EXERCISE 6.9

Q.1 By a rotation of axes, eliminate the $xy-$ term in each of the following equations. Identify the conic and find its elements.

(i) $4x^2-4xy+y^2-6=0$

Solution:

$4x^2-4xy+y^2-6=0$ ...(i)

Comparing (i) with

$ax^2+by^2+2gx+2fy+2hxy+c=0$

we get

$a=4, b=1, 2h=-4$

As we know that angle of rotation θ is given by

$$\tan 2\theta=\frac{2h}{a-b}=\frac{-4}{4-1}$$

$$\tan 2\theta=-\frac{4}{3}$$

$$\frac{2\tan\theta}{1-\tan^2\theta}=-\frac{4}{3}$$

$$6\tan\theta=-4+4\tan^2\theta$$

$$4\tan^2\theta-6\tan\theta-4=0$$

$$2\tan^2\theta-3\tan\theta-2=0$$

$$2\tan^2\theta-4\tan\theta+\tan\theta-2=0$$

$$2\tan\theta(\tan\theta-2)+1(\tan\theta-2)=0$$

$$(2\tan\theta+1)(\tan\theta-2)=0$$

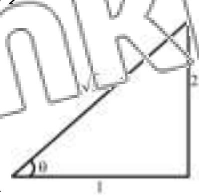
$$2\tan\theta+1=0, \tan\theta-2=0$$

$$\tan\theta=-\frac{1}{2} \text{ and } \tan\theta=2$$

Neglect $\tan\theta=-\frac{1}{2}$ because θ lies in first quadrant and in first quadrant $\tan\theta$ is positive.

Now consider $\tan\theta=2$

From right triangle



$$\sin\theta=\frac{2}{\sqrt5}, \cos\theta=\frac{1}{\sqrt5}$$

Using equations of transformations

$x=X\cos\theta-Y\sin\theta$

Using values of $\sin\theta$ and $\cos\theta$ we get

$$x=\frac{X}{\sqrt5}-\frac{2Y}{\sqrt5}=\frac{X-2Y}{\sqrt5} \quad \text{...(ii)}$$

And $y=X\sin\theta+Y\cos\theta$

$$y=\frac{2X}{\sqrt5}+\frac{Y}{\sqrt5}=\frac{2X+Y}{\sqrt5} \quad \text{...(iv)}$$

Put values of x and y in (i), we get

$$4\left(\frac{X-2Y}{\sqrt5}\right)^2-4\left(\frac{X-2Y}{\sqrt5}\right)\left(\frac{2X+Y}{\sqrt5}\right)+\left(\frac{2X+Y}{\sqrt5}\right)^2-6=0$$

$$\frac{4(X^2-4XY+4Y^2)}{5}-\frac{4(2X^2-3XY-2Y^2)}{5}+\frac{(4X^2+4XY+Y^2)}{5}-6=0$$

Multiply by 5

$$4(X^2-4XY+4Y^2)-4(2X^2-3XY-2Y^2)+(4X^2+4XY+Y^2)-30=0$$

$$4X^2-16XY+16Y^2-8X^2+12XY+8Y^2+4X^2+4XY+Y^2-30=0$$

$$25Y^2-30=0 \Rightarrow 25Y^2=30$$

$$Y^2=\frac{30}{25} \Rightarrow Y^2=\frac{6}{5}$$

So

$Y=\pm\sqrt{\frac{6}{5}}$ are the lines in $XY-$ plane.

To find their equations in xy - plane, we find Y in terms of 'x' and 'y'.

Using

$Y=-x\sin\theta+y\cos\theta$

Put values of $\sin\theta$ and $\cos\theta$

$$Y=-\frac{2x}{\sqrt5}+\frac{y}{\sqrt5}$$

Put $Y=\pm\sqrt{\frac{6}{5}}$

$$\pm\frac{\sqrt{6}}{\sqrt{5}}=\frac{-2x+y}{\sqrt{5}}$$

$$\Rightarrow -2x+y=\pm\sqrt{6}$$

$$-2x+y\pm\sqrt{6}=0$$

$$2x-y\mp\sqrt{6}=0$$

Hence (i) represents a pair of lines which are $2x-y+\sqrt{6}=0$ and $2x-y-\sqrt{6}=0$

(ii) $x^2-2xy+y^2-8x-8y=0$

Solution:

$x^2-2xy+y^2-8x-8y=0\ \ldots$(i)

Comparing (i) with

$ax^2+by^2+2gx+2fy+2hxy+c=0$,

we get

$a=1, b=1, 2h=-2$

As we know that angle of rotation is

$$\tan 2\theta=\frac{2h}{a-b}=\frac{-2}{1-1}=\frac{-2}{0}=\infty$$

$$\Rightarrow 2\theta=90^o\Rightarrow\theta=45^o$$

Using equations of transformations

$x=X\cos\theta-Y\sin\theta$

For

$$\theta=45^o\Rightarrow\cos\theta=\frac{1}{\sqrt{2}},\sin\theta=\frac{1}{\sqrt{2}}$$

$$x=\frac{X}{\sqrt{2}}-\frac{Y}{\sqrt{2}}=\frac{X-Y}{\sqrt{2}}\ \ldots\text{(ii)}$$

And $y=X\sin\theta+Y\cos\theta$

$$y=\frac{X}{\sqrt{2}}+\frac{Y}{\sqrt{2}}=\frac{X+Y}{\sqrt{2}}\ \ldots\text{(iii)}$$

Put values of $'x'$ and $'y'$ in (i) we get ;

$$\left(\frac{X-Y}{\sqrt{2}}\right)^2-2\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right)+\left(\frac{X+Y}{\sqrt{2}}\right)^2-8\left(\frac{X-Y}{\sqrt{2}}\right)-8\left(\frac{X+Y}{\sqrt{2}}\right)=0$$

$$\frac{(X-Y)^2}{2}-2\frac{(X^2-Y^2)}{2}+\frac{(X+Y)^2}{2}-8\frac{(X-Y)}{\sqrt{2}}-8\frac{(X+Y)}{\sqrt{2}}=0$$

Multiply by 2 on both sides

$$(X-Y)^2-2(X^2-Y^2)+(X+Y)^2-8\sqrt{2}(X-Y)-8\sqrt{2}(X+Y)=0$$

$$(X-Y)^2+(X+Y)^2-2(X^2-Y^2)$$

$$-8\sqrt{2}X+8\sqrt{2}Y-8\sqrt{2}X-8\sqrt{2}Y=0$$

$$2(X^2+Y^2)-2(X^2-Y^2)-16\sqrt{2}X=0$$

$$4Y^2-16\sqrt{2}X=0$$

$$Y^2-4\sqrt{2}X=0$$

$$Y^2=4\sqrt{2}X\ \ldots\text{(iv)}$$

As (iv) is parabola, hence (i) represents a parabola.

Next we convert 'X' and 'Y' in terms of 'x' and 'y'.

Using equations of transformations

$X=x\cos\theta+y\sin\theta$

For

$$\theta=45^o\Rightarrow\cos\theta=\frac{1}{\sqrt{2}},\sin\theta=\frac{1}{\sqrt{2}}$$

$$X=\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}=\frac{x+y}{\sqrt{2}}$$

$$X=\frac{x+y}{\sqrt{2}}\ \ldots\text{(v)}$$

and

$Y=-x\sin\theta+y\cos\theta$

For

$$\theta=45^o\Rightarrow\cos\theta=\frac{1}{\sqrt{2}},\sin\theta=\frac{1}{\sqrt{2}}$$

$$Y=\frac{-x}{\sqrt{2}}+\frac{y}{\sqrt{2}}$$

$$Y=-\frac{(x-y)}{\sqrt{2}}\ \ldots\text{(vi)}$$

Elements:

From (iv) $Y^2=4\sqrt{2}X$

Here $4a=4\sqrt{2}$

$a=\sqrt{2}$

Focus of (iv) is $F(a,0)=F\left(\sqrt{2},0\right)$

$X=\sqrt{2}$ and $Y=0$

From (ii) $x=\dfrac{X-Y}{\sqrt{2}}=\dfrac{\sqrt{2}-0}{\sqrt{2}}=1$

From (iii) $y=\dfrac{X+Y}{\sqrt{2}}=\dfrac{\sqrt{2}+0}{\sqrt{2}}=1$

Hence focus of (i) is $(1,1)$

Vertex of (iv) is $(0,0)$,

$X=0$ and $Y=0$

From (ii) $x=\dfrac{X-Y}{\sqrt{2}}=\dfrac{0-0}{\sqrt{2}}=0$

From (iii) $y=\dfrac{X+Y}{\sqrt{2}}=\dfrac{0+0}{\sqrt{2}}=0$

Hence vertex of (i) is $(0,0)$

Directrix of (iv) is given by

$X=-a\Rightarrow X=-\sqrt{2}$

$X=\dfrac{x+y}{\sqrt{2}}$

$X=-\sqrt{2}$ we get

$-\sqrt{2}=\dfrac{x+y}{\sqrt{2}}\Rightarrow x+y=-2$

Hence equation of directrix for (i) is $x+y+2=0$

Axis of (iv) is $Y=0$

$\Rightarrow \dfrac{-x+y}{\sqrt{2}}=0$ As $Y=\dfrac{-x+y}{\sqrt{2}}$

$-x+y=0$

$x-y=0$

Hence axis of (i) is $x-y=0$

(iii) $x^2+2xy+y^2+2\sqrt{2}x-2\sqrt{2}y+2=0$

Solution:

$x^2+2xy+y^2+2\sqrt{2}x-2\sqrt{2}y+2=0$ …(i)

