Ellipse:

An ellipse is the locus of a point which moves in a plane such that the ratio of its distance from fixed point (focus) and a fixed lines (Directrix) is a constant which is less than 1. This ratio is called eccentricity and is denoted by e

$0 < e < 1$

OR

An ellipse is the locus of a point that moves in such a way that the sum of its distance from two fixed points (Foci) is constant i.e. $|PF|+|PF'|=2a$

Equation of Ellipse in Standard Form:

Let $F(-c,0)$ be the focus and the line $x=\frac{-c}{e^2}$ be the Directrix of an ellipse with eccentricity $e,(o<e<1)$. Then equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

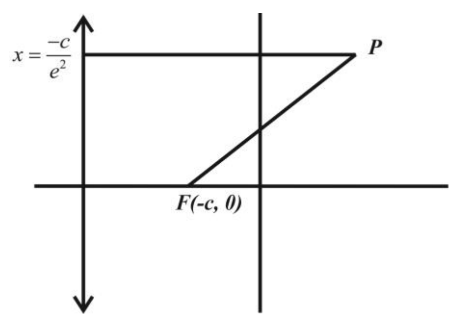
Proof:

Let $P(x,y)$ be any point on the ellipse and suppose $|PM|$ be the perpendicular distance of P from the Directrix. Then $|PM|=x+\frac{c}{e^2}$

By definition of ellipse

$$|PF|=e|PM| \qquad \text{where } 0<e<1$$

$$\sqrt{(x+c)^2+(y-0)^2}=e\left(x+\frac{c}{e^2}\right)$$



Taking square on both sides

$$x^2+c^2+2cx+y^2=e^2\left(x+\frac{c}{e^2}\right)^2$$

$$x^2+c^2+2cx+y^2=e^2\left(x^2+\frac{c^2}{e^4}+\frac{2cx}{e^2}\right)$$

$$x^2+c^2+2cx+y^2=e^2x^2+\frac{c^2}{e^2}+2cx$$

$$x^2+2cx-2cx+y^2-e^2x^2=\frac{c^2}{e^2}-c^2$$

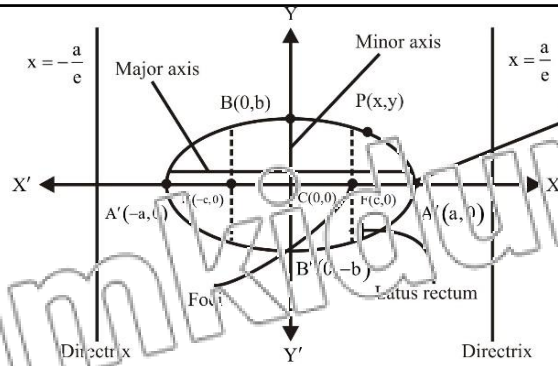
$$x^2\left(1-e^2\right)+y^2=\frac{c^2}{e^2}\left(1-e^2\right)$$

$$x^2\left(1-e^2\right)+y^2=a^2\left(1-e^2\right) \qquad \text{where } a=\frac{c}{e}$$

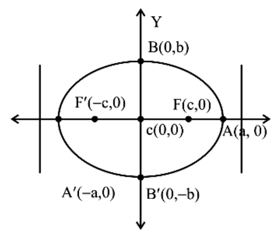
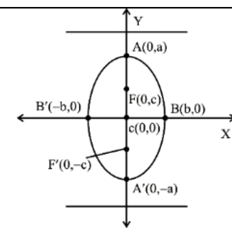
$$\frac{x^2\left(1-e^2\right)}{a^2\left(1-e^2\right)}+\frac{y^2}{a^2\left(1-e^2\right)}=1$$

$$\Rightarrow \boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1} \qquad \text{where } b^2=a^2\left(1-e^2\right) \text{ and } a>b$$

Graph:



Two Standard Equations of Ellipse:

Standard Equation	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$	$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1, a>b$
Shape of the Ellipse		
Centre	(0, 0)	(0, 0)
Equation of Major axis	$y = 0$	$x = 0$
Equation of Minor axis	$x = 0$	$y = 0$
Length of major axis	$2a$	$2a$
Length of minor axis	$2b$	$2b$
Foci	$(\pm c, 0)$	$(0, \pm c)$
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Equation of DTX	$x=\pm\frac{c}{e^2}$	$y=\pm\frac{c}{e^2}$
Eccentricity	$e=\sqrt{\frac{a^2-b^2}{a^2}}$	$e=\sqrt{\frac{a^2-b^2}{a^2}}$
Length of Latus Rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Parametric Equations	$(a\cos\theta, b\sin\theta)$	$(b\cos\theta, a\sin\theta)$
Symmetry	About both the axis and center	About both the axis and center

EXERCISE 6.5

Q.1 Find an equation of the ellipse with given data and sketch its graph.

