**Conic Section:**

Let $L$ be a fixed line in a plane and $F$ be a fixed point not on the line $L$. Suppose $|PM|$ denotes the distance of a point $P(x,y)$ from the line $L$. The set of all points $P$ in the plane such that

$\frac{|PF|}{|PM|} = e$ (a positive constant) is called a conic section.

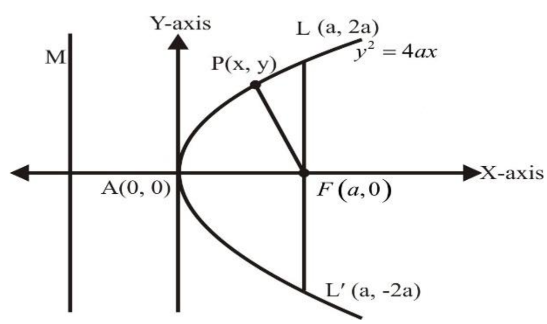
The fixed point is called the **Focus** and the fixed line is called the **Directrix** of the conic. The constant ratio is called the **Eccentricity** of the conic and is denoted by ***e***.

- If $e = 1$, then conic is called parabola.
- If $0 < e < 1$, then the conic is called ellipse.
- If $e > 1$, then the conic is called hyperbola
- If $e = 0$, then the conic is called circle
- If $e = \infty$, then the conic is called pair of straight lines.

**Parabola:**

A parabola is the locus of a point which moves in a plane such that its distance from a fixed point (called the focus) is equal to its distance form a fixed straight line (called the Directrix).

i.e. $\frac{|PF|}{|PM|} = 1$




**Standard form of equation of Parabola:**

If we take the focus of the parabola as $F(a,0)$, $a > 0$ and its Directrix as line $L$ whose equation is $x = -a$ then equation of parabola is $y^2 = 4ax$

**Proof:**

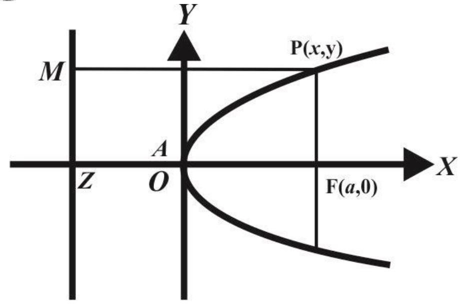
Let $P(x,y)$ be a point on the parabola.

So, by definition

$\frac{|PF|}{|PM|} = 1$

Or $|PF| = |PM|$ ...(i)

From figure



$|PM| = \sqrt{(x+a)^2 + (y-y)^2}$

$|PM| = x + a$ ...(ii)

And $|PF| = \sqrt{(x-a)^2 + (y-0)^2}$

$|PF| = \sqrt{(x-a)^2 + y^2}$

Putting in equation (i)

$x + a = \sqrt{(x-a)^2 + y^2}$

Taking square on both sides

$(x+a)^2 = (x-a)^2 + y^2$

or $y^2 = (x+a)^2 - (x-a)^2$

$y^2 = x^2 + 2ax + a^2 - (x^2 - 2ax + a^2)$

$y^2 = x^2 + 2ax + a^2 - x^2 + 2ax - a^2$

$\boxed{y^2 = 4ax}$

Similarly other three standard forms can be proved.

**Important Terms:**

**Axis:**

The straight line passing through the focus and perpendicular to the Directrix of the conic is known as its **axis**.

**Vertex:**

A point of intersection of a conic with its axis is known as **vertex** of the conic.

**Centre:**

The point which bisects every chord of the conic passing through it, is called the **centre** of the conic.

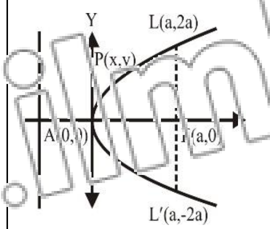
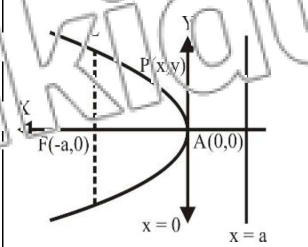
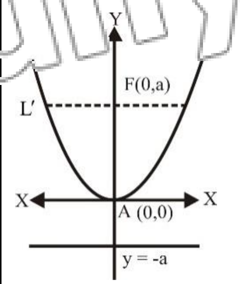
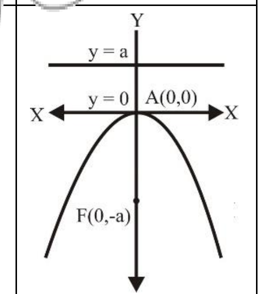
**Focal Chord:**

A chord passing through the focus is knows as **focal chord** of the conic.

**Latus Rectum:**

The focal chord which is perpendicular to the axis is known as latus rectum of the conic.

