Scalar Quantity:

All those quantities which requires only magnitude for complete description are called **scalar quantities** e.g. time, density, temperature and length etc.

Vector Quantity:

All those quantities which requires magnitude as well as direction for their complete description are called **vector quantities** e.g. weight, force, momentum, displacement, velocity etc.

Geometric Interpretation of Vector:

Geometrically, a vector is represented by a directed line segment $\overrightarrow{AB}$ with A its initial point and B its terminal point.

Magnitude of a Vector:

The magnitude or length or norm of vector $\overrightarrow{AB}$ is its absolute value and is written as $\left|\overrightarrow{AB}\right|$.

Unit Vector:

A unit vector $\hat{\underline{v}}$ (read as $\underline{v}$ hat) of a given vector $\underline{v}$ is a vector with magnitude one and direction same as vector $\underline{v}$. Mathematically

$$\text{Unit Vector}=\frac{\text{Vector}}{\text{Magnitude of Vector}}$$

i.e. $\hat{\underline{v}}=\dfrac{\underline{v}}{|\underline{v}|}$

Null Vector:

A vector whose terminal point coincides with its initial point is called **null** or **zero vector.**

Negative of a Vector:

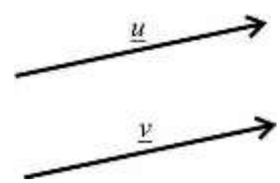
Two vectors $\underline{u}$ and $\underline{v}$ are called negative of each other, if they have same magnitude but opposite direction.

Multiplication of a Vector by a scalar (Number):

Multiplication of a vector $\underline{v}$ by a scalar 'n' is a vector whose magnitude is n times that of '$\underline{v}$' i.e. $n\underline{v}$.

(i) If n is positive, then $\underline{v}$ and $n\underline{v}$ are in the same direction.

(ii) If n is negative, then $\underline{v}$ and $n\underline{v}$ are in opposite directions.



Equal Vectors:

Two vectors $\underline{u}$ and $\underline{v}$ are called equal vectors, if they have same magnitude and same direction.

Parallel Vectors:

Two vectors $\underline{u}$ and $\underline{v}$ are parallel if and only if they are non-zero scalar multiple of each other i.e.

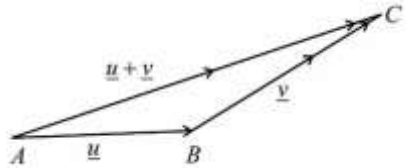
$$\underline{u} = \lambda \underline{v}, \lambda \neq 0$$

Triangle Law of Addition:

If two vectors $\underline{u}$ and $\underline{v}$ are represented by the two sides AB and BC of a triangle such that the terminal point of $\underline{u}$ coincide with the initial point of $\underline{v}$, then the third side AC of the triangle gives vector sum $\underline{u}+\underline{v}$, that is

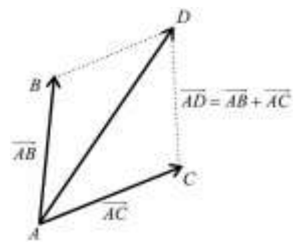
$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

$$\Rightarrow \quad \underline{u}+\underline{v} = \overrightarrow{AC}$$



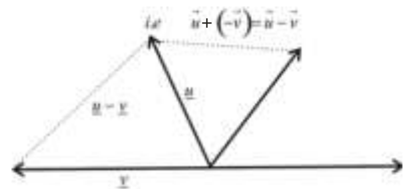
Parallelogram Law of Addition:

If two vector $\underline{u}$ and $\underline{v}$ are represented by two adjacent sides ***AB*** and ***AC*** of a parallelogram as shown in the figure, then diagonal ***AD*** give the sum or resultant of $\overrightarrow{AB}$ and $\overrightarrow{AC}$, that is $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{AC} = \underline{u}+\underline{v}$



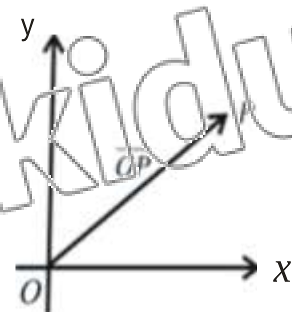
Subtraction of two vectors:

Let $\underline{u}$ and $\underline{v}$ are non-zero vector then subtraction of $\underline{v}$ and $\underline{u}$ is defined as the addition of $\underline{u}$ and $-\underline{v}$ i.e. $\vec{u} + (-\vec{v}) = \vec{u} - \vec{v}$



Position Vector:

A vector which describes the location of a point w.r.t origin is called position vector. The vector, whose initial point is the origin **O** and whose terminal point **P** is called the position vector of the point **P** and is written as $\overrightarrow{OP}$



Vector in a Plane:

Let R be a set of real numbers. The Cartesian plane is defined to be the $R^2 = \{(x, y) : x, y \in R\}$.

The Unit Vectors $\underline{i}$, $\underline{j}$:

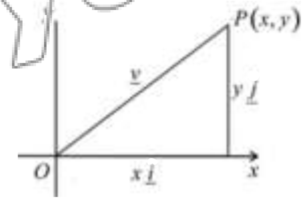
$\underline{i}$ and $\underline{j}$ are called unit vectors along *x*-axis and *y*-axis respectively.

They are written as

$$\underline{i} = [1,0],\ \underline{j} = [0,1]$$

$$|\underline{i}| = \sqrt{(1)^2 + (0)^2} = 1$$

$$|\underline{j}| = \sqrt{(0)^2 + (1)^2} = 1$$



A vector $\vec{v}$ can be written as

$$\underline{v} = [x, y] = [x, 0] + [0, y] = x[1,0] + y[0,1] = x\underline{i} + y\underline{j}$$

Similarly, sum of two vectors $\underline{u}$ and $\underline{v}$ can be written as

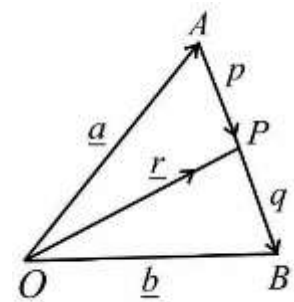
$$\underline{u} = [x_1, y_1],\quad \underline{v} = [x_2, y_2]$$

$$\underline{u} + \underline{v} = [x_1 + x_2,\ y_1 + y_2] = (x_1 + x_2)\,\underline{i} + (y_1 + y_2)\,\underline{j}$$

The Ratio Formula:

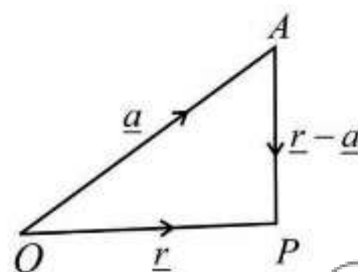
Let A and B are two points whose position vectors are $\underline{a}$ and $\underline{b}$ respectively. If a point P divides AB in the ratio $p:q$, then the position vector of P is given by

$$\underline{r} = \frac{q\underline{a} + p\underline{b}}{q + p}$$



Proof:

Given $\underline{a}$ and $\underline{b}$ are position vectors of the points A and B respectively. Let $\underline{r}$ be the position vector of the point P which divides the line segment $\overrightarrow{AB}$ in the ratio $p:q$.



That is

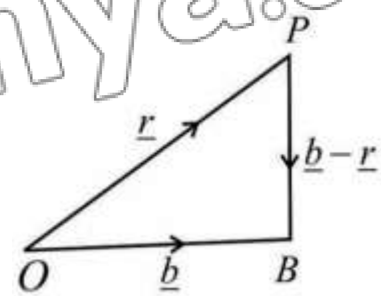
$$m\overline{AP} : m\overline{PB} = p : q$$

so

$$\frac{m\overline{AP}}{m\overline{PB}} = \frac{p}{q}$$

$$\Rightarrow \quad q\left(m\overline{AP}\right) = p\left(m\overline{PB}\right)$$

Thus $\quad q\left(\overline{AP}\right) = p\left(\overline{PB}\right)$



$$\Rightarrow \quad q(\underline{r} - \underline{a}) = p(\underline{b} - \underline{r})$$

$$\Rightarrow \quad q\underline{r} - q\underline{a} = p\underline{b} - p\underline{r}$$

$$\Rightarrow \quad p\underline{r} + q\underline{r} = p\underline{b} + q\underline{a}$$

$$\Rightarrow \quad \underline{r}(p + q) = q\underline{a} + p\underline{b}$$

$$\Rightarrow \quad \underline{r} = \frac{q\underline{a} + p\underline{b}}{q + p}$$

Corollary:

If P is the midpoint of AB, then $p:q=1:1$

$$\because \text{Position vector of } P = \underline{r} = \frac{\underline{a}+\underline{b}}{2}$$

Example:

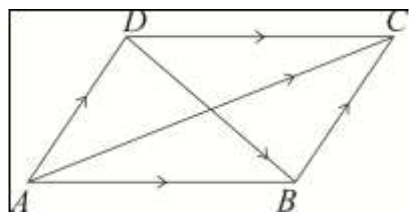
Use vectors, to prove that the diagonals of a parallelogram bisect each other.

Solution:

Let $A(\underline{a}), B(\underline{b}), C(\underline{c})$ and $D(\underline{d})$ be the vertices of parallelogram $ABCD$

$\therefore \overrightarrow{AB} = \overrightarrow{DC}$

$\underline{b} - \underline{a} = \underline{c} - \underline{d}$



Or $\underline{b} + \underline{d} = \underline{a} + \underline{c}$ …(i)

Let M be the mid-point of diagonal $\overrightarrow{AC}$

Then $M = \dfrac{\underline{a}+\underline{c}}{2}$

And N be the mid-point of diagonal $\overrightarrow{BD}$

$$N = \frac{\underline{b}+\underline{d}}{2}$$

If the diagonals of parallelogram bisect each other.

Then, mid-point of diagonal $\overrightarrow{AC}$ = mid-point of diagonal $\overrightarrow{BD}$

From equation (i) $\underline{b}+\underline{d} = \underline{a}+\underline{c}$

$$\Rightarrow \frac{\underline{b}+\underline{d}}{2} = \frac{\underline{a}+\underline{c}}{2}$$
$$\Rightarrow N = M$$

i.e. diagonals of parallelogram bisect each other.

EXERCISE 7.1

Q.1 Write the vector $\overrightarrow{PQ}$ in the form $x\underline{i}+y\underline{j}$.

(i) $P=(2,3)$, $Q(6,-2)$

Solution:

$P=(2,3)$, $Q(6,-2)$

$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}$ (Where O is origin)

$=(6\underline{i}-2\underline{j})-(2\underline{i}+3\underline{j})$

$=(6-2)\underline{i}+(-2-3)\underline{j}$

$\boxed{\overrightarrow{PQ}=4\underline{i}-5\underline{j}}$

(ii) $P=(0,5)$ $Q(-1,-6)$

$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}$

$=(-\underline{i}-6\underline{j})-(0\underline{i}+5\underline{j})$

$=(-1-0)\underline{i}+(-6-5)\underline{j}$

$\boxed{\overrightarrow{PQ}=-\underline{i}-11\underline{j}}$

Q.2 Find the magnitude of the vector $\underline{u}$:

(i) $\underline{u}=2\underline{i}-7\underline{j}$

Solution:

$\underline{u}=2\underline{i}-7\underline{j}$

$|\underline{u}|=|2\underline{i}-7\underline{j}|$

$=\sqrt{2^2+(-7)^2}$

$=\sqrt{4+49}$

$\boxed{|\underline{u}|=\sqrt{53}}$

(ii) $\underline{u}=\underline{i}+\underline{j}$

Solution:

$|\underline{u}|=|\underline{i}+\underline{j}|$

$=\sqrt{(1)^2+(1)^2}$

$\boxed{|\underline{u}|=\sqrt{2}}$

(iii) $\underline{u}=3\underline{i}-4\underline{j}$

Solution:

$|\underline{u}|=|3\underline{i}-4\underline{j}|$

$=\sqrt{(3)^2+(-4)^2}$

$=\sqrt{9+16}$

$=\sqrt{25}$

$\boxed{|\underline{u}|=5}$

Q.3 If $\underline{u}=2\underline{i}-7\underline{j}$, $\underline{v}=\underline{i}-6\underline{j}$ and $\underline{w}=-\underline{i}+\underline{j}$. Find the following vectors.

(i) $\underline{u}+\underline{v}-\underline{w}$

Solution:

$\underline{u}+\underline{v}-\underline{w}$

$=(2\underline{i}-7\underline{j})+(\underline{i}-6\underline{j})-(-\underline{i}+\underline{j})$

$=4\underline{i}-14\underline{j}$

(ii) $2\underline{u}-3\underline{v}+4\underline{w}$

Solution:

$2\underline{u}-3\underline{v}+4\underline{w}$

$=2(2\underline{i}-7\underline{j})-3(\underline{i}-6\underline{j})+4(-\underline{i}+\underline{j})$

$=4\underline{i}-14\underline{j}-3\underline{i}+18\underline{j}-4\underline{i}+4\underline{j}$

$=-3\underline{i}+8\underline{j}$

(iii) $\frac{1}{2}\underline{u}+\frac{1}{2}\underline{v}+\frac{1}{2}\underline{w}$

Solution:

$\frac{1}{2}\underline{u}+\frac{1}{2}\underline{u}+\frac{1}{2}\underline{w}$

$=\frac{1}{2}[\underline{u}+\underline{v}+\underline{w}]$

$=\frac{1}{2}\left[(2\underline{i}-7\underline{j})+(\underline{i}-6\underline{j})+(-\underline{i}+\underline{j})\right]$

$=\frac{1}{2}(2\underline{i}-12\underline{j})$

$=\underline{i}-6\underline{j}$

Q.4 Find the sum of the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ given the four points $A(1,-1)$, $B(2,0)$, $C(-1,3)$ and $D(-2,2)$.

Solution:

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$ where "O" is origin

$=(2\underline{i}+0\underline{j})-(\underline{i}-\underline{j})$

$=\underline{i}+\underline{j}$

$$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$$
$$= \left(-2\underline{i} + 2\underline{j}\right) - \left(-\underline{i} + 3\underline{j}\right)$$
$$= -\underline{i} - \underline{j}$$

Now,

$$\overrightarrow{AB} + \overrightarrow{CD} = (\underline{i} + \underline{j}) + (-\underline{i} - \underline{j})$$
$$= 0\underline{i} + 0\underline{j} \text{ Null vector}$$

Q.5 Find the vector from the point A to the origin where $\overrightarrow{AB} = 4\underline{i} - 2\underline{j}$ and B is the point (–2,5).

Solution:

$$\overrightarrow{OB} = -2\underline{i} + 5\underline{j}$$

We know that

$$\overrightarrow{AO} = \overrightarrow{AB} + \overrightarrow{BO}$$
$$= \overrightarrow{AB} - \overrightarrow{OB}$$
$$= 4\underline{i} - 2\underline{j} - (-2\underline{i} + 5\underline{j})$$
$$\boxed{\overrightarrow{AO} = 6\underline{i} - 7\underline{j}}$$

Q.6 Find a unit vector in the direction of the vector given below:

(i) $\underline{v} = 2\underline{i} - \underline{j}$

Solution:

$$\underline{v} = 2\underline{i} - \underline{j}$$
$$|\underline{v}| = \left|2\hat{i} - \hat{j}\right|$$
$$= \sqrt{(2)^2 + (-1)^2}$$
$$= \sqrt{4+1}$$
$$= \sqrt{5}$$
$$\hat{v} = \frac{\underline{v}}{|\underline{v}|}$$
$$= \frac{2\underline{i} - \underline{j}}{\sqrt{5}}$$
$$\boxed{\hat{v} = \frac{2}{\sqrt{5}}\underline{i} - \frac{1}{\sqrt{5}}\underline{j}}$$

(ii) $\underline{v} = \frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}$

Solution:

$$|\underline{v}| = \left|\frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}\right|$$
$$= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$
$$= \sqrt{\frac{1}{4} + \frac{3}{4}}$$
$$= \sqrt{\frac{4}{4}} = 1$$
$$\hat{v} = \frac{\underline{v}}{|\underline{v}|}$$
$$= \frac{\frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}}{1}$$
$$\boxed{\hat{v} = \frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}}$$

(iii) $\hat{v} = -\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}$

Solution:

$$|\underline{v}| = \sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2}$$
$$= \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{\frac{4}{4}} = 1$$
$$\hat{v} = \frac{\underline{v}}{|\underline{v}|} = \frac{-\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}}{1}$$
$$\boxed{\hat{v} = -\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}}$$

Q.7 If A, B and C are respectively the points $(2,-4), (4,0)$ and $(1,6)$. Use vector method to find the coordinates of the point D if:

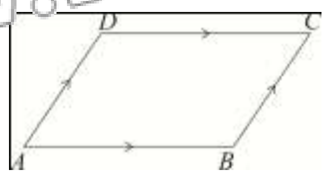
(i) $ABCD$ is parallelogram

Solution:

Let $D(a,b)$ be required vertex

Since $ABCD$ is parallelogram

$\therefore \overrightarrow{AD} = \overrightarrow{BC}$



$(a-2)\underline{i} + (b+4)\underline{j} = (1-4)\underline{i} + (6-0)\underline{j}$

$a-2 = -3 \qquad b+4 = 6$

$a = -3+2$ and $b = 6-4$

$a = -1 \ , \quad b = 2$

Thus $\boxed{D = (-1,2)}$

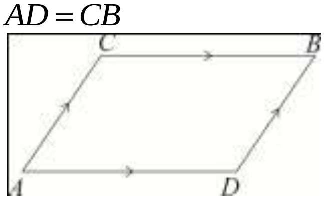
(ii) *ADBC* is parallelogram

Solution:

Let $D(a,b)$ be required vertex

Since $ADBC$ is parallelogram

$\therefore \overrightarrow{AD} = \overrightarrow{CB}$



$(a-2)\underline{i} + (b+4)\underline{j} = (4\underline{i} + 0\underline{j}) - (\underline{i} + 6\underline{j})$

$(a-2)\underline{i} + (b+4)\underline{j} = 3\underline{i} - 6\underline{j}$

$a-2 = 3$ and $b+4 = -6$

$a = 5$ and $b = -10$

Thus $\boxed{D(5,-10)}$

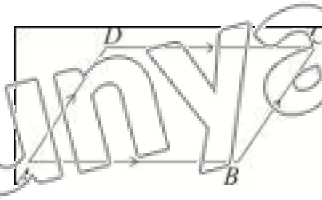
Q.8 If B, C and D are respectively $(4,1)$, $(-2,3)$ and $(-8,0)$. Use vector method to find the coordinates of the point.