Comparing (i) with

$ax^2+by^2+2hxy+2gx+2fy+c=0$, we get

$a=1, b=1, 2h=2$

As we know that angle of rotation is given by

$\tan 2\theta=\dfrac{2h}{a-b}=\dfrac{2}{1-1}=\dfrac{2}{0}=\infty$

$2\theta=90^o\Rightarrow\theta=45^o$

As we know that equations of transformation are

$x=X\cos\theta-Y\sin\theta$

and $y=X\sin\theta+Y\cos\theta$

For $\theta=45^o$ we have

$\cos\theta=\dfrac{1}{\sqrt{2}},\sin\theta=\dfrac{1}{\sqrt{2}}$

Thus

$$\begin{cases} x=X\left(\dfrac{1}{\sqrt{2}}\right)-Y\left(\dfrac{1}{\sqrt{2}}\right) \\ \text{and} \\ y=X\left(\dfrac{1}{\sqrt{2}}\right)+Y\left(\dfrac{1}{\sqrt{2}}\right)\end{cases}$$

$$\Rightarrow\begin{cases} x=\dfrac{X-Y}{\sqrt{2}}\text{...(ii)} \\ \text{and} \\ y=\dfrac{X+Y}{\sqrt{2}}\text{...(iii)}\end{cases}$$

Putting (ii) and (iii) in (i), we get

$$\left(\frac{X-Y}{\sqrt{2}}\right)^2+2\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right)+\left(\frac{X+Y}{\sqrt{2}}\right)^2+2\sqrt{2}\left(\frac{X-Y}{\sqrt{2}}\right)-2\sqrt{2}\left(\frac{X+Y}{\sqrt{2}}\right)+2=0$$

$$\frac{(X-Y)^2}{2}+\left(X^2-Y^2\right)+\frac{\left(X+Y\right)^2}{2}+2(X-Y)-2(X+Y)+2=0$$

Multiply by 2

$$(X+Y)^2+(X-Y)^2+2(X^2-Y^2)+4(X-Y)-4(X+Y)+4=0$$

$$\Rightarrow 2(X^2+Y^2)+2(X^2-Y^2)+4X-4Y-4X-4Y+4=0$$

$\Rightarrow 4X^2-8Y+4=0$

$\Rightarrow X^2-2Y+1=0$

$X^2=2Y-1$

$X^2=2\left(Y-\dfrac{1}{2}\right)$ …(iv)

Next we find 'X' and 'Y' in terms of 'x' and 'y'

Using equations of transformation

$X = x\cos\theta + y\sin\theta$

For

$\theta = 45^o \Rightarrow \cos\theta = \frac{1}{\sqrt{2}}, \sin\theta = \frac{1}{\sqrt{2}}$

$X = x\left(\frac{1}{\sqrt{2}}\right) + y\left(\frac{1}{\sqrt{2}}\right)$

$X = \frac{x+y}{\sqrt{2}}$ …(v)

and

$Y = -x\sin\theta + y\cos\theta$

$Y = -x\left(\frac{1}{\sqrt{2}}\right) + y\left(\frac{1}{\sqrt{2}}\right)$

$Y = \frac{-(x-y)}{\sqrt{2}}$ …(vi)

From (iv)

$X^2 = 2\left(Y - \frac{1}{2}\right)$

$4a = 2 \Rightarrow a = \frac{1}{2}$

For focus of (iv) we can write

$F\left(X, Y - \frac{1}{2}\right) = F(0, a)$

$X = 0$, $Y - \frac{1}{2} = a$

$X = 0$, $Y = \frac{1}{2} + \frac{1}{2}$

$X = 0$, $Y = 1$

So focus of (iv) is (0,1)

From (ii) $x = \frac{X-Y}{\sqrt{2}} = \frac{0-1}{\sqrt{2}} = \frac{-1}{\sqrt{2}}$

From (iii) $y = \frac{X+Y}{\sqrt{2}} = \frac{0+1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

Hence focus of (i) is $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

vertex of (iv) is

$A\left(X, Y - \frac{1}{2}\right) = A(0,0)$

$X = 0$, $Y - \frac{1}{2} = 0$

$X = 0$, $Y = \frac{1}{2}$

So vertex of (iv) is $\left(0, \frac{1}{2}\right)$

From (ii) $x = \frac{X-Y}{\sqrt{2}} = \frac{0-\frac{1}{2}}{\sqrt{2}} = \frac{-1}{2\sqrt{2}}$

From (iii) $y = \frac{X+Y}{\sqrt{2}} = \frac{0+\frac{1}{2}}{\sqrt{2}} = \frac{1}{2\sqrt{2}}$

Hence vertex of (i) is $\left(\frac{-1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$.

Equation of directrix for (iv) is

$Y - \frac{1}{2} = -a$

$Y = -a + \frac{1}{2} \Rightarrow Y = \frac{-1}{2} + \frac{1}{2} = 0$

Put $Y = 0$ in (vi) we get

$\frac{-(x-y)}{\sqrt{2}} = 0 \Rightarrow y = x$ is the equation of directrix for (i)

Axis of (vi) is $X = 0$

$\frac{x+y}{\sqrt{2}} = 0$ From (v)

$x + y = 0$

Hence axis of (i) is $x + y = 0$

(iv) $x^2 + xy + y^2 - 4 = 0$

Solution:

$x^2 + xy + y^2 - 4 = 0$ …(i)

Comparing (i) with

$ax^2 + by^2 + 2gx + 2fy + 2hxy + c = 0$, we get

$a = 1, b = 1, 2h = 1$

As we know that angle of rotation is given by

$\tan 2\theta = \frac{2h}{a-b}$.

$\tan 2\theta = \frac{1}{0-0} = \frac{1}{0} = \infty$

$2\theta = 90^o$

$\theta = 45^o$

Using equations of transformation

$x = X\cos\theta - Y\sin\theta$

$= X\cos 45^o - Y\sin 45^o$

$= \frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}$

$\Rightarrow x = \frac{X-Y}{\sqrt{2}} \ldots$ (ii)

And

$y = X\sin\theta + Y\cos\theta$

$= X\sin 45^\circ + Y\cos 45^\circ$

$\Rightarrow y = \frac{X+Y}{\sqrt{2}} \ldots$ (iii)

Putting values of "x" and "y" in given equation we get

$$\left(\frac{X-Y}{\sqrt{2}}\right)^2 + \left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right) + \left(\frac{X+Y}{\sqrt{2}}\right)^2 - 4 = 0$$

$$\frac{(X-Y)^2}{2} + \frac{(X^2-Y^2)}{2} + \frac{(X+Y)^2}{2} - 4 = 0$$

$$(X-Y)^2 + (X^2-Y^2) + (X+Y)^2 - 8 = 0$$

$$\left[(X+Y)^2 + (X-Y)^2\right] + (X^2-Y^2) - 8 = 0$$

$$2(X^2+Y^2) + X^2 - Y^2 - 8 = 0$$

$$2X^2 + 2Y^2 + X^2 - Y^2 - 8 = 0$$

$$3X^2 + Y^2 - 8 = 0$$

$$3X^2 + Y^2 = 8$$

$$\frac{3X^2}{8} + \frac{Y^2}{8} = 1$$

$$\frac{X^2}{\left(\frac{8}{3}\right)} + \frac{Y^2}{8} = 1 \ldots (iv)$$

As (iv) represents ellipse, hence (i) is ellipse

Elements:

Here $a^2 = 8, b^2 = \frac{8}{3}$ and

$c^2 = a^2 - b^2 = 8 - \frac{8}{3} = \frac{16}{3}$

$a = 2\sqrt{2}, b = \frac{2\sqrt{2}}{\sqrt{3}}, c = \frac{4}{\sqrt{3}}$

Next we convert "X" and "Y" in terms of "x" and "y", so using equations of transformation

$X = x\cos\theta + y\sin\theta$

For

$\theta = 45^o \Rightarrow \cos\theta = \frac{1}{\sqrt{2}}, \sin\theta = \frac{1}{\sqrt{2}}$

$X = x\left(\frac{1}{\sqrt{2}}\right) + y\left(\frac{1}{\sqrt{2}}\right)$

$X = \frac{x+y}{\sqrt{2}} \ldots$(v)

And

$Y = -x\sin\theta + y\cos\theta$

$Y = -x\left(\frac{1}{\sqrt{2}}\right) + y\left(\frac{1}{\sqrt{2}}\right)$

$Y = \frac{-(x-y)}{\sqrt{2}} \ldots$ (vi)

Center of (iv) is $(0,0)$

So coordinates of center are

$X = 0, Y = 0$

From (ii) $x = \frac{X-Y}{\sqrt{2}} = \frac{0-0}{\sqrt{2}} = 0$

From (iii) $y = \frac{X+Y}{\sqrt{2}} = \frac{0+0}{\sqrt{2}} = 0$

Hence center of (i) is $(0,0)$.