**Summary of Standard Parabolas:**

| | | | | |
|---|---|---|---|---|
| Standard Equation | $y^2 = 4ax,\ (a>0)$ | $y^2 = -4ax,\ (a>0)$ | $x^2 = 4ay, (a>0)$ | $x^2 = -4ay, (a>0)$ |
| Shape of the parabola |  |  |  |  |
| Vertex | $A(0, 0)$ | $A$ (0, 0`) | $A$ (0, 0) | $A$ (0, 0) |
| Focus | $F(a,0)$ | $F(-a,0)$ | $F(0,a)$ | $F(0,-a)$ |
| Equation of DTX | $x=-a$ | $x=a$ | $y=-a$ | $y=a$ |
| Equation of axis | $y = 0$ | $y = 0$ | $x = 0$ | $x = 0$ |
| Length of latus rectum | $4a$ | $4a$ | $4a$ | $4a$ |
| Extremities of latus rectum | $(a,\pm 2a)$ | $(-a,\pm 2a)$ | $(\pm 2a,a)$ | $(\pm 2a,-a)$ |
| Equation of latus rectum | $x = a$ | $x = -a$ | $y = a$ | $y = -a$ |
| Equation of tangent to vertex | $x = 0$ | $x = 0$ | $y = 0$ | $y = 0$ |
| Parametric co-ordinates | $(at^2, 2at)$ | $(-at^2,\ 2at)$ | $(2at,\ at^2)$ | $(2at, -at^2)$ |
| Eccentricity | 1 | 1 | 1 | 1 |
| Symmetry | about $x$ –axis | about $x$ – axis | about – $y$-axis | about – $y$-axis |

**Theorem:**
**The point of a parabola which is closest to the focus is the vertex of the parabola.**
**Proof:**

Let the parabola be
$x^2 = 4ay \qquad (a > 0)$
With focus at $F(0,a)$ and $P(x,y)$ be any point on the parabola

$$|PF| = \sqrt{(x-0)^2 + (y-a)^2}$$
$$= \sqrt{x^2 + y^2 - 2ay + a^2}$$
$$= \sqrt{4ay + y^2 - 2ay + a^2}$$
$$= \sqrt{y^2 + 2ay + a^2}$$
$$= \sqrt{(y+a)^2}$$
$$|PF| = y + a$$

Since $y$ can take up only non-negative values, $|PF|$ is minimum when $y = 0$. Thus $P$ coincides with *O*, thus the vertex of the parabola is closest to the focus.

**Example 1:**
**A comet has a parabolic orbit with the sun at the focus. When the comet is 100 million km from the sun, the line joining the sun and the comet makes an angle of 60° with the axis of the parabola. How close will the comet get to the sun?**

**Solution:**

Let the sun be at the focus, let the coordinates of the vertex be $(-a,0)$ and corresponding directrix line is $x = -2a$
If the comet is at $P(x,y)$
then $|PF| = |PM| \dots$ (i)

$$|PM| = \sqrt{(x+2a)^2 + (y-y)^2}$$
$$= \sqrt{(x+2a)^2}$$
$$|PM| = x + 2a \dots \text{(ii)}$$

Also $\dfrac{x}{|PF|} = \cos 60°$

$x = |PF|\cos 60°$
Put in equation (ii)
$|PM| = |PF|\cos 60° + 2a$
Equation (i) becomes
$$|PF| = |PF|\cos 60° + 2a$$
$$2a = |PF| - |PF|\cos 60°$$
$$2a = |PF|(1 - \cos 60°)$$

$$2a = 100,000,000\left(1-\frac{1}{2}\right)$$

$$a = \frac{100,000,000}{2}\left(\frac{1}{2}\right)$$

$$\boxed{a = 25,000,000}$$

**Example 2:**

**A suspension bridge with weight uniformly distributed along the length has two towers of 100 m height above the road surface and are 400 m apart. The cables are parabolic in shape and are tangent to road surface at the centre of the bridge. Find the height of the cables at a point 100 m from the centre.**

**Solution:**

The parabola formed by the cables has vertex at $(0,0)$ and focus on $y$-axis

The equation of this parabola is

$x^2 = 4ay \ldots$(i)

The point $(200,100)$ lies on the parabola

$\therefore \ (200)^2 = 4a(100)$

$40000 = 400a$

$a = 100$

Equation (i) becomes

$x^2 = 4(100)y$

$x^2 = 400y \ldots$(ii)

The height of cable when $x = 100$ is

$(100)^2 = 400y$

$10000 = 400y$

$$\boxed{y = 25m}$$

**Reflecting Property:**

If a light source is placed at the focus of a parabolic reflecting surface, then a light ray travelling from focus to any point on the parabola will be reflected in the direction parallel to the axis of parabola.

Another application of the parabola is in suspension bridge. The main cables are of parabolic shape. The total weight of the bridge is uniformly distributed along its length if the shape of the cables is parabolic. Cables in any other shape will not carry the weight evenly.

## EXERCISE 6.4

**Q.1 Find the focus vertex and Directrix of the parabola sketch its graph.**

**(i)** $y^2 = 8x$

**Solution:**

$y^2 = 8x \ldots$(i)

Compare it with $y^2 = 4ax$

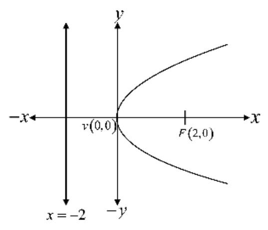
$4a = 8 \Rightarrow \boxed{a = 2}$

Focus of (i) is $F(a,0) = F(2,0)$

Vertex of (i) is $V(0,0)$

Equation of directrix is $x = -a$

$\Rightarrow x = -2$




Graph of $y^2 = 8x$

**(ii)** $x^2 = -16y$

**Solution:**

$x^2 = -16y \ldots$(i)

