The Hyperbola:

Let $e > 1$ and F be a fixed point and L be a line not containing F. Also let $P(x,y)$ be a point in the plane and $|PM|$ be the perpendicular distance of the P from L. The set of all points $P(x,y)$ such that $\frac{|PF|}{|PM|} = e > 1$ is called a hyperbola.

F and L are respectively **focus** and **directrix** of the hyperbola, e is the eccentricity.

OR

A hyperbola is the locus of pint which moves in such a way that the difference of its distances from two fixed points is a constant i.e. $|PF| - |PF'| = \pm 2a$

Standard Equation of Hyperbola:

Let $F(c,o)$ be the focus with $e > 0$ and $x = \frac{c}{e^2}$ be the directrix of the hyperbola, then equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Proof:

Let $P(x,y)$ be a point on the hyperbola and $|PM|$ be the perpendicular distance of P to the directrix. Then

$$|PM| = x - \frac{c}{e^2}$$

By definition of hyperbola

$$|PF| = e|PM| \text{ where } e > 1$$

$$\sqrt{(x-c)^2 + (y-0)^2} = e\left(x - \frac{c}{e^2}\right)$$

Taking square on both sides

$$x^2 + c^2 - 2cx + y^2 = e^2\left(x - \frac{c}{e^2}\right)^2$$

$$x^2 + c^2 - 2cx + y^2 = e^2\left(x^2 + \frac{c^2}{e^4} - \frac{2cx}{e^2}\right) \quad x^2 + c^2 - 2cx + y^2 = e^2x^2 + \frac{c^2}{e^2} - 2cx$$

$$x^2 - 2cx + y^2 + 2cx - e^2x^2 = \frac{c^2}{e^2} - c^2$$

$$x^2\left(1-e^2\right)+y^2=\frac{c^2}{e^2}\left(1-e^2\right)$$

$$x^2\left(e^2-1\right)-y^2=\frac{c^2}{e^2}\left(e^2-1\right)$$

$$x^2\left(e^2-1\right)-y^2=a^2\left(e^2-1\right)$$

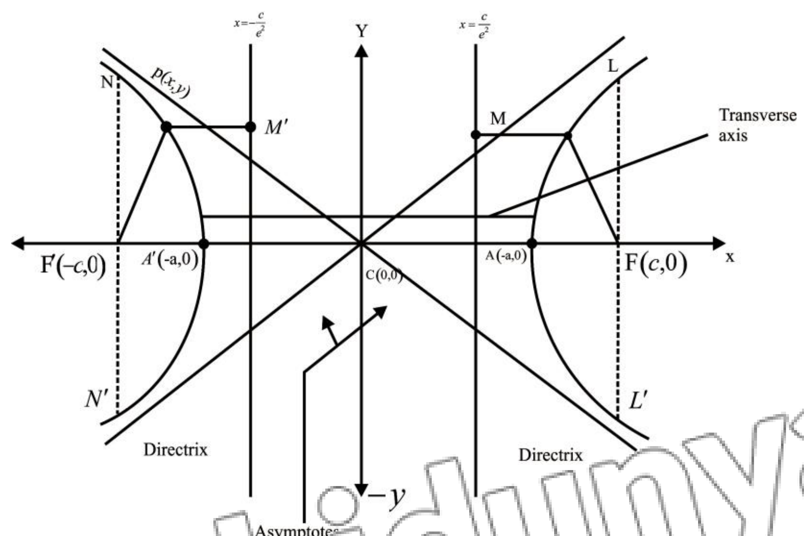
where $a=\frac{c}{e}$

$$\frac{x^2\left(e^2-1\right)}{a^2\left(e^2-1\right)}-\frac{y^2}{a^2\left(e^2-1\right)}=1$$

$$\frac{x^2}{a^2}-\frac{y^2}{a^2\left(e^2-1\right)}=1$$

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ where $b^2=a^2\left(e^2-1\right)$

$$\boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1}$$



Length of transverse axis = $2a = AA'$

Length of conjugate axis = $2b = BB'$

Vertices are $A(a,0)$ and $A'(-a,0)$

Ends of conjugate axes are B (0, b) and $B'(0,-b)$

Center of the hyperbola is C(0, 0)

Foci of the hyperbola are $F(c,0)$ and $F'(-c,0)$

LL' and NN' are the latra recta of the hyperbola.

Properties:

Axes of the Hyperbola:

The lines through AA' $i.e\left(x-axis\right)$ and BB' $i.e\left(y-axis\right)$ are called the **transverse** and **conjugate** axis respectively. Both together are called axes of the hyperbola.

Vertices of the Hyperbola:

The points A, A' where the curve meets the transverse axis of the hyperbola are called the **vertices** of the hyperbola.

Centre of the Hyperbola:

The mid-point $C(0, 0)$ of AA' is called the **center of the hyperbola**.

Latus Rectum:

The line passing through F or F' and perpendicular the transverse axis of the hyperbola is called the latus rectum of the hyperbola. LL' and NN' are two **latera recta** of the hyperbola.

Foci: The points $F(c, 0)$ and $F'\left(-c,0\right)$ are called **foci** of the hyperbola.

Focal Chord:

A chord of the hyperbola passing through its focus is called **focal chord**.

Eccentricity of the Hyperbola:

The eccentricity of the hyperbola = $e=\sqrt{1+\frac{b^2}{a^2}}>1$

Central Conics:

The ellipse and hyperbola are called central conics because each has a centre of symmetry.

Asymptotes:

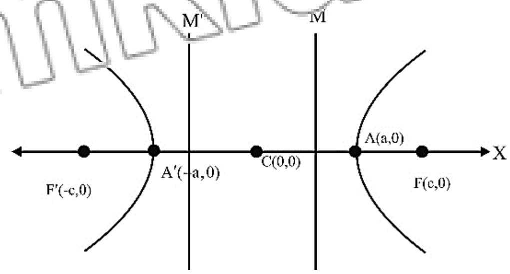
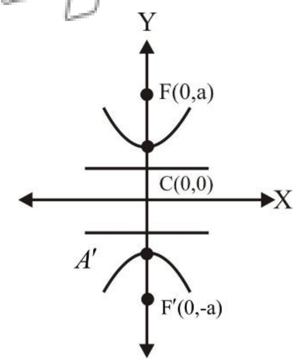
The lines which do not meet the curve but distance of any point on the curve from any of two lines approaches zero such lines are called asymptotes of curve.

