{}_1 - m^2a^2 - 2mx_1y_1 + y^2{}_1 - a^2 = 0$$

$$m^2\left(x^2{}_1 - a^2\right) - 2mx_1y_1 + \left(y^2{}_1 - a^2\right) = 0 \ldots \text{(ii)}$$

This being quadratic in m, gives two values of m and so there are two tangents from $P(x_1, y_1)$ to circle. These tangents are real and distinct, coincident or imaginary according as the roots of equation (ii) are real and distinct, coincident or imaginary. For nature of the roots of equation (ii), we examine the discriminant

Discriminant $= b^2 - 4ac$

Discriminant $= (-2x_1y_1)^2 - 4\left(x^2{}_1 - a^2\right)\left(y^2{}_1 - a^2\right)$

$$= 4x^2{}_1y^2{}_1 - 4\left[x^2{}_1y^2{}_1 - a^2x^2{}_1 - a^2y^2{}_1 + a^4\right]$$

$$= 4x^2{}_1y^2{}_1 - 4x^2{}_1y^2{}_1 + 4a^2x^2{}_1 + 4a^2y^2{}_1 - 4a^4$$

$$= 4a^2\left[x^2{}_1 + y^2{}_1 - a^2\right]$$

Roots of equation (ii) are real and distinct, coincident or imaginary according as:

$$= 4a^2\left[x^2{}_1 + y^2{}_1 - a^2\right] \gtreqless 0$$

$$x^2{}_1 + y^2{}_1 - a^2 \gtreqless 0$$

This shows $P(x_1, y_1)$ lies outside, on or inside the circle, accordingly.

Length of the tangent to a circle (Tangential Distance):

Let $P(x_1, y_1)$ be a point outside the circle

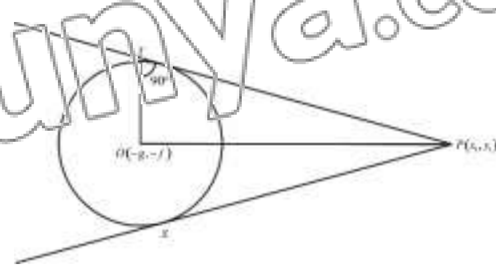
$x^2 + y^2 + 2gx + 2fy + c = 0 \ldots$(i)

Let $\overline{PT}$ be the length of Tangent or tangential distance then from right triangle CPT.

$$H^2 = B^2 + P^2$$

$$\left|\overline{CP}\right|^2 = \left|\overline{CT}\right|^2 + \left|\overline{PT}\right|^2$$

$$\left|\overline{PT}\right|^2 = \left|\overline{CP}\right|^2 - \left|\overline{CT}\right|^2$$



Length of tangent $= \left|\overline{PT}\right| = \sqrt{\left|\overline{CP}\right|^2 - \left|\overline{CT}\right|^2}$

$$= \sqrt{(x_1+g)^2 + (y_1+f)^2 - r^2}$$

$$= \sqrt{x_1^2 + g^2 + 2gx_1 + y_1^2 + f^2 + 2fy_1 - \left(g^2 + f^2 - c\right)}$$

$$= \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + g^2 + f^2 - g^2 - f^2 + c}$$
$$= \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

EXERCISE 6.2

Q.1 Write down equations of the tangent and normal to the circle

(i) $x^2 + y^2 = 25$ **at (4,3) and at** $(5\cos\theta, 5\sin\theta)$

Solution:

$x^2 + y^2 = 25$

Differentiating w.r.t. x,

$2x + 2y\dfrac{dy}{dx} = 0$

$x + y\dfrac{dy}{dx} = 0$

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{x}{y}$

The slope of tangent at $(4,3)$ is

$m = \left.\dfrac{dy}{dx}\right|_{(4,3)} = \dfrac{-4}{3}$

Thus an equation of tangent at $(4,3)$ is

$y - 3 = \dfrac{-4}{3}(x-4)$

$3y - 9 = -4x + 16$

$\Rightarrow \boxed{4x + 3y - 25 = 0}$

The slope of the normal at $(4,3)$ is

$m_1 = \dfrac{-1}{m} = \dfrac{-1}{-\frac{4}{3}} = \dfrac{3}{4}$

Thus an equation of normal at $(4,3)$ is

$y - 3 = \dfrac{3}{4}(x-4)$

$4y - 12 = 3x - 12$

$\Rightarrow \boxed{3x - 4y = 0}$

Now we find equations of tangent and normal at $(5\cos\theta, 5\sin\theta)$.

The slope of tangent at $P(5\cos\theta, 5\sin\theta)$ is

$m = \left.\dfrac{dy}{dx}\right|_P = -\dfrac{5\cos\theta}{5\sin\theta} = -\dfrac{\cos\theta}{\sin\theta}$

The equation of tangent to the circle is

$y - 5\sin\theta = -\dfrac{\cos\theta}{\sin\theta}(x - 5\cos\theta)$

Or

$y\sin\theta - 5\sin^2\theta = -x\cos\theta + 5\cos^2\theta$

$\Rightarrow x\cos\theta + y\sin\theta = 5\sin^2\theta + 5\cos^2\theta$

$\Rightarrow \boxed{x\cos\theta + y\sin\theta = 5}$

The slope of normal at $(5\cos\theta, 5\sin\theta)$

$m_1 = \dfrac{-1}{m} = \dfrac{-1}{\frac{-\cos\theta}{\sin\theta}} = \dfrac{\sin\theta}{\cos\theta}$

Thus an equation of normal is

$y - 5\sin\theta = \dfrac{\sin\theta}{\cos\theta}(x - 5\cos\theta)$

Or

$y\cos\theta - 5\sin\theta\cos\theta = x\sin\theta - 5\sin\theta\cos\theta$

$\Rightarrow \boxed{x\sin\theta - y\cos\theta = 0}$

(ii) $3x^2 + 3y^2 + 5x - 13y + 2 = 0$ at $\left(1, \dfrac{10}{3}\right)$

Solution:

$3x^2+3y^2+5x-13y+2=0$

Differentiating w.r.t. "x"

$$6x+6y\frac{dy}{dx}+5-13\frac{dy}{dx}=0$$

$$(6y-13)\frac{dy}{dx}=-(6x+5)$$

$$\Rightarrow \frac{dy}{dx}=-\frac{6x+5}{6y-13}$$

The slope of tangent at $P\left(1,\frac{10}{3}\right)$ is

$$m=\left.\frac{dy}{dx}\right|_P=-\frac{6(1)+5}{6\left(\frac{10}{3}\right)-13}$$

$$=-\frac{11}{20-13}=-\frac{11}{7}$$

The equation of tangent at $\left(1,\frac{10}{3}\right)$ is

$$y-\frac{10}{3}=-\frac{11}{7}(x-1)$$

$$\frac{3y-10}{3}=\frac{-11}{7}(x-1)$$

$$21y-70=-33x+33$$

$$\Rightarrow \boxed{33x+21y-103=0}$$

The slope of normal at $P\left(1,\frac{10}{3}\right)$ is

$$m_1=\frac{-1}{m}=\frac{-1}{\frac{-11}{7}}=\frac{7}{11}$$

Thus an equation of normal is

$$y-\frac{10}{3}=\frac{7}{11}(x-1)$$

$$\frac{3y-10}{3}=\frac{7}{11}(x-1)$$

$$33y-110=21x-21$$

$$\Rightarrow \boxed{21x-33y+89=0}$$

Q.2 Write down equation of the tangent and normal to the circle $4x^2+4y^2-16x+24y-117=0$ **at the point on the circle where abscissa is** -4**.**

Solution:

$4x^2+4y^2-16x+24y-117=0 \quad \ldots(i)$

Here abscissa is $x=-4$ first put $x=-4$ in given equation, we get

$$4(-4)^2+4y^2-16(-4)+24y-117=0$$

$$64+4y^2+64+24y-117=0$$

$$4y^2+24y+11=0$$

$$4y^2+22y+2y+11=0$$

$$2y(2y+11)+1(2y+11)=0$$

$$(2y+1)(2y+11)=0$$

$$2y+1=0,\ 2y+11=0$$

$$y=\frac{-1}{2},\ y=\frac{-11}{2}$$

So points $A\left(-4,\frac{-1}{2}\right)$ and $B\left(-4,\frac{-11}{2}\right)$ lie on the given circle

$4x^2+4y^2-16x+24y-117=0$

Differentiating w.r.t "x"

$$8x+8y\frac{dy}{dx}-16+24\frac{dy}{dx}=0$$

$$x+y\frac{dy}{dx}-2+3\frac{dy}{dx}=0$$

$$(y+3)\frac{dy}{dx}=2-x \Rightarrow \frac{dy}{dx}=\frac{2-x}{y+3}$$

The slope of tangent at $A\left(-4,\frac{-1}{2}\right)$ is

$$m=\left.\frac{dy}{dx}\right|_A=\frac{2-(-4)}{\frac{-1}{2}+3}=\frac{6}{\frac{5}{2}}=\frac{12}{5}$$

Thus an equation of tangent is

$$y+\frac{1}{2}=\frac{12}{5}(x+4)$$

$$\frac{2y+1}{2}=\frac{12}{5}(x+4)$$

$$10y+5=24x+96$$

$$\Rightarrow \boxed{24x-10y+91=0}$$

The slope of normal at $A\left(-4,\frac{-1}{2}\right)$ is

$$=m_1=\frac{-1}{m}=\frac{-1}{\frac{12}{5}}=\frac{-5}{12}$$

Thus equation of normal is

$$y+\frac{1}{2}=\frac{-5}{12}(x+4)$$

$$12y+6=-5x-20$$

$$\boxed{5x+12y+26=0}$$

The slope of tangent at $B\left(-4,\frac{-11}{2}\right)$ is

$$m=\left.\frac{dy}{dx}\right|_B=\frac{2-(-4)}{\frac{-11}{2}+3}=\frac{6}{\frac{-5}{2}}=\frac{-12}{5}$$

Thus an equation of tangent is

$$y+\frac{11}{2}=\frac{-12}{5}(x+4)$$

$$\frac{2y+11}{2}=\frac{-12}{5}(x+4)$$

$$10y+55=-24x-96$$

$$\Rightarrow \boxed{24x+10y+151=0}$$

The slope of normal at $B\left(-4,\frac{-11}{2}\right)$ is

$$m_1=\frac{-1}{m}=\frac{-1}{\frac{-12}{5}}=\frac{5}{12}$$

Thus equation of normal is

$$y+\frac{11}{2}=\frac{5}{12}(x+4)$$

$$12y+66=5x+20$$

$$\Rightarrow \boxed{5x-12y-46=0}$$

Q.3 Check the position of the point (5,6) with respect to the circle

(i) $x^2+y^2=81$

Solution:

$$x^2+y^2-81=0 \ldots\text{(i)}$$

Put $x=5, y=6$ in L.H.S of (i),

$$25+36-81<0$$

This shows that point lies inside the circle.

(ii) $2x^2+2y^2+12x-8y+1=0$

Solution:

$$2x^2+2y^2+12x-8y+1=0 \ldots\text{(i)}$$

Put $x=5, y=6$ in L.H.S of (i),

$$2(5)^2+2(6)^2+12(5)-8(6)+1$$

$$=50+72+60-48+1$$

$$=105>0$$

This shows that point lies outside the circle.