Compare it with $x^2 = -4ay$

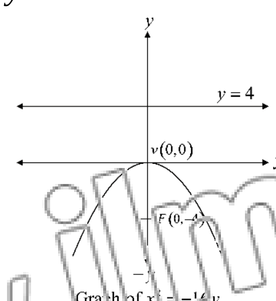
$4a = 16 \Rightarrow \boxed{a = 4}$

Focus of (i) is $F(0,-a) = F(0,-4)$

Vertex of (i) is $V(0,0)$

Equation of directrix is $y = a$

$\Rightarrow y = 4$




Graph of $x^2 = -16y$

**(iii)** $x^2 = 5y$

**Solution:**

$x^2 = 5y \ldots$(i)

Compare it with $x^2 = 4ay$

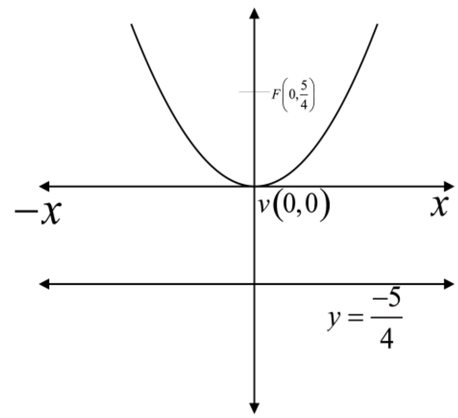
$4a = 5 \Rightarrow \boxed{a = \frac{5}{4}}$

Focus of (i) is $F(0,a) = F\left(0,\frac{5}{4}\right)$

Vertex of (i) is $V(0,0)$

Equation of directrix is $y = -a$

$\Rightarrow y = -\frac{5}{4}$




Graph of $x^2 = 5y$

**(iv)** $y^2 = -12x$

**Solution:**

$y^2 = -12x \ldots$(i)

Compare it with $y^2 = -4ax$

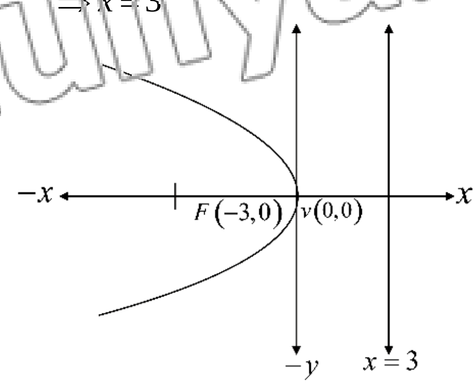
$4a = 12 \Rightarrow \boxed{a = 3}$

Focus of (i) is $F(-a,0) = F(-3,0)$

Vertex of (i) is $V(0,0)$

Equation of directrix $x = a$

$\Rightarrow x = 3$



**(v)** $x^2 = 4(y-1)$

**Solution:**

$x^2 = 4(y-1)$ …(i)

Let $x = X, y-1 = Y$

equation (i) becomes

$X^2 = 4Y$ …(ii)

Compare it with $X^2 = 4aY$

$4a = 4 \Rightarrow \boxed{a = 1}$

So focus of (ii) is $F(0,a) = F(0,1)$

$X = 0$ and $Y = 1$

$x = 0$ and $y-1 = 1 \Rightarrow y = 2$

i.e. $F(x,y) = F(0,2)$

vertex of (ii) is $V(0,0)$

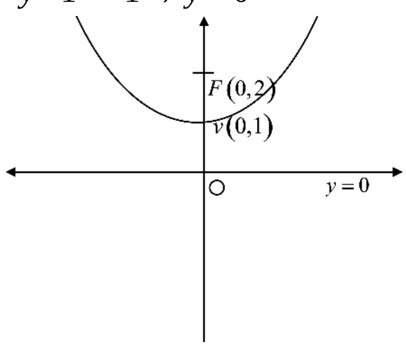
$X = 0$ and $Y = 0$

$x = 0$ and $y-1 = 0 \Rightarrow y = 1$

So vertex of (i) is $V(0,1)$

Directrix of (ii) is $Y = -a \Rightarrow Y = -1$

$y-1 = -1 \Rightarrow y = 0$




**(vi)** $y^2 = -8(x-3)$

**Solution:**

$y^2 = -8(x-3)$ …(i)

Let $y = Y$ and $x-3 = X$

equation (i) becomes

$Y^2 = -8X$ …(ii)

Compare it with $Y^2 = -4aX$

$4a = 8 \Rightarrow \boxed{a = 2}$

Focus of (ii) is $F(-a,0) = F(-2,0)$

$\Rightarrow X = -2$ and $Y = 0$

$x-3 = -2 \Rightarrow x = 1$ and $y = 0$

i.e. $F(x,y) = F(1,0)$

Vertex of (ii) is $V(0,0)$

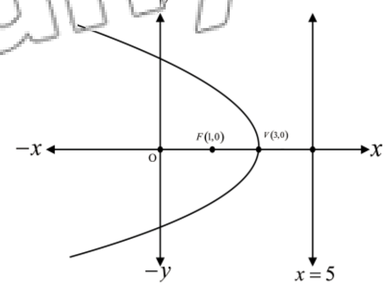
So $X = 0 \Rightarrow x-3 = 0 \Rightarrow x = 3$ and

$Y = 0 \Rightarrow y = 0$

So vertex of (i) is $V(3,0)$

Directrix of (ii) is $X = a \Rightarrow x-3 = 2$

$\Rightarrow x = 5$ is the directrix of (i)