$y=\pm\frac{b}{a}x$ are asymptotes.

Rectangular Hyperbola:

If $a=b$ in the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ then the equation $x^2-y^2=a^2$ is general form of rectangular hyperbola.

Summary of Standard Hyperbola:

Standard Equation	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
Shape of the Graph		
Vertices	$(\pm a,0)$	$(0,\pm a)$
Covertices	$(0,\pm b)$	$(\pm b,0)$
Foci	$(\pm c,0)$	$(0,\pm c)$
Equation of transverse axis	$y=0$	$x=0$
Center	(0, 0)	(0, 0)
Eccentricity	$e=\frac{c}{a}=\sqrt{1+\frac{b^2}{a^2}}>1$	$e=\frac{c}{a}=\sqrt{1+\frac{b^2}{a^2}}>1$
Directrices	$x=\pm\frac{c}{e^2}$	$y=\pm\frac{c}{e^2}$
Symmetry	About both the axis	About both the axis
Parametric equations	$(a\sec\theta, b\tan\theta)$	$(b\sec\theta, a\tan\theta)$

EXERCISE 6.6

Q.1 Find an equation of the hyperbola with the given data. Sketch graph of each.

(i) Center (0,0) **focus** (6,0) **vertex** (4,0).

Solution:

Centre $= C(h,k) = C(0,0)$

Focus $F(6,0) \Rightarrow c = 6$

Vertex $A(4,0) \Rightarrow a = 4$

$b^2 = c^2 - a^2 = 36 - 16 = 20$

$\because$ transverse axis lies on x-axis

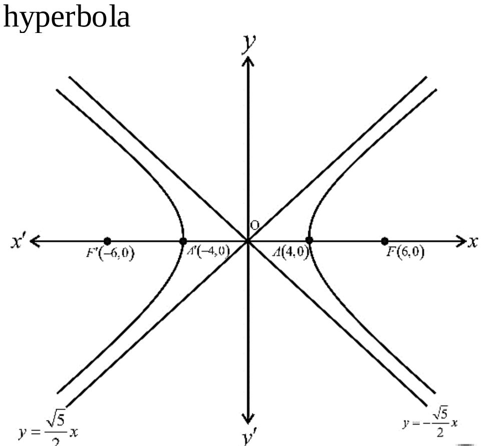
$$\therefore \quad \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\Rightarrow \boxed{\frac{x^2}{16} - \frac{y^2}{20} = 1}$$

Asymptotes:

$$\frac{x^2}{16} - \frac{y^2}{20} = 0$$

$\Rightarrow y = \pm\frac{2\sqrt{5}}{4}x$ are the asymptotes of hyperbola



(ii) Foci $(\pm5,0)$, **vertex** $(3,0)$

Solution:

Foci $F(5,0)$ and $F'(-5,0)$

$\Rightarrow c = 5$

Vertex $A(3,0) \Rightarrow a = 3$

$b^2 = c^2 - a^2 = 25 - 9 = 16$

Centre $= C(h,k) = C(0,0)$

$\because$ transverse axis lies on x-axis

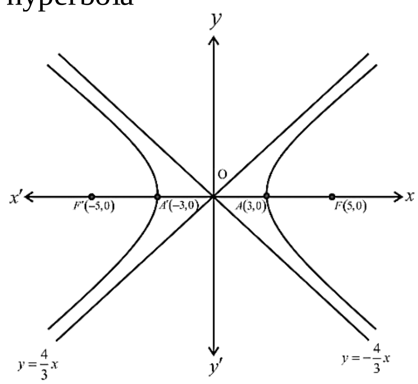
$$\therefore \quad \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\Rightarrow \boxed{\frac{x^2}{9} - \frac{y^2}{16} = 1}$$

Asymptotes:

$$\frac{x^2}{9} - \frac{y^2}{16} = 0$$

$\Rightarrow y = \pm\frac{4}{3}x$ are the asymptotes of hyperbola



(iii) Foci $\left(2\pm5\sqrt{2},-7\right)$, **length of the transverse axis 10.**

Solution:

Foci $F\left(2+5\sqrt{2},-7\right)$ and $F'\left(2-5\sqrt{2},-7\right)$

The mid point of FF' is

$$C(h,k) = C\left(\frac{2+5\sqrt{2}+2-5\sqrt{2}}{2}, \frac{-7-7}{2}\right)$$

$\Rightarrow C(h,k) = C(2,-7)$

Length of transverse of axis is $2a = 10$

$\Rightarrow \boxed{a = 5}$

$c = |CF| = \left|2+5\sqrt{2}-2\right| = 5\sqrt{2}$

$b^2 = c^2 - a^2 = 50 - 25 = 25$

$\because$ transverse axis lies on x-axis

$$\therefore \quad \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\Rightarrow \boxed{\frac{(x-2)^2}{25}-\frac{(y+7)^2}{25}=1}$$

Vertices:

For vertices of hyperbola

$(x-2, y+7)=(\pm5,0)$

$x-2=\pm5,\ y=-7$

$x=2\pm5,\ y=-7$

$x=7,-3,\ y=-7$

Hence vertices are

$A(7,-7), A'(-3,-7)$

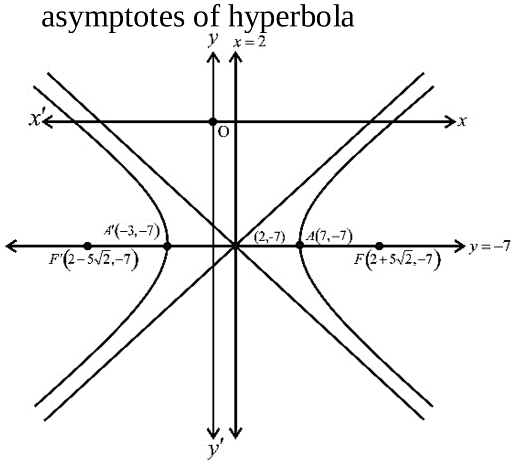
Asymptotes:

$$\frac{(x-2)^2}{25}-\frac{(y+7)^2}{25}=0$$

$\Rightarrow y=-7\pm(x-2)$

$y+7=x-2$ and $y+7=-(x-2)$

$y=x-9$ and $y=-x-5$ are the asymptotes of hyperbola



(iv) Foci $(0,\pm6), e=2$

Solution:

The mid-point of FF' is

$C(h,k)=C(0,0)$

$c=|CF|=6$

$e=\frac{c}{a}\Rightarrow a=\frac{c}{e}=\frac{6}{2}=3$

$b^2=c^2-a^2=36-9=27$

$\because$ transverse axis lies on y-axis

$$\therefore\ \frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

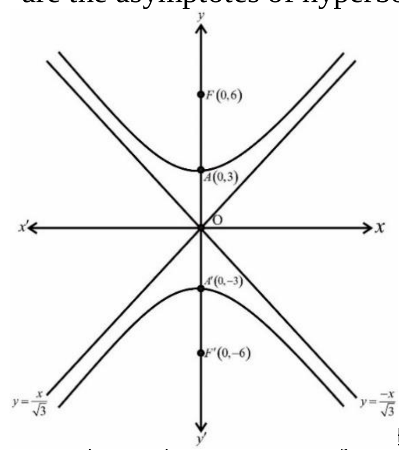
$$\Rightarrow \boxed{\frac{y^2}{9}-\frac{x^2}{27}=1}$$

Asymptotes:

$$\frac{y^2}{9}-\frac{x^2}{27}=0$$

$$y=\pm\frac{1}{\sqrt{3}}x$$

are the asymptotes of hyperbola



(v) ... ±4.

Solution:

The mid-point of FF' is

$C(h,k)=C(0,0)$

Now

$c=|CF|=9$

Equation of DTX $y=\pm4$

$y=\pm\frac{c}{e^2}$

$\frac{c}{e^2}=4$

$\Rightarrow \frac{9}{e^2}=4 \qquad \because c=9$

$\Rightarrow e^2=\frac{9}{4}$

$\Rightarrow e=\frac{3}{2}>1$

Now as $e=\frac{c}{a}$

So $\frac{3}{2}=\frac{9}{a}$

$a=6\Rightarrow \boxed{a^2=36}$

As $c^2=a^2+b^2$

$81=36+b^2$

$b^2=81-36$

$\Rightarrow \boxed{b^2 = 45}$

$\because$ transverse axis lies on y-axis

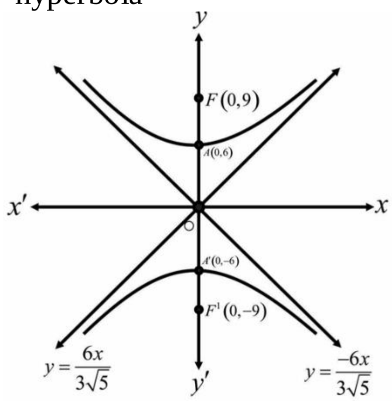
$\therefore \quad \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

$\Rightarrow \frac{y^2}{36} - \frac{x^2}{45} = 1$

Asymptotes:

$\Rightarrow \frac{y^2}{36} - \frac{x^2}{45} = 0$

$\Rightarrow y = \pm \frac{6}{3\sqrt{5}} x$ are the asymptotes of hyperbola



(vi) Centre (2,2), horizontal **transverse axis of length 6 and eccentricity** $e = 2$.

Solution:

Centre = $C(h,k) = C(2,2)$

The length transverse axis is

$2a = 6 \Rightarrow a = 3$

$\Rightarrow \boxed{a^2 = 9}$

$e = 2$.

$e = \frac{c}{a} = 2$

$\because \ a = 3$ and $e = 2, \ \Rightarrow c = 6$

As

$c^2 = a^2 + b^2$

$\Rightarrow 36 = 9 + b^2 \Rightarrow \boxed{b^2 = 27}$

$\because$ transverse axis lies on x-axis

$\therefore \quad \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

$$\boxed{\frac{(x-2)^2}{9} - \frac{(y-2)^2}{27} = 1}$$

Vertices:

$(x-2, y-2) = (\pm 3, 0)$

$x - 2 = \pm 3 \ , \ y - 2 = 0$

$x = 2 \pm 3 \ , \ y = 2$

$x = 5, -1 \ , \ y = 2$

So vertices are $A(5,2)$ and $A'(-1,2)$

Foci:

$(x-2, y-2) = (\pm 6, 0)$

$x - 2 = \pm 6 \ , \ y = 2$

$x = 2 \pm 6 \ , \ y = 2$

$x = 8, -4 \ , \ y = 2$

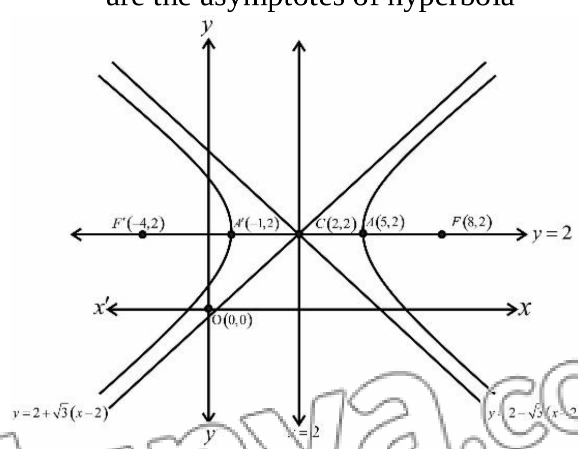
So foci are $F(8,2)$ and $F'(-4,2)$

Asymptotes:

$\frac{(x-2)^2}{9} - \frac{(y-2)^2}{27} = 0$

$\Rightarrow y = 2 \pm \sqrt{3}(x-2)$

are the asymptotes of hyperbola



on the curve.