Q.4 Find the length of the tangent drawn from the point (–5, 4) to the circle. $5x^2+5y^2-10x+15y-131=0$

Solution:

$$5x^2+5y^2-10x+15y-131=0 \ldots\text{(i)}$$

Dividing both sides by 5 we get

$$x^2+y^2-2x+3y-\frac{131}{5}=0$$

Hence length of tangent

$$=\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$$

$$=\sqrt{(-5)^2+(4)^2-2(-5)+3(4)-\frac{131}{5}}$$

$$=\sqrt{25+16+10+12-\frac{131}{5}}$$

$$=\sqrt{63-\frac{131}{5}}$$

$$=\sqrt{\frac{315-131}{5}}=\sqrt{\frac{184}{5}}$$

Q.5 Find the length of the chord cut off from the line $2x+3y=13$ by the circle $x^2+y^2=26$

Solution:

$$x^2+y^2=26\ldots\text{(i)}$$

$$2x+3y=13\ldots\text{(ii)}$$

From (ii) $y=\frac{13-2x}{3}$ put in (i)

$$x^2+\left(\frac{13-2x}{3}\right)^2=26$$

$$x^2+\frac{169+4x^2-52x}{9}=26$$

$$\frac{9x^2+169+4x^2-52x}{9}=26$$

$$13x^2-52x+169=234$$

$$13x^2-52x-65=0$$

$$x^2-4x-5=0$$

$$x^2-5x+x-5=0$$

$$x(x-5)+1(x-5)=0$$

$$(x+1)(x-5)=0$$

$\Rightarrow \quad x+1=0, \quad x-5=0$

$\Rightarrow \quad \boxed{x=-1,5}$

When $x=-1$ then

$$y=\frac{13-2(-1)}{3}=\frac{13+2}{3}=\frac{15}{3}=5$$

When $x=5$ then

$$y=\frac{13-2(5)}{3}=\frac{3}{3}=1$$

So points of intersection of given line and circle are

$A(-1,5)$ and $B(5,1)$

Hence length of the chord $\overline{AB}$ is

$$\left|\overline{AB}\right|=\sqrt{(5+1)^2+(1-5)^2}$$

$$=\sqrt{36+16}$$

$$=\sqrt{52}$$

$$=2\sqrt{13}$$

Q.6 Find the coordinates of the points of intersection of the line $x+2y=6$ **with the circle** $x^2+y^2-2x-2y-39=0$

Solution:

$x^2+y^2-2x-2y-39=0 \ldots$(i)

$x+2y=6 \ldots$(ii)

From (ii) $y=\frac{6-x}{2}$ put in (i), we get

$$x^2+\left(\frac{6-x}{2}\right)^2-2x-2\left(\frac{6-x}{2}\right)-39=0$$

$$x^2+\frac{1}{4}(36+x^2-12x)-2x-(6-x)-39=0$$

$$x^2+9+\frac{x^2}{4}-3x-2x-6+x-39=0$$

$$\frac{5}{4}x^2-4x-36=0$$

$$5x^2-16x-144=0$$

$$x=\frac{16\pm\sqrt{256+2880}}{10}=\frac{16\pm56}{10}$$

$$x=\frac{16+56}{10},\frac{16-56}{10}$$

$\Rightarrow \boxed{x=\frac{36}{5},-4}$

When $x=\frac{36}{5}$ then

$$y=\frac{1}{2}\left(6-\frac{36}{5}\right)=\frac{-3}{5}$$

When $x=-4$ then

$$y=\frac{1}{2}(6+4)=y=5$$

Hence points of intersection are

$\left(\frac{36}{5},\frac{-3}{5}\right)$ and $(-4,5)$

Q.7 Find equation of the tangents to the circle $x^2+y^2=2$

(i) Parallel to $x-2y+1=0$

Solution:

$x^2+y^2=2 \ldots$(i)

Let equation of tangent to (i) be

$$y=mx\pm a\sqrt{1+m^2}$$

Here $a=\sqrt{2}$

When tangent line is parallel to $x-2y+1=0$

Here slope of given line $m=\frac{-1}{-2}=\frac{1}{2}$

So, slope of tangent line $m=\frac{1}{2}$

Hence equation of tangent line is

$$y=\frac{1}{2}x\pm\sqrt{2}\sqrt{1+\frac{1}{4}}$$

$$y=\frac{1}{2}x\pm\sqrt{2}\sqrt{\frac{5}{4}}$$

$$y=\frac{1}{2}x\pm\frac{\sqrt{10}}{2}$$

$$2y=x\pm\sqrt{10}$$

Hence equations of tangent lines are

$$2y=x+\sqrt{10}\quad\Rightarrow\boxed{x-2y+\sqrt{10}=0}$$

$$2y=x-\sqrt{10}\quad\Rightarrow\boxed{x-2y-\sqrt{10}=0}$$

(ii) Perpendicular to the line

$3x+2y=6$

Solution:

$3x+2y=6$

slope of given line is $m_1=\frac{-3}{2}$

So slope of required tangent line is

$$m=-\frac{1}{m_1}=\frac{-1}{\frac{-3}{2}}=\frac{2}{3}$$

Hence equations of tangent line are

$$y=mx\pm a\sqrt{1+m^2}$$

put $a=\sqrt{2},m=\frac{2}{3}$, we get

$$y=\frac{2x}{3}\pm\sqrt{2}\times\sqrt{1+\left(\frac{2}{3}\right)^2}$$

$$y=\frac{2x}{3}\pm\sqrt{2}\times\frac{\sqrt{13}}{3}$$

$$3y=2x\pm\sqrt{26}$$

Hence equation of tangent lines are

$$3y=2x+\sqrt{26}$$

$$\Rightarrow\boxed{2x-3y+\sqrt{26}=0}$$

And $3y=2x-\sqrt{26}$

$$\Rightarrow\boxed{2x-3y-\sqrt{26}=0}$$

Q.8 Find equations of tangents drawn from

(i) $(0,5)$ **to** $x^2+y^2=16$

Also find the points of contact.