Graph of $y^2 = -8(x-3)$

**(vii)** $(x-1)^2 = 8(y+2)$

**Solution:**

$(x-1)^2 = 8(y+2)$ …(i)

Let $x-1 = X, y+2 = Y$

equation (i) becomes

$X^2 = 8Y$ …(ii)

Compare it with $X^2 = 4aY$

$4a = 8 \Rightarrow \boxed{a = 2}$

So focus of (ii) is $F(0,a) = F(0,2)$

$X = 0 \Rightarrow x-1 = 0 \Rightarrow x = 1,$

$Y = 2 \Rightarrow y+2 = 2 \Rightarrow y = 0$

i.e. $F(x,y) = F(1,0)$

Vertex of (ii) is $V(0,0)$

so here $X = 0 \Rightarrow x-1 = 0 \Rightarrow x = 1$

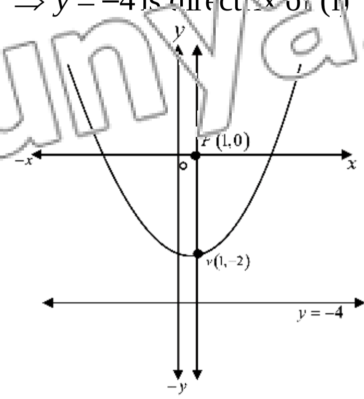
$Y = 0 \Rightarrow y+2 = 0 \Rightarrow x = -2$

so vertex of (i) is $(1,-2)$

Directrix of (ii) is

$Y = -a \Rightarrow y+2 = -2$

$\Rightarrow y = -4$ is directrix of (i)




Graph of $(x-1)^2 = 8(y+2)$

**(viii)** $y = 6x^2 - 1$

**Solution:**

$$y = 6x^2 - 1$$
$$y + 1 = 6x^2$$
$$x^2 = \frac{1}{6}(y+1) \ldots \text{(i)}$$

Let $x = X$ and $y + 1 = Y$

equation (i) becomes

$$X^2 = \frac{1}{6}Y \ldots \text{(ii)}$$

Compare it with $X^2 = 4aY$

$$4a = \frac{1}{6} \Rightarrow \boxed{a = \frac{1}{24}}$$

so focus of (ii) is $F(0, a) = F\left(0, \frac{1}{24}\right)$

$$X = 0 \Rightarrow x = 0,$$
$$Y = \frac{1}{24} \Rightarrow y + 1 = \frac{1}{24} \Rightarrow y = \frac{-23}{24}$$

i.e. $F(x, y) = F\left(0, \frac{-23}{24}\right)$

Vertex of (ii) is

$V(0,0) \Rightarrow X = 0 \Rightarrow x = 0$,

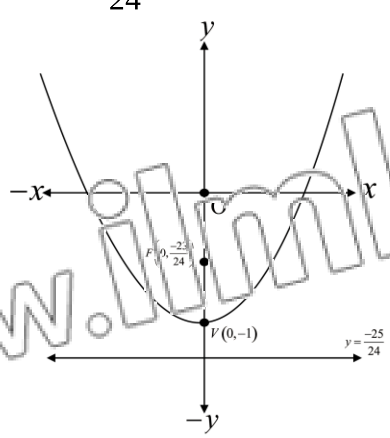
$Y = 0 \Rightarrow y + 1 = 0 \Rightarrow y = -1$

Hence vertex of (i) is $V(0, -1)$

Directrix of (ii) is

$$Y = -a \Rightarrow y + 1 = -\frac{1}{24} \Rightarrow -\frac{1}{24} - 1$$

$\Rightarrow y = \frac{-25}{24}$ is the directrix of (i)




Graph of $y = 6x^2 - 1$

**(ix)** $x + 8 - y^2 + 2y = 0$

**Solution:**

$$x + 8 - y^2 + 2y = 0$$
$$y^2 - 2y = x + 8$$
$$y^2 - 2y + 1 = x + 8 + 1$$
$$(y-1)^2 = x + 9 \ldots \text{(i)}$$

Let $y - 1 = Y$ and $x + 9 = X$

equation (i) becomes

$$Y^2 = X \ldots \text{(ii)}$$

Compare it with $Y^2 = 4ax$

$$4a = 1 \Rightarrow \boxed{a = \frac{1}{4}}$$

So focus of (ii) $F(a, 0) = F\left(\frac{1}{4}, 0\right)$

$$X = \frac{1}{4}$$
$$\Rightarrow x + 9 = \frac{1}{4} \Rightarrow x = \frac{1}{4} - 9 \Rightarrow x = \frac{-35}{4}$$

and

$Y = 0 \Rightarrow y - 1 = 0 \Rightarrow y = 1$

i.e. $F(x, y) = F\left(-\frac{35}{4}, 1\right)$

Vertex of (ii) is $V(0,0)$

$X = 0 \Rightarrow x + 9 = 0$

And $Y = 0 \Rightarrow y - 1 = 0 \Rightarrow y = 1$

Hence vertex of (i) is $V(-9, 1)$

Directrix of (ii) is

$$X = -a \Rightarrow x + 9 = -\frac{1}{4} \Rightarrow x = -\frac{1}{4} - 9$$

$\Rightarrow x = \frac{-37}{4}$ is directrix of (i)