Solution:

$A(2,3)$ and $A'(2,-3)$

$2a = |AA'| = 6$

$\Rightarrow \boxed{a = 3}$

The mid-point of AA' is

$C(h,k) = C(2,0)$

$\because$ transverse axis lies on y-axis

$\therefore \quad \dfrac{(y-k)^2}{a^2}-\dfrac{(x-h)^2}{b^2}=1 \quad$...(i)

distance between vertices is $2a$,

$2a=\sqrt{(2-2)^2+(3+(-3))^2}$

$2a=\sqrt{6^2}$

$2a=6$

$a=3 \Rightarrow a^2=9$

Put $(h,k)=(2,0)$ and $a=3$ in (i), we get;

$\dfrac{(y-0)^2}{9}-\dfrac{(x-2)^2}{b^2}=1 \quad$...(ii)

$\dfrac{y^2}{9}-\dfrac{(x-2)^2}{b^2}=1$

As (0,5) lies on the curve, so

$\dfrac{25}{9}-\dfrac{(0-2)^2}{b^2}=1$

$\dfrac{25}{9}-1=\dfrac{4}{b^2}$

$\dfrac{16}{9}=\dfrac{4}{b^2}$

$16b^2=36$

$b^2=\dfrac{36}{16}$

$b^2=\dfrac{9}{4}$

Put value of $b^2=\dfrac{9}{4}$ in (ii), we get;

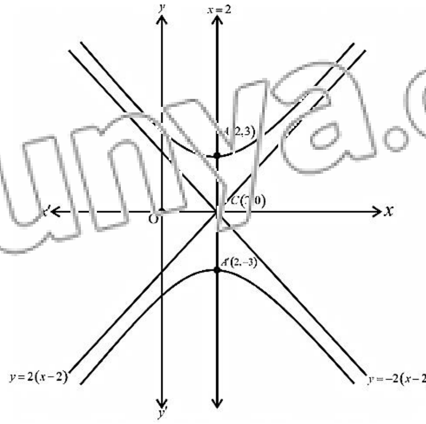
$\dfrac{y^2}{9}-\dfrac{(x-2)^2}{\frac{9}{4}}=1$

$\boxed{\dfrac{y^2}{9}-\dfrac{4(x-2)^2}{9}=1}$

Asymptotes:

$\dfrac{y^2}{9}-\dfrac{4(x-2)^2}{9}=0$

$\Rightarrow y=\pm 2(x-2)$ are the asymptotes of the hyperbola



(viii) Foci $(5,-2),(5,4)$ and one vertex (5,3).

Solution:

Foci $F(5,4)$ and $F'(5,-2)$

Mid-point of FF' is

$C(h,k)=\left(\dfrac{5+5}{2},\dfrac{-2+4}{2}\right)=C(5,1)$

$2c=|FF'|=6$

$\Rightarrow c=3$

Distance between vertices is

$2a=4 \quad a=2$

$b^2=c^2-a^2=9-4=5$

$\because$ transverse axis lies on *y*-axis

$\therefore \quad \dfrac{(y-k)^2}{a^2}-\dfrac{(x-h)^2}{b^2}=1$

$\boxed{\dfrac{(y-1)^2}{4}-\dfrac{(x-5)^2}{5}=1}$

Asymptotes:

$\dfrac{(y-1)^2}{4}-\dfrac{(x-5)^2}{5}=0$

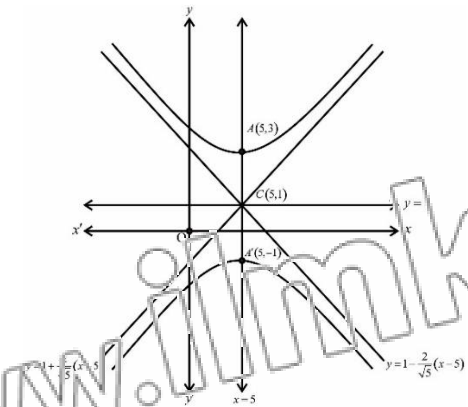
$\Rightarrow \dfrac{(y-1)^2}{4}=\dfrac{(x-5)^2}{5}$

$(y-1)^2=\dfrac{4}{5}(x-5)^2$

$y-1=\pm\dfrac{2}{\sqrt{5}}(x-5)$

$\Rightarrow y=1\pm\dfrac{2}{\sqrt{5}}(x-5)$

are the asymptotes of hyperbola



Q.2 Find the center, foci, eccentricity, vertices and equation of the directrices of each of the following

(i) $x^2 - y^2 = 9$

Solution:

$x^2 - y^2 = 9$

$\frac{x^2}{9} - \frac{y^2}{9} = 1$

$a^2 = b^2 = 9 \Rightarrow a = b = 3$

$c^2 = a^2 + b^2$

$c^2 = 9 + 9 = 18 \Rightarrow c = 3\sqrt{2}$

Centre $C(h,k) = C(0,0)$

Foci are $F(\pm c, 0) = F(\pm 3\sqrt{2}, 0)$

Eccentricity

$e = \frac{c}{a} \Rightarrow e = \frac{3\sqrt{2}}{3} = \sqrt{2} > 1$

Vertices $A(\pm a, 0) = A(\pm 3, 0)$

Directrices $x = \pm\frac{a}{e}$

$x = \pm\frac{3}{\sqrt{2}}$

(ii) $\frac{x^2}{4} - \frac{y^2}{9} = 1$

Solution:

$\frac{x^2}{4} - \frac{y^2}{9} = 1$

$a^2 = 4 \Rightarrow a = 4$

$b^2 = 9 \Rightarrow b = 3$

$c^2 = a^2 + b^2$

$c^2 = 4 + 9 = 13$

Centre $C(h,k) = C(0,0)$

Foci are $F(\pm c, 0) = F(\pm\sqrt{13}, 0)$

Eccentricity: $e = \frac{c}{a} \Rightarrow e = \frac{\sqrt{13}}{2} > 1$

Vertices $A(\pm a, 0) = A(\pm 2, 0)$

Directrices $x = \pm\frac{a}{e}$

$x = \pm\frac{2}{\sqrt{13}/2} \Rightarrow x = \pm\frac{4}{\sqrt{13}}$

(iii) $\frac{y^2}{16} - \frac{x^2}{9} = 1$

Solution:

$\frac{y^2}{16} - \frac{x^2}{9} = 1$

$a^2 = 16 \Rightarrow a = 4$

$b^2 = 9 \Rightarrow b = 3$

$c^2 = a^2 + b^2 \Rightarrow c^2 = 16 + 9 = 25$

$c = 5$

Centre $C(h,k) = C(0,0)$

Foci are $F(0, \pm c) = F(0, \pm 5)$

Eccentricity $e = \frac{c}{a} \Rightarrow e = \frac{5}{4} > 1$

Vertices $A(0, \pm a) = A(0, \pm 4)$

Directrices $y = \pm\frac{a}{e} \Rightarrow = \pm\frac{4}{5/4}$

$y = \pm\frac{16}{5}$

(iv) $\frac{y^2}{4} - x^2 = 1$

Solution:

$\frac{y^2}{4} - x^2 = 1$

$a^2 = 4 \Rightarrow a = 2$

$b^2 = 1 \Rightarrow b = 1$

$c^2 = a^2 + b^2$

$c^2 = 4 + 1 = 5$

$c = \sqrt{5}$

Centre $C(h,k) = C(0,0)$

Foci are $F(0,\pm c)=F(0,\pm\sqrt{5})$

Eccentricity $e=\frac{c}{a}\Rightarrow e=\frac{\sqrt{5}}{2}>1$

Vertices $A(0,\pm a)=A(0,\pm 2)$

Directrices $y=\pm\frac{a}{e}\Rightarrow y=\pm\frac{2}{\frac{\sqrt{5}}{2}}$

$y=\pm\frac{4}{\sqrt{5}}$

(v) $\frac{(x-1)^2}{2}-\frac{(y-1)^2}{9}=1$

Solution:

$\frac{(x-1)^2}{2}-\frac{(y-1)^2}{9}=$...(i)

Let $x-1=X$ and $y-1=Y$

then (i) becomes

$\frac{X^2}{2}-\frac{Y^2}{9}=1$...(ii)

$a^2=2\Rightarrow a=\sqrt{2}$

$b^2=9\Rightarrow b=3$

$c^2=a^2+b^2\Rightarrow c^2=2+9=11$

$c=\sqrt{11}$

Centre of (ii) $C(h,k)=C(0,0)$

$X=0$ and $Y=0$

$x-1=0$, $y-1=0$

$x=1$, $y=1$

Centre of (i) $C(h,k)=C(1,1)$

Foci of (ii) are

$F(\pm c,0)=F(\pm\sqrt{11},0)$

$X=\pm\sqrt{11}$, $Y=0$

$x-1=\pm\sqrt{11}$, $y-1=0$

$x=1\pm\sqrt{11}$, $y=1$

foci of (i) are $(1\pm\sqrt{11},1)$

Eccentricity of (i) is

$e=\frac{c}{a}\Rightarrow e=\frac{\sqrt{11}}{\sqrt{2}}=\sqrt{\frac{11}{2}}>1$

Vertices of (ii) are

$A(\pm a,0)=A(\pm\sqrt{2},0)$

$X=\pm\sqrt{2}$, $Y=0$

$x-1=\pm\sqrt{2}$, $y-1=0$

$x=1\pm\sqrt{2}$, $y=1$

vertices of (i) are $(1\pm\sqrt{2},1)$

Equations of directrices of (ii) are

$X=\pm\frac{a}{e}$

$X=\pm\frac{\sqrt{2}}{\sqrt{\frac{11}{2}}}$

$x-1=\pm\frac{2}{\sqrt{11}}$

$x=1\pm\frac{2}{\sqrt{11}}$ are equations of directrices of (i)

(vi) $\frac{(y+2)^2}{9}-\frac{(x-2)^2}{16}=1$

Solution:

$\frac{(y+2)^2}{9}-\frac{(x-2)^2}{16}=1$...(i)

Let $x-2=X$ and $y+2=Y$

then (i) becomes

$\frac{Y^2}{9}-\frac{X^2}{16}=1$...(ii)

$a^2=9\Rightarrow a=3$

$b^2=16\Rightarrow b=4$

$c^2=a^2+b^2$

$c^2=9+16\Rightarrow c^2=25\Rightarrow c=5$

Centre of (ii) $C(h,k)=C(0,0)$

$X=0$ and $Y=0$

$x-2=0$, $y+2=0$

$\Rightarrow x=2$, $y=-2$

Centre of (i) $C(h,k)=C(2,-2)$

Foci of (ii) are $F(0,\pm c)=F(0,\pm 5)$,

$X=0$, , $Y=\pm 5$

$x-2=0$, $y+2=\pm 5$

$x=2$, $y=3,-7$

$F(2,3)$ and $F'(2,-7)$ are the foci of (i)

Eccentricity of (i) is $e=\frac{c}{a}$

$e=\frac{5}{3}$

Vertices: of (ii) are

$A(0,\pm a)=A(0,\pm 3)$

$X=0\ ,\ Y=\pm 3$

$x-2=0\ ,\ y+2=\pm 3$

$\Rightarrow x=2\ ,\ y=-2\pm 3$

$y=1,-5$

Hence vertices of (i) are $A(2,1)$ and $A'(2,-5)$

Equations of directrices of (ii) are

$Y=\pm\frac{a}{e}\Rightarrow Y=\pm\frac{3}{5/3}$

$Y=\pm\frac{9}{5}$

$y+2=\pm\frac{9}{5}$

$y=-2\pm\frac{9}{5}$

$y=-2+\frac{9}{5}\ ,\ y=-2-\frac{9}{5}$

$y=\frac{-1}{5}\ ,\ y=\frac{-19}{5}$

directrices of (i) are $y=\frac{-1}{5}$ and $y=\frac{-19}{5}$

(vii) $9x^2-12x-y^2-2y+2=0$

Solution:

$9x^2-12x-y^2-2y+2=0$

$9x^2-12x-\left(y^2+2y\right)+2=0$

$9\left(x^2-\frac{4}{3}x\right)-\left(y^2+2y+1^2-1^2\right)+2=0$

$9\left(x^2-\frac{4}{3}x+\frac{4}{9}-\frac{4}{9}\right)-\left((y+1)^2-1\right)+2=0$

$9\left(\left(x-\frac{2}{3}\right)^2-\frac{4}{9}\right)-(y+1)^2+1+2=0$

$9\left(x-\frac{2}{3}\right)^2-4-(y+1)^2+3=0$

$\frac{\left(x-\frac{2}{3}\right)^2}{1/9}-\frac{(y+1)^2}{1}=1\ldots$(i)

Let $X=x-\frac{2}{3},Y=y+1$

Then (i) becomes

$\frac{X^2}{1/9}-\frac{Y^2}{1}=1\ldots$(ii)

$a^2=\frac{1}{9}\Rightarrow a=\frac{1}{3}$

$b^2=1\Rightarrow b=1$

$c^2=a^2+b^2$

$c^2=\frac{1}{9}+1=\frac{10}{9}\Rightarrow c=\frac{\sqrt{10}}{3}$

Centre of (ii) $C(h,k)=C(0,0)$

$X=0$ and $Y=0$

$x-\frac{2}{3}=0\ ,\ y+1=0$

$x=\frac{2}{3}\ ,\ y=-1$

Centre of (i) $C(h,k)=C\left(\frac{2}{3},-1\right)$

Foci of (ii) are

$F(\pm c,0)=F\left(\pm\frac{\sqrt{10}}{3},0\right)$

$X=\pm\frac{\sqrt{10}}{3}$

$x-\frac{2}{3}=\pm\frac{\sqrt{10}}{3}$

$x=\frac{2}{3}\pm\frac{\sqrt{10}}{3}$

$$=\frac{2\pm\sqrt{10}}{3}$$

Here $Y=0$

So $y+1=0\Rightarrow y=-1$

Hence foci of (1) are $\left(\frac{2\pm\sqrt{10}}{3},-1\right)$

Eccentricity: of (1) is

$$e=\frac{c}{a}\Rightarrow e=\frac{\sqrt{10}/3}{1/3}\Rightarrow e=\sqrt{10}>1$$

Vertices: of (2) are

$$(\pm a,0)=\left(\pm\frac{1}{3},0\right)$$

Here $X=\pm\frac{1}{3}$

so $x-\frac{2}{3}=\pm\frac{1}{3}\Rightarrow x=\frac{2}{3}\pm\frac{1}{3}$

$x=1,\frac{1}{3}$

Here $Y=0$

So $y+1=0\Rightarrow y=-1$

Hence vertices of (1) are $A(1,-1)$

and $A'\left(\frac{1}{3},-1\right)$

Equations of directrices for (2) are

$$X=\pm\frac{a}{e}$$

$$X=\pm\frac{1/3}{\sqrt{10}}$$

$$X=\pm\frac{1}{3\sqrt{10}}$$

So $x-\frac{2}{3}=\pm\frac{1}{3\sqrt{10}}$

$$x=\frac{2}{3}\pm\frac{1}{3\sqrt{10}}$$

(viii) $4y^2+12y-x^2+4x+1=0$

Solution

$$4y^2+12y-x^2+4x+1=0$$

$$\left(4y^2+12y\right)-\left(x^2-4x\right)+1=0$$

$$4\left(y^2+3y\right)-\left(x^2-4x\right)+1=0$$

$$4\left(y^2+3y+\frac{9}{4}-\frac{9}{4}\right)-\left(x^2-4x+4-4\right)+1=0$$

$$4\left[\left(y+\frac{3}{2}\right)^2-\frac{9}{4}\right]-\left(\left(x-2\right)^2-4\right)+1=0$$

$$4\left(y+\frac{3}{2}\right)^2-9-\left(x-2\right)^2+4+1=0$$

$$4\left(y+\frac{3}{2}\right)^2-\left(x-2\right)^2=4$$

$$\frac{\left(y+\frac{3}{2}\right)^2}{1}-\frac{\left(x-2\right)^2}{4}=1\text{...(i)}$$

Let $Y=y+\frac{3}{2}$ and $X=x-2$

then (i) becomes

$$\frac{Y^2}{1}-\frac{X^2}{4}=1\text{...(ii)}$$

$a^2=1\Rightarrow a=1$

$b^2=4\Rightarrow b=2$

$c^2=a^2+b^2$

$c^2=1+4=5\Rightarrow c=\sqrt{5}$

Center of (ii) is $C(h,k)=C(0,0)$

$X=0\quad,\quad Y=0$

$x-2=0,\ y+\frac{3}{2}=0$

$x=2,\ y=-\frac{3}{2}$

Center of (i) is $C(h,k)=C\left(2,\frac{-3}{2}\right)$

Foci (ii) are $F(0,\pm c)=F(0,\pm 5)$

$X=0\ ,\ Y=\pm\sqrt{5}$

$x-2=0,\ y+\frac{3}{2}=\pm\sqrt{5}$

$x=2\ ,\ y=\frac{-3}{2}\pm\sqrt{5}$

foci of (i) are $\left(2,\frac{-3}{2}\pm\sqrt{5}\right)$

Eccentricity of (i) is $e=\frac{c}{a}=\frac{\sqrt{5}}{1}$

$e=\sqrt{5}>1$

Vertices of (ii) are

$A(0,\pm a)=A(0,\pm 1)$

$X=0\ ,\ Y=\pm 1$

$x-2=0\ ,\ y+\frac{3}{2}=\pm 1$

$\Rightarrow x=2\ ,\ y=-\frac{3}{2}\pm 1$

$y=-\frac{3}{2}+1,\ y=-\frac{3}{2}-1$

$y=-\frac{1}{2},\ y=-\frac{5}{2}$

vertices of (i) are $A\left(2,-\frac{1}{2}\right)$ and

$A'\left(2,-\frac{5}{2}\right)$

Equations of directrices of (ii) are

$Y=\pm\frac{a}{e}\Rightarrow Y=\pm\frac{1}{\sqrt{5}}$

$y+\frac{3}{2}=\pm\frac{1}{\sqrt{5}}$

$y=\frac{-3}{2}\pm\frac{1}{\sqrt{5}}$ are the equations of directrices of (i)

