Chapter 2

VECTORS AND EQUILIBRIUM

LEARNING OBJECTIVES

At the end of this chapter the students will be able to:

Understand the definition of scalars and vectors.

Understand and use rectangular coordinate system.

Understand the idea of unit vector, null vector and position vector.

Represent a vector as two perpendicular components (rectangular components).

Understand multiplication of vectors and solve problems.

Define the moment of force or torque.

Appreciate the use of the torque due to a force.

Appreciate the applications of the principle of moments.

Q.1 Define scalars and vectors. Give examples.

Ans. **SCALAR QUANTITIES**

Those physical quantities which are completely described by magnitude with proper units are called scalar quantities. e.g. time, current, speed etc. Scalars are added, subtracted, divided and multiplied by ordinary arithmetic rules.

VECTOR QUANTITIES

Those physical quantities which are completely described by magnitude with proper units as well as direction are called vector quantities. e.g. force, torque etc. Vectors are not added, subtracted, divided and multiplied by ordinary arithmetic rules but it can be used as vector addition, vector multiplication and vector subtraction.

Q.2 Describe how a vector quantity is represented?

Ans. **REPRESENTATION OF A VECTOR**

A vector is usually represented by a bold face letter that is **A** or by a letter with an arrow drawn above or below it that is $\vec{A}$ or $\underline{A}$.

The magnitude of a vector is denoted by $|\vec{A}|$ (modulus) or A.

(ii) By Graphically

A vector is represented graphically by a directed line segment with an arrow-head in the direction of the vector. The length of the line segment, according to the suitable scale, corresponds to the magnitude of the vector. The length of the line is called its magnitude and arrow head indicates the direction.

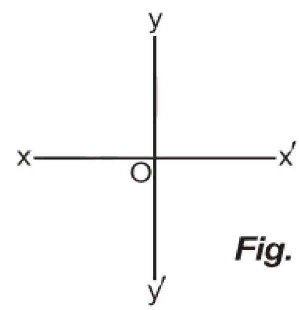


Fig. (i)

Q.3 Describe the rectangular coordinate system.

Ans. **RECTANGULAR COORDINATE SYSTEM**

Two reference lines drawn at right angles as shown in figure. They are known as coordinate axes and their point of intersection is known as origin. This system of coordinate axes is called Cartesian or rectangular coordinate system.

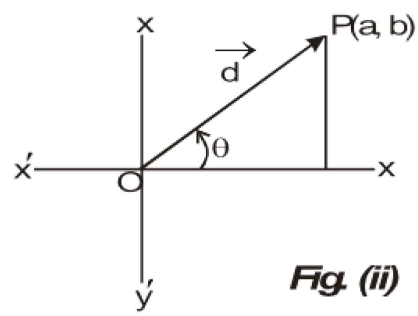
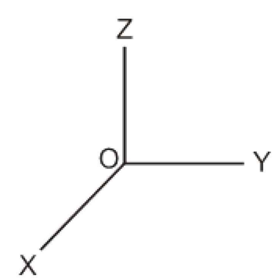


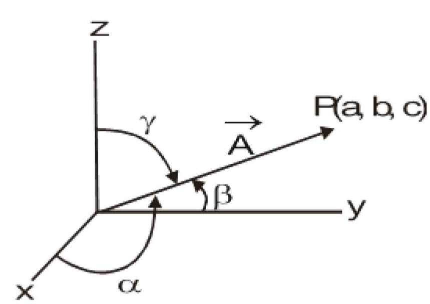
Fig. (ii)

One of the lines is named as x-axis, and the other the y-axis. Usually the x-axis is taken as the horizontal axis, with the positive direction to the right, and the y-axis as the vertical axis with the positive direction upward.

The direction of a vector in a plane is denoted by the angle which representative line of the vector makes with positive x-axis in the anti-clockwise direction as shown in Fig. (ii).



The direction of a vector in space requires another axis which is at right angle to both x and y axes, as shown in figure which is called z-axis.