Solution:

$x^2+y^2=16\ldots$(i)

Let (x_1,y_1) be any point on the circle then

$x_1^{\,2}+y_1^{\,2}=16\ldots$(ii)

Now tangent to the circle (i) at (x_1,y_1) is

$xx_1+yy_1=16\ldots$(iii)

If this tangent passes through the $(0,5)$

Then $0x_1+5y_1=16$

$\Rightarrow\boxed{y_1=\frac{16}{5}}$ Putting in (ii), we get

$$x_1^2+\left(\frac{16}{5}\right)^2=16$$

$$\Rightarrow x_1=\pm\sqrt{16-\left(\frac{16}{5}\right)^2}$$

$$\Rightarrow x_1=\pm\frac{12}{5}$$

Thus the required points of contact are $\left(\frac{12}{5},\frac{16}{5}\right)$ and $\left(-\frac{12}{5},\frac{16}{5}\right)$

At $(x_1,y_1)=\left(\frac{12}{5},\frac{16}{5}\right)$,

(iii) becomes

$$\frac{12}{5}x+\frac{16}{5}y=16$$

$\Rightarrow 12x+16y=5\times16$

$\Rightarrow \boxed{3x+4y=20}$

At $(x_1,y_1)=\left(-\frac{12}{5},\frac{16}{5}\right)$,

(iii) becomes

$-\frac{12}{5}x+\frac{16}{5}y=16$

$\Rightarrow -12x+16y=5\times16$

$\Rightarrow \boxed{3x-4y+20=0}$

(ii) $(-1,2)$ **to** $x^2+y^2+4x+2y=0$

Solution:

$x^2+y^2+4x+2y=0 \ldots$(i)

Let (x_1,y_1) be any point on the circle then

$x_1^2+y_1^2+4x_1+2y_1=0 \ldots$(ii)

Now tangent to the circle (i) at (x_1,y_1) is

$xx_1+yy_1+4\left(\frac{x+x_1}{2}\right)+2\left(\frac{y+y_1}{2}\right)=0$

$\Rightarrow xx_1+yy_1+2(x+x_1)+(y+y_1)=0 \ldots$(iii)

If this tangent passes through $(-1,2)$ then

$-1x_1+2y_1+2(x_1-1)+(y_1+2)=0$

$\Rightarrow x_1+3y_1=0$

$\Rightarrow \boxed{x_1=-3y_1}$ Putting in (ii), we get

$-3y_1^2+y_1^2-12y_1+2y_1=0$

$\Rightarrow 10y_1^2-10y_1$

$\Rightarrow y_1(y_1-1)=0$

$\Rightarrow \boxed{y_1=0,1}$

Since $x_1=-3y_1$

When $y_1=0$ then $x_1=0$ and

When $y_1=1$ then $x_1=-3$

At $(x_1,y_1)=(0,0)$,

(iii) becomes

$\Rightarrow \boxed{2x+y=0}$

Thus the required points of contact are $(0,0)$ and $(-3,1)$

At $(x_1,y_1)=(-3,1)$,

(iii) becomes

$-3x+y+2(x-3)+y+1=0$

$\Rightarrow \boxed{x-2y+5=0}$

(iii) $(-7,-2)$ **to** $(x+1)^2+(y-2)^2=26$

Solution:

$(x+1)^2+(y-2)^2=26$

$\Rightarrow x^2+y^2+2x-4y+1+4=26$

$\Rightarrow x^2+y^2+2x-4y=21 \ldots$ (i)

Let (x_1,y_1) be any point on the given circle.

Then

$x_1^2+y_1^2+2x_1-4y_1=21 \ldots$(ii)

Now tangent to (i) at (x_1,y_1) is

$xx_1+yy_1+2\left(\frac{x+x_1}{2}\right)-4\left(\frac{y+y_1}{2}\right)=21$

$\boxed{xx_1+yy_1+(x+x_1)-2(y+y_1)=21} \ldots$(iii)

If this tangent passes through $(-7,-2)$

Then

$-7x_1-2y_1+(-7+x_1)-2(-2+y_1)=21$

$\Rightarrow -6x_1-4y_1-7+4=21$

$\Rightarrow -6x_1-4y_1=24$

$\Rightarrow 3x_1+2y_1=-12$

$\Rightarrow y_1=-\left(\frac{3x_1+12}{2}\right) \ldots$(iv)

Putting in (ii), we get

$x_1^2+\left(\frac{3x_1+12}{2}\right)^2+2x_1+4\left(\frac{3x_1+12}{2}\right)=21$

$$x_1^2+\left(\frac{9x_1^2+72x_1+144}{4}\right)+2x_1+2(3x_1+12)=21$$

$$\Rightarrow 4x_1^2+9x_1^2+72x_1+144+8x_1+24x_1+96=84$$

$$13x_1^2+104x_1+240=84$$

$$13x_1^2+104x_1+156=0$$

$$x_1^2+8x_1+12=0$$

$$(x_1+6)(x_1+2)=0$$

$$\Rightarrow \quad \boxed{x_1=-2,-6}$$

Putting in (iv), we get

When $x_1=-2$ then	When $x_1=-6$ then
$y_1=-\frac{3(-2)+12}{2}=-3$	$y_1=-\frac{3(-6)+12}{2}=3$

Thus the points of contact are

$(-6,3)$ and $(-2,-3)$

Now

For $(x_1,y_1)=(-6,3)$,	For $(x_1,y_1)=(-2,-3)$,
(iii) becomes	(iii) becomes
$-6x+3y+(x-6)-2(y+3)=21$	$-2x-3y+(x-2)-2(y-3)=21$
Or $-5x+y-12=21$	Or $-x-5y+4=21$
$\Rightarrow \boxed{5x-y+33=0}$	$\Rightarrow \boxed{x+5y+17=0}$