(ix) $x^2-y^2+8x-2y-10=0$

Solution:

$x^2-y^2+8x-2y-10=0$

$(x^2+8x)-(y^2+2y)-10=0$

$(x^2+8x+4^2-4^2)-(y^2+2y+1^2-1^2)-10=0$

$((x+4)^2-16)-((y+1)^2-1)-10=0$

$(x+4)^2-(y+1)^2-16+1-10=0$

$(x+4)^2-(y+1)^2-25=0$

$(x+4)^2-(y+1)^2=25$

$\frac{(x+4)^2}{25}-\frac{(y+1)^2}{25}=1$ ...(i)

Let $x+4=X,\ y+1=Y$

Then (i) becomes

$\frac{X^2}{25}-\frac{Y^2}{25}=1$...(ii)

$a^2=25\Rightarrow a=5$

$b^2=25\Rightarrow b=5$

$c^2=a^2+b^2$, so $c^2=25+25=50$

$c=\sqrt{50}=5\sqrt{2}$

Center of (ii) is $C(h,k)=C(0,0)$

$X=0\ ,\ Y=0$

$x+4=0\ ,\ y+1=0$

$x=-4\ ,\ y=-1$

center of (i) is $C(h,k)=(-4,-1)$

Foci of (ii) are

$F(\pm c,0)=F(\pm 5\sqrt{2},0)$

$X=\pm 5\sqrt{2}\ ,\ Y=0$

$x+4=\pm 5\sqrt{2}\ ,\ y+1=0$

$x=-4\pm 5\sqrt{2}\ ,\ y=-1$

foci of (i) are $(-4\pm 5\sqrt{2},-1)$

Eccentricity of (i) is

$e=\frac{c}{a}=\frac{5\sqrt{2}}{5}=\sqrt{2}>1$

Vertices of (ii) are

$A(\pm c,0)=A(\pm 5,0)$

$X=\pm 5\ ,\ Y=0$

$x+4=\pm 5\ ,\ y+1=0$

$x=1,-9\ ,\ y=-1$

vertices of (i) are $A(1,-1)$ and

$A'(-9,-1)$

Equations of directrices of (ii) are

$X=\pm\frac{a}{e}$

$X=\pm\frac{5}{\sqrt{2}}$

$x+4=\pm\frac{5}{\sqrt{2}}$

$x=-4\pm\dfrac{5}{\sqrt{2}}$ are the equations of directrices for (i)

(x) $9x^2-y^2-36x-6y+18=0$

Solution:

$$9x^2-y^2-36x-6y+18=0$$
$$\left(9x^2-36x\right)-\left(y^2+6y\right)+18=0$$
$$9\left(x^2-4x\right)-\left(y^2+6y\right)+18=0$$
$$9\left(x^2-4x+2^2-2^2\right)-\left(y^2+6y+3^2-3^2\right)+18=0$$
$$9\left(\left(x-2\right)^2-4\right)-\left(\left(y+3\right)^2-9\right)+18=0$$
$$9(x-2)^2-36-(y+3)^2+9+18=0$$
$$9(x-2)^2-(y+3)^2-9=0$$
$$9(x-2)^2-(y+3)^2=9$$
$$\frac{(x-2)^2}{1}-\frac{(y+3)^2}{9}=1\ldots\text{(i)}$$

Let $x-2=X$ and $y+3=Y$

Then (i) becomes

$$\frac{X^2}{1}-\frac{Y^2}{9}=1\ldots\text{(ii)}$$
$$a^2=1\Rightarrow a=1$$
$$b^2=9\Rightarrow b=3$$
$$c^2=a^2+b^2$$
$$c^2=1+9=10\Rightarrow c=\sqrt{10}$$

Center of (ii) is $C(h,k)=C(0,0)$

$X=0$, $Y=0$

$x-2=0$, $y+3=0$

$x=2$, $y=-3$

center of (i) is $C(h,k)=(2,-3)$

Foci of (ii) are

$F(\pm c,0)=F\left(\pm\sqrt{10},0\right)$

$X=\pm\sqrt{10}$, $Y=0$

$x-2=\pm\sqrt{10}$, $y+3=0$

$x=2\pm\sqrt{10}$, $y=-3$

Foci of (i) are $\left(2\pm\sqrt{10},-3\right)$

Eccentricity of (i) is $e=\dfrac{c}{a}$

$e=\dfrac{\sqrt{10}}{1}=\sqrt{10}$

Vertices of (ii) are

$A(\pm a,0)=A(\pm 1,0)$

$X=\pm1$, $Y=0$

$x-2=\pm1$, $y+3=0$

$x-2=1$, $x-2=-1$, $y=-3$

$x=3,1$

vertices of (i) are $A(3,-3)$ and $A'(1,-3)$

Equations of directrices for (ii) are

$$X=\pm\frac{a}{e}$$
$$X=\pm\frac{1}{\sqrt{10}}$$
$$x-2=\pm\frac{1}{\sqrt{10}}$$

are $x=2\pm\dfrac{1}{\sqrt{10}}$ directrices of (i)

Q.3 Let and be the two fixed points. Show that the set of points such that

$|PF|-|PF'|=\pm2a$ **is the hyperbola**

$$\frac{x^2}{a^2}-\frac{y^2}{c^2-a^2}=1$$

(are foci of the hyperbola)

Proof:

$$|PF|-|PF'|=\pm2a$$
$$\sqrt{(x-c)^2+(y-0)^2}-\sqrt{(x+c)^2+(y-0)^2}=\pm2a$$
$$\sqrt{(x-c)^2+y^2}=\sqrt{(x+c)^2+y^2}\pm2a$$