(i) Foci $(\pm 3,0)$ and minor axis of length 10

Solution:

Foci: $F(3,0)$ and $F'(-3,0)$

The mid point of FF'

Centre $= C(h,k) = C(0,0)$

length of minor axis

$2b = 10 \Rightarrow \boxed{b=5}$

$2c = |FF'| = 6$

$\Rightarrow \boxed{c=3}$

As

$c^2 = a^2 - b^2$

$3^2 = a^2 - 5^2$

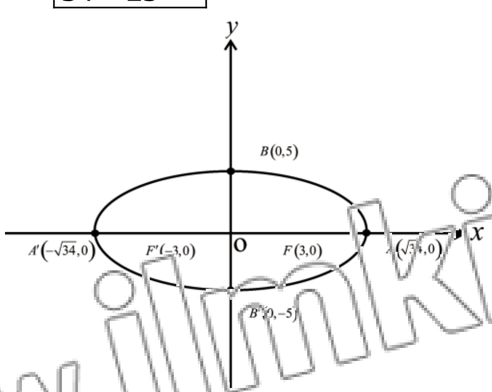
$9 + 25 = a^2 \Rightarrow a = \sqrt{34}$

Vertices are $A(\sqrt{34},0)$ and $A'(-\sqrt{34},0)$

$\because$ foci lies on x-axis

$\therefore \dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

$\Rightarrow \boxed{\dfrac{x^2}{34} + \dfrac{y^2}{25} = 1}$



(ii) Foci $(0,-1)$ and $(0,-5)$ and major axis of length 6.

Solution:

Foci: $F(0,-1)$ and $F'(0,-5)$

The mid-point of FF'

Centre $= C(h,k) = C(0,-3)$

length of major axis

$2a = 6 \Rightarrow \boxed{a=3}$

$2c = |FF'| = \sqrt{(0-0)^2 + (-1+5)^2}$

$= \sqrt{0+16} = 4$

$\Rightarrow \boxed{c=2}$

As

$c^2 = a^2 - b^2$

$2^2 = 3^2 - b^2$

$9 - 4 = b^2$

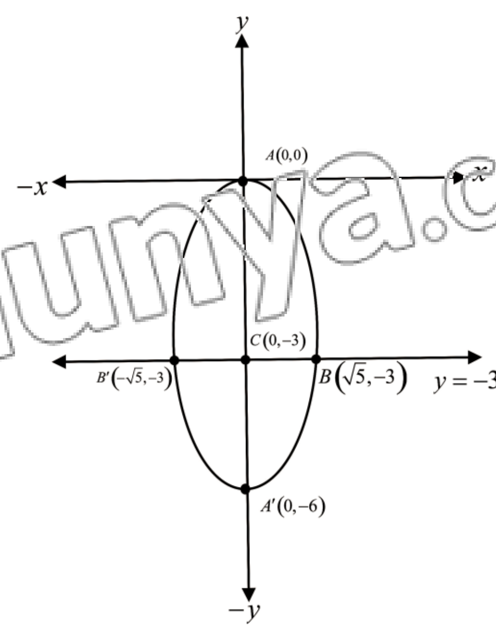
$\Rightarrow \boxed{b=\sqrt{5}}$

Vertices are $(0,0)$ and $(0,-6)$

$\because$ foci lies on y-axis

$\therefore \dfrac{(y-k)^2}{a^2} + \dfrac{(x-h)^2}{b^2} = 1$

$\Rightarrow \boxed{\dfrac{(y+3)^2}{9} + \dfrac{x^2}{5} = 1}$



(iii) Foci $\left(-3\sqrt{3},0\right)$ and vertices $\left(\pm 6,0\right)$

Solution:

Foci: $F\left(3\sqrt{3},0\right)$ and $F'\left(-3\sqrt{3},0\right)$

The mid-point of FF'

Centre $=C\left(h,k\right)=C\left(0,0\right)$

Vertices $A\left(6,0\right)$ and $A'\left(-6,0\right)$

$\boxed{a=6}$

$2c=\left|FF'\right|=6\sqrt{3}$

$\Rightarrow \boxed{c=3\sqrt{3}}$

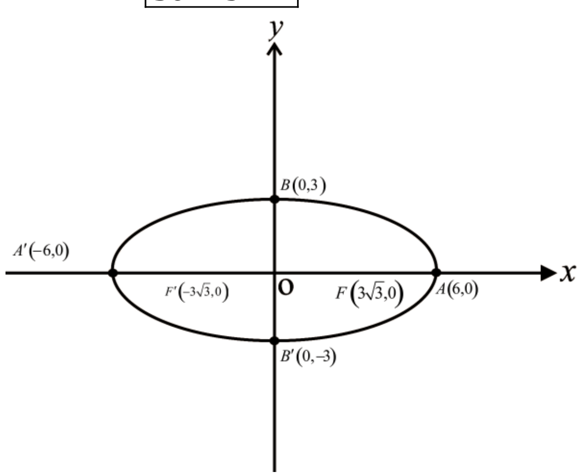
$b^2=a^2-c^2=36-27=9$

$\Rightarrow \boxed{b=3}$

$\because$ foci lies on x-axis

$\therefore \dfrac{\left(x-h\right)^2}{a^2}+\dfrac{\left(y-k\right)^2}{b^2}=1$

$\Rightarrow \boxed{\dfrac{x^2}{36}+\dfrac{y^2}{9}=1}$



(iv) Vertices $\left(-1,1\right),\left(5,1\right)$; Foci $\left(4,1\right)$ and $\left(0,1\right)$

Solution:

Foci: $F\left(4,1\right)$ and $F'\left(0,1\right)$

The mid-point of FF'