**(x)** $x^2-4x-8y+4=0$

**Solution:**

$x^2-4x-8y+4=0$

$(x^2-4x+4)=8y$

$(x-2)^2=8y\ \ldots\text{(i)}$

Let $x-2=X$ and $y=Y$

equation (i) becomes

$X^2=8Y\ \ldots\text{(ii)}$

Compare it with $X^2=4aY$

$4a=8\Rightarrow \boxed{a=2}$

Focus of (ii) is $F(0,a)=F(0,2)$

$X=0\Rightarrow x-2=0\Rightarrow x=2$,

$Y=2\Rightarrow y=2$

i.e. $F(x,y)=F(2,2)$

Vertex of (ii) is $V(0,0)$

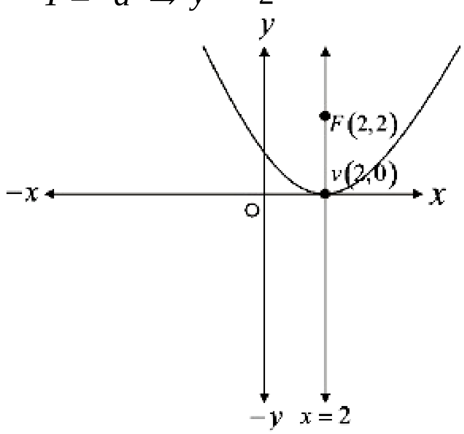
$X=0\Rightarrow x-2=0\Rightarrow x=2$

$Y=0\Rightarrow y=0$,

so vertex of (i) is $V(2,0)$

Directrix of (ii) is

$Y=-a\ \Rightarrow y=-2$




**Q.2 Write an equation of the parabola with given element.**

**(i) Focus $(-3,1)$, Directrix $x=3$**

**Solution:**

Let $P(x,y)$ be any point on the parabola. Length of the perpendicular from $P(x,y)$ to the directrix $x=3$ is

$|PM|=|x-3|$

By definition, we have

$|PF|=|PM|$

$\sqrt{(x+3)^2+(y-1)^2}=|x-3|$

Squaring on both sides we get

$\Rightarrow (x+3)^2+(y-1)^2=(x-3)^2$

$(y-1)^2=(x-3)^2-(x+3)^2$

$=x^2+9-6x-x^2-9-6x$

$=-12x$

$\boxed{(y-1)^2=-12x}$

**(ii) Focus** (2, 5)**, Directrix** $y=1$

**Solution:**

Let $P(x,y)$ be any point on the parabola. Length of the perpendicular from $P(x,y)$ to the directrix $y=1$ is

$|PM|=|y-1|$

By definition, we have

$|PF|=|PM|$

$\sqrt{(x-2)^2+(y-5)^2}=|y-1|$

Squaring on both sides we get

$(x-2)^2+(y-5)^2=(y-1)^2$

$(x-2)^2=(y-1)^2-(y-5)^2$

$(x-2)^2=y^2+1-2y-y^2-25+10y$

$(x-2)^2=-24+8y$

$\boxed{(x-2)^2=8(y-3)}$

**(iii) Focus $(-3,1)$, Directrix**

$x-2y-3=0$

**Solution:**

Let $P(x,y)$ be any point on the parabola. Length of the perpendicular from $P(x,y)$ to the directrix $x-2y-3=0$ is

$|PM|=\dfrac{|x-2y-3|}{\sqrt{(1)^2+(-2)^2}}$

$|PM|=\dfrac{|x-2y-3|}{\sqrt{5}}$

By definition, we have

$|PF|=|PM|$

$\Rightarrow \sqrt{(x+3)^2+(y-1)^2}=\dfrac{|x-2y-3|}{\sqrt{5}}$

$$\sqrt{5}\left[\sqrt{(x+3)^2+(y-1)^2}\right]=|x-2y-3|$$

Squaring on both sides, we get

$$5\left[(x+3)^2+(y-1)^2\right]=(x-2y-3)^2$$

$$5\left[x^2+9+6x+y^2+1-2y\right]$$

$$=x^2+4y^2+9-4xy+12y-6x$$

$$\Rightarrow 5\left[x^2+6x+y^2-2y+10\right]$$

$$=x^2+4y^2+9-4xy+12y-6x$$

$$\Rightarrow 5x^2+30x+5y^2-10y+50$$

$$=x^2+4y^2+9-4xy+12y-6x$$

$$\Rightarrow 5x^2-x^2+4xy+5y^2-4y^2+30x$$

$$+6x-10y-12y+50-9=0$$

$$\Rightarrow \boxed{4x^2+y^2+4xy+36x-22y+41=0}$$

is the required equation of parabola.

**(iv)** **Focus** (1, 2)**, vertex** (3, 2)

**Solution:**

The directrix line is parallel to $y$-axis and its equation is $x=5$

Let $P(x,y)$ be any point on the parabola. Length of the perpendicular from $P(x,y)$ to the directrix $x=5$ is $|PM|=|x-5|$

By definition, we have

$$|PF|=|PM|$$

$$\sqrt{(x-1)^2+(y-2)^2}=|x-5|$$

$$(x-1)^2+(y-2)^2=(x-5)^2$$

$$x^2+y^2-2x-4y+1+4=x^2-10x+25$$

$$\boxed{y^2-4y+8x-20=0}$$

**(v)** **Focus (–1,0), Vertex (–1,2)**

**Solution:**

The directrix line is parallel to $x$-axis and its equation is $y=4$

Let $P(x,y)$ be any point on the parabola. Length of the perpendicular from $P(x,y)$ to the directrix $y=4$ is $|PM|=|y-4|$