Squaring on both sides we get;

$$(x-c)^2-(x+c)^2+y^2-y^2=4a^2+4a\sqrt{(x+c)^2+y^2}$$
$$-4cx=4a^2+4a\sqrt{(x+c)^2+y^2}$$
$$-4cx-4a^2=4a\sqrt{(x+c)^2+y^2}$$
$$-\left(cx+a^2\right)=a\sqrt{(x+c)^2+y^2}$$

Again squaring on both sides, we get;

$$c^2x^2+a^4+2a^2cx=a^2\left[\left(x+c\right)^2+y^2\right]$$

$$c^2x^2+a^4+2a^2cx=a^2x^2+a^2c^2+2a^2cx+a^2y^2$$

$$c^2x^2-a^2x^2-a^2y^2=a^2c^2-a^4$$

$$x^2\left(c^2-a^2\right)-a^2y^2=a^2\left(c^2-a^2\right)$$

Dividing on both sides by $a^2\left(c^2-a^2\right)$ we get;

$$\boxed{\frac{x^2}{a^2}-\frac{y^2}{c^2-a^2}=1}$$

Q.4 Using problem (3), find an equation of the hyperbola with foci $(-5,-5)$ and $(5,5)$, vertices $\left(-3\sqrt{2},-3\sqrt{2}\right)$ and $\left(3\sqrt{2},3\sqrt{2}\right)$

Solution:

As $|PF|-|PF'|=\pm 2a$ …(i)

$2a=$ distance between vertices

$$2a=\sqrt{\left(3\sqrt{2}+3\sqrt{2}\right)^2+\left(3\sqrt{2}+3\sqrt{2}\right)^2}$$

$$2a=\sqrt{2\left(2\times 3\sqrt{2}\right)^2}$$

$$2a=\sqrt{2\times 4\times 9\times 2}$$

$$2a=2\times 2\times 3$$

$$a=6$$

Put value of a in (i), we get;

$$\sqrt{(x+5)^2+(y+5)^2}-\sqrt{(x-5)^2+(y-5)^2}=12$$

$$\sqrt{(x+5)^2+(y+5)^2}=\sqrt{(x-5)^2+(y-5)^2}+12$$

Squaring on both sides, we get;

$$(x+5)^2+(y+5)^2=(x-5)^2+(y-5)^2+144+24\sqrt{(x-5)^2+(y-5)^2}$$

$$(x+5)^2-(x-5)^2+(y+5)^2-(y-5)^2=144+24\sqrt{(x-5)^2+(y-5)^2}$$

$$20x+20y=144+24\sqrt{(x-5)^2+(y-5)^2}$$

$$5x+5y=36+6\sqrt{(x-5)^2+(y-5)^2}$$

$$5(x+y)-36=+6\sqrt{(x-5)^2+(y-5)^2}$$

Again squaring on both sides we get;

$$25(x+y)^2+1296-360(x+y)=36\left[(x-5)^2+(y-5)^2\right]$$

$$25\left(x^2+y^2+2xy\right)+1296-360x-360y=36\left[x^2+25-10x+y^2+25-10y\right]$$

$$25x^2+25y^2+50xy+1296-360x-360y=36\left[x^2+y^2-10x-10y+50\right]$$

$$25x^2+25y^2+50xy+1296-360x-360y=36x^2+36y^2-360x-360y+1800$$

$$\boxed{11x^2+11y^2-50xy+504=0}$$

Q.5 For any point on the hyperbola the difference of its distance from the points $(2,2)$ and $(10,2)$ is 6. Find an equation of the hyperbola.

Solution:

Let $P(x,y)$ be any point on the hyperbola then by definition difference of its distance from two fixed points is constant

$$|PF|-|PF'|=2a$$

But $2a=6$

$$|PF|-|PF'|=6$$

$$|PF|=6+|PF'|$$

$$\sqrt{(x-2)^2+(y-2)^2}=6+\sqrt{(x-10)^2+(y-2)^2}$$

Squaring on both sides, we get;

$$(x-2)^2+(y-2)^2=36+(x-10)^2+(y-2)^2+12\sqrt{(x-10)^2+(y-2)^2}$$

$$(x-2)^2+(y-2)^2-36-(x-10)^2-(y-2)^2=12\sqrt{(x-10)^2+(y-2)^2}$$

$$(x-2)^2-(x-10)^2-36=12\sqrt{(x-10)^2+(y-2)^2}$$

$$x^2+4-4x-x^2-100+20x-36=12\sqrt{(x-10)^2+(y-2)^2}$$

$$16x-132=12\sqrt{(x-10)^2+(y-2)^2}$$

$$4x-33=3\sqrt{(x-10)^2+(y-2)^2}$$

(Dividing both sides by 4)

Again squaring on both sides, we get;

$$16x^2+1089-264x=9\left[(x-10)^2+(y-2)^2\right]$$

$$16x^2+1089-264x=9\left[x^2+100-20x+y^2+4-4y\right]$$

$$16x^2+1089-264x=9x^2+9y^2-180x-36y+936$$

$$\boxed{7x^2-9y^2-84x+36y+153=0}$$

Q.6 Two listening posts hear the sound of an enemy gun. The difference in time is one second. If the listening posts are 1400 feet apart, write an equation of the hyperbola passing through the position of the enemy gun. (Sound travels at 1080 ft/sec)

Solution:

Let the two listening posts F and F' hear the sound of an enemy gun after t and $t-1$ seconds respectively.

It is given that the listening posts are 1400 feet apart, that is

$2c=1400\Rightarrow c=700$

If the position of enemy gun is at $P(x,y)$ on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\ldots\text{(i)}$$

Then $|PF|-|PF'|=2a\ldots$(ii)

As distance = speed ×time

Here $|PF|=1080\times t=1080t$

$|PF'|=1080(t-1)$

Now put values of $|PF|$ and $|PF'|$ in (ii) we get;

$$1080t-1080(t-1)=2a$$

$$1080[t-t+1]=2a$$

$$2a=1080$$

$a=540\Rightarrow a^2=291600$

$c^2=a^2+b^2$

$700^2=540^2+b^2$

$b^2=198400$

Put values of a^2 and b^2 in (i)

$$\boxed{\frac{x^2}{291600}-\frac{y^2}{198400}=1}$$