The direction of a vector in space is specified by the three angles which the representative line of the vector makes with x, y and z axes respectively as shown in figure. The point P of a vector A is thus denoted by three coordinates (a, b, c).

Q.4 Explain the following terms:

(iii) Vector subtraction ***(iv) Multiplication of a vector***

(v) Unit vector ***(vi) Null vector***

(vii) Equal vector

Ans.

ADDITION OF VECTORS

Consider two vectors $\vec{A}$ and $\vec{B}$ as shown in Fig. (i).

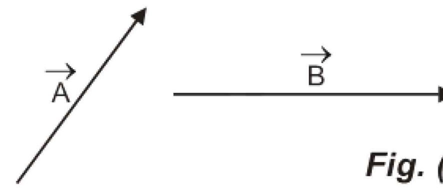


Fig. (i)

Their sum is obtained by drawing, their representative lines in such a way that tail of vector $\vec{B}$ coincides with the head of vector $\vec{A}$. Now if we join the tail of $\vec{A}$ to the head of $\vec{B}$ as shown in Fig. (ii).

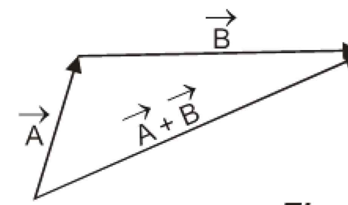


Fig. (ii)

If we join the tail of first vector with the head of last vector, it will represent the vector sum $\vec{A}+\vec{B}$ in magnitude and direction. This is known as head to tail rule of vector addition. Similarly the sum $\vec{B}+\vec{A}$ is represented by doted lines as shown in Fig. (iii). It is clear from Fig. (iii) that

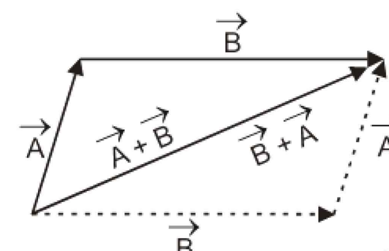


Fig. (iii)

$$\vec{A}+\vec{B} = \vec{B}+\vec{A}$$

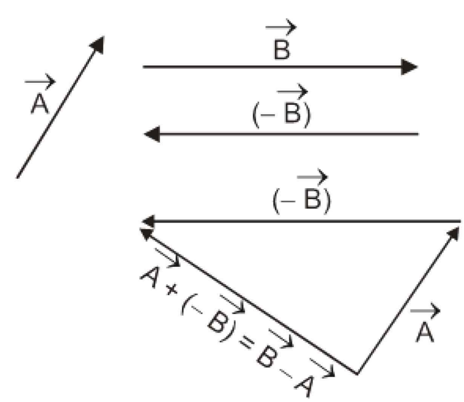
So the vector addition is said to be commutative. It means that when vectors are added, the result is same for any order of addition.

RESULTANT VECTOR

The resultant of number of vectors is that single vector which would have the same effect as all the original vectors taken together.

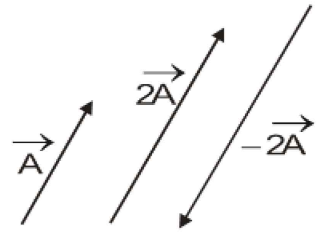
VECTOR SUBTRACTION

The subtraction of a vector is equal to the addition of the same vector with its direction reversed. To subtract vector $\vec{B}$ from vector $\vec{A}$ reverse the direction of $\vec{B}$ and add it to $\vec{A}$ as shown in figure.



MULTIPLICATION OF A VECTOR BY A SCALAR

The product of a vector $\vec{A}$ and a number $n > 0$ is defined to be a new vector $n\vec{A}$ having the same direction as $\vec{A}$ but a magnitude n times the magnitude of $\vec{A}$ as shown in figure.



If the vector is multiplied by a negative number, then its direction is reversed.