(i) A if $ABCD$ is a parallelogram

Solution:

Let $A(a,b)$ be required vertex

Since $ABCD$ is a parallelogram



$\therefore \overrightarrow{AD} = \overrightarrow{BC}$

$(-8-a)\underline{i} + (0-b)\underline{j} = (-2\underline{i} + 3\underline{j}) - (4\underline{i} + 1\underline{j})$

$(-8-a)\underline{i} + (0-b)\underline{j} = (-2-4)\underline{i} + (3-1)\underline{j}$

$(-8-a)\underline{i} - b\underline{j} = -6\underline{i} + 2\underline{j}$

$-8-a = -6 \ , \quad -b = 2$

$-8+6 = a \ , \quad b = -2$

$-2 = a$

Thus $\boxed{A(-2,-2)}$

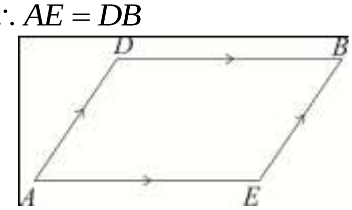
(ii) *E* if *AEBD* is a parallelogram

Solution:

Let $E(a,b)$ be required vertex

Since $AEBD$ is parallelogram

$\therefore \overrightarrow{AE} = \overrightarrow{DB}$



$(a\underline{i} + b\underline{j}) - (-2\underline{i} - 2\underline{j}) = (4\underline{i} + \underline{j}) - (-8\underline{i} + 0\underline{j})$

$(a+2)\underline{i} + (b+2)\underline{j} = (4+8)\underline{i} + (1-0)\underline{j}$

$(a+2)\underline{i} + (b+2)\underline{j} = 12\underline{i} + \underline{j}$

$a+2 = 12$ and $b+2 = 1$

$a = 10 \ , \quad b = -1$

Thus $\boxed{E(10,-1)}$

Q.9 If O is the origin and $\overrightarrow{OP} = \overrightarrow{AB}$, find the point *P* when *A* and *B* are $(-3,7)$ and $(1,0)$ respectively.

Solution:

Let $P(a,b)$ be required vertex

As $\overrightarrow{OP} = \overrightarrow{AB}$

$a\underline{i} + b\underline{j} = (\underline{i} + 0\underline{j}) - (-3\underline{i} + 7\underline{j})$

$a\underline{i}+b\underline{j}=(1+3)\underline{i}+(0-7)\underline{j}$

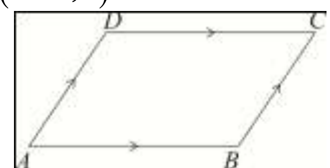
$a=4 \quad , \quad b=-7$

Thus $\boxed{P(4,-7)}$

Q.10 Use vectors to show that *ABCD* is a parallelogram when the points *A,B,C* and *D* are respectively $(0,0),(a,0),(b,c)$ and $(b-a,c)$.

Solution:

$A(0,0), B(a,0), C(b,c)$ and $D(b-a,c)$



$\overrightarrow{AB}=(a-0)\underline{i}+(0-0)\underline{j}$

$=a\underline{i}+0\underline{j}$

$\overrightarrow{DC}=(b-b+a)\underline{i}+(c-c)\underline{j}$

$=a\underline{i}+0\underline{j}$

$\overrightarrow{AD}=(b-a-0)\underline{i}+(c-0)\underline{j}$

$=(b-a)\underline{i}+c\underline{j}$

$\overrightarrow{BC}=(b-a)\underline{i}+(c-0)\underline{j}$

$=(b-a)\underline{i}+c\underline{j}$

$\because \overrightarrow{AB}=\overrightarrow{DC}$ and $\overrightarrow{AD}=\overrightarrow{BC}$

$\therefore \overrightarrow{AB}\parallel\overrightarrow{DC}$ and $\overrightarrow{AD}\parallel\overrightarrow{BC}$

Hence $ABCD$ is a parallelogram.

Q.11 If $\overrightarrow{AB}=\overrightarrow{CD}$. Find the coordinates of the point *A* when points *B,C,D* are (1,2), (–2,5), (4,11) respectively.

Solution:

Let $A(a,b)$ be required vertex

$\overrightarrow{AB}=\overrightarrow{CD}$

$(1\underline{i}+2\underline{j})-(a\underline{i}+b\underline{j})=(4\underline{i}+11\underline{j})-(-2\underline{i}+5\underline{j})$

$(1-a)\underline{i}+(2-b)\underline{j}=(4+2)\underline{i}+(11-5)\underline{j}$

$(1-a)\underline{i}+(2-b)\underline{j}=(6\underline{i}+6\underline{j})$

$1-a=6$ and $2-b=6$

$a=-5$ and $b=-4$

Thus $\boxed{A(-5,-4)}$

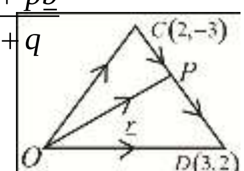
Q.12 Find the position vectors of the point of division of the line segment joining the following pair of points, in the given ratio:

(i) Point *C* with position vector $2\underline{i}-3\underline{j}$ and point *D* with position vector $3\underline{i}+2\underline{j}$ in the ratio 4:3

Solution:

Let $\underline{r}$ be the position vector of the point *P* which divides *CD* in ratio 4:3 i.e. $\overrightarrow{OP}=\underline{r}$

$\underline{r}=\dfrac{q\underline{a}+p\underline{b}}{p+q}$



$\underline{r}=\dfrac{3(2\underline{i}-3\underline{j})+4(3\underline{i}+2\underline{j})}{3+4}$

$\underline{r}=\dfrac{6\underline{i}-9\underline{j}+12\underline{j}+8\underline{j}}{7}=\dfrac{18\underline{i}-\underline{j}}{7}$

$\boxed{\underline{r}=\dfrac{18}{7}\underline{i}-\dfrac{1}{7}\underline{j}}$

(ii) Point *E* with position vector $5\underline{j}$ and point *F* with position vector $4\underline{i}+\underline{j}$ in ratio 2:5.

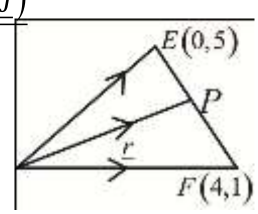
Solution:

Let $\underline{r}$ be the position vector of point *P* which divides *E* and *F* into ratio 2:5

$\underline{r}=\dfrac{5(0\underline{i}+5\underline{j})+2(4\underline{i}+\underline{j})}{2+5}$

$\underline{r}=\dfrac{(25\underline{j}+8\underline{i}+2\underline{j})}{7}$

$\boxed{\underline{r}=\dfrac{8}{7}\underline{i}+\dfrac{27}{7}\underline{j}}$



Q.13 Prove that the line segment joining the mid-point of two sides of a triangle is parallel to third side and half as long.

Solution:

Let $A(\underline{a}), B(\underline{b})$ and $C(\underline{c})$ be the vertices of triangle ABC

Let D and E are mid points of sides BC and AC respectively.

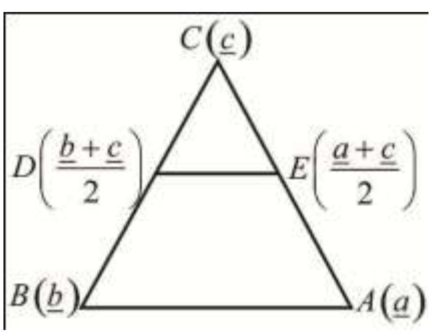
$D\left(\frac{\underline{b}+\underline{c}}{2}\right)$ and $E\left(\frac{\underline{a}+\underline{c}}{2}\right)$

Now $\overrightarrow{AB} = \underline{b}-\underline{a}$

$$\overrightarrow{ED} = \left(\frac{\underline{b}+\underline{c}}{2}\right) - \left(\frac{\underline{a}+\underline{c}}{2}\right)$$

$$\overrightarrow{ED} = \frac{\underline{b}+\underline{c}-\underline{a}-\underline{c}}{2}$$

$$= \frac{\underline{b}-\underline{a}}{2}$$



$$\overrightarrow{ED} = \frac{1}{2}(\underline{b}-\underline{a})$$

Thus $\overrightarrow{ED} = \frac{1}{2}\overrightarrow{AB}$ $\quad \because \overrightarrow{AB} = \underline{b}-\underline{a}$

Also $\overrightarrow{ED} \parallel \overrightarrow{AB}$

And $\left|\overrightarrow{ED}\right| = \frac{1}{2}\left|\overrightarrow{AB}\right|$

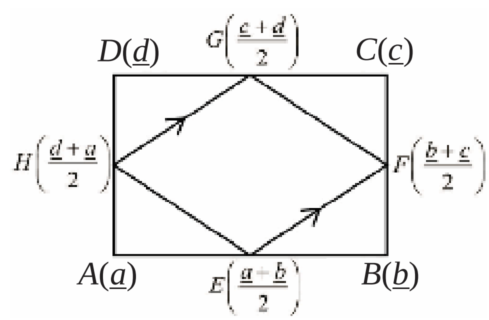
Q.14 Prove that the line segments joining the mid points of the sides of a quadrilateral taken in order form a parallelogram.

Solution:

Let $A(\underline{a}), B(\underline{b}), C(\underline{c})$ and $D(\underline{d})$ be the vertices of quadrilateral $ABCD$ respectively E, F, G and H are mid points of $\overline{AB}, \overline{BC}, \overline{CD}$ and $\overline{DA}$ respectively.

$E\left(\frac{\underline{a}+\underline{b}}{2}\right)$, $F\left(\frac{\underline{b}+\underline{c}}{2}\right)$

$G\left(\frac{\underline{c}+\underline{d}}{2}\right)$ and $H\left(\frac{\underline{d}+\underline{a}}{2}\right)$



$$\overrightarrow{EF} = \left(\frac{\underline{b}+\underline{c}}{2}\right) - \left(\frac{\underline{a}+\underline{b}}{2}\right)$$

$$= \frac{\underline{b}+\underline{c}-\underline{a}-\underline{b}}{2}$$

$$\overrightarrow{EF} = \frac{\underline{c}-\underline{a}}{2} \quad \dots\text{(i)}$$

$$\overrightarrow{HG} = \left(\frac{\underline{c}+\underline{d}}{2}\right) - \left(\frac{\underline{d}+\underline{a}}{2}\right)$$

$$= \frac{\underline{c}+\underline{d}-\underline{d}-\underline{a}}{2}$$

$$\overrightarrow{HG} = \frac{\underline{c}-\underline{a}}{2} \quad \dots\text{(ii)}$$

From (i) and (ii)

We have $\overrightarrow{EF} = \overrightarrow{HG}$

Also $\overrightarrow{FG} = \overrightarrow{EH}$

Hence $EFGH$ form a parallelogram.

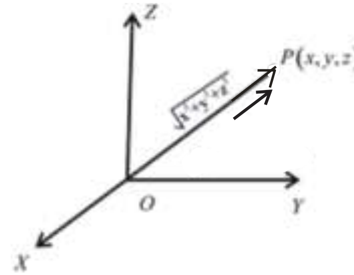
Vectors in Space:

The set $R^3 = \{(x, y, z) : x, y.z \in R\}$ is called the 3-dimensional space.

(i) Position Vector

The position vector of a point $P(x, y, z)$ in space, from the origin $O(0,0,0)$ is

$\overrightarrow{OP} = x\underline{i} + y\underline{j} + z\underline{k}$



The magnitude of $\overrightarrow{OP}$ is the distance of point P from the origin, i.e.

$|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$

(ii) The unit vectors $\underline{i}, \underline{j}, \underline{k}$

$\underline{i}, \underline{j}$ and $\underline{k}$ are called unit vectors along X, Y, Z axes respectively. They are written as:

$\underline{i} = [1,0,0] \qquad \underline{j} = [0,1,0] \qquad \underline{k} = [0,0,1]$

$|\ \underline{i}\ | = \sqrt{(1)^2 + (0)^2 + (0)^2} = 1$

$|\ \underline{j}\ | = \sqrt{(0)^2 + (1)^2 + (0)^2} = 1$

$|\ \underline{k}\ | = \sqrt{(0)^2 + (0)^2 + (1)^2} = 1$

A vector $\overline{v}$ can be written as:

$$\underline{v} = [x, y, z] = [x,0,0] + [0, y,0] + [0,0, z]$$

$$= x[1,0,0] + y[0,1,0] + z[0,0,1]$$

$$= x\underline{i} + y\underline{j} + z\underline{k}$$

Similarly, sum of two vectors $\underline{u}$ *and* $\underline{v}$ can be written as:

$$\underline{u} = [x_1, y_1, z_1], \ \underline{v} = [x_2, y_2, z_2]$$

$$\underline{u} + \underline{v} = [x_1 + x_2, y_1 + y_2, z_1 + z_2]$$

$$= (x_1 + x_2)\,\underline{i} + (y_1 + y_2)\,\underline{j} + (z_1 + z_2)\underline{k}$$

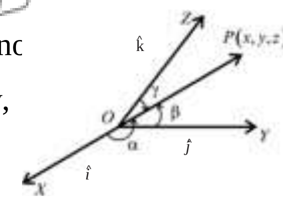
(iii) Distance between two points in space:

The distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in space is given by:

$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

(iv) Direction Angles and Direction Cosines of a vector:

Let $\underline{r} = \overrightarrow{OP} = x\underline{i} + y\underline{j} + z\underline{k}$ be a non-zero vector, let α, β and γ denote the angles between $\underline{r}$ and the unit coordinate vector $\underline{i}, \underline{j}$ and $\underline{k}$ respectively, $0 \le \alpha \le \pi$, $0 \le \beta \le \pi$ and $0 \le \gamma \le \pi$.



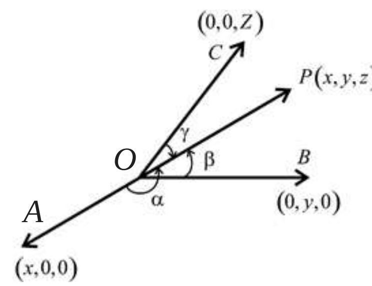
(i) The angles α, β, γ are called the direction angles.

(ii) The numbers $\cos\alpha, \cos\beta$ and $\cos\gamma$ are called direction cosines.

Important Result:

Prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$



Proof:

Let $\underline{r} = [x, y, z] = x\underline{i} + y\underline{j} + z\underline{k}$

$\therefore \; |\underline{r}| = \sqrt{x^2 + y^2 + z^2} = r$

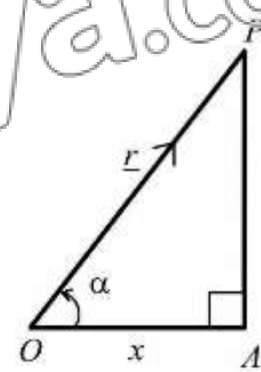
Then $\dfrac{\underline{r}}{|\underline{r}|} = \left[\dfrac{x}{r}, \dfrac{y}{r}, \dfrac{z}{r},\right]$ is the unit vector in the direction of the vector $\underline{r} = \overrightarrow{OP}$. It can be visualized that the triangle OAP is a right triangle with $\angle A = 90^o$. Therefore in right triangle OAP,

$$\cos\alpha = \frac{\overline{OA}}{\overline{OP}} = \frac{x}{r},$$

Similarly

$$\cos\beta = \frac{y}{r}, \; \cos\gamma = \frac{z}{r}$$

The numbers $\cos\alpha = \dfrac{x}{r}$, $\cos\beta = \dfrac{y}{r}$ and $\cos\gamma = \dfrac{z}{r}$ are called the **direction cosines of $\overrightarrow{OP}$**.



$$\therefore \; \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}$$

$$= \frac{x^2 + y^2 + z^2}{r^2} = \frac{r^2}{r^2}$$

$$\boxed{\therefore \; \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1}$$

EXERCISE 7.2

Q.1 **Let** $A=(2,5)$ $B=(-1,1)$ **and** $C\ (2,-6)$ **. Find**

(i) $\overrightarrow{AB}$

Solution:

$$\overrightarrow{AB}=\left(-\underline{i}+\underline{j}\right)-\left(2\underline{i}+5\underline{j}\right)$$
$$=(-1-2)\,\underline{i}+(1-5)\,\underline{j}$$
$$\boxed{\overrightarrow{AB}=-3\underline{i}-4\underline{j}}$$

(ii) $2\overline{AB}-\overline{CB}$

Solution:

$$\overrightarrow{AB}=\left(-\underline{i}+\underline{j}\right)-\left(2\underline{i}+5\underline{j}\right)$$
$$=-3\underline{i}-4\underline{j}$$
$$\overrightarrow{CB}=\left(-\underline{i}+\underline{j}\right)-\left(2\underline{i}-6\underline{j}\right)$$
$$=-3\underline{i}+7\underline{j}$$
$$2\overrightarrow{AB}-\overrightarrow{CB}=2\left(-3\underline{i}-4\underline{j}\right)-\left(-3\underline{i}+7\underline{j}\right)$$
$$=\left(-6\underline{i}-8\underline{j}\right)-\left(-3\underline{i}+7\underline{j}\right)$$
$$=(-6+3)\,\underline{i}+(-8-7)\,\underline{j}$$
$$\boxed{2\overrightarrow{AB}-\overrightarrow{CB}=-3\underline{i}-15\underline{j}}$$

(iii) $2\overrightarrow{CB}-2\overrightarrow{CA}$

Solution:

$$\overrightarrow{CB}=\left(-\underline{i}+\underline{j}\right)-\left(2\underline{i}-6\underline{j}\right)$$
$$=-3\underline{i}+7\underline{j}$$
$$\overrightarrow{CA}=\left(2\underline{i}+5\underline{j}\right)-\left(2\underline{i}-6\underline{j}\right)$$
$$=(2-2)\,\underline{i}+(5+6)\,j$$
$$=0\underline{i}+11\underline{j}$$
$$2\overrightarrow{CB}-2\overrightarrow{CA}$$
$$=2\left(-3\underline{i}+7\underline{j}\right)-2\left(0\underline{i}+11\underline{j}\right)$$
$$=\left(-6\underline{i}+14\underline{j}\right)-\left(0\underline{i}+22\underline{j}\right)$$
$$\boxed{2\overrightarrow{CB}-2\overrightarrow{CA}=-6\underline{i}-8\underline{j}}$$

Q.2 **Let** $\underline{u}=\underline{i}+2\underline{j}-\underline{k}$, $\underline{v}=3\underline{i}-2\underline{j}+2\underline{k}$, $\underline{w}=5\underline{i}-\underline{j}+3\underline{k}$ **. Find the indicated vector or number.**

(i) $\underline{u}+2\underline{v}+\underline{w}$

Solution:

$$\underline{u}+2\underline{v}+\underline{w}$$
$$=\left(\underline{i}+2\underline{j}-\underline{k}\right)+2\left(3\underline{i}-2\underline{j}+2\underline{k}\right)+\left(5\underline{i}-\underline{j}+3\underline{k}\right)$$
$$=\underline{i}+2\underline{j}-\underline{k}+6\underline{i}-4\underline{j}+4\underline{k}+5\underline{i}-\underline{j}+3\underline{k}$$
$$\boxed{\underline{u}+2\underline{v}+\underline{w}=12\underline{i}-3\underline{i}+6\underline{k}}$$