By definition, we have

$$|PF|=|PM|$$

$$\sqrt{(x+1)^2+(y-0)^2}=|y-4|$$

$$(x+1)^2+(y-0)^2=(y-4)^2$$

$$x^2+y^2+2x+1=y^2-8y+16$$

$$\boxed{x^2+2x+8y-15=0}$$

**(vi)** **Directrix** $x=-2$**, Focus (2,2)**

**Solution:**

Let $P(x,y)$ be any point on the parabola. Length of the perpendicular from $P(x,y)$ to the directrix $x=-2$ is $|PM|=|x+2|$

By definition, we have

$$|PF|=|PM|$$

$$\sqrt{(x-2)^2+(y-2)^2}=|x+2|$$

$$(x-2)^2+(y-2)^2=(x+2)^2$$

$$x^2+y^2-4x-4y+4+4=x^2+4x+4$$

$$\boxed{y^2-4y-8x+4=0}$$

**(vii)** **Directrix y=3; Vertex (2, 2)**

**Solution:**

Directrix $y=3$, $V(2,2)$

From figure Focus $F(2,1)$

Let $P(x,y)$ be any point on the parabola. Length of the perpendicular from $P(x,y)$ to the directrix $y=3$ is $|PM|=|y-3|$

By definition, we have

$$|PF|=|PM|$$

$$\sqrt{(x-2)^2+(y-1)^2}=|y-3|$$

$$(x-2)^2+(y-1)^2=(y-3)^2$$

$$x^2+y^2-4x-2y+4+1=y^2-6y+9$$

$$\boxed{x^2-4x+4y-4=0}$$

**(viii) Directrix $y=1$, length of the latus rectum is 8 open downward**

**Solution:**

Directrix $y=1$

And length of the latus rectum is

$4a=8 \Rightarrow a=2$

Let $V(h,k)$ be the vertex

Then equation of parabola open downwards is

$(x-h)^2=-4a(y-k)$

$\Rightarrow (x-h)^2=-8(y-k)$

$y$-coordinate of the vertex $=1-a$

$k=1-2=-1$

$(x-h)^2=-8(y+1)$

$x^2+h^2-2hx=-8y-8$

$\boxed{x^2+h^2-2hx+8y+8=0}$

**(ix) Axis $y=0$ through $(2,1)$ and $(11,2)$**

**Solution:**

Equation of parabola whose axis is $y=0$ and vertex $(h,k)$ is

$(y-k)^2=4a(x-h)$

$y^2=4a(x-h) \quad \because k=0 \ldots$(i)

It passes through $(2,1)$ and $(11,2)$

$\therefore \ 1=4a(2-h)$

$1=8a-4ah$

$4ah=-1+8a \ldots$(ii)

Also $(-2)^2=4a(11-h)$

$4=44a-4ah$

$4=44a-(-1+8a)$

$4=44a+1-8a$

$3=36a \Rightarrow a=\dfrac{1}{12}$

Equation (ii) becomes

$4\left(\dfrac{1}{12}\right)h=-1+8\left(\dfrac{1}{12}\right)$

$\dfrac{1}{3}h=-1+\dfrac{2}{3}$

$\dfrac{1}{3}h=-\dfrac{1}{3}$

$h=-1$

Equation (i) becomes

$y^2=4\left(\dfrac{1}{12}\right)(x+1)$

$y^2=\dfrac{1}{3}(x+1)$

**(x) Axis parallel to y-axis the points (0,3) (3,4) and (4,11) lies on the graph.**

**Solution:**

Equation of parabola whose axis is $x=0$ and vertex $(h,k)$ is

$(x-h)^2=4a(y-k) \ldots$(i)

It passes through (0,3) (3,4) and (4,11)

$\therefore \ (0-h)^2=4a(3-k)$

$h^2=12a-4ak \ldots$(ii)

$\therefore \ (3-h)^2=4a(4-k)$

$9-6h+h^2=16a-4ak \ldots$(iii)

$\therefore \ (4-h)^2=4a(11-k)$

$16-8h+h^2=44a-4ak \ldots$(iv)

Subtracting (iii) and (iv)

$28a+2h=7 \ldots$(v)

Subtracting (ii) and (iii)

$4a+6h=9 \ldots$(vi)

Solving (v) and (vi)

We get $h=\dfrac{7}{5}$ and $a=\dfrac{3}{20}$

Equation (ii) becomes

$\left(\dfrac{7}{5}\right)^2=12\left(\dfrac{3}{20}\right)-4\left(\dfrac{3}{20}\right)k$

$k = -\frac{4}{15}$

Equation (i)

$\left(x-\frac{7}{5}\right)^2 = 4\left(\frac{3}{20}\right)\left(y+\frac{4}{15}\right)$

$\boxed{\left(x-\frac{7}{5}\right)^2 = \frac{3}{5}\left(y+\frac{4}{15}\right)}$

**Q.3 Find an equation of the parabola having its focus at the origin and Directrix parallel to**
**(i) x-axis (ii) y-axis**