Centre $=C\left(h,k\right)=C\left(\dfrac{-1+5}{2},\dfrac{1+1}{2}\right)$

$=C\left(2,1\right)$

Vertices: $A\left(-1,1\right)$ and $A'\left(5,1\right)$

$2a=\left|AA'\right|=\sqrt{\left(5+1\right)^2+\left(1-1\right)^2}=6$

$\Rightarrow \boxed{a=3}$

$2c=\left|FF'\right|=\sqrt{\left(0-4\right)^2+\left(1-1\right)^2}=4$

$\Rightarrow \boxed{c=2}$

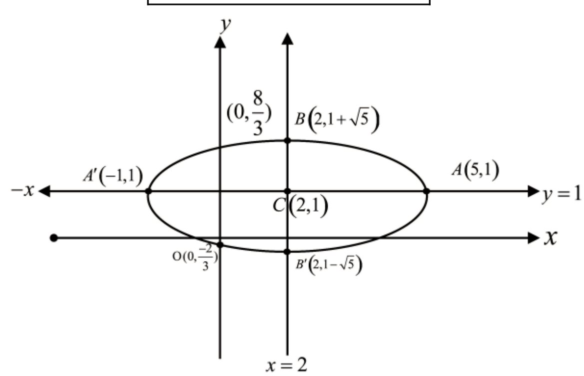
$b^2=a^2-c^2=9-4=5$

$\Rightarrow \boxed{b=\sqrt{5}}$

$\because$ foci lies on x-axis

$\therefore \dfrac{\left(x-h\right)^2}{a^2}+\dfrac{\left(y-k\right)^2}{b^2}=1$

$\Rightarrow \boxed{\dfrac{\left(x-2\right)^2}{9}+\dfrac{\left(y-1\right)^2}{5}=1}$



(v) Foci $\left(\pm\sqrt{5},0\right)$ and passing through the point $\left(\dfrac{3}{2},\sqrt{3}\right)$

Solution:

Foci $F\left(\sqrt{5},0\right)$ and $F'\left(-\sqrt{5},0\right)$

The mid-point of FF'

Centre $=C\left(h,k\right)=C\left(0,0\right)$

$2c=\left|FF'\right|=2\sqrt{5}$

$\Rightarrow \boxed{c=\sqrt{5}}$

Using $c^2=a^2-b^2$

$\left(\sqrt{5}\right)^2=a^2-b^2$

$5=a^2-b^2$ or $\boxed{b^2=a^2-5}$

$\because$ foci lies on x-axis

$$\therefore \frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-5}=1 \ldots \text{(i)}$$

But it passes through $P\left(\frac{3}{2},\sqrt{3}\right)$

$$\therefore \frac{\left(\frac{3}{2}\right)^2}{a^2}+\frac{\left(\sqrt{3}\right)^2}{a^2-5}=1$$

$$\frac{9}{4a^2}+\frac{3}{a^2-5}=1$$

$$9\left(a^2-5\right)+3\left(4a^2\right)=4a^2\left(a^2-5\right)$$

$$4a^4-41a^2+45=0$$

It is quadratic in a^2

$$a^2=\frac{41\pm\sqrt{1681-720}}{8}$$

$$a^2=\frac{41\pm\sqrt{981}}{8}$$

$$a^2=\frac{41\pm 31}{8}$$

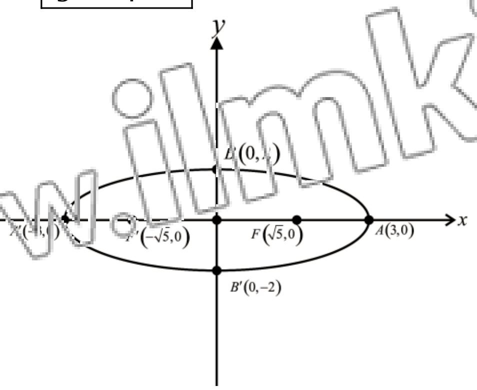
$$a^2=\frac{41+31}{8} \quad \text{and} \quad a^2=\frac{41-31}{8}$$

$$a^2=9 \quad \text{and} \quad a^2=\frac{10}{8}$$

$\boxed{a=3}$, $a=\frac{\sqrt{5}}{2}$ (not possible) $\because a>c$

$$\frac{x^2}{3^2}+\frac{y^2}{3^2-5}=1$$

$$\boxed{\frac{x^2}{9}+\frac{y^2}{4}=1}$$



(vi) Vertices $(0,\pm5)$, eccentricity $\frac{3}{5}$

Solution:

Vertices: $A(0,5)$ and $A'(0,-5)$

$$2a=|AA'|=10$$

$$\Rightarrow \boxed{a=5}$$

Eccentricity: $e=\frac{3}{5}$

$$c=ae=5\left(\frac{3}{5}\right)$$

$$\Rightarrow \boxed{c=3}$$

The mid-point of

$$AA'=C(h,k)=C(0,0)$$

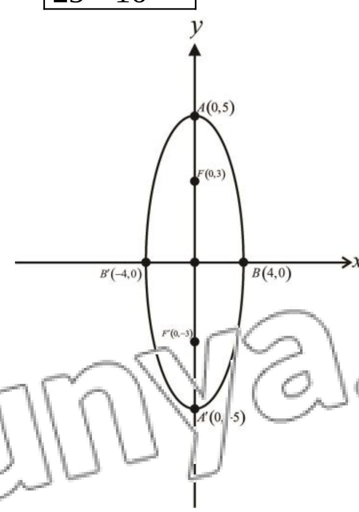
$$b^2=a^2-c^2=25-9=16$$

$$\Rightarrow \boxed{b=4}$$

$\because$ foci lies on y-axis

$$\therefore \frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$$

$$\Rightarrow \boxed{\frac{y^2}{25}+\frac{x^2}{16}=1}$$



(vii) Centre $(0,0)$, Focus $(0,3)$ and Vertex $(0,4)$

Solution:

$$C(h,k)=C(0,0)$$

Focus $F'(0,-c)=F'(0,-3)$

$$\boxed{c=3}$$

Vertex $A(0,a)=A(0,4)$

$\boxed{a=4}$

Using $c^2=a^2-b^2$

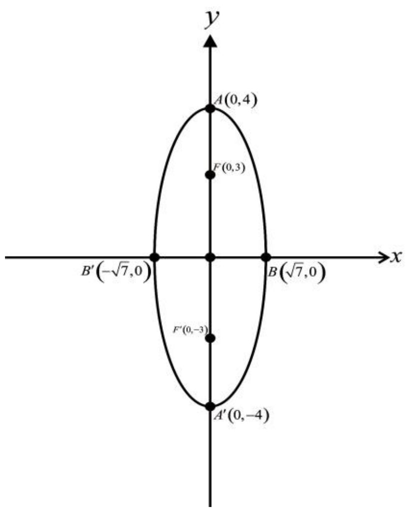
$3^2=4^2-b^2$

$b^2=7$

$\because$ foci lies on y-axis

$\therefore \dfrac{(y-k)^2}{a^2}+\dfrac{(x-h)^2}{b^2}=1$

$\Rightarrow \boxed{\dfrac{y^2}{16}+\dfrac{x^2}{7}=1}$



(viii) Centre (2, 2), major axis parallel to y-axis and of length 8 units, minor axis parallel to x-axis of length 6 units.

Solution:

Centre $C(h,k)=C(2,2)$

Major axis is parallel to x-axis

$2a=8\Rightarrow \boxed{a=4}$

Minor axis is parallel to x-axis

$2b=6\Rightarrow \boxed{b=3}$

$c=\sqrt{a^2-b^2}=\sqrt{16-9}=\sqrt{7}$

Vertices

$A(h,k+a)=A(2,2+4)=A(2,6)$

$A'(h,k-a)=A'(2,2-4)=A'(2,-2)$

Co-vertices

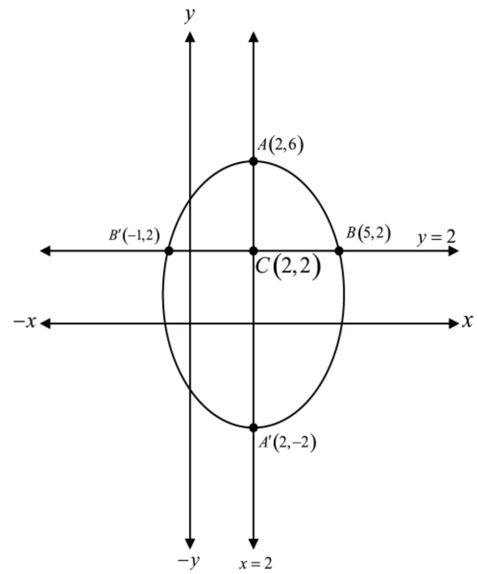
$B(h+b,k)=B(2+3,2)=B(5,2)$

$B'(h-b,k)=B'(2-3,2)=B'(-1,2)$

The major axis parallel to y-axis

$\dfrac{(y-k)^2}{a^2}+\dfrac{(x-h)^2}{b^2}=1$

$\Rightarrow \boxed{\dfrac{(y-2)^2}{16}+\dfrac{(x-2)^2}{9}=1}$



(ix) Centre (0, 0), symmetric with respect to both the axes and passing through the points (2, 3) and (6, 1)

Solution:

Centre $C(h,k)=C(0,0)$

The general equation of the ellipse (having centre at origin) is

$\dfrac{x^2}{\lambda^2}+\dfrac{y^2}{\mu^2}=1\ldots$(i)

Substituting $(2,3)$ and $(6,1)$ in (i), we get respectively

$\dfrac{4}{\lambda^2}+\dfrac{9}{\mu^2}=1\ldots$(ii)

$\dfrac{36}{\lambda^2}+\dfrac{1}{\mu^2}=1\ldots$(iii)

Multiply $(\text{ii})\times 9$ and subtract from (iii) we get

$\frac{80}{\mu^2} = 8 \Rightarrow \boxed{\mu^2 = 10}$ putting in (ii),

we get

$$\frac{4}{\lambda^2} + \frac{9}{10} = 1$$

$$\Rightarrow \frac{4}{\lambda^2} = \frac{1}{10} \Rightarrow \boxed{\lambda^2 = 40}$$

Since $\lambda^2 > \mu^2$

$\therefore a^2 = \lambda^2 = 40$ and $b^2 = \mu^2 = 10$

So (i) becomes $\boxed{\frac{x^2}{40} + \frac{y^2}{10} = 1}$

Vertices

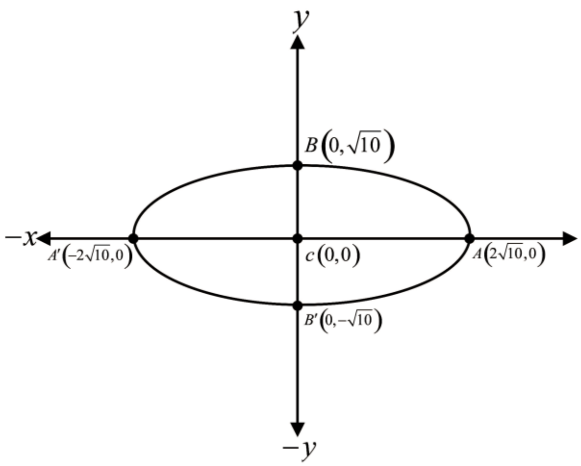
$$A(a,0) = A\left(2\sqrt{10},0\right)$$

$$A'(-a,0) = A'\left(-2\sqrt{10},0\right)$$

Co-vertices

$$B(0,b) = B\left(0,\sqrt{10}\right)$$

$$B'(0,-b) = B'\left(0,-\sqrt{10}\right)$$



(x) Centre (0,0), major axis horizontal, the points (3,1) (4,0) lies on the graph.