(ii) $\underline{v}-3\underline{w}$

Solution:

$$\underline{v}-3\underline{w}$$
$$=\left(3\underline{i}-2\underline{j}+2\underline{k}\right)-3\left(5\underline{i}-\underline{j}+3\underline{k}\right)$$
$$=\left(3\underline{i}-2\underline{j}+2\underline{k}\right)-15\underline{i}+3\underline{j}-9\underline{k}$$
$$\boxed{\underline{v}-3\underline{w}=-12\underline{i}+\underline{j}-7\underline{k}}$$

(iii) $\left|3\underline{v}+\underline{w}\right|$

Solution:

$$3\underline{v}+\underline{w}=3\left(3\underline{i}-2\underline{j}+2\underline{k}\right)+\left(5\underline{i}-\underline{j}+3\underline{k}\right)$$
$$=9\underline{i}-6\underline{j}+6\underline{k}+5\underline{i}-\underline{j}+3\underline{k}$$
$$=14\underline{i}-7\underline{j}+9\underline{k}$$
$$\left|3\underline{v}+\underline{w}\right|=\left|14\underline{i}+(-7)\underline{j}+9\underline{k}\right|$$
$$=\sqrt{(14)^2+(-7)^2+(9)^2}$$
$$=\sqrt{196+49+81}$$
$$\boxed{\left|3\underline{v}+\underline{w}\right|=\sqrt{326}}$$

Q.3 **Find the magnitude of the vector** $\underline{v}$ **and write the direction cosines of** $\underline{v}$**.**

(i) $\underline{v}=2\underline{i}+3\underline{j}+4\underline{k}$

Solution:

$$\underline{v}=2\underline{i}+3\underline{j}+4\underline{k}$$
$$\left|\underline{v}\right|=\sqrt{(2)^2+(3)^2+(4)^2}$$

$= \sqrt{4+9+16}$

$= \sqrt{29}$

Thus the direction cosines of $\underline{v}$ are

$$\boxed{\left[\frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}\right]}$$

(ii) $\underline{v} = \underline{i} - \underline{j} - \underline{k}$

Solution:

$|\underline{v}| = |\underline{i} - \underline{j} - \underline{k}|$

$= \sqrt{(1)^2 + (1)^2 + (1)^2}$

$= \sqrt{3}$

Thus the direction cosines of $\underline{v}$ are

$$\boxed{\left[\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right]}$$

(iii) $\underline{v} = 4\underline{i} - 5\underline{j}$

Solution:

$|\underline{v}| = |4\underline{i} - 5\underline{j}|$

$= \sqrt{(4)^2 + (5)^2}$

$= \sqrt{16+25}$

$= \sqrt{41}$

Thus the direction cosines of $\underline{v}$ are

$$\left[\frac{4}{\sqrt{41}}, \frac{-5}{\sqrt{41}}, 0\right]$$

Q.4 Find α so that

$|\alpha \underline{i} + (\alpha+1)\underline{j} + 2\underline{k}| = 3$

Solution:

$|\alpha \underline{i} + (\alpha+1)\underline{j} + 2\underline{k}| = 3$

$\sqrt{\alpha^2 + (\alpha+1)^2 + 2^2} = 3$

Taking square on both sides

$\alpha^2 + (\alpha+1)^2 + 4 = 9$

$\alpha^2 + \alpha^2 + 1 + 2\alpha + 4 - 9 = 0$

$2\alpha^2 + 2\alpha - 4 = 0$

$2(\alpha^2 + \alpha - 2) = 0$

$\alpha^2 + \alpha - 2 = 0$

$\alpha^2 + 2\alpha - \alpha - 2 = 0$

$\alpha(\alpha+2) - 1(\alpha+2) = 0$

$(\alpha-1)(\alpha+2) = 0$

Either $\alpha - 1 = 0$ or $\alpha + 2 = 0$

$\boxed{\alpha = 1}$ $\boxed{\alpha = -2}$

Q.5 Find a unit vector in the direction of $\underline{v} = \underline{i} + 2\underline{j} - \underline{k}$

Solution:

Let $\hat{\underline{v}}$ be unit vector in the direction of $\underline{v}$ then $\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|}$

$\underline{v} = \underline{i} - 2\underline{j} - \underline{k}$

$|\underline{v}| = |\underline{i} + 2\underline{j} - \underline{k}|$

$= \sqrt{1^2 + 2^2 + (-1)^2}$

$= \sqrt{6}$

$\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|} = \frac{\underline{i} + 2\underline{j} - \underline{k}}{\sqrt{6}}$

$$\boxed{\hat{\underline{v}} = \frac{1}{\sqrt{6}}\underline{i} + \frac{2}{\sqrt{6}}\underline{j} - \frac{1}{\sqrt{6}}\underline{k}}$$

Q.6 If $\underline{a} = 3\underline{i} - \underline{j} - 4\underline{k}$, $\underline{b} = -2\underline{i} - 4\underline{j} - 3\underline{k}$ and $\underline{c} = \underline{i} + 2\underline{j} - \underline{k}$. Find the unit vector parallel to $3\underline{a} - 2\underline{b} + 4\underline{c}$.

Solution:

Let $\underline{v} = 3\underline{a} - 2\underline{b} + 4\underline{c}$

$= 3(3\underline{i} - \underline{j} - 4\underline{k}) - 2(-2\underline{i} - 4\underline{j} - 3\underline{k})$

$+ 4(\underline{i} + 2\underline{j} - \underline{k})$

$= 9\underline{i} - 3\underline{j} - 12\underline{k} + 4\underline{i} + 8\underline{j} +$

$6\underline{k} + 4\underline{i} + 8\underline{j} - 4\underline{k}$

$\underline{v} = 17\underline{i} + 13\underline{j} - 10\underline{k}$

$|\underline{v}| = |17\underline{i} + 13\underline{j} - 10\underline{k}|$

$= \sqrt{(17)^2 + (13)^2 + (-10)^2}$

$= \sqrt{289 + 169 + 100}$

$= \sqrt{558}$

$\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|}$

$= \frac{17\underline{i} + 13\underline{j} - 10\underline{k}}{\sqrt{558}}$

$$\boxed{\hat{\underline{v}} = \frac{17}{\sqrt{558}}\underline{i} + \frac{13}{\sqrt{558}}\underline{j} - \frac{10}{\sqrt{558}}\underline{k}}$$

Q.7 Find a vector whose

(i) Magnitude is 4 and is parallel to $2\underline{i}-3\underline{j}+6\underline{k}$

Solution:

Let $\underline{v}=2\underline{i}-3\underline{j}+6\underline{k}$

$$|\underline{v}|=|2\underline{i}-3\underline{j}+6\underline{k}|$$
$$=\sqrt{2^2+(-3)^2+(6)^2}$$
$$=\sqrt{4+9+36}$$
$$=\sqrt{49}$$
$$=7$$

If $\hat{\underline{v}}$ is the unit vector in the direction of $\underline{v}$, then

$$\hat{\underline{v}}=\frac{\underline{v}}{|\underline{v}|}=\frac{2\underline{i}-3\underline{j}+6\underline{k}}{7}$$

Thus the vector

$$4\hat{\underline{v}}=4\left(\frac{2\underline{i}-3\underline{j}+6\underline{k}}{7}\right)$$

$$\boxed{4\hat{\underline{v}}=\frac{8}{7}\underline{i}-\frac{12}{7}\underline{j}+\frac{24}{7}\underline{k}}$$

(ii) Magnitude is 2 and is parallel to $-\underline{i}+\underline{j}+\underline{k}$

Solution:

Let $\underline{v}=-\underline{i}+\underline{j}+\underline{k}$

$$|\underline{v}|=|-\underline{i}+\underline{j}+\underline{k}|$$
$$=\sqrt{(-1)^2+(1)^2+(1)^2}$$
$$=\sqrt{3}$$

If $\hat{\underline{v}}$ is unit vector in the direction of $\underline{v}$ then ,

$$\hat{\underline{v}}=\frac{\underline{v}}{|\underline{v}|}=\frac{-\underline{i}+\underline{j}+\underline{k}}{\sqrt{3}}$$

Thus the vector

$$2\hat{\underline{v}}=2\left(\frac{-\underline{i}+\underline{j}+\underline{k}}{\sqrt{3}}\right)$$

$$\boxed{2\hat{\underline{v}}=\frac{-2}{\sqrt{3}}\underline{i}+\frac{2}{\sqrt{3}}\underline{j}+\frac{2}{\sqrt{3}}\underline{k}}$$

Q.8 If $\underline{u}=2\underline{i}+3\underline{j}+4\underline{k}$, $\underline{v}=-\underline{i}+3\underline{j}-\underline{k}$ and $\underline{w}=\underline{i}+6\underline{j}+z\underline{k}$ **represent the sides of a triangle, find the value of** z.

Solution:

Property of triangle in vector.

$$\underline{u}+\underline{v}=\underline{w}$$
$$2\underline{i}+3\underline{j}+4\underline{k}+\left(-\underline{i}+3\underline{j}-\underline{k}\right)=\underline{i}+6\underline{j}+z\underline{k}$$
$$\underline{i}+6\underline{j}+3\underline{k}=\underline{i}+6\underline{j}+z\underline{k}$$
$$\boxed{z=3}$$

Q.9 The position vectors of the points A, B, C, D **are** $2\underline{i}-\underline{j}+\underline{k}$, $3\underline{i}+\underline{j}$, $2\underline{i}+4\underline{j}-2\underline{k}$ **and** $-\underline{i}-2\underline{j}+\underline{k}$ **respectively. Show that** $\overrightarrow{AB}$ **is parallel to** $\overrightarrow{CD}$.

Solution:

$$\overrightarrow{AB}=3\underline{i}+\underline{j}-(2\underline{i}-\underline{j}+\underline{k})$$
$$=\underline{i}+2\underline{j}-\underline{k}$$
$$\overrightarrow{CD}=\left(-\underline{i}-2\underline{j}+\underline{k}\right)-\left(2\underline{i}+4\underline{j}-2\underline{k}\right)$$
$$=-3\underline{i}-6\underline{j}+3\underline{k}$$
$$=-3\left(\underline{i}+2\underline{j}+\underline{k}\right)$$
$$\overrightarrow{CD}=-3\ \overrightarrow{AB}$$

Hence $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel

Q.10 We say that two vectors $\underline{v}$ **and** $\underline{w}$ **in space are parallel if there is a scalar** c **such that** $\underline{v}=c\underline{w}$ **the vectors point in the same direction if** $c>0$**, and the vectors point in the opposite direction if** $c<0$

(a) Find two vectors of length 2 parallel to the vector $\underline{v}=2\underline{i}-4\underline{j}+4\underline{k}$

Solution:

$$\underline{v}=2\underline{i}-4\underline{j}+4\underline{k}$$
$$|\underline{v}|=|2\underline{i}-4\underline{j}+4\underline{k}|$$
$$=\sqrt{2^2+(-4)^2+(4)^2}$$

$= \sqrt{4+16+16}$
$= \sqrt{36}$
$= 6$

If it is unit vector parallel to $\underline{v}$ then,

$\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|}$

$= \frac{2\underline{i} - 4\underline{j} + 4\underline{k}}{6}$

$= \frac{2}{6}\underline{i} - \frac{4}{6}\underline{j} + \frac{4}{6}\underline{k}$

$= \frac{1}{3}\underline{i} - \frac{2}{3}\underline{j} + \frac{2}{3}\underline{k}$

The two vectors of length 2 and parallel to $\hat{v}$ are $2\hat{v}$ and $-2\hat{v}$

$2\hat{\underline{v}} = 2\left(\frac{1}{3}\underline{i} - \frac{2}{3}\underline{j} + \frac{2}{3}\underline{k}\right)$

and

$-2\hat{v} = -2\left(\frac{1}{3}\underline{i} - \frac{2}{3}\underline{j} + \frac{2}{3}\underline{k}\right)$

$\boxed{2\hat{\underline{v}} = \frac{2}{3}\underline{i} - \frac{4}{3}\underline{j} + \frac{4}{3}\underline{k}}$

and

$\boxed{-2\hat{\underline{v}} = -\frac{2}{3}\underline{i} + \frac{4}{3}\underline{j} - \frac{4}{3}\underline{k}}$

(b) Find the constant α so that the vectors $\underline{v} = \underline{i} - 3\underline{j} + 4\underline{k}$ an $\underline{w} = \alpha\underline{i} + 9\underline{j} - 12\underline{k}$ are parallel.

Solution:

If $\underline{v}$ and $\underline{w}$ are parallel then, $v = c\underline{w}$

$\underline{i} - 3\underline{j} + 4\underline{k} = c\left(\alpha\underline{i} + 9\underline{j} - 12\underline{k}\right)$

$\underline{i} - 3\underline{j} + 4\underline{k} = c\alpha\underline{i} + 9c\underline{j} - 12c\underline{k}$

Comparing both side

$\alpha c = 1$ ...(i)

$9c = -3$

$c = -\frac{1}{3}$

Put in (i)

$\boxed{\alpha = -3}$

(c) Find a vector of length 5 in the direction opposite that of $\underline{v} = \underline{i} - 2\underline{j} + 3\underline{k}$.

Solution:

$\underline{v} = \underline{i} - 2\underline{j} + 3\underline{k}$

$|\underline{v}| = |\underline{i} - 2\underline{j} + 3\underline{k}|$

$= \sqrt{(1)^2 + (-2)^2 + (3)^2}$

$= \sqrt{1+4+9}$

$= \sqrt{14}$

If $\hat{v}$ be unit vector in the direction of $\underline{v}$ then

$\hat{v} = \frac{\underline{v}}{|\underline{v}|} = \frac{\underline{i} - 2\underline{j} + 3\underline{k}}{\sqrt{14}}$

Thus the required vector of length 5 and direction opposite is:

$-5\hat{\underline{v}} = -5\left(\frac{\underline{i} - 2\underline{j} + 3\underline{k}}{\sqrt{14}}\right)$

$= \frac{-5}{\sqrt{14}}\underline{i} + \frac{10}{\sqrt{14}}\underline{j} - \frac{15}{\sqrt{14}}\underline{k}$

(d) Find *a* and *b* so that the vectors $3\underline{i} - \underline{j} + 4\underline{k}$ and $a\underline{i} + b\underline{j} - 2\underline{k}$ are Parallel.

Solution:

Let $\underline{v} = 3\underline{i} - \underline{j} + 4\underline{k}$ and $\underline{w} = a\underline{i} + b\underline{j} - 2\underline{k}$

$\underline{v}$ and $\underline{w}$ are parallel if

$\underline{w} = c\underline{v}$

$a\underline{i} + b\underline{j} - 2\underline{k} = c\left(3\underline{i} - \underline{j} + 4\underline{k}\right)$

$a\underline{i} + b\underline{j} - 2\underline{k} = 3c\underline{i} - c\underline{j} + 4c\underline{k}$

Comparing both sides

$a = 3c$ ...(i)

$b = -c$ ...(ii)

$-2 = 4c$ ...(iii)

$\Rightarrow c = -\frac{1}{2}$

Put value of c in equation (i) and (ii), we have

$a = 3\left(\frac{-1}{2}\right)$ and $b = -\left(\frac{-1}{2}\right)$

$\boxed{a = \frac{-3}{2}}$ and $\boxed{b = \frac{1}{2}}$

Q.11 Find the direction cosines for the given vector:

(i) $\underline{v}=3\underline{i}-\underline{j}+2\underline{k}$

Solution:

$\underline{v}=3\underline{i}-\underline{j}+2\underline{k}$

$|\underline{v}|=|3\underline{i}-\underline{j}+2\underline{k}|$

$=\sqrt{(3)^2+(-1)^2+(2)^2}$

$=\sqrt{9+1+4}$

$=\sqrt{14}$

The direction cosines of $\underline{v}$ are:

$$\left[\frac{3}{\sqrt{14}},\frac{-1}{\sqrt{14}},\frac{2}{\sqrt{14}}\right]$$

(ii) $6\underline{i}-2\underline{j}+\underline{k}$

Solution:

$\underline{v}=6\underline{i}-2\underline{j}+\underline{k}$

$|\underline{v}|=|6\underline{i}-2\underline{j}+\underline{k}|$

$=\sqrt{6^2+(-2)^2+(1)^2}$

$=\sqrt{36+4+1}$

$=\sqrt{41}$

The direction cosines of $\underline{v}$ are:

$$\left[\frac{6}{\sqrt{14}},\frac{-2}{\sqrt{41}},\frac{1}{\sqrt{41}}\right]$$

(iii) $\overrightarrow{PQ}$**, where** $P=(2,1,5)$ **and** $Q=(1,3,1)$

Solution:

$\overrightarrow{PQ}=(\underline{i}+3\underline{j}+\underline{k})-(2\underline{i}+\underline{j}+5\underline{k})$

$\overrightarrow{PQ}=-\hat{i}+2\hat{j}-4\hat{k}$

$|\overrightarrow{PQ}|=\sqrt{(-1)^2+(2)^2+(-4)^2}$

$=\sqrt{1+4+16}$

$=\sqrt{21}$

$|\overrightarrow{PQ}|=\sqrt{21}$

The direction cosines of $\overrightarrow{PQ}$ are:

$$\left[\frac{-1}{\sqrt{21}},\frac{2}{\sqrt{21}},\frac{-4}{\sqrt{21}}\right]$$

Q.12 Which of the following triples can be the direction angles of a single vector:

(i) **45°,45°,60°**

Solution:

45°,45°,60°

Let $\alpha=45°$, $\beta=45°$, $\gamma=60°$

We are to show that

$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

$\text{L.H.S}=\cos^2(45^\circ)+\cos^2(45^\circ)+\cos^2(60^\circ)$

$=\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{2}\right)^2$

$=\frac{1}{2}+\frac{1}{2}+\frac{1}{4}$

$=\frac{2+2+1}{4}$

$=\frac{5}{4}\neq 1$

Thus the given angles are not direction angles of a vector.

(ii) 30°,45°,60°

Solution:

Let $\alpha = 30°$, $\beta = 45°$, $\gamma = 60°$

We are to show that

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

L.H.S= $\cos^2(30) + \cos^2(45^\circ) + \cos^2(60^\circ)$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2$$

$$= \frac{3}{4} + \frac{1}{2} + \frac{1}{4}$$

$$= \frac{3+2+1}{4}$$

$$= \frac{6}{4} \neq 1$$

Thus the given angles are not direction angle of the vector .