Q.9 Find an equation of the chord of contact of the tangent drawn from (4,5) to the circle $2x^2+2y^2-8x+12y+21=0$

Solution:

$$2x^2+2y^2-8x+12y+21=0$$

Dividing both sides by 2, we get

$$x^2+y^2-4x+6y+\frac{21}{2}=0 \quad \ldots(i)$$

Let the points of contact of the two tangent be $P(x_1,y_1)$ and $Q(x_2,y_2)$

Now equation of the tangent at point $P(x_1,y_1)$ is

$$xx_1+yy_1-4\left(\frac{x+x_1}{2}\right)+6\left(\frac{y+y_1}{2}\right)+\frac{21}{2}=0$$

$$xx_1+yy_1-2(x+x_1)+3(y+y_1)+\frac{21}{2}=0$$

As $(4,5)$ lies on this line so,

$$4x_1+5y_1-2(4+x_1)+3(5+y_1)+\frac{21}{2}=0$$

$$8x_1+10y_1-16-4x_1+30+6y_1+21=0$$

$$4x_1+16y_1+35=0 \quad \ldots(ii)$$

Similarly equation of tangent at point $Q(x_2,y_2)$ is

$$xx_2+yy_2-2(x+x_2)+3(y+y_2)+\frac{21}{2}=0$$

As $(4,5)$ lies on this line so

$$4x_2+5y_2-2(4+x_2)+3(5+y_2)+\frac{21}{2}=0$$

$$8x_2+10y_2-16-4x_2+30+6y_2+21=0$$

$$4x_2+16y_2+35=0 \quad \ldots(iii)$$

From (ii) and (iii)

It is clear that both $P(x_1,y_1)$ and $Q(x_2,y_2)$ lie on the same line $4x+16y+35=0$

So, $4x+16y+35=0$ is the required equation of chord of contacts of the tangents.

ANALYTIC PROOFS OF IMPORTANT PROPERTIES OF A CIRCLE:

Theorem 1: Length of a diameter of the circle $x^2+y^2=a^2$ is $2a$.

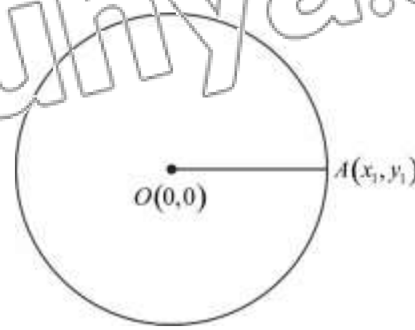
Proof:

Equation of circle with radius a is

$x^2+y^2=a^2$

Let $A(x_1,y_1)$ be any point on the circle

then $x_1^2+y_1^2=a^2$



Now $|OA|=\sqrt{(x_1-0)^2+(y_1-0)^2}$

$=\sqrt{x_1^2+y_1^2}$

$=\sqrt{a^2}$

$|OA|=a$

but the diameter of circle is

$d=2(\text{radius})=2|OA|$

$d=2a$

Theorem 2: Perpendicular dropped from the centre of a circle on a chord bisects the chord.

Proof:

Consider a circle (with centre at $O(0,0)$ and having radius "a")

$x^2+y^2=a^2$ …(i)

Consider the chord joining the points $P(a,0)$ and $Q(0,-a)$

Draw perpendicular OL from centre to the chord

Slope of chord $PQ=\dfrac{0-(-a)}{a-0}=\dfrac{a}{a}=1$

Slope of $OL=-1$

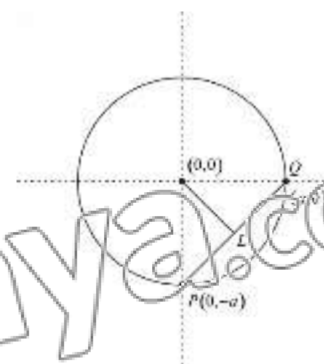
equation of OL is

$y-y_1=m(x-x_1)$

$y-0=-1(x-0)$

$y=-x$

$x+y=0$



Mid-point of PQ has coordinates $\left(\dfrac{0+a}{2},\dfrac{-a+0}{2}\right)=\left(\dfrac{a}{2},-\dfrac{a}{2}\right)$

The line OL passes through mid-point of PQ if the point $\left(\dfrac{a}{2},-\dfrac{a}{2}\right)$ will satisfy its equation

i.e. $x+y=0$

$\dfrac{a}{2}-\dfrac{a}{2}=0$

Hence proved

Theorem 3:

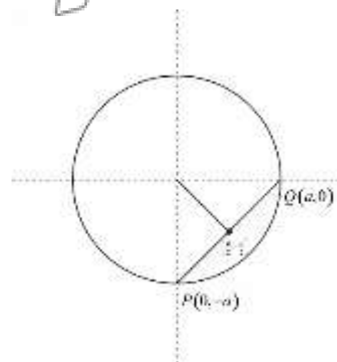
The perpendicular bisector of any chord of a circle passes through the centre of the circle.