Solution:

Centre $C(h,k) = C(0,0)$

Major axis is horizontal

The equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ldots \text{(i)}$$

Put $(3,1)$ in (i), we get

$$\frac{9}{a^2} + \frac{1}{b^2} = 1 \ldots \text{(ii)}$$

Putting $(4,0)$ in (i), we get

$$\frac{16}{a^2} = 1 \Rightarrow a^2 = 16 \Rightarrow \boxed{a = 4}$$

Putting in (ii), we get

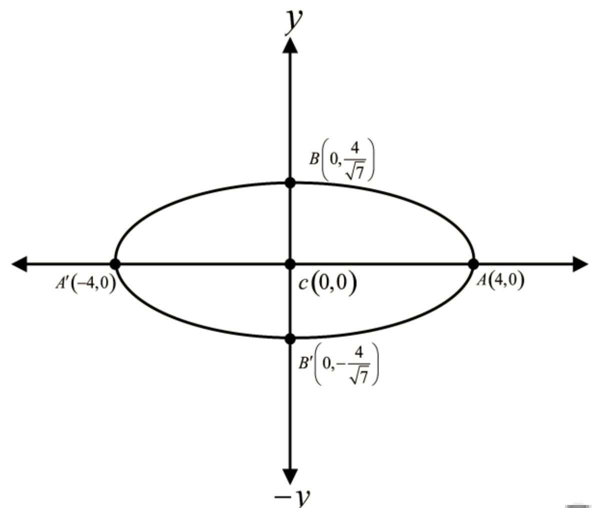
$$\frac{9}{16} + \frac{1}{b^2} = 1$$

$$\Rightarrow \frac{1}{b^2} = \frac{7}{16} \Rightarrow b^2 = \frac{16}{7} \Rightarrow \boxed{b = \frac{4}{\sqrt{7}}}$$

Equation (i) becomes

$$\frac{x^2}{16} + \frac{y^2}{\frac{16}{7}} = 1$$

$$\Rightarrow \boxed{\frac{x^2}{16} + \frac{7y^2}{16} = 1}$$



Q.2 Find the center, foci, eccentricity, vertices and directrices of the ellipse whose equation is given.

(i) $x^2 + 4y^2 = 16$

Solution:

$$\frac{x^2}{16} + \frac{4y^2}{16} = 1 \Rightarrow \frac{x^2}{16} + \frac{y^2}{4} = 1$$

$$a^2 = 16 \Rightarrow a = 4$$

$$b^2 = 4 \Rightarrow b = 2$$

so $c^2 = a^2 - b^2 = 16 - 4 = 12$

$c = 2\sqrt{3}$

Centre $C(h,k) = C(0,0)$

Foci are $F(\pm c, 0) = F(\pm 2\sqrt{3}, 0)$

Eccentricity

$e = \frac{c}{a} = \frac{2\sqrt{3}}{4} \Rightarrow e = \frac{\sqrt{3}}{2} < 1$

Vertices

$A(\pm a, 0) = A(\pm 4, 0)$

Directrices

$x = \pm\frac{c}{e^2} \Rightarrow x = \pm\frac{2\sqrt{3}}{\frac{3}{4}}$

$\Rightarrow x = \pm 2\sqrt{3} \times \frac{4}{3}$

$\Rightarrow x = \pm\frac{8}{\sqrt{3}}$

(ii) $9x^2 + y^2 = 18$

Solution:

$\frac{9x^2}{18} + \frac{y^2}{18} = 1 \Rightarrow \frac{x^2}{2} + \frac{y^2}{18} = 1$

$a^2 = 18 \Rightarrow a = 3\sqrt{2}$

$b^2 = 2 \Rightarrow b = \sqrt{2}$

$c^2 = a^2 - b^2 = 18 - 2$

$\Rightarrow c^2 = 16 \Rightarrow c = 4$

Centre $C(h,k) = C(0,0)$

Foci are $F(0, \pm c) = F(0, \pm 4)$

Eccentricity $e = \frac{c}{a} = \frac{4}{3\sqrt{2}} < 1$

Vertices

$A(0, \pm a) = A(0, \pm 3\sqrt{2})$

Directrices $y = \pm\frac{a}{e}$

$y = \pm\frac{3\sqrt{2}}{\frac{4}{3\sqrt{2}}} \Rightarrow y = \pm\frac{18}{4} \Rightarrow y = \pm\frac{9}{2}$

(iii) $25x^2 + 9y^2 = 225$

Solution:

$25x^2 + 9y^2 = 225$

$\frac{25x^2}{225} + \frac{9y^2}{225} = 1$

$\frac{x^2}{9} + \frac{y^2}{25} = 1$

$a^2 = 25 \Rightarrow a = 5$

$b^2 = 9 \Rightarrow b = 3$

$c^2 = a^2 - b^2$

$\Rightarrow c^2 = 25 - 9 \Rightarrow c^2 = 16 \Rightarrow c = 4$

Centre $C(h,k) = C(0,0)$

Foci are $F(0, \pm c) = F(0, \pm 4)$

Eccentricity $e = \frac{c}{a} \Rightarrow e = \frac{4}{5} < 1$

Vertices $A(0, \pm a) = A(0, \pm 5)$

Directrices $y = \pm\frac{a}{e}$

$\Rightarrow y = \pm\frac{5}{\frac{4}{5}} \Rightarrow y = \pm\frac{25}{4}$

(iv) $\frac{(2x-1)^2}{4} + \frac{(y+2)^2}{16} = 1$

Solution:

$\frac{(2x-1)^2}{4} + \frac{(y+2)^2}{16} = 1$

$\Rightarrow 4\frac{\left(x - \frac{1}{2}\right)^2}{4} + \frac{(y+2)^2}{16} = 1$

$\Rightarrow \frac{\left(x - \frac{1}{2}\right)^2}{1} + \frac{(y+2)^2}{16} = 1 \ldots$(i)

Let $x - \frac{1}{2} = X$ and $y + 2 = Y$

So (i) becomes $\frac{X^2}{1} + \frac{Y^2}{16} = 1$ (ii)

$a^2 = 16 \Rightarrow a = 4$

$b^2 = 1 \Rightarrow b = 1$

$c^2 = a^2 - b^2 = 16 - 1 = 15 \Rightarrow c = \sqrt{15}$

Centre of (ii) $C(h,k) = C(0,0)$

$X = 0$ and $Y = 0$

$x - \frac{1}{2} = 0$ and $y + 2 = 0$

$x = \frac{1}{2}$ and $y = -2$

Centre of (i) $C(h,k) = C\left(\frac{1}{2}, -2\right)$

Foci of (ii) are

$F(0, \pm c) \Rightarrow F\left(0, \pm\sqrt{15}\right)$

$X = 0$ and $Y = \pm\sqrt{15}$

$x - \frac{1}{2} = 0$ and $y + 2 = \pm\sqrt{15}$

$x = \frac{1}{2}$ and $y = -2 \pm \sqrt{15}$

Foci of (i) are $F\left(\frac{1}{2}, -2 \pm \sqrt{15}\right)$

Eccentricity of (i) is $e = \frac{c}{a} = \frac{\sqrt{15}}{4} < 1$

Vertices of (ii) are

$A(0, \pm a) = A(0, \pm 4)$

$X = 0$ and $Y = \pm 4$

$\Rightarrow x - \frac{1}{2} = 0$ | $\Rightarrow y + 2 = \pm 4$

$\Rightarrow x = \frac{1}{2}$ | $y = -2 \pm 4$, $y = -6, 2$

vertices of (i) are $A\left(\frac{1}{2}, -6\right)$ and

$A'\left(\frac{1}{2}, 2\right)$

Equation of Directrices of (ii) are

$Y = \pm\frac{a}{e}$

$Y = \pm\frac{4}{\frac{\sqrt{15}}{4}} \Rightarrow Y = \pm\frac{16}{\sqrt{15}}$

$y + 2 = \pm\frac{16}{\sqrt{15}} \Rightarrow y = -2 \pm \frac{16}{\sqrt{15}}$ are the equation of directrices of (i)

(v) $x^2 + 16x + 4y^2 - 16y + 76 = 0$

Solution:

$x^2 + 16x + 4y^2 - 16y + 76 = 0$

$\left(x^2 + 16x\right) + 4\left(y^2 - 4y\right) + 76 = 0$

$\left(x^2 + 16x + 64 - 64\right) + 4\left(y^2 - 4y + 4 - 4\right) + 76 = 0$

$(x+8)^2 - 64 + 4\left((y-2)^2 - 4\right) + 76 = 0$

$(x+8)^2 + 4(y-2)^2 - 16 + 76 - 64 = 0$

$(x+8)^2 + 4(y-2)^2 = 4$

$\frac{(x+8)^2}{4} + \frac{(y-2)^2}{1} = 1 \ldots \text{(i)}$

Let $x + 8 = X$ and $y - 2 = Y$ then (i) will become;

$\frac{X^2}{4} + \frac{Y^2}{1} = 1 \ldots \text{(ii)}$

$a^2 = 4 \Rightarrow a = 2$

$b^2 = 1 \Rightarrow b = 1$

$c^2 = a^2 - b^2 = 4 - 1 = 3 \Rightarrow c = \sqrt{3}$

Centre of (ii) $C(h,k) = C(0,0)$

$X = 0$ and $Y = 0$

$x + 8 = 0$ and $y - 2 = 0$

$x = -8$ and $y = 2$

centre of (i) is $(-8, 2)$

Foci of (ii) are

$F(\pm c, 0) = F\left(\pm\sqrt{3}, 0\right)$

$X = \pm\sqrt{3}$ and $Y = 0$

$x + 8 = \pm\sqrt{3}$ and $y - 2 = 0$

$x = -8 \pm \sqrt{3}$ and $y = 2$

foci of (i) are $\left(-8 \pm \sqrt{3}, 2\right)$

Vertices of (ii) are

$A(\pm a, 0) = A(\pm 2, 0)$

$X = \pm 2$ and $Y = 0$

$\Rightarrow x+8=\pm 2$ and $y-2=0$

$x=-8\pm 2$ and $y=2$

$x=-10,-6$

vertices of (i) are

$A(-10,2), A'(-6,2)$.