(iii) 45°,60°,60°

Solution:

Let $\alpha = 45°$, $\beta = 60°$, $\gamma = 60°$

We are to show that

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

L.H.S $= \cos^2(45) + \cos^2(60^\circ) + \cos^2(60^\circ)$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{4}$$

$$= \frac{2+1+1}{4}$$

$$= \frac{4}{4}$$

$$= 1$$

Thus given angles are direction angles of a vector.

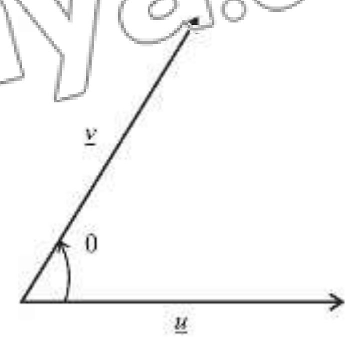
THE SCALAR PRODUCT OF TWO VECTORS:

Definition: 1

The scalar or dot product of two vectors $\underline{u}$ and $\underline{v}$ in a plane or in space is

$\underline{u}.\underline{v} = |\underline{u}||\underline{v}|\cos\theta$

where θ is the angle between $\underline{u}$ and $\underline{v}$ and $0 \le \theta \le \pi$.



The unit vectors $\underline{i}, \underline{j}, \underline{k}$:

(a) $\underline{i}\,.\,\underline{i} = |\underline{i}||\underline{i}|\cos 0^o = (1)(1)(1) = 1$ as $\cos 0 = 1$

$\underline{j}\,.\,\underline{j} = |\underline{j}||\underline{j}|\cos 0^o = (1)(1)(1) = 1$

$\underline{k}\,.\,\underline{k} = |\underline{k}||\underline{k}|\cos 0^o = (1)(1)(1) = 1$

(b) $\underline{i}\,.\,\underline{j} = |\underline{i}||\underline{j}|\cos 90^o = (1)(1)(0) = 0$ as $\cos 90^o = 0$

$\underline{j}\,.\,\underline{k} = |\underline{j}||\underline{k}|\cos 90^o = (1)(1)(0) = 0$

$\underline{k}\,.\,\underline{i} = |\underline{k}||\underline{i}|\cos 90^o = (1)(1)(0) = 0$

(c) $\underline{u}.\underline{v} = \underline{v}.\underline{u}$ (Dot product of two vectors is commutative)

Definition: 2

(a) If $\underline{u} = a_1\underline{i} + b_1\underline{j}$ *and* $\underline{v} = a_2\underline{i} + b_2\underline{j}$ are two vectors in a plane, then the dot product of $\underline{u}$ and $\underline{v}$ is

$$\underline{u}.\underline{v} = (a_1\underline{i} + b_1\underline{j}).(a_2\underline{i} + b_2\underline{j})$$

$$= a_1a_2(\underline{i}.\underline{i}) + a_1b_2(\underline{i}.\underline{j}) + a_2b_1(\underline{j}.\underline{i}) + (b_1b_2)(\underline{j}.\underline{j})$$

$$= a_1a_2(1) + a_1b_2(0) + a_2b_1(0) + b_1b_2(1)$$

$$\underline{u}.\underline{v} = a_1a_2 + b_1b_2$$

(b) If $\underline{u} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}$ and $\underline{v} = a_2\underline{i} + b_2\underline{j} + c_2\underline{k}$ are two vectors in space, then the dot product of $\underline{u}$ and $\underline{v}$ is

$$\underline{u}.\underline{v} = (a_1\underline{i} + b_1\underline{j} + c_1\underline{k}).(a_2\underline{i} + b_2\underline{j} + c_2\underline{k})$$

$$= a_1a_2(\underline{i}.\underline{i}) + a_1b_2(\underline{i}.\underline{j}) + a_1c_2(\underline{i}.\underline{k}) + a_2b_1(\underline{j}.\underline{i})$$

$$+ b_1b_2(\underline{j}.\underline{j}) + b_1c_2(\underline{j}.\underline{k}) + a_2c_1(\underline{k}.\underline{i}) + b_2c_1(\underline{k}.\underline{j}) + c_1c_2(\underline{k}.\underline{k})$$

$$= a_1a_2(1) + a_1b_2(0) + a_1c_2(0) + a_2b_1(0) + b_1b_2(1)$$

$$= b_1c_2(0) + a_2c_1(0) + b_2c_1(0) + c_1c_2(1)$$

$$\underline{u}.\underline{v} = a_1a_2 + b_1b_2 + c_1c_2$$

Perpendicular (Orthogonal) Vectors:

Two vectors $\underline{u}$ and $\underline{v}$ are perpendicular if and only if $\underline{u}.\underline{v} = 0$

The angle between $\underline{u}$ and $\underline{v}$ is $\dfrac{\pi}{2}$

$\therefore \underline{u}.\underline{v} = |\underline{u}||\underline{v}|\cos\frac{\pi}{2} = |\underline{u}||\underline{v}|(0)$

$\underline{u}.\underline{v} = 0$

Angle between two vectors:

The angles between two vectors $\underline{u}$ and $\underline{v}$ is

(a) $\underline{u}.\underline{v} = |\underline{u}||\underline{v}|\cos\theta$ where $0 \le \theta \le \pi$

$\therefore \quad \cos\theta = \frac{\underline{u}.\underline{v}}{|\underline{u}||\underline{v}|}$

(b) If $\underline{u} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}$ and $\underline{v} = a_1\underline{i} + b_2\underline{j} + c_2\underline{k}$

Then $\underline{u}.\underline{v} = a_1a_2 + b_1b_2 + c_1c_2$

$|\underline{u}| = \sqrt{a_1^2 + b_1^2 + c_1^2} \qquad |\underline{v}| = \sqrt{a_2^2 + b_2^2 + c_2^2}$

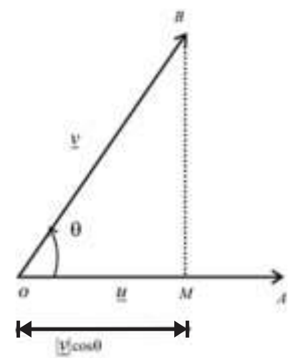
$\because \quad \cos\theta = \frac{\underline{u}.\underline{v}}{|\underline{u}||\underline{v}|}$

$\therefore \quad \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$

Projection of one vector upon another vector:

Let $\underline{u} = \overrightarrow{OA}$ and $\underline{v} = \overrightarrow{OB}$ and θ be the angle between them. Where $0 \le \theta \le \pi$.

Draw $\overline{BM} \perp \overline{OA}$. Then $\overline{OM}$ is called the projection of $\underline{v}$ along $\underline{u}$. In right triangle OMB



$\cos\theta = \frac{\overline{OM}}{\overline{OB}}$

$\Rightarrow \overline{OM} = \overline{OB}\cos\theta$

$\therefore \overline{OM} = |\underline{v}|\cos\theta$...(i)

Also $\cos\theta = \frac{\underline{u}.\underline{v}}{|\underline{u}||\underline{v}|}$...(ii)

putting (ii) in (i)

$\overline{OM} = |\underline{v}| \frac{\underline{u}.\underline{v}}{|\underline{u}||\underline{v}|}$

$= \frac{\underline{u}.\underline{v}}{|\underline{u}|}$

$= \frac{|\underline{u}||\underline{v}|\cos\theta}{|\underline{u}|} = |\underline{v}|\cos\theta$

i.e. projection of $\underline{v}$ along $\underline{u} = \frac{\underline{u}.\underline{v}}{|\underline{u}|} = |\underline{v}|\cos\theta$

Similarly

Projection of $\underline{u}$ along $\underline{v} = \frac{\underline{u}.\underline{v}}{|\underline{v}|} = |\underline{u}|\cos\theta$.

EXERCISE 7.3

Q.1 Find the cosine of the angle θ between $\underline{u}$ and $\underline{v}$

(i) $\underline{u}=3\underline{i}+\underline{j}-\underline{k}$, $\underline{v}=2\underline{i}-\underline{j}+\underline{k}$

Solution:

$\underline{u}=3\underline{i}+\underline{j}-\underline{k}$, $\underline{v}=2\underline{i}-\underline{j}+\underline{k}$

$$\cos\theta=\frac{\underline{u}.\underline{v}}{|\underline{u}||\underline{v}|}$$

$$\cos\theta=\frac{\left(3\underline{i}+\underline{j}-\underline{k}\right).\left(2\underline{i}-\underline{j}+\underline{k}\right)}{\sqrt{3^2+(1)^2+(-1)^2}\sqrt{(2)^2+(1)^2+(1)^2}}$$

$$=\frac{3(2)+(1)(-1)+(-1)(1)}{\sqrt{9+1+1}\sqrt{4+1+1}}$$

$$=\frac{6-1-1}{\sqrt{11}\sqrt{6}}$$

$$=\frac{4}{\sqrt{11}\sqrt{6}}$$

$$\boxed{\cos\theta=\frac{4}{\sqrt{66}}}$$

(ii) $\underline{u}=\underline{i}-3\underline{j}+4\underline{k}$, $\underline{v}=4\underline{i}-\underline{j}+3\underline{k}$

Solution:

$\underline{u}=\underline{i}-3\underline{j}+4\underline{k}$, $\underline{v}=4\underline{i}-\underline{j}+3\underline{k}$

$$\cos\theta=\frac{\underline{u}.\underline{v}}{|\underline{u}||\underline{v}|}$$

$$\underline{u}.\underline{v}=\frac{\left(\underline{i}-3\underline{j}+4\underline{k}\right).\left(4\underline{i}-\underline{j}+3\underline{k}\right)}{\sqrt{(1)^2+(-3)^2+(4)^2}\sqrt{(4)^2+(-1)^2+(3)^2}}$$

$$=\frac{(1)(4)+(-3)(-1)+(4)(3)}{\sqrt{1+9+16}\sqrt{16+1+9}}$$

$$=\frac{4+3+12}{\sqrt{26}\sqrt{26}}$$

$$\boxed{\cos\theta=\frac{19}{\sqrt{26}\sqrt{26}}=\frac{19}{26}}$$

(iii) $\underline{u}=[-3,5]$, $\underline{v}=[6,-2]$

Solution:

$\underline{u}=-3\underline{i}+5\underline{j}$, $\underline{v}=6\underline{i}-2\underline{j}$

$$\cos\theta=\frac{\underline{u}.\underline{v}}{|\underline{u}||\underline{v}|}$$

$$\cos\theta=\frac{\left(-3\underline{i}+5\underline{j}\right).\left(6\underline{i}-2\underline{j}\right)}{\sqrt{(-3)^2+(5)^2}\sqrt{(6)^2+(-2)^2}}$$

$$=\frac{-18-10}{\sqrt{9+25}\sqrt{36+4}}$$

$$=\frac{-28}{\sqrt{34}\sqrt{40}}$$

$$=\frac{-28}{\sqrt{34\times40}}$$

$$=\frac{-28}{\sqrt{1360}}$$

$$=\frac{-28}{\sqrt{16\times85}}$$

$$=\frac{-28}{4\sqrt{85}}$$

$$\boxed{\cos\theta=\frac{-7}{\sqrt{85}}}$$

(iv) $\underline{u}=[2,-3,1]$, $\underline{v}=[2,4,1]$

Solution:

$\underline{u}=2\underline{i}-3\underline{j}+\underline{k}$, $\underline{v}=2\underline{i}+4\underline{j}+\underline{k}$

$$\cos\theta=\frac{\underline{u}.\underline{v}}{|\underline{u}||\underline{v}|}$$

$$=\frac{\left(2\underline{i}-3\underline{j}+\underline{k}\right).\left(2\underline{i}+4\underline{j}+\underline{k}\right)}{\sqrt{(2)^2+(-3)^2+(1)^2}\sqrt{(2)^2+(4)^2+(1)^2}}$$

$$=\frac{2(2)+-3(4)+(1)(1)}{\sqrt{4+9+1}\sqrt{4+16+1}}$$

$$=\frac{4-12+1}{\sqrt{14}\sqrt{21}}$$

$$=\frac{-7}{\sqrt{14}\sqrt{21}}$$

$$=\frac{-7}{7\sqrt{6}}$$

$$\boxed{\cos\theta=\frac{-1}{\sqrt{6}}}$$

Q.2 Calculate the projection of $\underline{a}$ along $\underline{b}$ and projection of $\underline{b}$ along $\underline{a}$ when

(i) $\underline{a} = \underline{i} - \underline{k}$, $\underline{b} = \underline{j} + \underline{k}$

Solution:

$\underline{a} = \underline{i} - \underline{k} \Rightarrow |\underline{a}| = \sqrt{1+1} = \sqrt{2}$

$\underline{b} = \underline{j} + \underline{k} \Rightarrow |\underline{b}| = \sqrt{1+1} = \sqrt{2}$

Projection of $\underline{a}$ along $\underline{b} = \dfrac{\underline{a}.\underline{b}}{|\underline{b}|}$

$$= \frac{(\underline{i} - \underline{k}).(\underline{j} + \underline{k})}{\sqrt{2}}$$

$$= \frac{0+0-1}{\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

Projection of $\underline{b}$ along $\underline{a} = \dfrac{\underline{a}.\underline{b}}{|\underline{a}|} = \dfrac{-1}{\sqrt{2}}$

(ii) $\underline{a} = 3\underline{i} + \underline{j} - \underline{k}$, $\underline{b} = -2\underline{i} - \underline{j} + \underline{k}$

Solution:

$\underline{a} = 3\underline{i} + \underline{j} - \underline{k}$

$|\underline{a}| = \sqrt{(3)^2 + (1)^2 + (-1)^2} = \sqrt{11}$

$\underline{b} = -2\underline{i} - \underline{j} + \underline{k}$

$|\underline{b}| = \sqrt{(-2)^2 + (-1)^2 + (1)^2} = \sqrt{6}$

Projection of $\underline{a}$ along $\underline{b} = \dfrac{\underline{a}.\underline{b}}{|\underline{b}|}$

$$= \frac{(3\underline{i} + \underline{j} - \underline{k}).(-2\underline{i} - \underline{j} + \underline{k})}{\sqrt{6}}$$

$$= \frac{(3)(-2) + (1)(-1) + (1)(-1)}{\sqrt{6}}$$

$$= \frac{-8}{\sqrt{6}}$$

Projection of $\underline{b}$ along $\underline{a} = \dfrac{\underline{a}.\underline{b}}{|\underline{a}|} = \dfrac{-8}{\sqrt{11}}$

Q.3 Find a real number α so that the vectors $\underline{u}$ and $\underline{v}$ are perpendicular.

(i) $\underline{u} = 2\alpha\underline{i} + \underline{j} - \underline{k}$, $\underline{v} = \underline{i} + \alpha\underline{j} + 4\underline{k}$

Solution:

The vectors $\underline{u}$ and $\underline{v}$ are perpendicular

So $\underline{u}.\underline{v} = 0$

$(2\alpha\underline{i} + \underline{j} - \underline{k}).(\underline{i} + \alpha\underline{j} + 4\underline{k}) = 0$

$2\alpha + \alpha - 4 = 0$

$3\alpha - 4 = 0$

$$\boxed{\alpha = \frac{4}{3}}$$

(ii) $\underline{u} = \alpha\underline{i} + 2\alpha\underline{j} - \underline{k}$, $\underline{v} = \underline{i} + \alpha\underline{j} + 3\underline{k}$

Solution:

The vectors $\underline{u}$ and $\underline{v}$ are perpendicular

So $\underline{u}.\underline{v} = 0$

$(\alpha\underline{i} + 2\alpha\underline{j} - \underline{k}).(\underline{i} + \alpha\underline{j} + 3\underline{k}) = 0$

$\alpha + 2\alpha^2 - 3 = 0$

$2\alpha^2 + \alpha - 3 = 0$

$2\alpha^2 + 3\alpha - 2\alpha - 3 = 0$

$\alpha(2\alpha + 3) - 1(2\alpha + 3) = 0$

$(\alpha - 1)(2\alpha + 3) = 0$

Either $\alpha - 1 = 0$ or $2\alpha + 3 = 0$

$\boxed{\alpha = 1}$ or $\boxed{\alpha = \dfrac{-3}{2}}$

Q.4 Find the number z so that the triangle with vertices $A(1,-1,0)$, $B(-2,2,1)$ and $C(0,2,z)$ is right triangle with right angle at C.

Solution:

$\overrightarrow{AC} = (0\underline{i} + 2\underline{j} + z\underline{k}) - (\underline{i} - \underline{j} + 0\underline{k})$

$= (-\underline{i} + 3\underline{j} + z\underline{k})$

$\overrightarrow{BC} = (0\underline{i} + 2\underline{j} + z\underline{k}) - (-2\underline{i} + 2\underline{j} + \underline{k})$

$= (2\underline{i} + 0\underline{j} + (z-1)\underline{k})$

As $\overrightarrow{AC}.\overrightarrow{BC} = 0$

$\left(-\underline{i}+3\underline{j}+z\underline{k}\right).\left(2\underline{i}+0\underline{j}+(z-1)\underline{k}\right)=0$

$-2+0+z(z-1)=0$

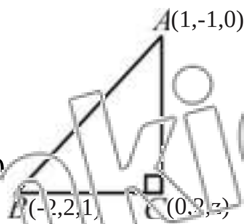
$-2+z^2-z=0$

$z^2-z-2=0$

$z^2-2z+z-2=0$

$z(z-2)+1(z-2)=0$

$(z+1)(z-2)=0$



Either $z+1=0$ or $z-2=0$

$\boxed{z=-1}$ or $\boxed{z=2}$

Q.5 If $\underline{v}$ is a vector for which $\underline{v}.\underline{i}=0, \underline{v}.\underline{j}=0, \underline{v}.\underline{k}=0$, find $\underline{v}$.

Solution:

Let $\underline{v}=x\underline{i}+y\underline{j}+z\underline{k}$ …(i)

$\underline{v}.\underline{i}=\left(x\underline{i}+y\underline{j}+z\underline{k}\right).\underline{i}$

$0=x(\underline{i}.\underline{i}) \qquad \because \underline{i}.\underline{i}=1$

$0=x$

$\underline{v}.\underline{j}=\left(x\underline{i}+y\underline{j}+z\underline{k}\right).\underline{j}$

$0=y(\underline{j}.\underline{j}) \qquad \because \underline{j}.\underline{j}=1$

$0=y$

$\underline{v}.\underline{k}=\left(x\underline{i}+y\underline{j}+z\underline{k}\right).\underline{k}$

$0=z\left(\underline{k}.\underline{k}\right) \qquad \because \underline{k}.\underline{k}=1$

$0=z$

Put the value of x, y, z in (i)

$\underline{v}=x\underline{i}+y\underline{j}+z\underline{k}$

$\boxed{\underline{v}=0\underline{i}+0\underline{j}+0\underline{k}}$

Q.6

(i) Show that the vectors $3\underline{i}-2\underline{j}+\underline{k}$, $\underline{i}-3\underline{j}+5\underline{k}$ and $2\underline{i}+\underline{j}-4\underline{k}$ form a right triangle.