For Example

When velocity (vector) is multiplied by scalar i.e. mass m, the product is a new vector quantity called momentum having same dimensions as product of mass and velocity.

UNIT VECTOR

A unit vector in a given direction is a vector with magnitude one in that direction.

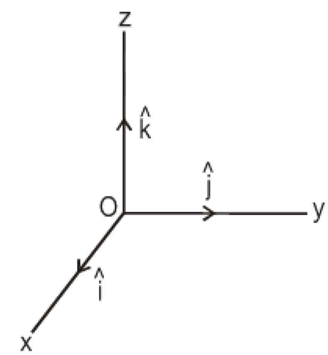
It is used to represent the direction of a vector.

A unit vector in the direction of $\vec{A}$ is written as $\hat{A}$.

For information

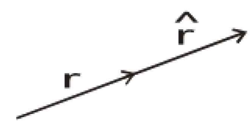
$$\left|\hat{A}\right| = \frac{\left|\vec{A}\right|}{A} = \frac{A}{A} = 1$$

As
$$\vec{A} = A\hat{A}$$
$$\hat{A} = \frac{\vec{A}}{A}$$

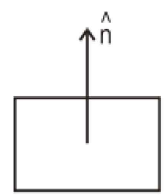


The direction along x, y and z axes is represented by unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$ respectively. Two of the more frequently used unit vectors are:

(i) The vector $\hat{r}$ which represents the direction of the vector r as shown.



(ii) The vector $\hat{n}$ which represents the direction of a normal drawn on a surface as shown.



NULL VECTORS

It is a vector of zero magnitude and arbitrary direction.

For example, sum of a vector and its negative vector is a null vector.

$$\vec{A} + \left(-\vec{A}\right) = \vec{0}$$

EQUAL VECTORS

Two vectors $\vec{A}$ and $\vec{B}$ are said to be equal if they have the same magnitude and direction, regardless of the position of their initial points.

This means that parallel vectors of the same magnitude are equal to each other.

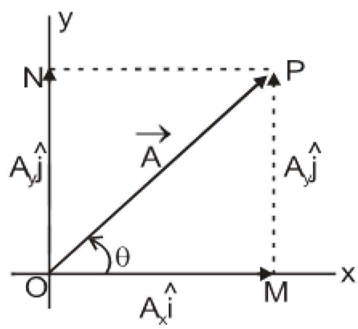
Q.5 Define rectangular components of a vector. How will you find resultant vector of the rectangular components are given?

Ans. ## RECTANGULAR COMPONENTS OF A VECTOR

The splitting up of a vector into its parts is called the resolution of a vector. Usually a vector can be resolved into two parts called the components of a vector. **(OR)** A component of a vector is its

Explanation

Consider a vector $\vec{A}$ represented by $\vec{OP}$ making an angle θ with x-axis. Draw projection OM of vector $\vec{OP}$ on x-axis and projection ON of vector OP on y-axis as shown in figure Projection OM being along x-direction is represented by $A_x\hat{i}$ and projection ON = MP along y-direction is represented by $A_y\hat{j}$.



By head to tail rule

$$\vec{A} = A_x\hat{i} + A_y\hat{j}$$

Here $A_x\hat{i}$ and $A_y\hat{j}$ are the components of $\vec{A}$. Since these components are at right angle to each other, hence, are called rectangular components.

Considering the right angle triangle OPM

$$\frac{\text{Base}}{\text{Hypotenuse}} = \cos\theta \quad \Rightarrow \quad \cos\theta = \frac{OM}{OP}$$

$$\frac{A_x}{A} = \cos\theta$$

$$A_x = A\cos\theta$$

It is the magnitude of x-component of vector $\vec{A}$.