Vertices of (iv) are

$(0, \pm a) = (0, \pm 2\sqrt{2})$

Here coordinates of vertices are

$X = 0$ and $Y = \pm 2\sqrt{2}$

For $Y = 2\sqrt{2}, X = 0$

From (ii) $x = \frac{X-Y}{\sqrt{2}} = \frac{0-2\sqrt{2}}{\sqrt{2}} = -2$

From (iii) $y=\dfrac{X+Y}{\sqrt{2}}=\dfrac{0+2\sqrt{2}}{\sqrt{2}}=2$

For $Y=-2\sqrt{2}, X=0$

From (ii) $x=\dfrac{X-Y}{\sqrt{2}}=\dfrac{0+2\sqrt{2}}{\sqrt{2}}=2$

From (iii) $y=\dfrac{X+Y}{\sqrt{2}}=\dfrac{0-2\sqrt{2}}{\sqrt{2}}=-2$

Hence vertices of (i) are $(-2,2)$ and $(2,-2)$

Foci of (iv) are $(0,\pm c)=\left(0,\pm\dfrac{4}{\sqrt{3}}\right)$

Here coordinates of foci are

$X=0$ and $Y=\pm\dfrac{4}{\sqrt{3}}$

For $Y=\dfrac{4}{\sqrt{3}}, X=0$

From (ii)

$$x=\frac{X-Y}{\sqrt{2}}=\frac{0-\frac{4}{\sqrt{3}}}{\sqrt{2}}=\frac{-4}{\sqrt{2}\sqrt{3}}=\frac{-2\sqrt{2}}{\sqrt{3}}$$

From (iii)

$$y=\frac{X+Y}{\sqrt{2}}=\frac{0+\frac{4}{\sqrt{3}}}{\sqrt{2}}=\frac{4}{\sqrt{2}\sqrt{3}}=\frac{2\sqrt{2}}{\sqrt{3}}$$

For $Y=\dfrac{-4}{\sqrt{3}}, X=0$

From (ii)

$$x=\frac{X-Y}{\sqrt{2}}=\frac{0+\frac{4}{\sqrt{3}}}{\sqrt{2}}=\frac{4}{\sqrt{2}\sqrt{3}}=\frac{2\sqrt{2}}{\sqrt{3}}$$

From (iii)

$$y=\frac{X+Y}{\sqrt{2}}=\frac{0-\frac{4}{\sqrt{3}}}{\sqrt{2}}=\frac{-4}{\sqrt{2}\sqrt{3}}\ \frac{-2\sqrt{2}}{\sqrt{3}}$$

Here coordinates of foci of (i) are $\left(\dfrac{-2\sqrt{2}}{\sqrt{3}},\dfrac{2\sqrt{2}}{\sqrt{3}}\right)$ and $\left(\dfrac{2\sqrt{2}}{\sqrt{3}},\dfrac{-2\sqrt{2}}{\sqrt{3}}\right)$

Major axis of (iv) is $X=0$

From (v) $X=\dfrac{x+y}{\sqrt{2}}$

$0=\dfrac{x+y}{\sqrt{2}}$

$x+y=0$

So major axis of (i) is given by the line $y+x=0\Rightarrow y=-x$

Minor axis of (iv) is $Y=0$

From (vi) $Y=\dfrac{-(x-y)}{\sqrt{2}}$, so put

$Y=0$

We get $\dfrac{-(x-y)}{\sqrt{2}}=0\Rightarrow y=x$ which is the minor axis of (i)

Eccentricity of (i) is $e=\dfrac{c}{a}$

$$e=\frac{\frac{4}{\sqrt{3}}}{2\sqrt{2}}=\frac{4}{2\sqrt{6}}=\frac{2}{\sqrt{6}}$$

(v) $7x^2-6\sqrt{3}xy+13y^2-16=0$

Solution:

$7x^2-6\sqrt{3}xy+13y^2-16=0\ \dots$(i)

Comparing (i) with

$ax^2+by^2+2gx+2fy+2hxy+c=0$, we get

$a=7, b=13, 2h=-6\sqrt{3}$

As we know that angle of rotation is given by

$$\tan 2\theta=\frac{2h}{a-b}=\frac{-6\sqrt{3}}{7-13}=\frac{-6\sqrt{3}}{-6}$$

$\tan 2\theta=\sqrt{3}\Rightarrow 2\theta=60^\circ\Rightarrow\theta=30^\circ$

Using equations of transformation

$x=X\cos\theta-Y\sin\theta$, put $\theta=30^\circ$

$x=X\cos 30^\circ-Y\sin 30^\circ$

$x = \frac{X\sqrt{3}}{2} - \frac{Y}{2}$

$x = \frac{\sqrt{3}X - y}{2}$ … (ii)

And

$y = X\sin\theta + Y\cos\theta$; put $\theta = 30^o$

$y = X\sin 30^o + Y\cos 30^o$

$y = \frac{X}{2} + \frac{\sqrt{3}Y}{2}$

$y = \frac{X + \sqrt{3}Y}{2}$ … (iii)

Putting (ii) and (iii) in (i), we get

$$7\left(\frac{\sqrt{3}X - Y}{2}\right)^2 - 6\sqrt{3}\left(\frac{\sqrt{3}X - Y}{2}\right)\left(\frac{X + \sqrt{3}Y}{2}\right) + 13\left(\frac{X + \sqrt{3}Y}{2}\right)^2 - 16 = 0$$

$$7\left[\frac{3X^2 + Y^2 - 2\sqrt{3}XY}{4}\right] - 6\sqrt{3}\frac{\left(\sqrt{3}X^2 + 2XY - \sqrt{3}Y^2\right)}{4} + \frac{13\left(X^2 + 3Y^2 + 2\sqrt{3}XY\right)}{4} - 16 = 0$$

$$7\left(3X^2 + Y^2 - 2\sqrt{3}XY\right) - 6\sqrt{3}\left(\sqrt{3}X^2 + 2XY - \sqrt{3}Y^2\right) + 13\left(X^2 + 3Y^2 + 2\sqrt{3}XY\right) - 64 = 0$$

$$21X^2 + 7Y^2 - 14\sqrt{3}XY - 18X^2 - 12\sqrt{3}XY + 18Y^2 + 13X^2 + 39Y^2 + 26\sqrt{3}XY - 64 = 0$$

$16X^2 + 64Y^2 - 64 = 0$

$16X^2 + 64Y^2 = 64$

$\frac{X^2}{4} + \frac{Y^2}{1} = 1$ …(iv)

Here $a^2 = 4, b^2 = 1$ and $c^2 = a^2 - b^2 = 4 - 1 = 3$

$\Rightarrow a = 2, b = 1, c = \sqrt{3}$

Next we convert "X" and "Y" in terms of "x" and "y"

Using equations of transformations

$$\begin{cases} X = x\cos\theta + y\sin\theta \\ \text{and} \\ Y = -x\sin\theta + y\cos\theta \end{cases}$$

$$\Rightarrow \begin{cases} X = x\cos 30^\circ + y\sin 30^\circ = \frac{\sqrt{3}x + y}{2} \text{...(v)} \\ \text{and} \\ Y = -x\sin 30^\circ + y\cos 30^\circ = \frac{-x + \sqrt{3}y}{2} \text{...(vi)} \end{cases}$$

Centre of (iv) is (0,0)

So coordinates of center are $X = 0$ and $Y = 0$

From (ii)

$$x = \frac{\sqrt{3}X - Y}{2} = \frac{\sqrt{3}(0) - 0}{2} = 0$$

From (iii)

$$y = \frac{X + \sqrt{3}Y}{2} = \frac{0 + \sqrt{3}(0)}{2} = 0$$

Hence center of (i) is (0,0).

Vertices of (iv) are $(\pm a, 0) = (\pm 2, 0)$

So coordinates of vertices are

$X = \pm 2, Y = 0$

For $X = 2, Y = 0$

From (ii)

$$x = \frac{\sqrt{3}X - Y}{2} = \frac{\sqrt{3}(2) - 0}{2} = \sqrt{3}$$

From (iii)

$$y = \frac{X + \sqrt{3}Y}{2} = \frac{2 + \sqrt{3}(0)}{2} = 1$$

For $X = -2, Y = 0$

From (ii)

$$x = \frac{\sqrt{3}X - Y}{2} = \frac{\sqrt{3}(-2) - 0}{2} = -\sqrt{3}$$

From (iii)

$$y = \frac{X + \sqrt{3}Y}{2} = \frac{-2 + \sqrt{3}(0)}{2} = -1$$

Hence vertices of (i) are $(\sqrt{3}, 1)$ and

Foci of (iv) are $(\pm c, 0) = (\pm\sqrt{3}, 0)$

Here coordinates of foci are

$X = \pm\sqrt{3}, Y = 0$

For $X = \sqrt{3}, Y = 0$

From (ii)

$$x = \frac{\sqrt{3}X - Y}{2} = \frac{\sqrt{3}(\sqrt{3}) - 0}{2} = \frac{3}{2}$$

From (iii)

$$y = \frac{X + \sqrt{3}Y}{2} = \frac{\sqrt{3} + \sqrt{3}(0)}{2} = \frac{\sqrt{3}}{2}$$

For $X = -\sqrt{3}, Y = 0$

From (ii)

$$x = \frac{\sqrt{3}X - Y}{2} = \frac{\sqrt{3}(-\sqrt{3}) - 0}{2} = -\frac{3}{2}$$

From (iii)

$$y = \frac{X + \sqrt{3}Y}{2} = \frac{-\sqrt{3} + \sqrt{3}(0)}{2} = -\frac{\sqrt{3}}{2}$$

Hence foci of (i) are $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{-3}{2}, \frac{-\sqrt{3}}{2}\right)$

Minor axis of (iv) is $X = 0$

Put value of X in (v), we get

$\frac{\sqrt{3}x + y}{2} = 0 \Rightarrow y = -\sqrt{3}x$ is the

Equation of minor axis of (i)

Major axis of (iv) is $Y = 0$

Put value of Y in (vi), we get

$$\frac{-x + \sqrt{3}y}{2} = 0 \Rightarrow -x + \sqrt{3}y = 0$$

$y = \frac{1}{\sqrt{3}}x$ is the equation of major axis of (i)

Eccentricity of (i) is $e = \frac{c}{a}$

$$\Rightarrow e = \frac{\sqrt{3}}{2}$$

(vi) $4x^2 - 4xy + 7y^2 + 12x + 6y - 9 = 0$

Solution:

$4x^2 - 4xy + 7y^2 + 12x + 6y - 9 = 0$ ... (1)