Solution:

Let $\underline{u}\left(3\underline{i}-2\underline{j}+\underline{k}\right), \underline{v}\left(\underline{i}-3\underline{j}+5\underline{k}\right)$ and $\underline{w}\left(2\underline{i}+\underline{j}-4\underline{k}\right)$ are vectors along sides of the triangle,

$\underline{u}.\underline{w}=\left(3\underline{i}-2\underline{j}+\underline{k}\right).\left(2\underline{i}+\underline{j}-4\underline{k}\right)$

$=3(2)-2(1)+1(-4)$

$=6-2-4$

$\underline{u}.\underline{w}=0$

$\Rightarrow \underline{u}\perp\underline{w}$

So $\underline{u},\underline{v}$ and $\underline{w}$ form a right triangle.

(ii) Show that the set of points $P=(1,3,2)$, $Q(4,1,4)$ and $R(6,5,5)$ form a right triangle.

Solution:

$\overrightarrow{PQ}=\left(4\underline{i}+\underline{j}+4\underline{k}\right)-\left(\underline{i}+3\underline{j}+2\underline{k}\right)$

$=3\underline{i}-2\underline{j}+2\underline{k}$

$\overrightarrow{QR}=\left(6\underline{i}+5\underline{j}+5\underline{k}\right)-\left(4\underline{i}+\underline{j}+4\underline{k}\right)$

$=2\underline{i}+4\underline{j}+\underline{k}$

$\overrightarrow{PQ}.\overrightarrow{QR}=\left(3\underline{i}-2\underline{j}+2\underline{k}\right).\left(2\underline{i}+4\underline{j}+\underline{k}\right)$

$=6-8+2$

$=0$

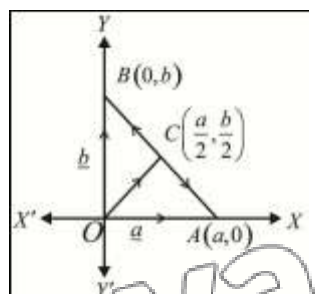
$\Rightarrow \overrightarrow{PQ}\perp\overrightarrow{QR}$

Thus PQR is a right triangle.

Q.7 Show that the midpoint of hypotenuse of a right triangle is equidistant from its vertices.

Solution:

Consider a right triangle AOB with "O" is origin and $A(a,0)$, $B(0,b)$ be its other vertices.



C be the midpoint of hypotenuse $\overline{AB}$ of a right triangle

Then $C\left(\frac{a}{2},\frac{b}{2}\right)$

We have to show that

$\left|\overrightarrow{OC}\right|=\left|\overrightarrow{AC}\right|=\left|\overrightarrow{BC}\right|$

$\overrightarrow{AC}=\left(\frac{a}{2}\underline{i}+\frac{b}{2}\underline{j}\right)-(a\underline{i}+0\underline{j})$

$=-\frac{a}{2}\underline{i}+\frac{b}{2}\underline{j}$

$$|\overrightarrow{AC}| = \sqrt{\left(-\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}$$

$$= \sqrt{\frac{a^2}{4} + \frac{b^2}{4}}$$

$$= \sqrt{\frac{a^2+b^2}{4}}$$

$$= \frac{\sqrt{a^2+b^2}}{2} \ldots\text{(i)}$$

$$\overrightarrow{BC} = \left(\frac{a}{2}\underline{i} + \frac{b}{2}\underline{j}\right) - (0\underline{i} + b\underline{j})$$

$$= \frac{a}{2}\underline{i} - \frac{b}{2}\underline{j}$$

$$|\overrightarrow{BC}| = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}}$$

$$= \frac{\sqrt{a^2+b^2}}{2} \ldots\text{(ii)}$$

$$\overrightarrow{OC} = \left(\frac{a}{2}\underline{i} + \frac{b}{2}\underline{j}\right) - \left(0\underline{i} + 0\underline{j}\right)$$

$$= \left(\frac{a}{2}\underline{i} + \frac{b}{2}\underline{j}\right)$$

$$|\overrightarrow{OC}| = \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}$$

$$= \frac{\sqrt{a^2+b^2}}{2} \ldots\text{(iii)}$$

From (i), (ii), (iii) $|\overrightarrow{AC}| = |\overrightarrow{BC}| = |\overrightarrow{OC}|$

(The midpoint of hypotenuse of a right triangle is equidistance from its vertices)

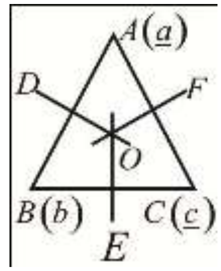
Q.8 Prove that perpendicular bisector of the sides of a triangle are concurrent.

Solution:

Let $A(\underline{a}), B(\underline{b})$ and $C(\underline{c})$ be the vertices of ΔABC

D, E and F be the mid points of $\overline{AB}, \overline{BC}$ and $\overline{CA}$ respectively.

$\overrightarrow{OD}\left(\frac{\underline{a}+\underline{b}}{2}\right)$, $\overrightarrow{OE}\left(\frac{\underline{b}+\underline{c}}{2}\right)$ and $\overrightarrow{OF}\left(\frac{\underline{c}+\underline{a}}{2}\right)$ are right bisector of sides $\overrightarrow{AB}, \overrightarrow{BC}$ and $\overrightarrow{AC}$ respectively.



Now $\overrightarrow{AB} = \underline{b} - \underline{a}$

As $\overrightarrow{OD} \perp \overrightarrow{AB}$

$\Rightarrow \overrightarrow{OD}.\overrightarrow{AB} = 0$

$$\left(\frac{\underline{a}+\underline{b}}{2}\right)(\underline{b}-\underline{a}) = 0$$

$$(\underline{b}+\underline{a}).(\underline{b}-\underline{a}) = 0$$

$$b^2 - a^2 = 0 \ldots\text{(i)}$$

Similarly $\overrightarrow{OE} \perp \overrightarrow{BC}$

$\Rightarrow \overrightarrow{OE}.\overrightarrow{BC} = 0$

$$\left(\frac{\underline{b}+\underline{c}}{2}\right).(\underline{c}-\underline{b}) = 0$$

$$(\underline{c}+\underline{b}).(\underline{c}-\underline{b}) = 0$$

$$c^2 - b^2 = 0 \ldots\text{(ii)}$$

Adding (i) and (ii)

$$b^2 - a^2 + c^2 - b^2 = 0$$

$$c^2 - a^2 = 0$$

$$(\underline{c}+\underline{a}).(\underline{c}-\underline{a}) = 0$$

$$\left(\frac{\underline{c}+\underline{a}}{2}\right).(\underline{c}-\underline{a}) = 0$$

$$\overrightarrow{OF}.\overrightarrow{AC} = 0$$

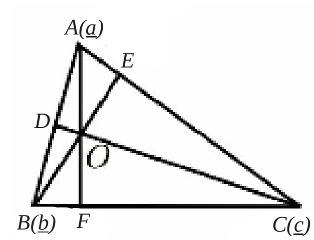
$$\overrightarrow{OF} \perp \overrightarrow{AC}$$

So the right bisectors of the sides of a triangle are concurrent.

Q.9 Prove that the altitude of a triangle are concurrent.

Solution:

Let $A(\underline{a}), B(\underline{b})$ and $C(\underline{c})$ be the vertices of ΔABC



$\overrightarrow{CD}, \overrightarrow{BE}$ and $\overrightarrow{AF}$ be the altitudes along sides $\overrightarrow{AB}, \overrightarrow{CA}$ and $\overrightarrow{BC}$

Now $\overrightarrow{CD} \perp \overrightarrow{AB} \quad \Rightarrow \overrightarrow{CO} \perp \overrightarrow{AB}$

$\therefore \overrightarrow{CO}.\overrightarrow{AB} = 0$

$-\underline{c}.(\underline{b} - \underline{a}) = 0$

$\Rightarrow \underline{c}.(\underline{b} - \underline{a}) = 0$

$\underline{b}.\underline{c} - \underline{a}.\underline{c} = 0$

$\underline{b}.\underline{c} = \underline{a}.\underline{c} \ldots$(i)

Also $\overrightarrow{BE} \perp \overrightarrow{CA} \quad \Rightarrow \overrightarrow{BO} \perp \overrightarrow{CA}$

$\therefore \overrightarrow{BO}.\overrightarrow{CA} = 0$

$-\underline{b}.(\underline{a} - \underline{c}) = 0$

$\Rightarrow \underline{b}.(\underline{a} - \underline{c}) = 0$

$\underline{a}.\underline{b} - \underline{b}.\underline{c} = 0$

$\underline{a}.\underline{b} = \underline{b}.\underline{c} \ldots$ (ii)

From (i) and (ii)

$\underline{a}.\underline{b} = \underline{a}.\underline{c}$

$\underline{a}.\underline{b} - \underline{a}.\underline{c} = 0$

$-\underline{a}.(\underline{c} - \underline{b}) = 0$

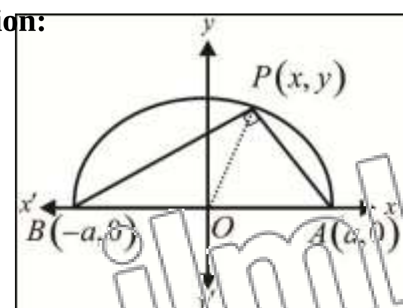
$\overrightarrow{AO}.\overrightarrow{BC} = 0$

$\Rightarrow \overrightarrow{AO} \perp \overrightarrow{BC} \Rightarrow \overrightarrow{AF} \perp \overrightarrow{BC}$

Thus the altitude of a triangle are concurrent.

Q.10 Prove that the angle in a semi circle is a right angle.

Solution:



Let AOB be semicircle with radius a and centre at origin O. whereas $x-axis$ is taken along the line AB.

Let $P(x, y)$ be any point on semicircle. Join A and B with P.

Also join O and P.

Now $\quad \overrightarrow{OA} = a\underline{i} \quad \overrightarrow{OB} = -a\underline{i}$

$\overrightarrow{OP} = x\underline{i} + y\underline{j} \Rightarrow \left|\overrightarrow{OP}\right| = \sqrt{x^2 + y^2}$

Also $\left|\overrightarrow{OP}\right| = a$

$\Rightarrow \left|\overrightarrow{OP}\right|^2 = a^2$

$x^2 + y^2 = a^2 \ldots$(i)

$\overrightarrow{AP} = (x\underline{i} + y\underline{j}) - (a\underline{i})$

$= (x - a)\underline{i} + y\underline{j}$

$\overrightarrow{BP} = (x\underline{i} + y\underline{j}) - (-a\underline{i})$

$= (x + a)\underline{i} + y\underline{j}$

$\overrightarrow{AP}.\overrightarrow{BP} = \left((x-a)\underline{i} + y\underline{j}\right)\left((x+a)\underline{i} + y\underline{j}\right)$

$= (x-a)(x+a) + y^2$

$= x^2 - a^2 + y^2$

$= x^2 + y^2 - a^2$

$a^2 - a^2 = 0$ from (i)

$\overrightarrow{AP}.\overrightarrow{BP} = 0$

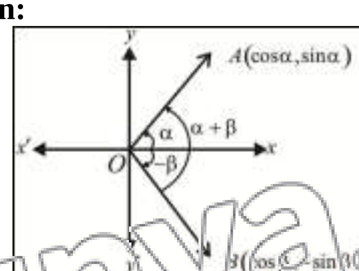
$\overrightarrow{AP} \perp \overrightarrow{BP}$

$\Rightarrow m\angle APB = 90°$

Q.11 Prove that

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Solution:



Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are unit vectors in xy-plane making angles α and $-\beta$ with the positive $x-axis$ respectively.

so that $\angle AOB = \alpha + \beta$

then $\overrightarrow{OA} = \cos\alpha\,\underline{i} + \sin\alpha\,\underline{j}$

and $\overrightarrow{OB} = \cos\beta\,\underline{i} - \sin\beta\,\underline{j}$

$\overrightarrow{OB}.\overrightarrow{OA} = \left(\cos\beta\,\underline{i} - \sin\beta\,\underline{j}\right).\left(\cos\alpha\,\underline{i} + \sin\alpha\,\underline{j}\right)$

$$\left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right|\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

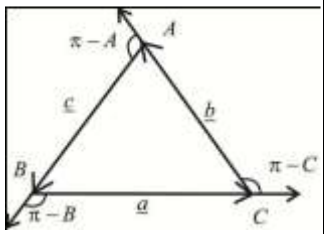
$$\boxed{\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta}\quad \left|\overrightarrow{OA}\right|=\left|\overrightarrow{OB}\right|=1$$

Q.12 Prove that in any triangle ABC.

(i) $b=c\cos A+a\cos C$

Solution:

Let the vector $\underline{a},\underline{b}$ and $\underline{c}$ are along the sides BC, CA and AB of the triangle ABC



$\therefore \underline{a}+\underline{b}+\underline{c}=0$

$\underline{b}=-(\underline{a}+\underline{c})$

Now taking dot product with $\underline{b}$

$\underline{b}.\underline{b}=-(\underline{a}+\underline{c}).\underline{b}$

$b^2=-(a.\underline{b}+\underline{c}.\underline{b})$

$=-\underline{a}.\underline{b}-\underline{c}.\underline{b}$

$=-ab\cos(\pi-C)-bc\cos(\pi-A)$

$=-ab(-\cos C)-bc(-\cos A)$

$b^2=ab\cos C+bc\cos A$

$$\boxed{b=a\cos C+c\cos A}$$

(ii) $c=a\cos B+b\cos A$

Solution:

Let the vectors $\underline{a},\underline{b}$ and $\underline{c}$ are along the sides BC, CA and AB of the triangle ABC

$\therefore \underline{a}+\underline{b}+\underline{c}=0$

$\underline{c}=-\underline{a}-\underline{b}$

Now taking dot product with $\underline{c}$

$\underline{c}.\underline{c}=-(\underline{a}+\underline{b}).\underline{c}$

$c^2=-(\underline{a}.c+\underline{b}.\underline{c})$

$=-\underline{a}.c-\underline{b}.\underline{c}$

$=-ac\cos(\pi-B)-bc\cos(\pi-A)$

$=-ac\cos(\pi-B)-bc\cos(\pi-A)$

$=-ac(-\cos B)-bc(-\cos A)$

$c^2=ac\cos B+bc\cos A$

$$\boxed{c=a\cos B+b\cos A}$$

(iii) $b^2=c^2+a^2-2ca\cos B$

Solution:

Let the vectors $\underline{a},\underline{b}$ and $\underline{c}$ along the Sides BC, CA and AB of triangle ABC.

$\Rightarrow \underline{a}+\underline{b}+\underline{c}=0$

$\underline{b}=-(\underline{a}+\underline{c})$

$\underline{b}.\underline{b}=-(\underline{a}+\underline{c}).\underline{b}$

$b^2=(-(\underline{a}+\underline{c}).(-(\underline{a}+\underline{c}))$

$=(\underline{a}+\underline{c}).(\underline{a}+\underline{c})$

$=\underline{a}.\underline{a}+\underline{a}.\underline{c}+\underline{c}.\underline{a}+\underline{c}.\underline{c}$

$=a^2+2\underline{a}.\underline{c}+c^2$

$=a^2+c^2+2\underline{c}.\underline{a}$

$=a^2+c^2+2ca\cos(\pi-B)$

$$\boxed{b^2=a^2+c^2-2ca\cos B}$$

(iv) $c^2=a^2+b^2-2ab\cos C$

Let $\underline{a},\underline{b}$ and $\underline{c}$ are vectors along sides $\overrightarrow{BC},\overrightarrow{CA}$ and $\overrightarrow{AB}$ of triangle ABC

$\underline{a}+\underline{b}+\underline{c}=0$

$\underline{c}=-\underline{a}-\underline{b}$

$\underline{c}=-(\underline{a}+\underline{b})$

$\underline{c}.\underline{c}=(-(\underline{a}+\underline{b})).(-(\underline{a}+\underline{b}))$

$c^2=(\underline{a}+\underline{b}).(\underline{a}+\underline{b})$

$=\underline{a}.\underline{a}+\underline{a}.\underline{b}+\underline{b}.\underline{a}+\underline{b}.\underline{b}$

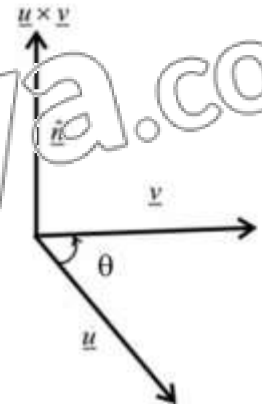
$=a^2+2\underline{a}.\underline{b}+\underline{b}$

$=a^2+2\underline{a}.\underline{b}+b^2$

$=a^2+b^2+2ab\cos(\pi-C)$

$$\boxed{c^2=a^2+b^2-2ab\cos C}$$

The Cross Product or Vectors Product of Two Vector:



(i) Definition 1:

The vector or cross product of two vectors $\underline{u}$ and $\underline{v}$ in space is

$$\underline{u}\times\underline{v}=|\underline{u}|\,|\underline{v}|\sin\theta\,\underline{n}$$

where θ is the angle between $\underline{u}$ and $\underline{v}$ and $0\le\theta\le\pi$. $\underline{n}$ is a unit vector perpendicular to the plane of $\underline{u}$ and $\underline{v}$ with direction given by the "right hand rule" stated below.

"If the fingers of right hand are curled in a direction from $\underline{u}$ towards $\underline{v}$, then the thumb will point in the direction of $\underline{n}$ which is $\underline{u}\times\underline{v}$. It is important to note that $\underline{u}\times\underline{v}\ne\underline{v}\times\underline{u}$, rather $\underline{u}\times\underline{v}=-(\underline{v}\times\underline{u})$.

(ii) The Unit Vectors $\underline{i},\underline{j},\underline{k}$:

(a)

$$\underline{i}\times\underline{i}=|\underline{i}|\,|\underline{i}|\sin 0^o\,\underline{n}=(1)(1)(0)\underline{n}=0 \quad \because \sin 0^o=0$$

$$\underline{j}\times\underline{j}=|\underline{j}|\,|\underline{j}|\sin 0^o\,\underline{n}=(1)(1)(0)\underline{n}=0$$

$$\underline{k}\times\underline{k}=|\underline{k}|\,|\underline{k}|\sin 0^o\,\underline{n}=(1)(1)(0)\underline{n}=0$$

(b)

$$\underline{i}\times\underline{j}=|\underline{i}|\,|\underline{j}|\sin 90^o\,\underline{k}=(1)(1)(1)\underline{k}=\underline{k} \quad \sin 90^o=1$$

$$\underline{j}\times\underline{k}=|\underline{j}|\,|\underline{k}|\sin 90^o\,\underline{i}=(1)(1)(1)\,\underline{i}=\underline{i}$$

$$\underline{k}\times\underline{i}=|\underline{k}|\,|\underline{i}|\sin 90^o\,\underline{j}=(1)(1)(1)\,\underline{j}=\underline{j}$$

Definition 2:

If $\underline{u}=a_1\underline{i}+b_1\underline{j}+c_1\underline{k}$ and $\underline{v}=a_2\underline{i}+b_2\underline{j}+c_2\underline{k}$ are two vectors in space, then cross product of $\underline{u}$ and $\underline{v}$ is :

$$\underline{u}\times\underline{v}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}$$

which is called the "Determinant formula" for $\underline{u}\times\underline{v}$.