Proof:

Consider a circle (with centre at $O(0,0)$ and having radius "a")

$x^2+y^2=a^2 \ldots$(i)

Consider the chord PQ



Mid-point of PQ has coordinates $\left(\frac{0+a}{2},\frac{0-a}{2}\right)=\left(\frac{a}{2},-\frac{a}{2}\right)$

Slope of chord $PQ=\frac{0-(-a)}{a-0}=\frac{a}{a}=1$

$\therefore$ slope of the perpendicular bisector = –1

For equation of perpendicular bisector

$y-y_1=m(x-x_1)$

$y+\frac{a}{2}=-1\left(x-\frac{a}{2}\right)$

$y+\frac{a}{2}=-x+\frac{a}{2}$

$y=-x$

$x-y=0$

Clearly it passes through origin

Hence proved

Theorem 4:

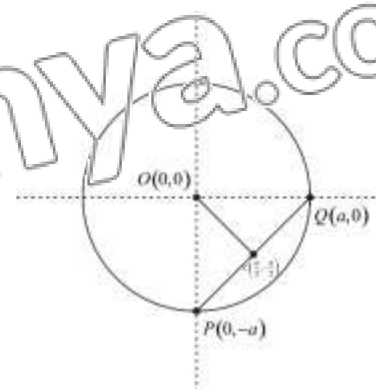
The line joining the centre of a circle to the midpoint of a chord is perpendicular to the chord.

Proof:

Consider a circle (with centre at $O(0,0)$ and having radius "a")

$x^2+y^2=a^2 \ldots$(i)

Consider the chord PQ



Mid-point of PQ has coordinates $\left(\frac{0+a}{2},\frac{0-a}{2}\right)=\left(\frac{a}{2},-\frac{a}{2}\right)$

Slope of chord $PQ=\frac{0-(-a)}{a-0}=\frac{a}{a}=1$

Slope of $OC=\frac{\frac{-a}{2}-0}{\frac{a}{2}-0}=-1$

(Slope of chord PQ) $\times$ (slope of OC) $=1\times-1=-1$

$\therefore\ \overline{OC} \perp \overline{PQ}$

Hence proved

Theorem 5:

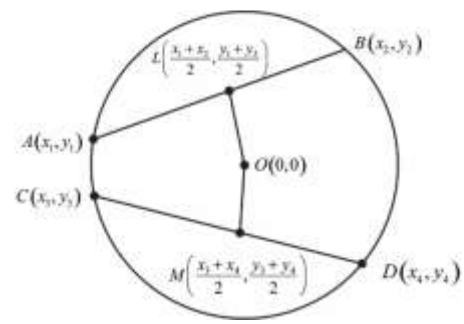
Congruent chords of a circle are equidistant from the centre.

Proof:

Consider a circle (with centre at $O(0,0)$ and having radius "a")

$x^2 + y^2 = a^2$ ... (i)

Consider two congruent chords $|AB|$ and $|CD|$



Then $|AB| = |CD|$

$|AB|^2 = |CD|^2$

Using distance formula

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = (x_3 - x_4)^2 + (y_3 - y_4)^2$$

$$x_1^2 + y_1^2 + x_2^2 + y_2^2 - 2x_1x_2 - 2y_1y_2 = x_3^2 + y_3^2 + x_4^2 + y_4^2 - 2x_3x_4 - 2y_3y_4$$

$$a^2 + a^2 - 2x_1x_2 - 2y_1y_2 = a^2 + a^2 - 2x_3x_4 - 2y_3y_4$$

$$-2(x_1x_2 + y_1y_2) = -2(x_3x_4 + y_3y_4)$$

$$(x_1x_2 + y_1y_2) = (x_3x_4 + y_3y_4) \ldots \text{(i)}$$

Now draw perpendicular from centre to the chord $|AB|$ and $|CD|$

We know that perpendicular drawn from centre to the chord bisects the chord

Now $|OL|^2 = \left(\frac{x_1 + x_2}{2} - 0\right)^2 + \left(\frac{y_1 + y_2}{2} - 0\right)^2$

$$= \frac{x_1^2 + y_1^2 + x_2^2 + y_2^2 + 2x_1x_2 + 2y_1y_2}{4}$$

$$= \frac{a^2 + a^2 + 2(x_1x_2 + y_1y_2)}{4}$$

$$= \frac{2(a^2 + x_1x_2 + y_1y_2)}{4}$$

$$= \frac{(a^2 + x_1x_2 + y_1y_2)}{2} \ldots \text{(ii)}$$

$$|OM|^2 = \left(\frac{x_3 + x_4}{2} - 0\right)^2 + \left(\frac{y_3 + y_4}{2} - 0\right)^2$$

$$= \frac{x_3^2 + y_3^2 + x_4^2 + y_4^2 + 2x_3x_4 + 2y_3y_4}{4}$$

$$= \frac{a^2 + a^2 + 2(x_3x_4 + y_3y_4)}{4}$$

$$= \frac{2(a^2 + x_1x_2 + y_1y_2)}{4} \qquad \because (x_1x_2 + y_1y_2) = (x_3x_4 + y_3y_4)$$

$$= \frac{(a^2 + x_1x_2 + y_1y_2)}{2} \quad \ldots(iii)$$

From (ii) and (iii)

$$|OL|^2 = |OM|^2$$

$$|OL| = |OM|$$

i.e. congruent chords are equidistant from centre

Theorem 6:

Show that measure of the central angle of a minor arc is double the measure of the angle subtended in the corresponding major arc.

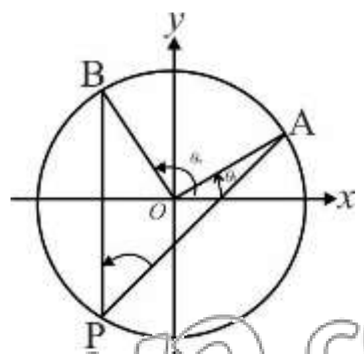
Proof:

Consider a circle (with centre at $O(0,0)$ and having radius "a")