Comparing (1) with

$ax^2 + by^2 + 2gx + 2fy + 2hxy + C = 0$, we get

$a = 4, b = 7, 2h = -4$

As we know that angle of rotation is given by

$$\tan 2\theta = \frac{2h}{a - b}$$

$$\tan 2\theta = \frac{-4}{4 - 7} = \frac{-4}{-3} = \frac{4}{3}$$

$$\tan 2\theta = \frac{4}{3}$$

$$\frac{2\tan\theta}{1-\tan^2\theta} = \frac{4}{3}$$

$$6\tan\theta = 4 - 4\tan^2\theta$$

$$4\tan^2\theta + 6\tan\theta - 4 = 0$$

$$2\tan^2\theta + 3\tan\theta - 2 = 0$$

$$2\tan^2\theta + 4\tan\theta - \tan\theta - 2 = 0$$

$$2\tan\theta(\tan\theta + 2) - 1(\tan\theta + 2) = 0$$

$$(2\tan\theta - 1)(\tan\theta + 2) = 0$$

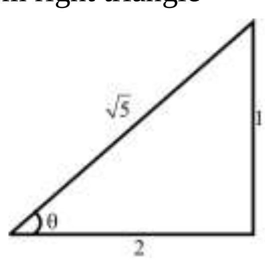
$$2\tan\theta - 1 = 0, \tan\theta + 2 = 0$$

$$\tan\theta = \frac{1}{2}, \tan\theta = -2$$

As θ lies in 1st quadrant and in 1st quadrant $\tan\theta$ is positive, so neglect $\tan\theta = -2$

Now consider $\tan\theta = \frac{1}{2}$

From right triangle



$$\sin\theta = \frac{1}{\sqrt{5}}, \cos\theta = \frac{2}{\sqrt{5}}$$

Using equations of transformation

$$x = X\cos\theta - Y\sin\theta$$

Using values of $\sin\theta$ and $\cos\theta$ we get;

$$x = \frac{2X}{\sqrt{5}} - \frac{Y}{\sqrt{5}}$$

$$x = \frac{2X - Y}{\sqrt{5}} \qquad (2)$$

And $y = X\sin\theta + Y\cos\theta$

Using values of $\sin\theta$ and $\cos\theta$ we get;

$$y = \frac{X}{\sqrt{5}} + \frac{2Y}{\sqrt{5}}$$

$$y = \frac{X + 2Y}{\sqrt{5}} \qquad (3)$$

Put values of x and y in (1)

$$4x^2 - 4xy + 7y^2 + 12x + 6y - 9 = 0$$

$$4\left(\frac{2X - Y}{\sqrt{5}}\right)^2 - 4\left(\frac{2X - Y}{\sqrt{5}}\right)$$

$$+7\left(\frac{X + 2Y}{\sqrt{5}}\right)^2 + 12\left(\frac{2X - Y}{\sqrt{5}}\right)$$

$$+6\left(\frac{X + 2Y}{\sqrt{5}}\right) - 9 = 0$$

$$\frac{4(4X^2 + Y^2 - 4XY)}{5} - 4\frac{(2X^2 + 3XY - 2Y^2)}{5} + 7\frac{(X^2 + 4Y^2 + 4XY)}{5}$$

$$+\frac{12(2X - Y)}{\sqrt{5}} + \frac{6(X + 2Y)}{\sqrt{5}} - 9 = 0$$

Multiplying by (5) we get

$$4(4X^2 + Y^2 - 4XY) - 4(2X^2 + 3XY - 2Y^2) + 7(X^2 + 4Y^2 + 4XY)$$

$$+12\sqrt{5}(2X - Y) + 6\sqrt{5}(X + 2Y) - 45 = 0$$

$$(16 - 8 + 7)X^2 + (4 + 8 + 28)y^2 + (-16 - 12 + 28)XY$$

$$+(24\sqrt{5} + 6\sqrt{5})X + (-12\sqrt{5} + 12\sqrt{5})Y - 45 = 0$$

$$\Rightarrow 15X^2 + 40Y^2 + 30\sqrt{5}X - 45 = 0$$

$$\Rightarrow 3X^2 + 8Y^2 + 6\sqrt{5}X - 9 = 0$$

$$\Rightarrow 3(X^2 + 2\sqrt{5}X) + 8Y^2 - 9 = 0$$

$$3\left(X^2 + 2\sqrt{5}X + (\sqrt{5})^2 - (\sqrt{5})^2\right) + 8Y^2 - 9 = 0$$

$$3\left[\left(X + \sqrt{5}\right)^2 - 5\right] + 8Y^2 - 9 = 0$$

$$3\left(X + \sqrt{5}\right)^2 - 15 + 8Y^2 - 9 = 0$$

$$3\left(X + \sqrt{5}\right)^2 + 8Y^2 = 24$$

$$\frac{\left(X + \sqrt{5}\right)^2}{8} + \frac{Y^2}{3} = 1 \qquad (4)$$

Here

$$a^2 = 8, b^2 = 3, c^2 = a^2 - b^2 = 8 - 3 = 5$$

$a = 2\sqrt{2}, b = 3, c = \sqrt{5}$

As (4) represents ellipse, hence (1) is ellipse

Next we convert X and Y in terms of x and y. So using equations of transformation

$X = x\cos\theta + y\sin\theta$

Using values of $\sin\theta$ and $\cos\theta$, we get

$$X = \frac{2x}{\sqrt{5}} + \frac{y}{\sqrt{5}}$$

$$X = \frac{2x+y}{\sqrt{5}} \qquad (5)$$

$Y = -x\sin\theta + y\cos\theta$

Using values of $\sin\theta$ and $\cos\theta$, we get

$$Y = \frac{-x}{\sqrt{5}} + \frac{2y}{\sqrt{5}}$$

$$Y = \frac{-x+2y}{\sqrt{5}} \qquad (6)$$

Elements

<u>Center</u>:

For center of (4), we can write

$\left(X+\sqrt{5}, Y\right) = (0,0)$

$X + \sqrt{5} = 0 \quad , \quad Y = 0$

$X = -\sqrt{5} \quad , \quad Y = 0$

Center of (4) is $C(-\sqrt{5}, 0)$

From (2)

$$x = \frac{2X - Y}{\sqrt{5}} = \frac{2\left(-\sqrt{5}\right) - 0}{\sqrt{5}} = -2$$

From (3)

$$y = \frac{X + 2Y}{\sqrt{5}} = \frac{-\sqrt{5} + 2(0)}{\sqrt{5}} = -1$$

Hence center of (1) is $(-2, -1)$

<u>Major Axis</u>:

Equation of major axis of (4) is $Y = 0$, put $Y = 0$ in (6) we get

$$\frac{-x+2y}{\sqrt{5}} = 0 \Rightarrow -x + 2y = 0$$

$y = \frac{1}{2}x$ is the equation of major axis of (1)

<u>Minor axis</u>:

Equation of minor axis of (4) is

$X + \sqrt{5} = 0 \Rightarrow X = -\sqrt{5}$

Put $X = -\sqrt{5}$ in (5) we get

$$\frac{2x+y}{\sqrt{5}} = -\sqrt{5}$$

$2x + y = -5$

$2x + y + 5 = 0$ is the equation of minor axis of (1)

<u>Vertices</u>:

For vertices of (4) we can write

$\left(X + \sqrt{5}, Y\right) = (\pm a, 0)$

$X + \sqrt{5} = \pm a \quad , \quad Y = 0$

$X + \sqrt{5} = \pm 2\sqrt{2} \quad , \quad Y = 0$

$X = -\sqrt{5} \pm 2\sqrt{2} \quad , \quad Y = 0$

So vertices of (4) are $\left(-5 \pm 2\sqrt{2}, 0\right)$

For $X = -\sqrt{5} + 2\sqrt{2}, Y = 0$

From (2)

$$x = \frac{2X - Y}{\sqrt{5}} = \frac{2\left(-\sqrt{5} + 2\sqrt{2}\right) - 0}{\sqrt{5}}$$

$$= \frac{-2\sqrt{5} + 4\sqrt{2}}{\sqrt{5}} = \frac{-10 + 4\sqrt{10}}{5}$$

From (3)

$$y = \frac{X + 2Y}{\sqrt{5}} = \frac{-\sqrt{5} + 2\sqrt{2} - 2(0)}{\sqrt{5}} = \frac{-5 + 2\sqrt{10}}{5}$$

For $X = -\sqrt{5} - 2\sqrt{2}, Y = 0$

From (2)

$$x = \frac{2X - Y}{\sqrt{5}} = \frac{2\left(-\sqrt{5} - 2\sqrt{2}\right) - 0}{\sqrt{5}}$$

$$= \frac{-2\sqrt{5} - 4\sqrt{2}}{\sqrt{5}} = \frac{-10 - 4\sqrt{10}}{5}$$

From (3)

$$y=\frac{X+2Y}{\sqrt{5}}=\frac{-\sqrt{5}-2\sqrt{2}-2(0)}{\sqrt{5}}=\frac{-5-2\sqrt{10}}{5}$$

Hence vertices of (1) are

$$\left(\frac{-10+4\sqrt{10}}{5},\frac{-5+2\sqrt{10}}{5}\right) \text{ and}$$

$$\left(\frac{-10-4\sqrt{10}}{5},\frac{-5-2\sqrt{10}}{5}\right)$$

Or $\left(-2+\sqrt{\frac{32}{5}},-1+\sqrt{\frac{8}{5}}\right)$ and

$$\left(-2-\sqrt{\frac{32}{5}},-1-\sqrt{\frac{8}{5}}\right)$$

Foci:

For foci of (4) we can write

$$\left(X+\sqrt{5},Y\right)=\left(\pm c,0\right)$$

$$X+\sqrt{5}=\pm c \quad , \quad Y=0$$

$$X+\sqrt{5}=\pm\sqrt{5} \quad , \quad Y=0$$

$$X=-\sqrt{5}\pm\sqrt{5} \quad , \quad Y=0$$

$$X=0,-2\sqrt{5} \quad , \quad Y=0$$

So foci of (4) are $(0,0)$ and

$$\left(-2\sqrt{5},0\right)$$

For $X=0, Y=0$

From (2)

$$x=\frac{2X-Y}{\sqrt{5}}=\frac{2(0)-0}{\sqrt{5}}=\frac{0}{\sqrt{5}}=0$$

From (3)

$$y=\frac{X+2Y}{\sqrt{5}}=\frac{0-2(0)}{\sqrt{5}}=0$$

For $X=-2\sqrt{5}, Y=0$

From (2)

$$x=\frac{2X-Y}{\sqrt{5}}=\frac{2\left(-2\sqrt{5}\right)-0}{\sqrt{5}}=-4$$

From (3)

$$y=\frac{X+2Y}{\sqrt{5}}=\frac{-2\sqrt{5}-2(0)}{\sqrt{5}}=-2$$

Hence foci of (1) are $(0,0)$ and $(-4,-2)$

Eccentricity:

Eccentricity of (1) is

$$e=\frac{c}{a}=\frac{\sqrt{5}}{2\sqrt{2}}=\sqrt{\frac{5}{8}}$$

(vii) $xy-4x-2y=0$

Solution: Consider

$$xy-4x-2y=0 \qquad (1)$$

Comparing (1) with

$ax^2+by^2+2gx+2fy+2hxy+c=0$, we get

$$a=0,b=0,2h=1$$

We know that angle of rotation is given by

$$\tan 2\theta=\frac{2h}{a-b}=\frac{1}{0-0}=\frac{1}{0}=\infty$$

$$2\theta=90^\circ \Rightarrow \theta=45^\circ$$

Using equations of transformation

$$x=X\cos\theta-Y\sin\theta$$

For

$$\theta=45^\circ \Rightarrow \cos\theta=\frac{1}{\sqrt{2}},\sin\theta=\frac{1}{\sqrt{2}}$$

$$x=X\left(\frac{1}{\sqrt{2}}\right)-Y\left(\frac{1}{\sqrt{2}}\right)$$

$$x=\frac{X-Y}{\sqrt{2}} \qquad (2)$$

And $y=X\sin\theta+y\cos\theta$

$$y=X\left(\frac{1}{\sqrt{2}}\right)+Y\left(\frac{1}{\sqrt{2}}\right)$$

$$y=\frac{X+Y}{\sqrt{2}} \qquad (3)$$

Putting values of x and y in (1), we get

$$\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right)-4\left(\frac{X-Y}{\sqrt{2}}\right)-2\left(\frac{X+Y}{\sqrt{2}}\right)=0$$

$$\frac{X^2-Y^2}{2}-\frac{4(X-Y)}{\sqrt{2}}-\frac{2(X+Y)}{\sqrt{2}}=0$$

Multiply both sides by 2

$$X^2-Y^2-4\sqrt{2}(X-Y)-2\sqrt{2}(X+Y)=0$$

$$X^2-Y^2-6\sqrt{2}X+2\sqrt{2}Y=0$$

$$(X^2-6\sqrt{2}X)-(Y^2-2\sqrt{2}Y)=0$$
$$\Rightarrow\left(X^2-6\sqrt{2}X+(3\sqrt{2})^2-(3\sqrt{2})^2\right)$$
$$-\left(Y^2-2\sqrt{2}Y+(\sqrt{2})^2-(\sqrt{2})^2\right)=0$$
$$\Rightarrow(X-3\sqrt{2})^2-18-(Y-\sqrt{2})^2+2=0$$
$$\Rightarrow(X-3\sqrt{2})^2-(Y-\sqrt{2})^2=16$$
$$\Rightarrow\frac{(X-3\sqrt{2})^2}{16}-\frac{(Y-\sqrt{2})^2}{16}=1 \ldots (4)$$

As (4) represents hyperbola, hence (1) is hyperbola.

Here $a^2=16, b^2=16$ and

$c^2=a^2+b^2=32$

$\Rightarrow a=4, b=4, c=4\sqrt{2}$

Now we convert X and Y in terms of x and y. Using equations of transformation

$X=x\cos\theta+y\sin\theta$

For

$$\theta=45^\circ\Rightarrow\cos\theta=\frac{1}{\sqrt{2}},\sin\theta=\frac{1}{\sqrt{2}}$$

$$X=\frac{x+y}{\sqrt{2}} \qquad (5)$$

And $Y=-x\sin\theta+y\cos\theta$

For

$$\theta=45^\circ\Rightarrow\cos\theta=\frac{1}{\sqrt{2}},\sin\theta=\frac{1}{\sqrt{2}}$$

$$Y=\frac{-x+y}{\sqrt{2}} \qquad (6)$$

Elements:

Center:

For center of (4), we can write

$X-3\sqrt{2}=0$, $Y-\sqrt{2}=0$

$X=3\sqrt{2}$, $Y=\sqrt{2}$

So center of (4) is $\left(3\sqrt{2},\sqrt{2}\right)$

From (2) $x=\dfrac{X-Y}{\sqrt{2}}=\dfrac{3\sqrt{2}-\sqrt{2}}{\sqrt{2}}=2$

From (3) $y=\dfrac{X+Y}{\sqrt{2}}=\dfrac{3\sqrt{2}+\sqrt{2}}{\sqrt{2}}=4$

Hence center of (1) is $(2,4)$

Transverse axis:

Equation of transverse axis of (4) is

$Y-\sqrt{2}=0\Rightarrow Y=\sqrt{2}$

Put value of Y in (6)

$$\frac{-x+y}{\sqrt{2}}=\sqrt{2}\Rightarrow -x+y=2$$

$x-y+2=0$ is the equation of transverse axis for (1)

Conjugate axis:

Conjugate axis of (4) is given by

$X-3\sqrt{2}=0\Rightarrow X=3\sqrt{2}$

Put value of X in (5)

$$\frac{x+y}{\sqrt{2}}=3\sqrt{2}$$

$x+y=6$

$x+y-6=0$ is the equation of conjugate axis for (1)

Vertices:

For vertices of (4), we can write

$X-3\sqrt{2}=\pm a$, $Y-\sqrt{2}=0$

$X-3\sqrt{2}=\pm 4$, $Y-\sqrt{2}=0$

$X=3\sqrt{2}\pm 4$, $Y=\sqrt{2}$

So vertices of (4) are $\left(3\sqrt{2}\pm 4,\sqrt{2}\right)$

For $X=3\sqrt{2}+4, Y=\sqrt{2}$

From (2)

$$x=\frac{X-Y}{\sqrt{2}}=\frac{3\sqrt{2}+4-\sqrt{2}}{\sqrt{2}}=\frac{2\sqrt{2}+4}{\sqrt{2}}=2+2\sqrt{2}$$

From (3)

$$y=\frac{X+Y}{\sqrt{2}}=\frac{3\sqrt{2}+4+\sqrt{2}}{\sqrt{2}}=\frac{4\sqrt{2}+4}{\sqrt{2}}=4+2\sqrt{2}$$

For $X=3\sqrt{2}-4, Y=\sqrt{2}$

From (2)

$$x=\frac{X-Y}{\sqrt{2}}=\frac{3\sqrt{2}-4-\sqrt{2}}{\sqrt{2}}$$
$$=\frac{2\sqrt{2}-4}{\sqrt{2}}=2-2\sqrt{2}$$

From (3)

$$y=\frac{X+Y}{\sqrt{2}}=\frac{3\sqrt{2}-4+\sqrt{2}}{\sqrt{2}}$$

$$=\frac{4\sqrt{2}-4}{\sqrt{2}}=4-2\sqrt{2}$$

Hence vertices of (1) are

$\left(2+2\sqrt{2},4+2\sqrt{2}\right)$ and

$\left(2-2\sqrt{2},4-2\sqrt{2}\right)$

Foci

For foci of (4), we can write

$X-3\sqrt{2}=\pm c$, $Y-\sqrt{2}=0$

$X-3\sqrt{2}=\pm 4\sqrt{2}$, $Y-\sqrt{2}=0$

$X=3\sqrt{2}\pm 4\sqrt{2}$, $Y=\sqrt{2}$

$X=7\sqrt{2},-\sqrt{2}$, $Y=\sqrt{2}$

So foci of (4) are $\left(7\sqrt{2},\sqrt{2}\right)$ and $\left(-\sqrt{2},\sqrt{2}\right)$

For $X=7\sqrt{2}, Y=\sqrt{2}$

From (2)

$$x=\frac{X-Y}{\sqrt{2}}=\frac{7\sqrt{2}-\sqrt{2}}{\sqrt{2}}=\frac{6\sqrt{2}}{\sqrt{2}}=6$$

From (3)

$$y=\frac{X+Y}{\sqrt{2}}=\frac{7\sqrt{2}+\sqrt{2}}{\sqrt{2}}=\frac{8\sqrt{2}}{\sqrt{2}}=8$$

For $X=-\sqrt{2}, Y=\sqrt{2}$

From (2)

$$x=\frac{X-Y}{\sqrt{2}}=\frac{-\sqrt{2}-\sqrt{2}}{\sqrt{2}}=\frac{-2\sqrt{2}}{\sqrt{2}}=-2$$

From (3)

$$y=\frac{X+Y}{\sqrt{2}}=\frac{-\sqrt{2}+\sqrt{2}}{\sqrt{2}}=\frac{0}{\sqrt{2}}=0$$

Hence foci of (1) are (6,8) and (−2,0)

Eccentricity:

Eccentricity of (1) is

$$e=\frac{c}{a}=\frac{4\sqrt{2}}{4}=\sqrt{2}>1$$

(viii) $x^2+4xy-2y^2-6=0$

Solution: Consider

$x^2+4xy-2y^2-6=0$ (1)

Comparing (1) with

$ax^2+by^2+2gx+2fy+2hxy+c=0$,

we get

$a=1, b=-2, 2h=4$

As we know that angle of rotation is given by

$$\tan 2\theta=\frac{2h}{a-b}=\frac{4}{1-(-2)}=\frac{4}{3}$$

$$\frac{2\tan\theta}{1-\tan^2\theta}=\frac{4}{3}$$

$6\tan\theta=4-4\tan^2\theta$

$4\tan^2\theta+6\tan\theta-4=0$

$2\tan^2\theta+3\tan\theta-2=0$

(Dividing by 2)