Parallel Vectors:

If two vectors $\underline{u}$ and $\underline{v}$ are parallel, then

$$\underline{u}\times\underline{v}=|\underline{u}|\,|\underline{v}|\sin 0^o\,\underline{n}=|\underline{u}|\,|\underline{v}|(0)\,\underline{n}$$

$$\therefore\ \underline{u}\times\underline{v}=0$$

Angle between two Vectors:

The angle θ between two vectors $\underline{u}$ and $\underline{v}$ is

$$\underline{u}\times\underline{v}=|\underline{u}|\,|\underline{v}|\sin\theta^o\,\underline{n}$$

$$\therefore\ \frac{\underline{u}\times\underline{v}}{\underline{n}}=|\underline{u}|\,|\underline{v}|\sin\theta$$

$$\therefore\ |\underline{u}\times\underline{v}|=|\underline{u}|\,|\underline{v}|\sin\theta$$

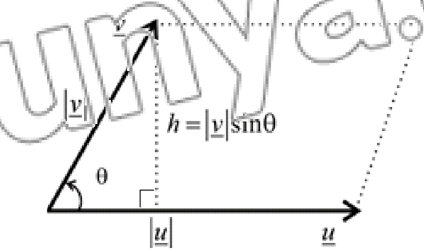
$$\therefore \quad \sin\theta = \frac{|\underline{u}\times\underline{v}|}{|\underline{u}|\,|\underline{v}|}$$

Area of Parallelogram:

Area of Parallelogram $= Base \times Height$

$= |\underline{u}|\,|\underline{v}|\sin\theta$

Area of Parallelogram $= |\underline{u}\times\underline{v}|$

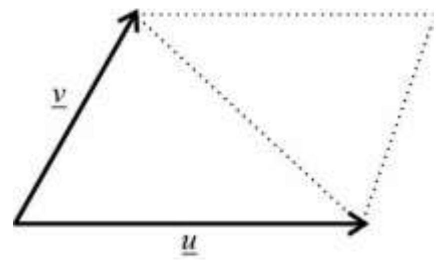


Area of a Triangle:

From the figure, it is clear Area of triangle

Area of Triangle $= \frac{1}{2}$ (Area of Parallelogram)

$= \frac{1}{2}|u\times v|$ where $\underline{u}$ and $\underline{v}$ are vectors along to adjacent sides of triangle.



EXERCISE 7.4

Q.1 Compute the cross product $\underline{a}\times\underline{b}$ and $\underline{b}\times\underline{a}$. Check your answer by showing that each $\underline{a}$ and $\underline{b}$ is perpendicular to $\underline{a}\times\underline{b}$ and $\underline{b}\times\underline{a}$.

(i) $\underline{a} = 2\underline{i}+\underline{j}-\underline{k}$, $\underline{b} = \underline{i}-\underline{j}+\underline{k}$

Solution:

$\underline{a} = 2\underline{i}+\underline{j}-\underline{k}$, $\underline{b} = \underline{i}-\underline{j}+\underline{k}$

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$= \underline{i}(1-1) - \underline{j}(2+1) + \underline{k}(-2-1)$

$= 0\underline{i} - 3\underline{j} - 3\underline{k}$

Now

$\underline{a}.(\underline{a}\times\underline{b}) = (2\underline{i}+\underline{j}-\underline{k}).(0\underline{i}-3\underline{j}-3\underline{k})$

$= 0-3+3$

$= 0$

$\Rightarrow \quad \underline{a}\perp\underline{a}\times\underline{b}$

Also

$\underline{b}.(\underline{a}\times\underline{b}) = (\underline{i}-\underline{j}+\underline{k}).(0\underline{i}-3\underline{j}-3\underline{k})$

$= (1)(0)+(-1)(-3)+(1)(-3)$

$= 0+3-3 = 0$

$\Rightarrow \quad \underline{b}\perp\underline{a}\times\underline{b}$

Thus $\underline{a}\times\underline{b}$ is perpendicular to both vectors $\underline{a}$ and $\underline{b}$.

$$\underline{b}\times\underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$= \underline{i}(1-1) - \underline{j}(-1-2) + \underline{k}(1+2)$

$= 0\underline{i} + 3\underline{j} + 3\underline{k}$

Now

$\underline{a}.(\underline{b}\times\underline{a}) = (2\underline{i}+\underline{j}-\underline{k}).(0\underline{i}+3\underline{j}+3\underline{k})$

$= 2(0)+(1)(3)+(-1)(3)$

$= 3-3$

$= 0$

$\Rightarrow \quad \underline{a}\perp\underline{b}\times\underline{a}$

Now

$\underline{b}.(\underline{b}\times\underline{a}) = (\underline{i}-\underline{j}+\underline{k}).(0\underline{i}+3\underline{j}+3\underline{k})$

$= (1)(0)+(-1)(3)+(1)(3)$

$= 0-3+3 = 0$

Thus $\underline{b}\times\underline{a}$ is perpendicular to both the vectors $\underline{a}$ and $\underline{b}$.

(ii) $\underline{a} = \underline{i} + \underline{j}$, $\underline{b} = \underline{i} - \underline{j}$

Solution:

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= \underline{i}(0-0) - \underline{j}(0-0) + \underline{k}(-1-1)$$

$$= -2\underline{k}$$

$$\underline{a}.(\underline{a}\times\underline{b}) = (\underline{i}+\underline{j}).(-2\underline{k}) = 0$$

Also $\underline{b}.(\underline{a}\times\underline{b}) = (\underline{i}-\underline{j}).(-2\underline{k}) = 0$

Thus $\underline{a}\times\underline{b}$ is perpendicular to both vector $\underline{a}$ and $\underline{b}$.

$$\underline{b}\times\underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= \underline{i}(0) - \underline{j}(0) + \underline{k}(1+1)$$

$$= 2\underline{k}$$

$$\underline{a}.(\underline{b}\times\underline{a}) = (\underline{i}+\underline{j}).(2\underline{k})$$

Also $\underline{b}.(\underline{b}\times\underline{a}) = (\underline{i}-\underline{j}).(2\underline{k}) = 0$

Thus $\underline{b}\times\underline{a}$ is perpendicular to both the vectors $\underline{a}$ and $\underline{b}$.

(iii) $\underline{a} = 3\underline{i} - 2\underline{j} + \underline{k}$, $\underline{b} = \underline{i} + \underline{j}$

Solution:

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -2 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= \underline{i}(0-1) - \underline{j}(0-1) + \underline{k}(3+2)$$

$$= -\underline{i} + \underline{j} + 5\underline{k}$$

$$\underline{a}.(\underline{a}\times\underline{b}) = (3\underline{i} - 2\underline{j} + \underline{k}).(-\underline{i} + \underline{j} + 5\underline{k})$$

$$= 3(-1) + (-2)(1) + 1(5)$$

$$= 0$$

$\Rightarrow \quad \underline{a} \perp \underline{a}\times\underline{b}$

$$\underline{b}.(\underline{a}\times\underline{b}) = (\underline{i}+\underline{j}).(-\underline{i} + \underline{j} - 5\underline{k})$$

$$= (1)(-1) + (1)(1)$$

$$= 0$$

$\Rightarrow \quad \underline{b} \perp \underline{a}\times\underline{b}$

Thus $\underline{a}\times\underline{b}$ is perpendicular to both the vectors $\underline{a}$ and $\underline{b}$.

$$\underline{b}\times\underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 3 & -2 & 1 \end{vmatrix}$$

$$= \underline{i}(1-0) - \underline{j}(1-0) + \underline{k}(-2-3)$$

$$= \underline{i} - \underline{j} - 5\underline{k}$$

$$\underline{a}.(\underline{b}\times\underline{a}) = (3\underline{i} - 2\underline{j} + \underline{k}).(\underline{i} - \underline{j} - 5\underline{k})$$

$$= 3(1) + (2)(-1) + (1)(-5)$$

$$= 3 + 2 - 5$$

$$= 0$$

$\Rightarrow \quad \underline{a} \perp \underline{b}\times\underline{a}$

$$\underline{b}.(\underline{b}\times\underline{a}) = (\underline{i}+\underline{j}).(\underline{i} - \underline{j} - 5\underline{k})$$

$$= (1)(1) + (1)(-1) + 0(-5)$$

$$= 1 - 1 = 0$$

$\Rightarrow \quad \underline{b} \perp \underline{b}\times\underline{a}$

Thus $\underline{b}\times\underline{a}$ is perpendicular to both the vectors $\underline{a}$ and $\underline{b}$.

(iv) $\underline{a} = -4\underline{i} + \underline{j} - 2\underline{k}$, $\underline{b} = 2\underline{i} + \underline{j} + \underline{k}$

Solution:

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -4 & 1 & -2 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= \underline{i}(1+2) - \underline{j}(-4+4) + \underline{k}(-4-2)$$

$$= 3\underline{i} - 0\underline{j} - 6\underline{k}$$

$$\underline{a}.(\underline{a}\times\underline{b}) = (-4\underline{i} + \underline{j} - 2\underline{k}).(3\underline{i} - 6\underline{k})$$

$$= (-4)(3) + 0 + (-2)(-6) = 0$$

$\Rightarrow \quad \underline{a} \perp \underline{a}\times\underline{b}$

Now

$$\underline{b}.(\underline{a}\times\underline{b}) = \left(2\underline{i} + \underline{j} + \underline{k}\right).\left(3\underline{i} + 0\underline{j} - 6\underline{k}\right)$$

$$= (2)(3) + (1)(0) + (1)(-6)$$

$$= 6 + 0 - 6$$

$$= 0$$

$\Rightarrow \quad \underline{b} \perp \underline{a}\times\underline{b}$

Thus $\underline{a}\times\underline{b}$ is perpendicular to both the vectors $\underline{a}$ and $\underline{b}$.

$$\underline{b}\times\underline{a}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 2 & 1 & 1\\ -4 & 1 & -2\end{vmatrix}$$

$= \underline{i}(-2-1)-\underline{j}(-4+4)+\underline{k}(2+4)$

$= -3\underline{i}+6\underline{k}$

Now

$\underline{a}.(\underline{b}\times\underline{a}) = (-4\underline{i}+\underline{j}-2\underline{k}).(-3\underline{i}+0\underline{j}+6\underline{k})$

$= (-4)(-3)+1(0)+(-2)(6)$

$= 0$

$\Rightarrow \quad \underline{a}\perp\underline{b}\times\underline{a}$

Also $\underline{b}.(\underline{b}\times\underline{a}) = (2\underline{i}+\underline{j}+\underline{k}).(-3\underline{i}+0\underline{j}+6\underline{k})$

$= (2)(-3)+1(0)+(1)(6)$

$= 0$

$\Rightarrow \quad \underline{b}\perp\underline{b}\times\underline{a}$

Thus $\underline{b}\times\underline{a}$ is perpendicular to both the vectors $\underline{a}$ and $\underline{b}$.

Q.2 Find a unit vector perpendicular to the plane containing $\underline{a}$ and $\underline{b}$. Also find sine of the angle between them.

(i) $\underline{a}=2\underline{i}-6\underline{j}-3\underline{k}$, $\underline{b}=4\underline{i}+3\underline{j}-\underline{k}$

Solution:

$\underline{a}=2\underline{i}-6\underline{j}-3\underline{k}$, $\underline{b}=4\underline{i}+3\underline{j}-\underline{k}$

$$\underline{a}\times\underline{b}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 2 & -6 & -3\\ 4 & 3 & 1\end{vmatrix}$$

$= \underline{i}(6+9)-\underline{j}(-2+12)+\underline{k}(6+24)$

$= 15\underline{i}-10\underline{j}+30\underline{k}$

$|\underline{a}\times\underline{b}| = |15\underline{i}-10\underline{j}+30\underline{k}|$

$= \sqrt{(15)^2+(-10)^2+(30)^2}$

$= \sqrt{225+100+900}$

$= \sqrt{1225}$

$= 35$

Let $\hat{n}$ is unit vector

$$\hat{n}=\frac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|}=\frac{15\underline{i}-10\underline{j}+30\underline{k}}{35}$$

$$\hat{n}=\frac{3}{7}\underline{i}-\frac{2}{7}\underline{j}+\frac{6}{7}\underline{k}$$

$$\sin\theta=\frac{|\underline{a}\times\underline{b}|}{|\underline{a}||\underline{b}|}$$

$$=\frac{35}{\sqrt{(2)^2+(-6)^2+(-3)^2}\sqrt{(4)^2+(3)^2+(-1)^2}}$$

$$=\frac{35}{\sqrt{4+36+9}\sqrt{16+9+1}}$$

$$=\frac{35}{7\sqrt{26}}$$

$$\boxed{\sin\theta=\frac{5}{\sqrt{26}}}$$

(ii) $\underline{a}=-\underline{i}-\underline{j}-\underline{k}$, $\underline{b}=2\underline{i}-3\underline{j}+4\underline{k}$

Solution:

$\underline{a}=-\underline{i}-\underline{j}-\underline{k}$, $\underline{b}=2\underline{i}-3\underline{j}+4\underline{k}$

$$\underline{a}\times\underline{b}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ -1 & -1 & -1\\ 2 & -3 & 4\end{vmatrix}$$

$= \underline{i}(-4-3)-\underline{j}(-4+2)+\underline{k}(3+2)$

$= -7\underline{i}+2\underline{j}+5\underline{k}$

$|\underline{a}\times\underline{b}| = \sqrt{(-7)^2+(2)^2+(5)^2}$

$= \sqrt{49+4+25}$

$= \sqrt{78}$

Let $\hat{n}$ is unit vector.

$$\hat{n}=\frac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|}=\frac{-7\underline{i}+2\underline{j}-5\underline{k}}{\sqrt{78}}$$

$$\hat{n}=\frac{-7}{\sqrt{78}}\underline{i}+\frac{2}{\sqrt{78}}\underline{j}\frac{5}{\sqrt{78}}\underline{k}$$

$$\sin\theta=\frac{|\underline{a}\times\underline{b}|}{|\underline{a}||\underline{b}|}$$

$$=\frac{\sqrt{78}}{\sqrt{1+1+1}\sqrt{4+9+16}}$$

$$=\frac{\sqrt{78}}{\sqrt{3}\sqrt{29}}$$

$$=\frac{\sqrt{3}\sqrt{26}}{\sqrt{3}\sqrt{29}}$$

$$\boxed{\sin\theta=\frac{\sqrt{26}}{\sqrt{29}}}$$

(iii) $\underline{a}=2\underline{i}-2\underline{j}+4\underline{k}$, $\underline{b}=-\underline{i}+\underline{j}-2\underline{k}$

Solution:

$\underline{a}=2\underline{i}-2\underline{j}+4\underline{k}$, $\underline{b}=-\underline{i}+\underline{j}-2\underline{k}$

$$\underline{a}\times\underline{b}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 2 & -2 & 4\\ -1 & 1 & -2\end{vmatrix}$$

$=\underline{i}(4-4)-\underline{j}(-4+4)+\underline{k}(2-2)$

$=0\underline{i}+0\underline{j}+0\underline{k}$

$|\underline{a}\times\underline{b}|=0$

It is not possible to find out required unit vector.

$$\sin\theta=\frac{|\underline{a}\times\underline{b}|}{|\underline{a}||\underline{b}|}$$

$$=\frac{0}{\sqrt{(2)^2+(-2)^2+(4)^2}\sqrt{(-1)^2+(1)^2+(-2)^2}}$$

$$=\frac{0}{\sqrt{24}\sqrt{6}}=0$$

$\boxed{\sin\theta=0^\circ}$

(iv) $\underline{a}=\underline{i}+\underline{j}$, $\underline{b}=\underline{i}-\underline{j}$

$$\underline{a}\times\underline{b}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 1 & 1 & 0\\ 1 & -1 & 0\end{vmatrix}$$

$=\underline{i}(0-0)-\underline{j}(0-0)+\underline{k}(-1-1)$

$=-2\underline{k}$

$|\underline{a}\times\underline{b}|=|-2\underline{k}|$

$=2$

Let $\hat{n}$ is unit vector

$$\hat{n}=\frac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|}=\frac{-2\underline{k}}{2}=-\underline{k}$$

$$\sin\theta=\frac{|\underline{a}\times\underline{b}|}{|\underline{a}||\underline{b}|}$$

$$=\frac{2}{\sqrt{1+1}\sqrt{1+1}}$$

$$=\frac{2}{\sqrt{2}\sqrt{2}}$$

$$=\frac{2}{2}$$

$\boxed{\sin\theta=1}$

Q.3 Find the area of the triangle, determined by the point P,Q and R.