$x^2 + y^2 = a^2 \ldots(i)$

$A(a\cos\theta_1, a\sin\theta_1)$ and B $(a\cos\theta_2, a\sin\theta_2)$ be end points of a minor arc AB. Let $P(a\cos\theta, a\sin\theta)$ be a point on the major arc and $m\angle AOB = \theta_2 - \theta_1$

we have to show that $m\angle APB = \frac{1}{2}(\theta_2 - \theta_1)$



$m_1 =$ Slope of $\overline{AP} = \frac{a(Sin\theta - Sin\theta_1)}{a(\cos\theta - \cos\theta_1)}$

$$= \frac{2\cos\left(\frac{\theta+\theta_1}{2}\right)\sin\left(\frac{\theta-\theta_1}{2}\right)}{-2\sin\left(\frac{\theta+\theta_1}{2}\right)sin\left(\frac{\theta-\theta_1}{2}\right)}$$

$$= -\cot\left(\frac{\theta+\theta_1}{2}\right)$$

$$= \tan\left(\frac{\pi}{2} + \frac{\theta+\theta_1}{2}\right)$$

Similarly,

$m_2 =$ Slope of $\overline{BP} = \tan\left(\frac{\pi}{2} + \frac{\theta+\theta_2}{2}\right)$

$$\tan(\angle APB) = \frac{m_2 - m_1}{1 + m_1m_2}$$

$$=\frac{\tan\left(\frac{\pi}{2}+\frac{\theta+\theta_2}{2}\right)-\tan\left(\frac{\pi}{2}+\frac{\theta+\theta_1}{2}\right)}{1+\tan\left(\frac{\pi}{2}+\frac{\theta+\theta_1}{2}\right)\tan\left(\frac{\pi}{2}+\frac{\theta+\theta_2}{2}\right)}$$

$$\because \quad \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}=\tan(\alpha-\beta)$$

$$=\tan\left(\frac{\pi}{2}+\frac{\theta+\theta_2}{2}-\frac{\pi}{2}-\frac{\theta+\theta_1}{2}\right)$$

$$\tan(<APB)=\tan\left(\frac{\theta_2-\theta_1}{2}\right)$$

$$m<APB=\frac{1}{2}(\theta_2-\theta_1)$$

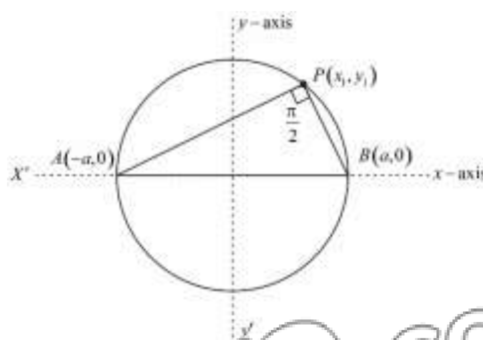
Hence $m\angle APB=\frac{1}{2}m\angle AOB$

Theorem 7:
An angle in a semi circle is a right angle
Proof:

Consider a circle (with centre at $O(0,0)$ and having radius "a")

$x^2+y^2=a^2$

Let $P(x_1,y_1)$ be any point of circle then $x_1^2+y_1^2=a^2$ …(i)



Slope of $\overline{AP}=m_1=\frac{y_1-0}{x_1+a}=\frac{y_1}{x_1+a}$

Slope of $\overline{PB}=m_2=\frac{y_1-0}{x_1-a}=\frac{y_1}{x_1-a}$

(Slope of AP) (Slop of PB) $=\frac{y_1}{x_1+a}\times\frac{y_1}{x_1-a}$

$$=\frac{y_1^2}{x_1^2-a^2}$$

$$=\frac{a^2-x_1^2}{x_1^2-a^2}$$

$$=-\frac{x_1^2-a^2}{x_1^2-a^2}=-1$$

$\therefore \overline{AP}\perp\overline{PB}$

Hence APB is a right triangle

Theorem 8:

The tangent to a circle at any point of the circle is perpendicular to the radial segment at that point.

Proof:

Let AB be the tangent to the circle $x^2+y^2=a^2$

at any point $A(x_1,y_1)$ laying on it.

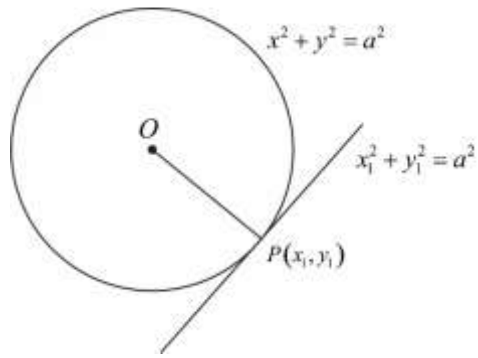
Now we have to show that $\overline{OA}\perp\overline{AB}$

$x^2+y^2=a^2$

Differentiate w.r.t 'x'

$2x+2y\dfrac{dy}{dx}=0$

$\dfrac{dy}{dx}=\dfrac{-x}{y}$



$m_1=$ slope of the tangent at $A=\left.\dfrac{dy}{dx}\right|_{A(x_1,y_1)}=-\dfrac{x_1}{y_1}$

$m_2=$ slope of $\overline{OA}=\dfrac{y_1-0}{x_1-0}=\dfrac{x_1}{y_1}$

$m_1m_2=\dfrac{-x_1}{y_1}\times\dfrac{y_1}{x_1}=-1$

Thus $\overline{OA}\perp\overline{AB}$

Theorem 9:

The perpendicular at the outer end of a radial segment in tangent to the circle.

Proof:

Consider a circle (with centre at $O(0,0)$ and having radius "a")

$x^2+y^2=a^2$

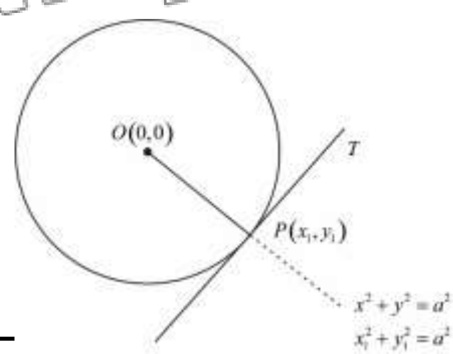
Let $P(x_1,y_1)$ be any point of circle then $x_1^2+y_1^2=a^2$ …(i)

Join O to P, Then we have radial segment $\overline{OP}$ and P is the outer end of the radial segment

Draw perpendicular $\overline{PT}$ through P at the radial segment.