Eccentricity of (i) is $e=\frac{c}{a}=\frac{\sqrt{3}}{2}<1$

Directrices of (ii) are

$X=\pm\frac{a}{e}$

$\Rightarrow X=\pm\frac{2}{\frac{\sqrt{3}}{2}} \Rightarrow X=\pm\frac{4}{\sqrt{3}}$

$x+8=\pm\frac{4}{\sqrt{3}}$

$\Rightarrow x=-8\pm\frac{4}{\sqrt{3}}$ are the equation of directrices of (i)

(vi) $25x^2+4y^2-250x-16y+541=0$

Solution:

$25x^2+4y^2-250x-16y+541=0$

$(25x^2-250x)+(4y^2-16y)+541=0$

$25(x^2-10x)+4(y^2-4y)+541=0$

$25(x^2-10x+25-25)+4(y^2-4y+4-4)+541=0$

$25((x-5)^2-25)+4((y-2)^2-4)+541=0$

$25(x-5)^2-625+4(y-2)^2-16+541=0$

$25(x-5)^2+4(y-2)^2-641+541=0$

$25(x-5)^2+4(y-2)^2=100$

$\frac{(x-5)^2}{4}+\frac{(y-2)^2}{25}=1 \ldots$(i)

put $x-5=X$ and $y-2=Y$

(i) will become $\frac{X^2}{4}+\frac{Y^2}{25}=1 \ldots$(ii)

$a^2=25 \Rightarrow a=5,$

$b^2=4 \Rightarrow b=2$

$c^2=a^2-b^2=25-4 \Rightarrow c=\sqrt{21}$

Centre of (ii) $C(h,k)=C(0,0)$

$X=0$ and $Y=0$

$x-5=0$ and $y-2=0$

$x=5$ and $y=2$

centre of (i) is (5,2)

Vertices of (ii) are

$A(0,\pm a)=A(0,\pm 5)$

$X=0$ and $Y=\pm 5$

$x-5=0$ and $y-2=\pm 5$

$x=5$ and $y=2\pm 5$

$y=-3,7$

Hence vertices of (i) are $A(5,-3)$ and $A'(5,7)$

Foci of (ii) are

$F(0,\pm c)=F(0,\pm\sqrt{21})$

$X=0$ and $Y=\pm\sqrt{21}$

$x-5=0$ and $y-2=\pm\sqrt{21}$

$x=5$ and $y=2\pm\sqrt{21}$

foci of (i) are $F(5,2\pm\sqrt{21})$

Eccentricity of (i) is $e=\frac{c}{a}=\frac{\sqrt{21}}{5}<1$

Directrices of (ii) are

$Y=\pm\frac{a}{e}=\pm\frac{5}{\frac{\sqrt{21}}{5}}$

$Y=\pm\frac{25}{\sqrt{21}}$

$y-2=\pm\frac{25}{\sqrt{21}} \Rightarrow y=2\pm\frac{25}{\sqrt{21}}$ are the equation of directrices of (i)

Q.3 Let 'a' be a positive number and $0 < c < a$. Let $F(-c,0)$ and $F'(c,0)$ be two given points. Prove that the locus of the points $P(x,y)$ such that $|PF| + |PF'| = 2a$ is an ellipse.

Proof: We have to show that $|PF| + |PF'| = 2a$ is an equation of ellipse.

As $|PF| + |PF'| = 2a$

$$\sqrt{(x+c)^2 + (y-0)^2} + \sqrt{(x-c)^2 + (y-0)^2} = 2a$$

$$\sqrt{(x+c)^2 + (y-0)^2} = 2a - \sqrt{(x-c)^2 + (y-0)^2}$$

Squaring both sides

$$x^2 + c^2 + 2cx + y^2 = 4a^2 + x^2 + c^2 - 2cx + y^2 - 4a\sqrt{(x-c)^2 + y^2}$$

$$4cx - 4a^2 = -4a\sqrt{(x-c)^2 + y^2}$$

$$cx - a^2 = -a\sqrt{(x-c)^2 + y^2}$$

Again squaring

$$c^2x^2 + a^4 - 2a^2cx = a^2x^2 + a^2c^2 - 2a^2cx + a^2y^2$$

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

Dividing both sides by $a^2(a^2 - c^2)$

$$\frac{(a^2 - c^2)x^2}{a^2(a^2 - c^2)} + \frac{a^2y^2}{a^2(a^2 - c^2)} = \frac{a^2(a^2 - c^2)}{a^2(a^2 - c^2)}$$

$$\frac{x^2}{a^2} + \frac{y^2}{(a^2 - c^2)} = 1$$

Let $a^2 - c^2 = b^2$

$$\Rightarrow \boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

Q.4 Use problem 3 to find equations of the ellipse as locus of the point $P(x,y)$ such that the sum of the distance from P to the point (0,0) and (1,1) is 2.

Solution: Using problem (3) here

$$|PF| + |PF'| = 2$$

$$\sqrt{x^2 + y^2} + \sqrt{(x-1)^2 + (y-1)^2} = 2$$

$$\sqrt{x^2 + y^2} = 2 - \sqrt{(x-1)^2 + (y-1)^2}$$

Squaring both sides

$$x^2 + y^2 = 4 + x^2 + 1 - 2x + y^2 + 1 - 2y - 4\sqrt{(x-1)^2 + (y-1)^2}$$

$$2x + 2y - 6 = 4\sqrt{(x-1)^2 + (y-1)^2}$$

$$x + y - 3 = 2\sqrt{(x-1)^2 + (y-1)^2}$$

Again squaring

$$x^2 + y^2 + 9 - 6x - 6y + 2xy = 4x^2 - 8x + 4 + 4y^2 - 8y + 4$$

$$\boxed{3x^2 + 3y^2 - 2xy - 2x - 2y - 1 = 0}$$

Q.5 Prove that the latusrectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{2b^2}{a}$

Proof:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Let $|LL'|$ be length of latus rectum.