(i) $P(0,0,0)$, $Q(2,3,2)$ $R(-1,1,4)$

Solution:

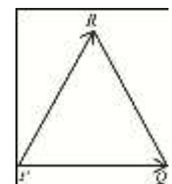
$P(0,0,0)$, $Q(2,3,2)$ $R(-1,1,4)$

$\overrightarrow{PQ}=(2\underline{i}+3\underline{j}+2\underline{k})-(0\underline{i}+0\underline{j}+0\underline{k})$

$=2\underline{i}+3\underline{j}+2\underline{k}$

$\overrightarrow{PR}=(-\underline{i}+\underline{j}+4\underline{k})-(0\underline{i}+0\underline{j}+0\underline{k})$

$=-\underline{i}+\underline{j}+4\underline{k}$



$$\text{Now } \overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 2 & 3 & 2\\ -1 & 1 & 4\end{vmatrix}$$

$=\underline{i}(12-2)-\underline{j}(8+2)+\underline{k}(2+3)$

$=10\underline{i}-10\underline{j}+5\underline{k}$

$|\overrightarrow{PQ}\times\overrightarrow{PR}|=\sqrt{(10)^2+(-10)^2+(5)^2}$

$=\sqrt{100+100+25}$

$=\sqrt{225}$

$=15$

$\therefore$ Area of triangle PQR is

$$\Delta=\frac{1}{2}|\overrightarrow{PQ}\times\overrightarrow{PR}|$$

$$=\frac{1}{2}(15)=\frac{15}{2}\text{ sq. unit}$$

(ii) $P(1,-1,-1)$, $Q(2,0,-1)$, $R(0,2,1)$

Solution:

$\overrightarrow{PQ}=(2\underline{i}+0\underline{j}-\underline{k})-(\underline{i}-\underline{j}-\underline{k})$

$=\underline{i}+\underline{j}+0\underline{k}$

$\overrightarrow{PR}=(0\underline{i}+2\underline{j}+1\underline{k})-(\underline{i}-\underline{j}-\underline{k})$

$=-\underline{i}+3\underline{j}+2\underline{k}$

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 1 & 1 & 0\\ -1 & 3 & 2\end{vmatrix}$$

$=\underline{i}(2-0)-\underline{j}(2-0)+\underline{k}(3+1)$

$=2\underline{i}-2\underline{j}+4\underline{k}$

$|\overrightarrow{PQ}\times\overrightarrow{PR}|=|2\underline{i}-2\underline{j}+4\underline{k}|$

$$= \sqrt{(2)^2 + (-2)^2 + (4)^2}$$
$$= \sqrt{4+4+16}$$
$$= \sqrt{24}$$
$$= 2\sqrt{6}$$

Area of triangle PQR is

$$\Delta = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|$$
$$= \frac{1}{2}(2\sqrt{6})$$
$$= \sqrt{6} \text{ sq. unit}$$

Q.4 Find the area of parallelogram, whose vertices are:

(i) $A(0,0,0)$, $B(1,2,3)$, $C\ (2,-1,1)$, $D\ (3,1,4)$

Solution:

$$\overrightarrow{AB} = \left(\underline{i} + 2\underline{j} + 3\underline{k}\right) - \left(0\underline{i} + 0\underline{j} + 0\underline{k}\right)$$
$$= \underline{i} + 2\underline{j} + 3\underline{k}$$
$$\overrightarrow{AC} = \left(2\underline{i} - \underline{j} + \underline{k}\right) - \left(0\underline{i} + 0\underline{j} + 0\underline{k}\right)$$
$$= 2\underline{i} - \underline{j} + \underline{k}$$
$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix}$$
$$= \underline{i}(2+3) - \underline{j}(1-6) + \underline{k}(-1-4)$$
$$= 5\underline{i} + 5\underline{j} - 5\underline{k}$$

Area of parallelogram $ABCD$ is

$$\Delta = \left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$
$$= \sqrt{(5)^2 + (5)^2 + (5)^2}$$
$$= \sqrt{25+25+25}$$
$$= \sqrt{75}$$
$$= 5\sqrt{3} \text{ sq. unit}$$

(ii) $A(1,2,-1)$, $B(4,2,-3)$, $C(6,-5,2)$, $D\ (9,-5,0)$

Solution:

$$\overrightarrow{AB} = \left(4\underline{i} + 2\underline{j} - 3\underline{k}\right) - \left(\underline{i} + 2\underline{j} - \underline{k}\right)$$
$$= 3\underline{i} + 0\underline{j} - 2\underline{k}$$
$$\overrightarrow{AC} = \left(6\underline{i} - 5\underline{j} + 2\underline{k}\right) - \left(\underline{i} + 2\underline{j} - \underline{k}\right)$$
$$= 5\underline{i} - 7\underline{j} + 3\underline{k}$$
$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 0 & -2 \\ 5 & -7 & 3 \end{vmatrix}$$
$$= \underline{i}(0-14) - \underline{j}(9+10) + \underline{k}(-21-0)$$
$$= -14\underline{i} - 19\underline{j} - 21\underline{k}$$

Area of parallelogram $ABCD$ is

$$\Delta = \left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$
$$= \sqrt{(-14)^2 + (-19)^2 + (-21)^2}$$
$$= \sqrt{196+361+441}$$
$$= \sqrt{998} \text{ sq. unit}$$

(iii) $A(-1,1,1)$, $B(-1,2,2)$, $C(-3,4,-5)$, $D(-3,5,-4)$

Solution:

$$\overrightarrow{AB} = \left(-\underline{i} + 2\underline{j} + 2\underline{k}\right) - \left(-\underline{i} + \underline{j} + \underline{k}\right)$$
$$= 0\underline{i} + \underline{j} + \underline{k}$$
$$\overrightarrow{AC} = \left(-3\underline{i} + 4\underline{j} - 5\underline{k}\right) - \left(-\underline{i} + \underline{j} + \underline{k}\right)$$
$$= -2\underline{i} + 3\underline{j} - 6\underline{k}$$
$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 1 & 1 \\ -2 & 3 & -6 \end{vmatrix}$$
$$= \underline{i}(-6-3) - \underline{j}(0+2) + \underline{k}(0+2)$$
$$= -9\underline{i} - 2\underline{j} + 2\underline{k}$$

Area of parallelogram $ABCD$ is

$$\Delta = \left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$
$$= \sqrt{(-9)^2 + (-2)^2 + (2)^2}$$
$$= \sqrt{81+4+4}$$
$$= \sqrt{89} \text{ sq. unit}$$

Q.5 Which vectors, if any, are perpendicular or parallel.

(i) $\underline{u}=5\underline{i}-\underline{j}+\underline{k}$; $\underline{v}=\underline{j}-5\underline{k}$, $\underline{w}=-15\underline{i}+3\underline{j}-3\underline{k}$

Solution:

$\underline{w}=-15\underline{i}+3\underline{j}-3\underline{k}$

$=-3(5\underline{i}-\underline{j}+\underline{k})$

$=-3\underline{u}$

$\underline{u}$ and $\underline{w}$ are parallel

(ii) $\underline{u}=\underline{i}+2\underline{j}-\underline{k}$, $\underline{v}=-\underline{i}+\underline{j}+\underline{k}$, $\underline{w}=-\frac{\pi}{2}\underline{i}-\pi\underline{j}+\frac{\pi}{2}\underline{k}$

Solution:

$\underline{u}.\underline{v}=(\underline{i}+2\underline{j}-\underline{k}).(-\underline{i}+\underline{j}+\underline{k})$

= (1) (–1) + 2(1) + (1) (1)

= –1 + 2–1

= 0

The vectors $\underline{u}$ and $\underline{v}$ are perpendicular

$\underline{v}.\underline{w}=(-\underline{i}+\underline{j}+\underline{k}).\left(\frac{-\pi}{2}\underline{i}-\pi\underline{j}+\frac{\pi}{2}\underline{k}\right)$

$=\frac{\pi}{2}-\pi+\frac{\pi}{2}$

= 0

The vectors $\underline{v}$ and $\underline{w}$ are perpendicular

Q.6 Prove that

$\underline{a}\times(\underline{b}+\underline{c})+\underline{b}\times(\underline{c}+\underline{a})+\underline{c}\times(\underline{a}+\underline{b})=0$

Solution:

L.H.S $=\underline{a}\times(\underline{b}+\underline{c})+\underline{b}\times(\underline{c}+\underline{a})+\underline{c}\times(\underline{a}+\underline{b})$

$=\underline{a}\times\underline{b}+\underline{a}\times\underline{c}+\underline{b}\times\underline{c}+\underline{b}\times\underline{a}+\underline{c}\times\underline{a}+\underline{c}\times\underline{b}$

$=\underline{a}\times\underline{b}+\underline{a}\times\underline{c}+\underline{b}\times\underline{c}+\left(-(\underline{a}\times\underline{b})\right)+\left(-(\underline{a}\times\underline{c})\right)+\left(-(\underline{b}\times\underline{c})\right)$

$=\underline{a}\times\underline{b}+\underline{a}\times\underline{c}+\underline{b}\times\underline{c}-\underline{a}\times\underline{b}-\underline{a}\times\underline{c}-\underline{b}\times\underline{c}$

= 0

= R.H.S

Q.7 If $=\underline{a}+\underline{b}+\underline{c}=0$**, then prove that** $\underline{a}\times\underline{b}=\underline{b}\times\underline{c}=\underline{c}\times\underline{a}$

Solution:

$\underline{a}+\underline{b}+\underline{c}=0$

Taking cross product with $\underline{a}$

$\underline{a}\times(\underline{a}+\underline{b}+\underline{c})=0$

$\underline{a}\times\underline{a}+\underline{a}\times\underline{b}+\underline{a}\times\underline{c}=0$

$0+\underline{a}\times\underline{b}-(\underline{c}\times\underline{a})=0$

$\underline{a}\times\underline{b}=\underline{c}\times\underline{a}$ ...(i)

Taking cross product with $\underline{b}$

$\underline{b}\times(\underline{a}+\underline{b}+\underline{c})=0$

$\underline{b}\times\underline{a}+\underline{b}\times\underline{b}+\underline{b}\times\underline{c}=0$

$-\underline{a}\times\underline{b}+0+\underline{b}\times\underline{c}=0$

$\underline{b}\times\underline{c}=\underline{a}\times\underline{b}$ ...(ii)

From (i) and (ii)

$\boxed{\underline{a}\times\underline{b}=\underline{b}\times\underline{c}=\underline{c}\times\underline{a}}$

Q.8 Prove that $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

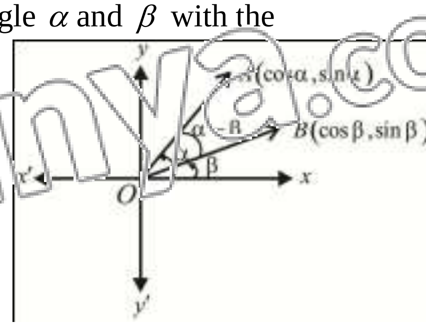
Solution:

Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are unit vectors in xy-plane making angle α and β with the positive $x-axis$ respectively.

So that $m\angle BOA=\alpha-\beta$

Now $\overrightarrow{OA}=\cos\alpha\,\underline{i}+\sin\alpha\,\underline{j}$

$\overrightarrow{OB}=\cos\beta\,\underline{i}+\sin\beta\,\underline{j}$



$\overrightarrow{OB}\times\overrightarrow{OA}=\left|\overrightarrow{OB}\right|\left|\overrightarrow{OA}\right|\sin(\alpha-\beta)\hat{k}$ $\therefore \overrightarrow{OB}\times\overrightarrow{OA}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ \cos\beta & \sin\beta & 0\\ \cos\alpha & \sin\alpha & 0\end{vmatrix}$

$$\left|\overrightarrow{OB}\right|\left|\overrightarrow{OA}\right|\sin(\alpha-\beta)\underline{k}=\underline{i}(0-0)-\underline{j}(0-0)+\underline{k}(\cos\beta\sin\alpha-\sin\beta\cos\alpha) \quad \therefore \left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right|=1$$

$$\sin(\alpha-\beta)\hat{k}=(\sin\alpha\cos\beta-\cos\alpha\sin\beta)\hat{k}$$

$$\boxed{\sin(\alpha-\beta)=(\sin\alpha\cos\beta-\cos\alpha\sin\beta)}$$

Q.9 If $\underline{a}\times\underline{b}=0$ and $\underline{a}.\underline{b}=0$, what conclusion can be drawn about $\underline{a}$ or $\underline{b}$?

Solution:

If $\underline{a}\times\underline{b}=0$ then $\underline{a}\parallel\underline{b}$

If $\underline{a}.\underline{b}=0$ then $\underline{a}$ and $\underline{b}$ are perpendicular, but it is only possible if either $\underline{a}=0$ or $b=0$ (null vector).

Scalar Triple Product:

For any three vectors $\underline{u},\underline{v}$ *and* $\underline{w}$, the dot product of one vector with cross product of remaining two vectors is called "Scalar Triple Product" of vectors $\underline{u},\underline{v}$ *and* $\underline{w}$. It is written as $\underline{u}.(\underline{v}\times\underline{w})$

If $\underline{u}=a_1\underline{i}+b_1\underline{j}+c_1\underline{k}$, $\underline{v}=a_2\underline{i}+b_2\underline{j}+c_2\underline{k}$ and $\underline{w}=a_3\underline{i}+b_3\underline{j}+c_3\underline{k}$ then

$$\underline{v}\times\underline{w}=\begin{vmatrix}\underline{i}&\underline{j}&\underline{k}\\a_2&b_2&c_2\\a_3&b_3&c_3\end{vmatrix}$$

$$=\underline{i}(b_2c_3-c_2b_3)-\underline{j}(a_2c_3-c_2a_3)+\underline{k}(a_2b_3-b_2a_3)$$

$$\underline{u}.(\underline{v}\times\underline{w})=(a_1\underline{i}+b_1\underline{j}+c_1\underline{k}).\left[(b_2c_3-c_2b_3)\underline{i}-(a_2c_3-c_2a_3)\underline{j}+(a_2b_3-b_2a_3)\underline{k}\right]$$

$$=a_1(b_2c_3-c_2b_3)-b_1(a_2c_3-c_2a_3)+c_1(a_2b_3-b_2a_3)$$

$$\underline{u}.(\underline{v}\times\underline{w})=\begin{vmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{vmatrix}$$

It is important to note that

$$\underline{u}.(\underline{v}\times\underline{w})=\underline{v}.(\underline{w}\times\underline{u})=\underline{w}.(\underline{u}\times\underline{v})$$

The Volume of the Parallelepiped:

The Scalar triple product i.e. $\underline{u}.(\underline{v}\times\underline{w})$ is **volume of a parallelepiped**. Hence it is a scalar.

The Volume of Tetrahedron:

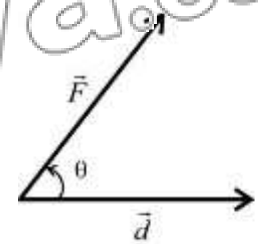
The volume of Tetrahedron $=\frac{1}{6}\ \underline{u}.(\underline{v}\times\underline{w})$

The properties of Scalar Triple Product:

(i) If $\underline{u},\underline{v}$ and $\underline{w}$ are coplanar then the volume of the parallelepiped is zero that is $(\underline{u}\times\underline{v}).\underline{w}=0$

(ii) If any two vector of scalar triple product are equal, then its values is zero i.e $\left[\underline{u}\ \underline{v}\ \underline{v}\right]=0$

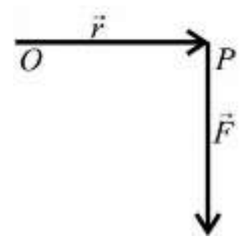
Work done by a Force:



If a constant Force $\vec{F}$ acts on a body, at an angle θ to the direction of motion, then work done by $\vec{F}$ is define to the product of the component of $\vec{F}$ in the direction of the displacement and the distance that the body moves.

Work done $=\underline{F}.\underline{d}=\left(\underline{F}\cos\theta\right)\underline{d}=\underline{F}\underline{d}\cos\theta$

Moment of a Force (Torque):



The turning effect produced by a force is called "Torque" or "Moment" of that Force.

Moment = Perpendicular distance between point of application of force and point of rotation $\times$ Force applied.

Moment of $\vec{F}$ about $O=\overrightarrow{OP}\times\vec{F}$

$=\vec{r}\times\vec{F}$

EXERCISE 7.5

Q.1 Find the volume of the parallelepiped for which the given vectors are three edges.

(i) $\underline{u}=3\underline{i}+2\underline{k}$, $\underline{v}=\underline{i}+2\underline{j}+\underline{k}$

$\underline{w}=-\underline{j}+4\underline{k}$

Solution:

Volume of parallelepiped is

$$\underline{u}.\left(\underline{v}\times\underline{w}\right)=\begin{vmatrix}3 & 0 & 2\\ 1 & 2 & 1\\ 0 & -1 & 4\end{vmatrix}$$

$=3\left(8+1\right)-0+2\left(-1-0\right)$

$=27-2$

=25 cubic unit

(ii) $\underline{u}=\underline{i}-4\underline{j}-\underline{k}$, $\underline{v}=\underline{i}-\underline{j}-2\underline{k}$,

$\underline{w}=2\underline{i}-3\underline{j}+\underline{k}$

Solution:

Volume of parallelepiped is

$$\underline{u}.\left(\underline{v}\times\underline{w}\right)=\begin{vmatrix}1 & -4 & -1\\ 1 & -1 & -2\\ -2 & -3 & 1\end{vmatrix}$$

= 1(–1–6) +4(1+4)–1(–3+2)

= –7+20+1

= 14 cubic unit

(iii) $\underline{u}=\underline{i}-2\underline{j}+3\underline{k}$, $\underline{v}=2\underline{i}-\underline{j}-\underline{k}$

$\underline{w}=\underline{j}+\underline{k}$

Solution:

Volume of parallelepiped is

$$\underline{u}.\left(\underline{v}\times\underline{w}\right)=\begin{vmatrix}1 & -2 & 3\\ 2 & -1 & -1\\ 0 & 1 & 1\end{vmatrix}$$

= 1(–1+1)+2(2+0)+3(2+0)

= 0+4+6

= 10 cubic unit

Q.2 Verify that $\underline{a}.\underline{b}\times\underline{c}=\underline{b}.\underline{c}\times\underline{a}=\underline{c}.\underline{a}\times\underline{b}$ **if** $\underline{a}=3\underline{i}-\underline{j}+5\underline{k}$, $\underline{b}=4\underline{i}+3\underline{j}-2\underline{k}$ **and** $\underline{c}=2\underline{i}+5\underline{j}+\underline{k}$

Solution:

$$\underline{a}.\underline{b}\times\underline{c}=\begin{vmatrix}3 & -1 & 5\\ 4 & 3 & -2\\ 2 & 5 & 1\end{vmatrix}$$

= 3(3+10)+1(4+4)+5(20–6)

= 39+8+70

$= 117$

$$\underline{b}.\underline{c}\times\underline{a} = \begin{vmatrix} 4 & 3 & -2 \\ 2 & 5 & 1 \\ 3 & -1 & 5 \end{vmatrix}$$

$= 4(25+1)-3(10-3)-2(-2-15)$

$= 104-21+34$

$= 117$

$$\underline{c}.\underline{a}\times\underline{b} = \begin{vmatrix} 2 & 5 & 1 \\ 3 & -1 & 5 \\ 4 & 3 & -2 \end{vmatrix}$$

$= 2(2-15)-5(-6-20)+1(9+4)$

$= -26+130+13$

$= 117$

Hence $\underline{a}.\underline{b}\times\underline{c} = \underline{b}.\underline{c}\times\underline{a} = \underline{c}.\underline{a}\times\underline{b}$

Q.3 Prove that the vectors $\underline{i}-2\underline{j}+3\underline{k}$, $-2\underline{i}+3\underline{j}-4\underline{k}$ and $\underline{i}-3\underline{j}+5\underline{k}$ are coplanar.