$2\tan^2\theta+4\tan\theta-\tan\theta-2=0$

$2\tan\theta(\tan\theta+2)-1(\tan\theta+2)=0$

$(2\tan\theta-1)(\tan\theta+2)=0$

$2\tan\theta-1=0, \tan\theta+2=0$

$$\tan\theta=\frac{1}{2}, \tan\theta=-2$$

As $0<\theta<90^\circ$ and $\tan\theta$ is positive in 1st quadrant, so neglect $\tan\theta=-2$

Consider $\tan\theta=\frac{1}{2}$

From the right triangle

$$\sin\theta=\frac{1}{\sqrt{5}}, \cos\theta=\frac{2}{\sqrt{5}}$$

Using equations of transformation

$x=X\cos\theta-Y\sin\theta$

Using values of $\sin\theta$ and $\cos\theta$, we get

$$x=\frac{2X}{\sqrt{5}}-\frac{Y}{\sqrt{5}}=\frac{2X-Y}{\sqrt{5}} \qquad (2)$$

And $y=X\sin\theta+y\cos\theta$

$$y=\frac{X}{\sqrt{5}}+\frac{2Y}{\sqrt{5}}$$

$$y=\frac{X+2Y}{\sqrt{5}} \qquad (3)$$

Substituting values of x and y in (1), we get

$$\left(\frac{2X-Y}{\sqrt{5}}\right)^2+4\left(\frac{2X-Y}{\sqrt{5}}\right)-2\left(\frac{X+2Y}{\sqrt{5}}\right)^2-6=0$$

$$\left(\frac{4X^2+Y^2-4XY}{5}\right)+4\left(\frac{2X^2+3XY-2Y^2}{5}\right)-2\left(\frac{X^2+4Y^2+4XY}{5}\right)-6=0$$

Multiplying by 5, we get

$$4X^2+Y^2-4XY+8X^2+12XY$$
$$-8Y^2-2X^2-8Y^2-8XY-30=0$$
$$10X^2-15Y^2-30=0$$
$$2X^2-3Y^2-6=0$$
$$2X^2-3Y^2=6$$
$$\frac{X^2}{3}-\frac{Y^2}{2}=1 \qquad (4)$$

Which represents a hyperbola, hence (1) represents hyperbola

Here

$$a^2=3,b^2=2,c^2=a^2+b^2=2+3=5$$
$$\Rightarrow a=\sqrt{3},b=\sqrt{2},c=\sqrt{5}$$

Next we convert X and Y in terms of x and y

As $X=x\cos\theta+y\sin\theta$

$$X=\frac{2x}{\sqrt{5}}+\frac{y}{\sqrt{5}}$$
$$X=\frac{2x+y}{\sqrt{5}} \qquad (5)$$

And $Y=-x\sin\theta+y\cos\theta$

$$Y=\frac{-x}{\sqrt{5}}+\frac{2y}{\sqrt{5}}$$
$$Y=\frac{-x+2y}{\sqrt{5}} \qquad (6)$$

Elements:

Center of (4) is (0,0) so coordinates of center of (4) are $X=0,Y=0$

From (2)

$$x=\frac{2X-Y}{\sqrt{5}}=\frac{2(0)-(0)}{\sqrt{5}}=0$$

From (3)

$$y=\frac{X+2Y}{\sqrt{5}}=\frac{0+2(0)}{\sqrt{5}}=0$$

Hence center of (1) is (0,0)

<u>Transvers axis</u>:

Transvers axis of (4) is $Y=0$

Put $Y=0$ in (6), we get $\frac{-x+2y}{\sqrt{5}}=0$

$\Rightarrow -x+2y=0 \Rightarrow y=\frac{1}{2}x$ is the transvers axis of (1)

<u>Conjugate axis</u>:

Conjugate axis of (4) is $X=0$

Put $X=0$ in (5), we get $\frac{2x+y}{\sqrt{5}}=0$

$\Rightarrow 2x+y=0 \Rightarrow y=-2x$ is the equation of conjugate axis of (1)

<u>Vertices</u>:

Vertices of (4) are $(\pm a,0)=\left(\pm\sqrt{3},0\right)$

Here coordinates of vertices are

$X=\pm\sqrt{3}$ and $Y=0$

For $X=\sqrt{3},Y=0$

From (2)

$$x=\frac{2X-Y}{\sqrt{5}}=\frac{2\left(\sqrt{3}\right)-(0)}{\sqrt{5}}=2\sqrt{\frac{3}{5}}$$

From (3)

$$y=\frac{X+2Y}{\sqrt{5}}=\frac{\sqrt{3}+2(0)}{\sqrt{5}}=\sqrt{\frac{3}{5}}$$

For $X=-\sqrt{3},Y=0$

From (2)

$$x=\frac{2X-Y}{\sqrt{5}}=\frac{2\left(-\sqrt{3}\right)-(0)}{\sqrt{5}}=-2\sqrt{\frac{3}{5}}$$

From (3)

$$y=\frac{X+2Y}{\sqrt{5}}=\frac{-\sqrt{3}+2(0)}{\sqrt{5}}=-\sqrt{\frac{3}{5}}$$

Hence vertices of (1) are

$$\left(2\sqrt{\frac{3}{5}},\sqrt{\frac{3}{5}}\right) \text{ and } \left(-2\sqrt{\frac{3}{5}},-\sqrt{\frac{3}{5}}\right)$$

Foci:

Foci of (4) are $(\pm c,0)=\left(\pm\sqrt{5},0\right)$

Here coordinates of foci are

$X=\pm\sqrt{5}, Y=0$

For $X=\sqrt{5}, Y=0$

From (2)

$$x=\frac{2X-Y}{\sqrt{5}}=\frac{2\left(\sqrt{5}\right)-(0)}{\sqrt{5}}=2$$

From (3)

$$y=\frac{X+2Y}{\sqrt{5}}=\frac{\sqrt{5}+2(0)}{\sqrt{5}}=1$$

For $X=-\sqrt{5}, Y=0$

From (2)

$$x=\frac{2X-Y}{\sqrt{5}}=\frac{2\left(-\sqrt{5}\right)-(0)}{\sqrt{5}}=-2$$

From (3)

$$y=\frac{X+2Y}{\sqrt{5}}=\frac{-\sqrt{5}+2(0)}{\sqrt{5}}=-1$$

Hence foci of (1) are $(2,1),(-2,-1)$

Eccentricity:

$e=\frac{c}{a}=\frac{\sqrt{5}}{\sqrt{3}}=\sqrt{\frac{5}{3}}$ is the eccentricity of (1)

(ix) $x^2-4xy-2y^2+10x+4y=0$

Solution: Consider

$x^2-4xy-2y^2+10x+4y=0$ (1)

Comparing (1) with

$ax^2+by^2+2gx+2fy+2hxy+c=0$, we get

$a=1, b=-2, 2h=-4$

As we know that angle of rotation is given by

$$\tan 2\theta=\frac{2h}{a-b}=\frac{-4}{1+2}=-\frac{4}{3}$$

$$\frac{2\tan\theta}{1-\tan^2\theta}=\frac{-4}{3}$$

$$\frac{\tan\theta}{1-\tan^2\theta}=\frac{-2}{3}$$

$3\tan\theta=-2+2\tan^2\theta$

$2\tan^2\theta-3\tan\theta-2=0$

$2\tan^2\theta-4\tan\theta+\tan\theta-2=0$

$2\tan\theta(\tan\theta-2)+1(\tan\theta-2)=0$

$(\tan\theta-2)(2\tan\theta+1)=0$

$\tan\theta-2=0, 2\tan\theta+1=0$

$\tan\theta=2,\ \tan\theta=-\frac{1}{2}$

As $0<\theta<90^\circ$ and $\tan\theta$ is positive in 1st quadrant, so neglect

$\tan\theta=-\frac{1}{2}$

Consider $\tan\theta=2$

From the right triangle

$$\sin\theta=\frac{2}{\sqrt{5}},\cos\theta=\frac{1}{\sqrt{5}}$$

Using equation of transformation

$x=X\cos\theta-Y\sin\theta$

$$x=\frac{X}{\sqrt{5}}-\frac{2Y}{\sqrt{5}}$$

$$x=\frac{X-2Y}{\sqrt{5}} \qquad (2)$$

And $y=X\sin\theta+Y\cos\theta$

$$y=\frac{2X}{\sqrt{5}}+\frac{Y}{\sqrt{5}}$$

$$y=\frac{2X+Y}{\sqrt{5}} \qquad (3)$$

Put (2) and (3) in (1), we get

$$\left(\frac{X-2Y}{\sqrt{5}}\right)^2-4\left(\frac{X-2Y}{\sqrt{5}}\right)\left(\frac{2X+Y}{\sqrt{5}}\right)-2\left(\frac{2X+Y}{\sqrt{5}}\right)^2+10\left(\frac{X-2Y}{\sqrt{5}}\right)+4\left(\frac{2X+Y}{\sqrt{5}}\right)=0$$

$$\frac{(X^2+4Y^2-4XY)}{5}-4\frac{(2X^2-3XY-2Y^2)}{5}-2\frac{(4X^2+Y^2+4XY)}{5}$$