Since the point (c, y_1) lies on the ellipse

$$\therefore \frac{c^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

$$\frac{y_1^2}{b^2} = 1 - \frac{c^2}{a^2}$$

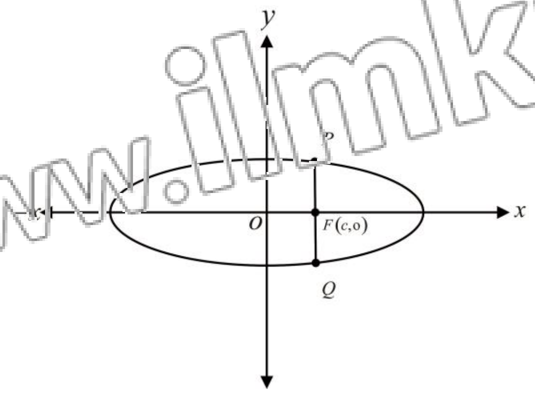
$$y_1^2 = b^2\left(\frac{a^2 - c^2}{a^2}\right)$$

$$y_1^2 = \frac{b^2}{a^2}b^2$$

$$y_1 = \frac{b^2}{a}$$

$$|LF'| = \frac{b^2}{a}$$

$$\therefore\ |LL'| = 2|LF'| = \frac{2b^2}{a}$$



Q.6 The major axis of an ellipse in standard form lies along the x-axis has length $4\sqrt{2}$ the distance between the foci equal the length of the minor axis. Write an equation of the ellipse

Solution:

Length of major axis = $4\sqrt{2}$

$\Rightarrow 2a = 4\sqrt{2} \Rightarrow a = 2\sqrt{2}$

It is given that distance between foci that is $2c$ is equal to length of minor axis that is $2b$.

$\Rightarrow 2c = 2b \Rightarrow c = b$

$c^2 = a^2 - b^2 \Rightarrow b^2 = \left(2\sqrt{2}\right)^2 - b^2$

$\Rightarrow 2b^2 = 8$

$\Rightarrow b^2 = 4$

Hence the required equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\boxed{\frac{x^2}{8} + \frac{y^2}{4} = 1}$$

Q.7 An astroid has elliptic orbit with the sun at one focus. Its distance from the sun ranges from 17 million miles to 183 million miles. Write an equation of the orbit of the astroid.

Solution:

Consider the sun is at $F'(c,0)$ and let the astroid is first at $A(a,0)$ and then at $A'(-a,0)$ then

$a - c = 17 \ldots$(i)

$a + c = 183 \ldots$(ii)

Adding (i) and (ii)

$2a = 200 \Rightarrow a = 100$

Put $a = 100$ in (i) we get;

$100 - c = 17 \Rightarrow c = 100 - 17$

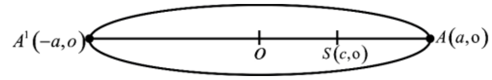
$\Rightarrow c = 83$

$c^2 = a^2 - b^2 \Rightarrow (83)^2 = (100)^2 - b^2$

$\Rightarrow b^2 = 10000 - 6889 = 3111$

Equation of the orbit of the astroid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{10000} + \frac{y^2}{3111} = 1$$



Q.8 An arch in the shape of semi ellipse is 90 meter wide at the base and 30 meter high at the centre. At what distance from the centre is the arch $20\sqrt{2}$ meter high?

Solution:

From figure

$2a = 90 \Rightarrow a = 45$

$b = 30$

equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{(45)^2} + \frac{y^2}{(30)^2} = 1 \ldots \text{(i)}$$

$\because \left(x_1, 20\sqrt{2}\right)$ lies on the ellipse

$$\therefore \frac{x_1^2}{(45)^2}+\frac{\left(20\sqrt{2}\right)^2}{(30)^2}=1$$

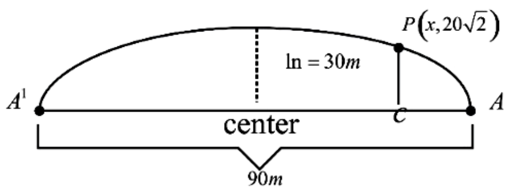
$$\frac{x_1^2}{(45)^2}+\frac{800}{900}=1$$

$$\frac{x_1^2}{(45)^2}+\frac{800}{900}=1$$

$$\frac{x^2}{2025}=1-\frac{8}{9} \Rightarrow x^2=\frac{2025}{9}$$

$$\Rightarrow x=\pm\frac{45}{3} \quad \Rightarrow x=\pm 15 \text{ meter}$$

But distance should be positive so $x=15$m.



Q.9 The moon orbits the earth in an elliptic path with earth at one focus. The major and minor axis of the orbit are 768, 806 km and 767, 746 km respectively. Find the greatest and the least distance (in Astronomy called the apogee and perigee) of the moon from the earth.

Solution: Let $F(a,0)$ shows earth as a focus

It is given that length of major axis is 768, 806 km

$\Rightarrow 2a = 786,806 \Rightarrow a = 384,403$ km

length of minor axis is 767,746 km

That is

$2b = 767,746 \Rightarrow b = 383,873$ km

As

$c^2=a^2-b^2=(384,403)^2-(383,873)^2=407,280$

$\Rightarrow c = 20178.85725$

So, $c \simeq 20179$

Now Greatest distance

$= a + c = 384,403 + 20179 = 404582$ km.

Least distance

$= a - c = 384,403 - 20179$

$= 364,224$ km