Solution:

$\underline{a} = \underline{i}-2\underline{j}+3\underline{k}$, $\underline{b} = -2\underline{i}+3\underline{j}-4\underline{k}$

$\underline{c} = \underline{i}-3\underline{j}+5\underline{k}$

If vectors $\underline{a}, \underline{b}$ and $\underline{c}$ are coplanar, then we have to show that $\underline{a}.\underline{b}\times\underline{c} = 0$

$$\underline{a}.\underline{b}\times\underline{c} = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix}$$

$= 1(15-12)+2(-10+4)+3(6-3)$

$= 3-12+9 = 0$

Thus the given vectors are coplanar.

Q.4 Find the constant α such that the vectors are coplanar.

(i) $\underline{i}-\underline{j}+\underline{k}$, $\underline{i}-2\underline{j}-\underline{k}$ and $3\underline{i}-\alpha\underline{j}+5\underline{k}$

Solution:

Let $\underline{u} = \underline{i}-\underline{j}+\underline{k}$, $\underline{v} = \underline{i}-2\underline{j}-3\underline{k}$ and $\underline{w} = 3\underline{i}-\alpha\underline{j}+5\underline{k}$

the vectors $\underline{u}, \underline{v}$ and $\underline{w}$ are coplanar.

If $\underline{u}.\underline{v}\times\underline{\omega} = 0$

$$\begin{vmatrix} 1 & -1 & 1 \\ 1 & -2 & -3 \\ 3 & -\alpha & 5 \end{vmatrix} = 0$$

$(-10+3\alpha)+1(5+9)+1(-\alpha+6) = 0$

$-10-3\alpha+14-\alpha+6 = 0$

$-4\alpha+10 = 0$

$-4\alpha = -10 \Rightarrow \boxed{\alpha = \frac{5}{2}}$

(ii) $\underline{i}-2\alpha\underline{j}-\underline{k}$, $\underline{i}-\underline{j}+2\underline{k}$ and $\alpha\underline{i}-\underline{j}+\underline{k}$

Solution:

Let $\underline{u} = \underline{i}-2\alpha\underline{j}-\underline{k}$, $\underline{v} = \underline{i}-\underline{j}+2\underline{k}$, $\underline{w} = \alpha\underline{i}-\underline{j}+\underline{k}$

the vectors $\underline{u}, \underline{v}$ and $\underline{\omega}$ are coplanar

If $\underline{u}.\underline{v}\times\underline{\omega} = 0$

$$\begin{vmatrix} 1 & -2\alpha & -1 \\ 1 & -1 & 2 \\ \alpha & -1 & 1 \end{vmatrix} = 0$$

$1(-1+2)+2\alpha(1-2\alpha)-1(-1+\alpha) = 0$

$1+2\alpha-4\alpha^2+1-\alpha = 0$

$-4\alpha^2+\alpha+2 = 0$

$4\alpha^2-\alpha-2 = 0$

$a = 4,\ b = -1,\ c = -2$

$$\alpha = \frac{-(-1)\pm\sqrt{(-1)^2-4(4)(-2)}}{2(4)}$$

$$= \frac{1\pm\sqrt{1+32}}{8}$$

$$\boxed{\alpha = \frac{1\pm\sqrt{33}}{8}}$$

Q.5

(a) Find the value of:

(i) $2\underline{i}\times 2\underline{j}.\underline{k}$

Solution:

$2\underline{i}\times 2\underline{j}.\underline{k}$

$= 2\underline{i}\times 2\underline{j}.\underline{k}$

$= 4(\underline{i}\times\underline{j}).\underline{k}$

$= 4(\underline{k}.\underline{k})$

$= 4(1)$

$= 4$

(ii) $3\underline{j}.\underline{k}\times\underline{i}$

Solution:

$3\underline{j}.(\underline{k}\times\underline{i})$

$=3\underline{j}.(\underline{j})$

$=3(1)$

$=3$

(iii) $\left[\underline{k}\quad\underline{i}\quad\underline{j}\right]$

Solution:

$\underline{k}.\left(\underline{i}\times\underline{j}\right)$

$=\underline{k}.\underline{k}$

$=1$

(iv) $\left[\underline{i}\quad\underline{i}\quad\underline{k}\right]$

Solution:

$\underline{i}.(\underline{j}\times\underline{k})$

$=\underline{i}.(-\underline{j})$

$=-(\underline{i}.\underline{j})$

$=0$

(b) **Prove that**

$\underline{u}(\underline{v}\times\underline{w})+\underline{v}(\underline{w}\times\underline{v})+\underline{w}.(\underline{u}.\underline{v})$

$=3\underline{u}(\underline{v}\times\underline{\omega})$

Solution:

L.H.S $=\underline{u}.(\underline{v}\times\underline{w})+\underline{v}.(\underline{w}\times\underline{u})+\underline{w}.(\underline{u}\times\underline{v})$

we know that

$\underline{u}.(\underline{v}\times\underline{w})=\underline{v}.(\underline{w}\times\underline{u})=\underline{w}.(\underline{u}\times\underline{v})$

$=\underline{u}.(\underline{v}\times\underline{w})+\underline{u}.(\underline{v}\times\underline{w})+\underline{u}.(\underline{v}\times\underline{w})$

$=3\underline{u}.(\underline{v}\times\underline{w})$

= R.H.S

Q.6 **Find volume of the Tetrahedron with the vertices:**

(i) **(0,1,2),(3,2,1),(1,2,1) and (5,5,6)**

Solution:

Let

$A(0,1,2),B(3,2,1),C(1,2,1),D(5,5,6)$

$\overrightarrow{AB}=\left(3\underline{i}+2\underline{j}+\underline{k}\right)-\left(0\underline{i}+\underline{j}+2\underline{k}\right)$

$=3\underline{i}+\underline{j}-\underline{k}$

$\overrightarrow{AC}=\left(\underline{i}+2\underline{j}+\underline{k}\right)-\left(0\underline{i}+\underline{j}+2\underline{k}\right)$

$=\underline{i}+\underline{j}-\underline{k}$

$\overrightarrow{AD}=\left(5\underline{i}+5\underline{j}+6\underline{k}\right)-\left(0\underline{i}+\underline{j}+2\underline{k}\right)$

$=5\underline{i}+4\underline{j}+4\underline{k}$

Volume of Tetrahedron

$=\frac{1}{6}\left[\overrightarrow{AB}\,\overrightarrow{AC}\,\overrightarrow{AD}\right]$

$=\frac{1}{6}\begin{vmatrix}3 & 1 & -1\\ 1 & 1 & -1\\ 5 & 4 & 4\end{vmatrix}$

$=\frac{1}{6}\left[3(4+4)-1(4+5)-1(4-5)\right]$

$=\frac{1}{6}\left[24-9+1\right]$

$=\frac{1}{6}\left[16\right]=\frac{8}{3}$ cubic units

(ii) $(2,1,8),(3,2,9),(2,1,4)$ **and** $(3,3,10)$

Solution:

Let

$A(2,1,8),B(3,2,9),C(2,1,4),D(3,3,10)$

$\overrightarrow{AB}=\left(3\underline{i}+2\underline{j}+9\underline{k}\right)-\left(2\underline{i}+\underline{j}+8\underline{k}\right)$

$=\underline{i}+\underline{j}+\underline{k}$

$\overrightarrow{AC}=\left(2\underline{i}+\underline{j}+4\underline{k}\right)-\left(2\underline{i}+\underline{j}+8\underline{k}\right)$

$=0\underline{i}+0\underline{j}-4\underline{k}$

$\overrightarrow{AD}=\left(3\underline{i}+3\underline{j}+10\underline{k}\right)-\left(2\underline{i}+\underline{j}+8\underline{k}\right)$

$=\underline{i}+2\underline{j}+2\underline{k}$

Volume of Tetrahedron

$=\frac{1}{6}\left[\overrightarrow{AB}\,\overrightarrow{AC}\,\overrightarrow{AD}\right]$

$=\frac{1}{6}\begin{vmatrix}1 & 1 & 1\\ 0 & 0 & -4\\ 2 & 2 & 2\end{vmatrix}$

$=\frac{1}{6}\left(1(0+8)-1(0+4)+1(0-0)\right)$

$=\frac{1}{6}(8-4)$

$=\frac{1}{6}\times 4$

$=\frac{2}{3}$ cubic unit

Q.7 Find the work done, if the point at which the constant force $\underline{F} = 4\underline{i} + 3\underline{j} + 5\underline{k}$ is applied to an object, moves from $P_1(3,1,-2)$ to $P_2(2,4,6)$

Solution:

$$\vec{d} = \overrightarrow{P_1P_2}$$
$$= (2\underline{i} + 4\underline{j} + 6\underline{k}) - (3\underline{i} + \underline{j} - 2\underline{k})$$
$$= -\underline{i} + 3\underline{j} + 8\underline{k}$$

Work done $= \underline{\boldsymbol{F}}.\underline{d}$

$$= (4\underline{i} + 3\underline{j} + 5\underline{k}).(-\underline{i} + 3\underline{j} + 8\underline{k})$$
$$= 4(-1) + 3(3) + 5(8)$$
$$= -4 + 9 + 40$$
$$= 45$$

Q.8 A particle, acted by constant forces $4\underline{i} + \underline{j} - 3\underline{k}$ and $3\hat{i} - \hat{j} - \hat{k}$ is displaced from A(1,2,3) to B(5,4,1). Find the work done.

Solution:

$$\vec{d} = \overrightarrow{AB}$$
$$= (5\underline{i} + 4\underline{j} + \underline{k}) - (\underline{i} + 2\underline{j} + 3\underline{k})$$
$$= 4\underline{i} + 2\underline{j} - 2\underline{k}$$
$$\underline{\boldsymbol{F}} = \underline{\boldsymbol{F_1}} + \underline{\boldsymbol{F_2}}$$
$$= (4\underline{i} + \underline{j} - 3\underline{k}) + (3\underline{i} - \underline{j} - \underline{k})$$
$$= 7\underline{i} + 0\underline{j} - 4\underline{k}$$

Work done $= \underline{\boldsymbol{F}}.\underline{d}$

$$= (7\underline{i} + 0\underline{j} - 4\underline{k}).(4\underline{i} + 2\underline{j} - 2\underline{k})$$
$$= 7(4) + 0(2) + (-4)(-2)$$
$$= 28 + 0 + 8$$
$$= 36$$

Q.9 A particle is displaced from the point $A(5,-5,-7)$ to the point $B(6,2,-2)$. Under the action of constant forces defined by $10\underline{i} - \underline{j} + 11\underline{k}$, $4\underline{i} + 5\underline{j} + 9\underline{k}$ and $-2\underline{i} + \underline{j} - 9\underline{k}$. Show that the total work done by the forces is 102 units.

Solution:

$$\vec{d} = \overrightarrow{AB}$$
$$= (6\underline{i} + 2\underline{j} - 2\underline{k}) - (5\underline{i} - 5\underline{j} - 7\underline{k})$$
$$= \underline{i} + 7\underline{j} + 5\underline{k}$$

$\underline{\boldsymbol{F}}$ = sum of forces

$$= (10\underline{i} - \underline{j} + 11\underline{k}) + (4\underline{i} + 5\underline{j} + 9\underline{k}) + (-2\underline{i} + \underline{j} - 9\underline{k})$$
$$= 12\underline{i} + 5\underline{j} + 11\underline{k}$$

Work done $= \vec{F}.\vec{d}$

$$= (12\underline{i} + 5\underline{j} + 11\underline{k}).(\underline{i} + 7\underline{j} + 5\underline{k})$$
$$= (12)(1) + 5(7) + 11(5)$$
$$= 12 + 35 + 55$$
$$= 102 \text{ units}$$

Q.10 A force of magnitude 6 units acting parallel to $2\underline{i} - 2\underline{j} + \underline{k}$ displaces the point of application from (1,2,3) to (5,3,7). Find the work done.

Solution:

Let $\underline{\boldsymbol{F_1}} = 2\underline{i} - 2\underline{j} + \underline{k}$

$$|\underline{\boldsymbol{F_1}}| = \sqrt{(2)^2 + (-2)^2 + (1)^2}$$
$$= \sqrt{4+4+1} = \sqrt{9}$$
$$= 3$$

If $\hat{F}_1$ is the unit vector in the direction of $\underline{\boldsymbol{F_1}}$

Then $\underline{\hat{\boldsymbol{F}}_1} = \dfrac{\boldsymbol{F_1}}{|\boldsymbol{F_1}|}$

$$= \frac{2\underline{i} - 2\underline{j} + \underline{k}}{3}$$

The required Force $\underline{\boldsymbol{F}} = 6\underline{\hat{\boldsymbol{F}}_1}$

$$= 6\left(\frac{2\underline{i} - 2\underline{j} + \underline{k}}{3}\right)$$

$$= 4\underline{i} - 4\underline{j} + 4\underline{k}$$

$$\underline{d} = \overrightarrow{AB}$$

$$= (5\underline{i} + 3\underline{j} + 7\underline{k}) - (\underline{i} + 2\underline{j} + 3\underline{k})$$

$$= 4\underline{i} + \underline{j} + 4\underline{k}$$

Work done $= \underline{\boldsymbol{F}}.\underline{d}$

$$= (4\underline{i} - 4\underline{j} + 2\underline{k}).(4\underline{i} + \underline{j} + 4\underline{k})$$

$$= 4(4) + (-4)(1) + 2(4)$$

$$= 16 - 4 + 8$$

$$= 20$$

Q.11 A force $\underline{\boldsymbol{F}} = 3\underline{i} + 2\underline{j} - 4\underline{k}$ is applied at the point (1,–1,2). Find the moment of the force about the point (2,–1,3).

Solution:

Let $P(1,-1,2)$, $Q(2,-1,3)$

$$\underline{r} = \overrightarrow{QP}$$

$$= (\underline{i} - \underline{j} + 2\underline{k}) - (2\underline{i} - \underline{j} + 3\underline{k})$$

$$= -\underline{i} + 0\underline{j} - \underline{k}$$

Moment of force of $\underline{\boldsymbol{F}}$ about Q is

$$\underline{r} \times \underline{\boldsymbol{F}} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & 0 & -1 \\ 3 & 2 & -4 \end{vmatrix}$$

$$= \underline{i}(0+2) - \underline{j}(+4+3) + \underline{k}(-2-0)$$

$$= 2\underline{i} - 7\underline{j} - 2\underline{k}$$

Q.12 A force $\underline{\boldsymbol{F}} = 4\underline{i} - 3\underline{k}$, passes through the point $A(2,-2,5)$. Find the moment of $\overline{F}$ about the point $B(1,-3,1)$.

Solution:

$$\underline{r} = \overrightarrow{BA}$$

$$= (2\underline{i} - 2\underline{j} + 5\underline{k}) - (\underline{i} - 3\underline{j} + \underline{k})$$

$$= \underline{i} + \underline{j} + 4\underline{k}$$

Moment of $\underline{\boldsymbol{F}}$ about B is

$$\underline{r} \times \underline{\boldsymbol{F}} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 4 \\ 4 & 0 & -3 \end{vmatrix}$$

$$= \underline{i}(-3-0) - \underline{j}(-3-16) + \underline{k}(0-4)$$

$$= -3\underline{i} + 19\underline{j} - 4\underline{k}$$

Q.13 Given a force $\underline{\boldsymbol{F}} = 2\underline{i} + \underline{j} - 3\underline{k}$ acting at a point $A(1,-2,1)$. Find the moment of $\underline{\boldsymbol{F}}$ about the point $B\ (2,0,-2)$.

Solution:

$$\underline{r} = \overrightarrow{BA}$$

$$= (\underline{i} - 2\underline{j} + \underline{k}) - (2\underline{i} + 0\underline{j} + 2\underline{k})$$

$$= -\underline{i} - 2\underline{j} + 3\underline{k}$$

$$\underline{\boldsymbol{F}} = 2\underline{i} + \underline{j} - 3\underline{k}$$

Moment of force $\underline{\boldsymbol{F}}$ about B is

$$\underline{r} \times \underline{\boldsymbol{F}} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -2 & 3 \\ 2 & 1 & -3 \end{vmatrix}$$

$$= \underline{i}(6-3) - \underline{j}(3-6) + \underline{k}(1-4)$$

$$= 3\underline{i} + 3\underline{j} + 3\underline{k}$$

Q.14 Find the moment about $A(1,1,1)$ of each of the concurrent forces $\underline{i}-2\underline{j}$, $3\underline{i}+2\underline{j}-\underline{k}$, $5\underline{j}-2\underline{k}$, where $P(2,0,1)$ is their point of concurrency.

Solution:

Let $\underline{\boldsymbol{F}}_1=\underline{i}-2\underline{j}+0\underline{k}$

$\underline{\boldsymbol{F}}_2=3\underline{i}+2\underline{j}-\underline{k}$

$\underline{\boldsymbol{F}}_3=0\underline{i}+5\underline{j}-2\underline{k}$

$\underline{\boldsymbol{F}}=\underline{\boldsymbol{F}}_1+\underline{\boldsymbol{F}}_2+\underline{\boldsymbol{F}}_3$

$=(\underline{i}-2\underline{j}+0\underline{k})+(3\underline{i}+2\underline{j}-\underline{k})+(0\underline{i}+5\underline{j}+2\underline{k})$ $\quad \underline{\boldsymbol{F}}=4\underline{i}+5\underline{j}+\underline{k}$

$\underline{r}=\overrightarrow{AP}$

$=\left(2\underline{i}+0\underline{j}+\underline{k}\right)-\left(\underline{i}+\underline{j}+\underline{k}\right)$

$\underline{r}=\underline{i}-\underline{j}+0\underline{k}$

Moment of force $\underline{\boldsymbol{F}}$ about A is

$$\underline{r}\times\underline{\boldsymbol{F}}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 1 & -1 & 0\\ 4 & 5 & 1\end{vmatrix}$$

$=\underline{i}(-1-0)-\underline{j}(1-0)+\underline{k}(5+4)$

$=-\underline{i}-\underline{j}+9\underline{k}$

Q.15 A force $\underline{\boldsymbol{F}}=7\underline{i}+4\underline{j}-3\underline{k}$ is applied at $P(1,-2,3)$. Find its moment about the point $Q(2,1,1)$.

Solution:

$\underline{r}=\overrightarrow{QP}$

$=\left(\underline{i}-2\underline{j}+3\underline{k}\right)-\left(2\underline{i}+\underline{j}+\underline{k}\right)$

$\underline{r}=-\underline{i}-3\underline{j}+2\underline{k}$

$\underline{\boldsymbol{F}}=7\underline{i}+4\underline{j}-3\underline{k}$

Moment of force $\underline{\boldsymbol{F}}$ about Q is

$$\underline{r}\times\underline{\boldsymbol{F}}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ -1 & -3 & 2\\ 7 & 4 & -3\end{vmatrix}$$

$=\underline{i}(9-8)-\underline{j}(3-14)+\underline{k}(-4+21)$

$=\underline{i}+11\underline{j}+17\underline{k}$