$$+10\frac{(X-2Y)}{\sqrt{5}}+4\left(\frac{2X+Y}{\sqrt{5}}\right)=0$$

Multiplying by 5, we get

$X^2+4Y^2-4XY-8X^2+12XY$

$+8Y^2-8X^2-2Y^2-8XY+10\sqrt{5}X$

$-20\sqrt{5}Y+8\sqrt{5}X+4\sqrt{5}Y=0$

$$-15X^2+10Y^2+18\sqrt{5}X-16\sqrt{5}Y=0$$
$$10Y^2-16\sqrt{5}Y-15X^2+18\sqrt{5}X=0$$
$$10\left(Y^2-\frac{8}{\sqrt{5}}Y\right)-15\left(X^2-\frac{6}{\sqrt{5}}X\right)=0$$
$$10\left(Y^2-\frac{8}{\sqrt{5}}Y+\left(\frac{4}{\sqrt{5}}\right)^2-\left(\frac{4}{\sqrt{5}}\right)^2\right)$$
$$-15\left(X^2-\frac{6}{\sqrt{5}}X+\left(\frac{3}{\sqrt{5}}\right)^2-\left(\frac{3}{\sqrt{5}}\right)^2\right)=0$$
$$10\left(\left(Y-\frac{4}{\sqrt{5}}\right)^2-\frac{16}{5}\right)$$
$$-15\left(\left(X-\frac{3}{\sqrt{5}}\right)^2-\frac{9}{5}\right)=0$$
$$10\left(Y-\frac{4}{\sqrt{5}}\right)^2-15\left(X-\frac{3}{\sqrt{5}}\right)^2$$
$$-32+27=0$$
$$10\left(Y-\frac{4}{\sqrt{5}}\right)^2-15\left(X-\frac{3}{\sqrt{5}}\right)^2=5$$
$$\frac{\left(Y-\frac{4}{\sqrt{5}}\right)^2}{\frac{1}{2}}-\frac{\left(X-\frac{3}{\sqrt{5}}\right)^2}{\frac{1}{3}}=1 \quad ..(4)$$

(4) represents hyperbola, hence (1) represents hyperbola

Here $a^2=\frac{1}{2}, b^2=\frac{1}{3}, c^2=a^2+b^2=\frac{5}{6}$

$$\Rightarrow a=\frac{1}{\sqrt{2}}, b=\frac{1}{\sqrt{3}}, c=\sqrt{\frac{5}{6}}$$

Next we convert X and Y in terms of x and y. Using equations of transformation

$$X=x\cos\theta+y\sin\theta$$
$$X=\frac{x}{\sqrt{5}}+\frac{2y}{\sqrt{5}}$$
$$X=\frac{x+2y}{\sqrt{5}} \quad (5)$$

And $Y=-x\sin\theta+y\cos\theta$

$$Y=\frac{-2x}{\sqrt{5}}+\frac{y}{\sqrt{5}}$$
$$Y=\frac{-2x+y}{\sqrt{5}} \quad (6)$$

Elements:

Center:

For center of (4), we can write

$$X-\frac{3}{\sqrt{5}}=0 \quad , \quad Y-\frac{4}{\sqrt{5}}=0$$
$$X=\frac{3}{\sqrt{5}} \quad , \quad Y=\frac{4}{\sqrt{5}}$$

So center of (4) is $\left(\frac{3}{\sqrt{5}},\frac{4}{\sqrt{5}}\right)$

From (2)

$$x=\frac{X-2Y}{\sqrt{5}}=\frac{\frac{3}{\sqrt{5}}-2\left(\frac{4}{\sqrt{5}}\right)}{\sqrt{5}}$$
$$=\frac{3-8}{5}=-1$$

From (3)

$$y=\frac{2X+Y}{\sqrt{5}}=\frac{2\left(\frac{3}{\sqrt{5}}\right)+\frac{4}{\sqrt{5}}}{\sqrt{5}}=\frac{6+4}{5}=2$$

Hence center of (1) is (–1,2)

Transverse axis:

Transverse axis of (4) is

$$X-\frac{3}{\sqrt{5}}=0\Rightarrow X=\frac{3}{\sqrt{5}}$$

Put $X=\frac{3}{\sqrt{5}}$ in (5), we get

$$\frac{x+2y}{\sqrt{5}}=\frac{3}{\sqrt{5}}$$

$x+2y=3$ is the transverse axis of (1)

Conjugate axis:

Conjugate axis of (4) is

$$Y-\frac{4}{\sqrt{5}}=0\Rightarrow Y=\frac{4}{\sqrt{5}}$$

Put $Y=\frac{4}{\sqrt{5}}$ in (6), we get

$$\frac{-2x+y}{\sqrt{5}}=\frac{4}{\sqrt{5}}$$
$$-2x+y=4$$

$2x-y+4=0$ is the conjugate axis for (1)

Foci:

For foci of (4), we can write

$X-\frac{3}{\sqrt{5}}=0 \quad , \quad Y-\frac{4}{\sqrt{5}}=\pm c$

$X-\frac{3}{\sqrt{5}}=0 \quad , \quad Y-\frac{4}{\sqrt{5}}=\pm\sqrt{\frac{5}{6}}$

$X=\frac{3}{\sqrt{5}} \quad , \quad Y=\frac{4}{\sqrt{5}}\pm\sqrt{\frac{5}{6}}$

So foci of (4) are given by

$\left(\frac{3}{\sqrt{5}},\frac{4}{\sqrt{5}}\pm\sqrt{\frac{5}{6}}\right)$

For $X=\frac{3}{\sqrt{5}},Y=\frac{4}{\sqrt{5}}+\sqrt{\frac{5}{6}}$

From (2)

$$x=\frac{X-2Y}{\sqrt{5}}=\frac{\frac{3}{\sqrt{5}}-2\left(\frac{4}{\sqrt{5}}+\frac{\sqrt{5}}{\sqrt{6}}\right)}{\sqrt{5}}$$

$$=\frac{1}{\sqrt{5}}\left(\frac{3}{\sqrt{5}}-\frac{8}{\sqrt{5}}-\frac{2\sqrt{5}}{\sqrt{6}}\right)$$

$$=\frac{3}{5}-\frac{8}{5}-\frac{2}{\sqrt{6}}=-1-\frac{2}{\sqrt{6}}$$

From (3)

$$y=\frac{2X+Y}{\sqrt{5}}=\frac{1}{\sqrt{5}}\left(2\left(\frac{3}{\sqrt{5}}\right)+\frac{4}{\sqrt{5}}+\frac{\sqrt{5}}{\sqrt{6}}\right)$$

$$=\frac{6}{5}+\frac{4}{5}+\frac{1}{\sqrt{6}}=2+\frac{1}{\sqrt{6}}$$

For $X=\frac{3}{\sqrt{5}},Y=\frac{4}{\sqrt{5}}-\sqrt{\frac{5}{6}}$

From (2)

$$x=\frac{X-2Y}{\sqrt{5}}=\frac{\frac{3}{\sqrt{5}}-2\left(\frac{4}{\sqrt{5}}-\frac{\sqrt{5}}{\sqrt{6}}\right)}{\sqrt{5}}$$

$$=\frac{1}{\sqrt{5}}\left(\frac{3}{\sqrt{5}}-\frac{8}{\sqrt{5}}+\frac{2\sqrt{5}}{\sqrt{6}}\right)$$

$$=\frac{3}{5}-\frac{8}{5}+\frac{2}{\sqrt{6}}=-1+\frac{2}{\sqrt{6}}$$

$$y=\frac{2X+Y}{\sqrt{5}}=\frac{1}{\sqrt{5}}\left(2\left(\frac{3}{\sqrt{5}}\right)+\frac{4}{\sqrt{5}}-\frac{\sqrt{5}}{\sqrt{6}}\right)$$

$$=\frac{6}{5}+\frac{4}{5}-\frac{1}{\sqrt{6}}=2-\frac{1}{\sqrt{6}}$$

Hence foci of (1) are

$\left(-1-\frac{2}{\sqrt{6}},2+\frac{1}{\sqrt{6}}\right)$ and

$\left(-1+\frac{2}{\sqrt{6}},2-\frac{1}{\sqrt{6}}\right)$

Vertices:

For vertices of (4), we can write

$X-\frac{3}{\sqrt{5}}=0 \quad , \quad Y-\frac{4}{\sqrt{5}}=\pm a$

$X-\frac{3}{\sqrt{5}}=0 \quad , \quad Y=\frac{4}{\sqrt{5}}\pm\frac{1}{\sqrt{2}}$

So vertices of (4) are

$\left(\frac{3}{\sqrt{5}},\frac{4}{\sqrt{5}}\pm\frac{1}{\sqrt{2}}\right)$

For $X=\frac{3}{\sqrt{5}},Y=\frac{4}{\sqrt{5}}+\frac{1}{\sqrt{2}}$

From (2)

$$x=\frac{X-2Y}{\sqrt{5}}=\frac{\frac{3}{\sqrt{5}}-2\left(\frac{4}{\sqrt{5}}+\frac{1}{\sqrt{2}}\right)}{\sqrt{5}}$$

$$=\frac{1}{\sqrt{5}}\left(\frac{3}{\sqrt{5}}-\frac{8}{\sqrt{5}}-\frac{2}{\sqrt{2}}\right)$$

$$=\frac{3}{5}-\frac{8}{5}-\frac{2}{\sqrt{10}}=-1-\frac{2}{\sqrt{10}}$$

From (3)

$$y=\frac{2X+Y}{\sqrt{5}}=\frac{1}{\sqrt{5}}\left(2\left(\frac{3}{\sqrt{5}}\right)+\frac{4}{\sqrt{5}}+\frac{1}{\sqrt{2}}\right)$$

$$=\frac{6}{5}+\frac{4}{5}+\frac{1}{\sqrt{10}}=2+\frac{1}{\sqrt{10}}$$

For $X=\frac{3}{\sqrt{5}}, Y=\frac{4}{\sqrt{5}}-\frac{1}{\sqrt{2}}$

From (2)

$$x=\frac{X-2Y}{\sqrt{5}}=\frac{\frac{3}{\sqrt{5}}-2\left(\frac{4}{\sqrt{5}}-\frac{1}{\sqrt{2}}\right)}{\sqrt{5}}$$

$$=\frac{1}{\sqrt{5}}\left(\frac{3}{\sqrt{5}}-\frac{8}{\sqrt{5}}+\frac{2}{\sqrt{2}}\right)$$

$$=\frac{3}{5}-\frac{8}{5}+\frac{2}{\sqrt{10}}=-1+\frac{2}{\sqrt{10}}$$

From (3)

$$y=\frac{2X+Y}{\sqrt{5}}=\frac{1}{\sqrt{5}}\left(2\left(\frac{3}{\sqrt{5}}\right)+\frac{4}{\sqrt{5}}-\frac{1}{\sqrt{2}}\right)$$

$$=\frac{6}{5}+\frac{4}{5}-\frac{1}{\sqrt{10}}=2-\frac{1}{\sqrt{10}}$$

Hence vertices of (1) are

$\left(-1-\frac{2}{\sqrt{10}},2+\frac{1}{\sqrt{10}}\right)$ and

$\left(-1+\frac{2}{\sqrt{10}},2-\frac{1}{\sqrt{10}}\right)$

<u>Eccentricity</u>:

Eccentricity of 1 is

$$e=\frac{c}{a}\Rightarrow e=\frac{\sqrt{5/6}}{1/\sqrt{2}}\Rightarrow e=\sqrt{\frac{5}{3}}$$

Q.2 Show that

(i) $10xy+8x-15y-12=0$ and

(ii) $6x^2+xy-y^2-21x-8y+9=0$

each represents a pair of straight lines and find an equation of each line.