Slope of $\overline{OP}=\dfrac{y_1-0}{x_1-0}=\dfrac{y_1}{x_1}$

Slope of $\overline{PT}=-\dfrac{x_1}{y_1}$



Equation of PT through the point $P(x_1, y_1)$ is

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1)$$

$$yy_1 - y_1^2 = -xx_1 + x_1^2$$

$$x_1^2 + y_1^2 = xx_1 + yy_1$$

$$a^2 = xx_1 + yy_1 \quad (\text{Which is equation of tangent at } P(x_1, y_1))$$

Hence proved

Note:

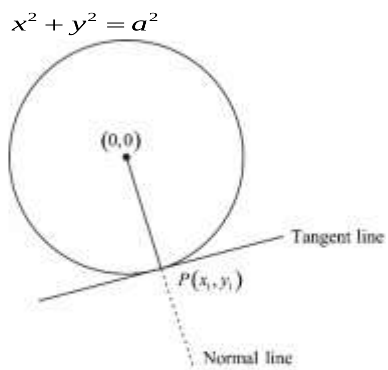
(i) A number G is said to be geometric mean (or mean proportional) between two numbers a and b if a, G, b are in G.P.

(ii) G.M between two numbers a and b is given by $G = \pm\sqrt{ab}$.

EXERCISE 6.3

Q.1 Prove that normal lines of a circle pass through the centre of circle.

Proof: Consider a circle (with centre at $O(0,0)$ and having radius "a")



Equation of normal at $P(x_1, y_1)$ is

$$xy_1 - x_1y = 0$$

It will pass through origin $(0,0)$, if it satisfy its equation

$$\text{L.H.S.} = xy_1 - x_1y$$

$$= (0)y_1 - x_1(0)$$

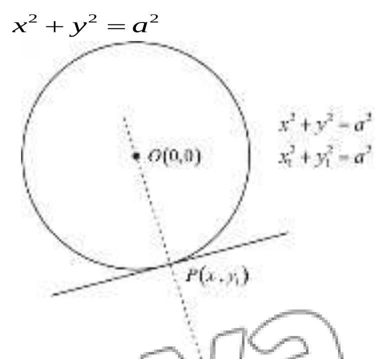
$$= 0$$

Hence proved

Q.2 Prove that the straight line drawn from the center of a circle perpendicular to the tangent passes through the point of tangency.

Proof:

Consider a circle (with centre at $O(0,0)$ and having radius "a")



Differentiate w.r.t. x

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

$m =$ slope of tangent at $P(x_1, y_1)$

$$\left.\frac{dy}{dx}\right|_P = -\frac{x_1}{y_1}$$

Slope of line perpendicular to tangent is

$$m_1 = -\frac{1}{m} = \frac{y_1}{x_1}$$

Equation through O and perpendicular to the tangent is

$$y - 0 = \frac{y_1}{x_1}(x - 0)$$

$$x_1 y = y_1 x$$

It will passes through $P(x_1, y_1)$

If it satisfy its equation

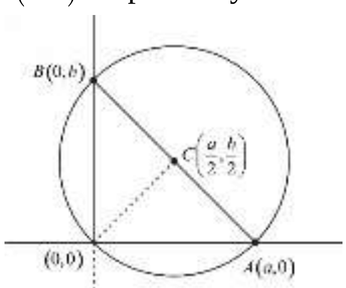
i.e. $x_1 y_1 = y_1 x_1$

L.H.S. = R.H.S.

Q.3 Prove that the midpoint of the hypotenuse of a right angle triangle is the circumcentre of the triangle.

Proof: Consider OAB is a right triangle with right angle at origin $O(0,0)$.

Consider a circumcircle passes through the vertices $O(0,0), A(a,0)$ and $B(0,b)$ respectively



AB is hypotenuse then mid point of AB is $C\left(\frac{a}{2}, \frac{b}{2}\right)$

Now we show that $|\overline{OC}| = |\overline{AC}| = |\overline{BC}|$

Here

$$|OC| = \sqrt{\left(0 - \frac{a}{2}\right)^2 + \left(0 - \frac{b}{2}\right)^2}$$

$$= \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} = \frac{\sqrt{a^2 + b^2}}{2}$$

$$|AC| = \sqrt{\left(a - \frac{a}{2}\right)^2 + \left(0 - \frac{b}{2}\right)^2}$$

$$= \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2} = \frac{\sqrt{a^2 + b^2}}{2}$$

$$|BC| = \sqrt{\left(0 - \frac{a}{2}\right)^2 + \left(b - \frac{b}{2}\right)^2}$$

$$= \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2} = \frac{\sqrt{a^2 + b^2}}{2}$$

As $|OC| = |AC| = |BC|$

This shows that the midpoint of the hypotenuse is a circumcentre of the triangle.

Q.4 Prove that the perpendicular dropped from a point of a circle on a diameter is a mean proportional between the segments into which it divides the diameter.

Proof:

Consider a circle (with centre at $O(0,0)$ and having radius "a")

$$x^2 + y^2 = a^2$$

Draw perpendicular PQ on the diameter and let its divides the diameter into two segments AQ and BQ

Take $x^2 + y^2 = a^2$

$$y^2 = a^2 - x^2$$

$$|PQ||PQ| = (a - x)(a + x)$$

$$|PQ||PQ| = |AQ||BQ|$$

$$\frac{|PQ|}{|AQ|} = \frac{|BQ|}{|PQ|}$$

Hence prove