**(i) When Directrix is parallel to the x-axis**

**Solution:**

Focus is $F(0,0)$ suppose that equation of Directrix is $y = h$

Let $P(x,y)$ be any point on the parabola, then

By definition, we have

$|PF| = |PM|$

$\Rightarrow \sqrt{(x-0)^2+(y-0)^2} = |y-h|$

$\Rightarrow x^2+y^2 = (y-h)^2$

$x^2+y^2 = y^2-2yh+h^2$

$\boxed{x^2+2yh-h^2=0}$

**(ii) When Directrix is parallel to the y-axis**

**Solution:**

Focus is $F(0,0)$ suppose that equation of Directrix is $x = h$

Let $P(x,y)$ be any point on the parabola, then

By definition, we have

$|PF| = |PM|$

$\Rightarrow \sqrt{(x-0)^2+(y-0)^2} = |x-h|$

Squaring on both sides, we get

$x^2+y^2 = (x-h)^2$

$\Rightarrow x^2+y^2 = x^2+h^2-2hx$

$\boxed{y^2+2hx-h^2=0}$

**Q.4 Show that an equation of the parabola with focus at $(a\cos\alpha, a\sin\alpha)$ and directrix $x\cos\alpha + y\sin\alpha + a = 0$ is $(x\sin\alpha - y\cos\alpha)^2 = 4a(x\cos\alpha + y\sin\alpha)$**

**Proof:** Let $P(x,y)$ by any point on the parabola then by definition $|PF| = |PM|$

$\sqrt{(x-a\cos\alpha)^2+(y-a\sin\alpha)^2} = |x\cos\alpha + y\sin\alpha + a|$

Squaring on both sides we get

$(x-a\cos\alpha)^2+(y-a\sin\alpha)^2 = (x\cos\alpha + y\sin\alpha + a)^2$

$x^2+a^2\cos^2\alpha - 2ax\cos\alpha + y^2 + a^2\sin^2\alpha - 2ay\sin\alpha$

$= x^2\cos^2\alpha + y^2\sin^2\alpha + a^2 + 2xy\sin\alpha\cos\alpha + 2ay\sin\alpha + 2ax\cos\alpha$

$x^2 + a^2\left(\cos^2\alpha + \sin^2\alpha\right) - 2ax\cos\alpha + y^2 - 2ay\sin\alpha =$

$x^2\cos^2\alpha + y^2\sin^2\alpha + a^2 + 2xy\sin\alpha\cos\alpha + 2ay\sin\alpha + 2ax\cos\alpha$

$x^2 + y^2 + a^2 - x^2\cos^2\alpha - y^2\sin^2\alpha - 2xy\sin\alpha\cos\alpha =$

$a^2 + 2ay\sin\alpha + 2ax\cos\alpha + 2ax\cos\alpha + 2ay\sin\alpha$

$x^2\left(1-\cos^2\alpha\right) + y^2\left(1-\sin^2\alpha\right) - 2\left(x\sin\alpha\right)\left(y\cos\alpha\right) = 4ax\cos\alpha + 4ay\sin\alpha$

$x^2\sin^2\alpha + y^2\cos^2\alpha - 2\left(x\sin\alpha\right)\left(y\cos\alpha\right) = 4a\left(x\cos\alpha + y\sin\alpha\right)$

$\boxed{\left(x\sin\alpha - y\cos\alpha\right)^2 = 4a\left(x\cos\alpha + y\sin\alpha\right)}$

**Q.5 Show that the ordinate at any point of the parabola is a mean proportional between the length of the latus rectum and the abscissa of *P*.**

**Solution:**

Consider the equation of parabola be $y^2 = 4ax$

Draw perpendicular $PQ$ from point $P$ on axis of parabola

Then $y^2 = 4ax$

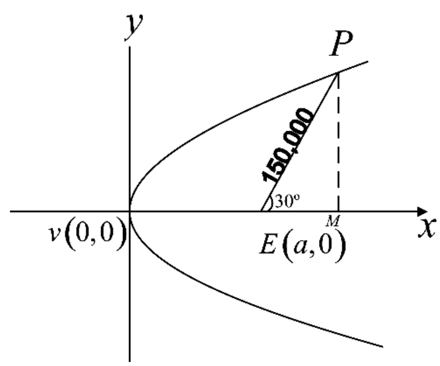
$y.y = 4ax$

$|PQ||PQ| = 4a|AQ|$

$\frac{|PQ|}{|AQ|} = \frac{4a}{|PQ|}$

Hence proved

**Q.6 A comet has a parabolic orbit with the earth at the focus, when the comet is 150,0000km from the earth the line joining the comet and the earth makes an angle of $30^\circ$ with the axis of the parabola. How close will the comet come to the earth.**




**Solution:**

Consider the comet is at point $P$ take focus at origin then the vertex is at $(-a,0)$ and corresponding directrix line is at $x = -2a$

By definition of parabola $|PF| = |PM|$

$|PF| = x + 2a$

$|PF| = |PF|\cos 30^\circ + 2a$

$|PF| - |PF|\cos 30^\circ = 2a$

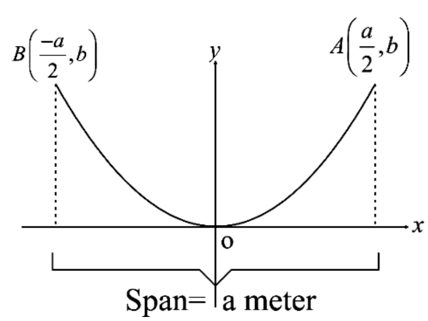
$|PF|(1 - \cos 30^\circ) = 2a$

$150,000\left(1 - \frac{\sqrt{3}}{2}\right) = 2a$

$75000\left(\frac{2-\sqrt{3}}{2}\right) = a$

$\boxed{37500\left(2-\sqrt{3}\right)km = a}$

**Q.7 Find an equation of the parabola formed by the cables of a suspension bridge whose span is "a" meter and the vertical height of the supporting tower is "b" meter.**