(i)

Solution:

$10xy+8x-15y-12=0 \; (1)$

Comparing (1) with

$ax^2+by^2+2gx+2fy+2hxy+c=0$,

we get

$a=0, b=0, 2h=10, 2g=8\Rightarrow g=4$,

$2f=-15\Rightarrow f=\frac{-15}{2}, c=-12$

(1) represents a pair of straight lines if

$$\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$$

Consider

$$\Delta=\begin{vmatrix} 0 & 5 & 4 \\ 5 & 0 & \frac{-15}{2} \\ 4 & \frac{-15}{2} & -12 \end{vmatrix}$$

$$\Delta=0\begin{vmatrix} 0 & \frac{-15}{2} \\ \frac{-15}{2} & -12 \end{vmatrix}-5\begin{vmatrix} 5 & \frac{-15}{2} \\ 4 & -12 \end{vmatrix}+4\begin{vmatrix} 5 & 0 \\ 4 & \frac{-15}{2} \end{vmatrix}$$

$$\Delta=0-5(-60+30)+4\left(\frac{-75}{2}-0\right)$$

$$\Delta=150-150=0$$

Hence (1) represents pair of straight lines

Now we find the pair of lines.

(1) can be written as

$(10xy-15y)+(8x-12)=0$

$5y(2x-3)+4(2x-3)=0$

$(5y+4)(2x-3)=0$

$2x-3=0$ and $5y+4=0$ are the required equations of straight lines

(ii) $6x^2+xy-y^2-21x-8y+9=0 \quad (1)$

Solution: Comparing (1) with

$ax^2+by^2+2gx+2fy+2hxy+c=0$,

we get

$a=6, b=-1, 2h=1\Rightarrow h=\frac{1}{2}$

$2g=-21 \Rightarrow g=-\frac{21}{2}$,

$2f=-8 \Rightarrow f=-4$, $c=9$

(1) represents a pair of straight lines if

$$\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$$

Consider

$$\Delta=\begin{vmatrix} 6 & \frac{1}{2} & \frac{-21}{2} \\ \frac{1}{2} & -1 & -4 \\ \frac{-21}{2} & -4 & 9 \end{vmatrix}$$

Expanding from R_1, we get

$$\Delta=6\begin{vmatrix} -1 & -4 \\ -4 & 9 \end{vmatrix}-\frac{1}{2}\begin{vmatrix} \frac{1}{2} & -4 \\ \frac{-21}{2} & 9 \end{vmatrix}-\frac{21}{2}\begin{vmatrix} \frac{1}{2} & -1 \\ \frac{-21}{2} & -4 \end{vmatrix}$$

$$\Delta=6(-9-16)-\frac{1}{2}\left(\frac{9}{2}-42\right)-\frac{21}{2}\left(-2-\frac{21}{2}\right)$$

$$\Delta=6(-25)-\frac{1}{2}\left(\frac{-75}{2}\right)-\frac{21}{2}\left(\frac{-25}{2}\right)$$

$$\Delta=-150+\frac{75}{4}+\frac{525}{4}$$

$$\Delta=-150+150=0$$

Hence (1) represents a pair of straight lines

Next we find pair of straight lines

(1) can be written as

$$y^2-xy+8y-6x^2+21x-9=0$$

$$y^2-(x-8)y-3(2x^2-7x+3)=0$$

It is quadratic in *y*, so

$$y=\frac{(x-8)\pm\sqrt{(x-8)^2-4(1)(-3)(2x^2-7x+3)}}{2}$$

$$y=\frac{(x-8)\pm\sqrt{x^2-16x+64+12(2x^2-7x+3)}}{2}$$

$$y=\frac{(x-8)\pm\sqrt{x^2-16x+64+24x^2-84x+36}}{2}$$

$$y=\frac{(x-8)\pm\sqrt{25x^2-100x+100}}{2}$$

$$y=\frac{(x-8)\pm5\sqrt{x^2-4x+4}}{2}$$

$$y=\frac{(x-8)\pm5\sqrt{(x-2)^2}}{2}$$

$$y=\frac{(x-8)\pm5(x-2)}{2}$$

$$y=\frac{x-8+5(x-2)}{2}, y=\frac{x-8-5(x-2)}{2}$$

$$y=\frac{6x-18}{2}, y=\frac{-4x+2}{2}$$

$$y=3x-9, y=-2x+1$$

Hence pair of straight lines are

$3x-y-9=0$ and $2x+y-1=0$

Q.3 Find an equation of the tangent to each of the conics at the indicated point.

(i) $3x^2-7y^2+2x-y-48=0$ at (4,1)

Solution: Equation of the tangent at point (x_1,y_1) is given by

$$3xx_1-7yy_1+2\left(\frac{x+x_1}{2}\right)-\left(\frac{y+y_1}{2}\right)-48=0$$

Put $x_1=4, y_1=1$

$$3x(4)-7y(1)+(x+4)-\left(\frac{y+1}{2}\right)-48=0$$

$$12x-7y+x+4-\frac{1}{2}(y+1)-48=0$$

$$13x-7y+4-\frac{1}{2}(y+1)-48=0$$

$$26x-14y+8-y-1-96=0$$

$26x-15y-89=0$ is the required equation of tangent

(ii) $x^2+5xy-4y^2+4=0$ at $y=-1$

Solution:

First we find *x* for given $y=-1$

Put $y=-1$ in given equation, we get

$$x^2-5x-4+4=0$$

$$x^2-5x=0 \Rightarrow x(x-5)=0$$

$$x=0, x=5$$

Hence points of contact are $A(0,-1)$ and $B(5,-1)$

Now equation of tangent to given conic at point (x_1, y_1) is

$$xx_1 + 5\left(\frac{xy_1 + yx_1}{2}\right) - 4yy_1 + 4 = 0 \quad ..(1)$$

For $A(0,-1)$, put $x_1 = 0, y_1 = -1$ in (1)

$$x(0) + 5\left(\frac{x(-1) + y(0)}{2}\right) - 4y(-1) + 4 = 0$$

$$\frac{-5x}{2} + 4y + 4 = 0$$

$$-5x + 8y + 8 = 0$$

$5x - 8y - 8 = 0$ is equation of tangent at $A(0,-1)$

For $B(5,-1)$, put $x_1 = 5, y_1 = -1$ in (1)

$$x(5) + 5\frac{(x(-1) + y(5))}{2} - 4y(-1) + 4 = 0$$

$$10x - 5x + 25y + 8y + 8 = 0$$

(multiplying by 2)

$5x + 33y + 8 = 0$ is the required equation of tangent at $B(5,-1)$

(iii) $x^2 + 4xy - 3y^2 - 5x - 9y + 6 = 0$ at $x = 3$

Solution:

First we find y for given $x = 3$

Put $x = 3$ in given equation, we get

$$(3)^2 + 4(3)y - 3y^2 - 5(3) - 9y + 6 = 0$$

$$9 + 12y - 3y^2 - 15 - 9y + 6 = 0$$

$$-3y^2 + 3y = 0$$

$$-3y(y - 1) = 0$$

$$y = 0, y = 1$$

Hence points of contact are $A(3,0)$ and $B(3,1)$

Now equation of tangent to given equation at point (x_1, y_1) is

$$xx_1 + 4\left(\frac{xy_1 + yx_1}{2}\right) - 3yy_1 - 5\left(\frac{x + x_1}{2}\right) - 9\left(\frac{y + y_1}{2}\right) + 6 = 0 \quad (1)$$

For $A(3,0)$, put $x_1 = 3, y_1 = 0$ in (1)

$$x(3) + \frac{4}{2}\left(x(0) + y(3)\right) - 3y(0) - \frac{5}{2}(x + 3) - \frac{9}{2}(y + 0) + 6 = 0$$

$$6x + 12y - 5x - 15 - 9y + 12 = 0$$

(multiplying by 2)

$x + 3y - 3 = 0$ is the equation of tangent at $A(3,0)$

For $B(3,1)$, put $x_1 = 3, y_1 = 1$ in (1)

$$x(3) + \frac{4}{2}\left(x(1) + y(3)\right) - 3y(1) - \frac{5}{2}(x + 3) - \frac{9}{2}(y + 1) + 6 = 0$$

$$6x + 4x + 12y - 6y - 5x - 15 - 9y + 12 = 0$$

(multiplying by 2)

$5x - 3y - 12 = 0$ is the required equation of tangent at $B(3,1)$