**Solution:**

Let equation of parabola be

$x^2 = 4\lambda y \ldots(i)$

We have to find $\lambda$

As span is a meter, so half of span will be $\frac{a}{2}$ meter

Hence, put $A\left(\frac{a}{2}, b\right)$ in (i), we get

$\Rightarrow \left(\frac{a}{2}\right)^2 = 4\lambda(b) \Rightarrow \frac{a^2}{4} = 4\lambda b$

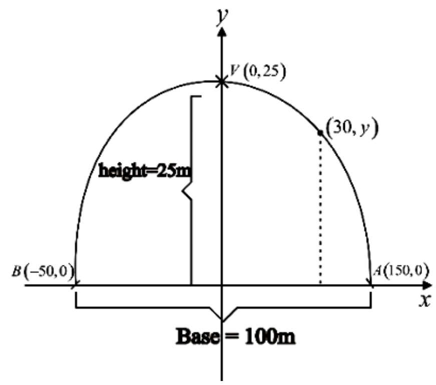
$\Rightarrow \lambda = \frac{a^2}{16b}$ putting in (i), we get

$x^2 = 4\left(\frac{a^2}{16b}\right)y$

$\boxed{x^2 = \frac{a^2}{4b}y}$

**Q.8 A parabolic arch has 100m base and height 25m. Find the height of the arch at the point 30m from the center of the base.**

**Solution:**




Consider the equation of parabola is

$x^2 = -4ay \ldots$(i)

The point $(50,-25)$ lies on it

$\therefore\ (50)^2 = -4a(-25)$

$2500 = 100a$

$a = 25$

Equation (i) $x^2 = -4(25)y$

$x^2 = -100y$

When $x = 30$, $(30)^2 = -100y$

$900 = -100y$

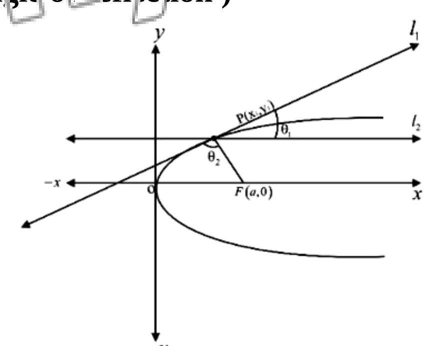
$y = -9$

Height from base $= 25 - 9$

$= 16m$

**Q.9 Show that the tangent at any point $P$ of a parabola makes equal angles with the line $\overline{PF}$ and the line through $P$ parallel to the axis of the parabola $F$ being focus. (These angels are called respectively angle of incident and angle of reflection )**

**Solution:** Let equation of parabola be




$y^2 = 4ax$ (i)

As $P(x_1, y_1)$ lie on (I), So

$y_1^2 = 4ax_1$ (II)

Let $l_1$ be the equation of tangent passes through $P(x_1, y_1)$

Now for slope of the tangent line $l_1$

we differentiate (i) w.r.t $x$, we get

$2y\dfrac{dy}{dx} = 4a \Rightarrow \dfrac{dy}{dx} = \dfrac{2a}{y}$ putting $P(x_1, y_1)$ we get

Slope of tangent line

i.e. $m_1 = \dfrac{2a}{y_1}$

Next $l_2$ is a line parallel to the axis of the parabola passes through $P(x_1, y_1)$

So, its slope is $m_2 = 0$

Next $l_3$ is a line through points P and F So,

slope of line $\overrightarrow{PF}$ is $m_3 = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{y_1 - 0}{x_1 - a} = \dfrac{y_1}{x_1 - a}$

Now let $\theta_1$ is the angle between $l_1$ and $l_2$ then

$$\tan\theta_1 = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\frac{2a}{y_1} - 0}{1 + \left(\frac{2a}{y_1}\right)(0)} = \frac{\frac{2a}{y_1}}{1} = \frac{2a}{y_1} \qquad \text{(iii)}$$

Now let $\theta_2$ is the angle between $l_2$ and $l_3$ so

$$\tan\theta_2 = \frac{m_3 - m_1}{1 + m_3 m_1} = \frac{\frac{y_1}{x_1 - a} - \frac{2a}{y_1}}{1 + \left(\frac{y_1}{x_1 - a}\right)\left(\frac{2a}{y_1}\right)} = \frac{\frac{(y^2{}_1 - 2ax_1 + 2a^2)}{y_1(x_1 - a)}}{\frac{y_1(x_1 - a) + 2ay_1}{(x_1 - a)y_1}}$$

$$= \frac{4ax_1 - 2ax_1 + 2a^2}{x_1 y_1 - ay_1 + 2ay_1}$$

From (ii)

$$= \frac{2ax_1 + 2a^2}{x_1 y_1 + ay_1}$$

$$\tan\theta_2 = \frac{2a(x_1 + a)}{y_1(x_1 + a)} = \frac{2a}{y_1} \qquad \text{(iv)}$$

Form (iii) and (iv) it is clear that

$\tan\theta_1 = \tan\theta_2$

$\Rightarrow \theta_1 = \theta_2$ which is required proof