Now
$$\frac{\text{Perpendicular}}{\text{Hypotenuse}} = \sin\theta$$

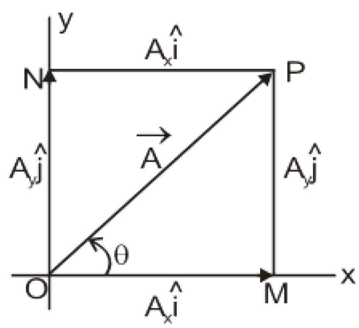
$$\frac{A_y}{A} = \sin\theta$$

$$A_y = A\sin\theta$$

It is the magnitude of y-component of $\vec{A}$.

Determination of a Vector from its Rectangular Components

If rectangular components $A_x\hat{i}$ and $A_y\hat{j}$ vectors are given. Then we can find the magnitude and direction of the vector.



Using Pythagorean theorem.

$$(H)^2 = (B)^2 + (P)^2$$

$$A^2 = A_x^2 + A_y^2$$

The magnitude of resultant

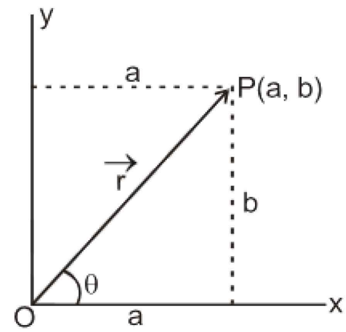
$$A = \sqrt{A_x^2 + A_y^2}$$

The direction of resultant

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$

Q.6 Define position vector.

Ans. POSITION VECTOR



It is a vector that describes the location of a particle with respect to the origin. It is represented by a straight line drawn in such a way that its tail coincides with the origin and the head with point P (a, b) as shown in figure. The projection of position vector r on the x and y-axes are the coordinates a and b and they are rectangular components of the vector r.

$$\vec{r} = a\,\hat{i} + b\,\hat{j}$$

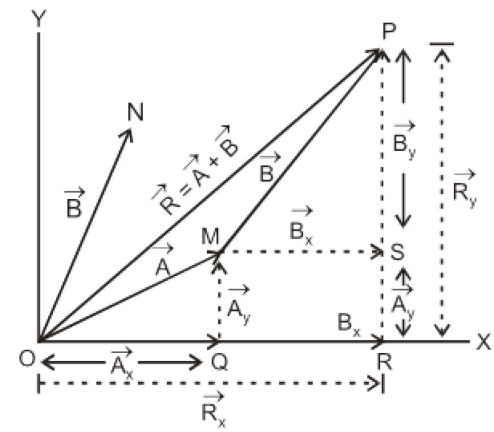
and $$r = \sqrt{a^2 + b^2}$$

In three dimension $\vec{r} = a\,\hat{i} + b\,\hat{j} + c\,\hat{k}$

$$r = \sqrt{a^2 + b^2 + c^2}$$

Q.7 Explain the addition of vectors by rectangular components.

Ans. VECTOR ADDITION BY RECTANGULAR COMPONENTS



Consider two vectors $\vec{A}$ and $\vec{B}$ which are represented by two directed lines OM and ON respectively. The $\vec{B}$ is added to $\vec{A}$ by head to tail rule as shown in figure. The resultant vector $\vec{R} = \vec{A} + \vec{B}$ is given in direction and magnitude by the $\overrightarrow{OP}$.

Resolving $\vec{A}$, $\vec{B}$ and $\vec{R}$ vector into their rectangular components.

In the figure, $\vec{A}_x$, $\vec{B}_x$ and $\vec{R}_x$ are the x–components of the vectors $\vec{A}$, $\vec{B}$, $\vec{R}$ and their magnitude are given by the lines OQ, MS, OR respectively.

From figure:

$$OR = OQ + QR$$

$\therefore$ $$QR = MS$$

$$OR = OQ + MS$$

$$\vec{R}_x = R_x\,\hat{i} = A_x\,\hat{i} + B_x\,\hat{i}$$

$$R_x\,\hat{i} = (A_x + B_x)\,\hat{i}$$

Equation (i) means that the sum of the magnitude of x–components of two vectors, which are to be added is equal to the x–component of the resultant.

Similarly $RP = RS + SP$

$\therefore$ $RS = QM$

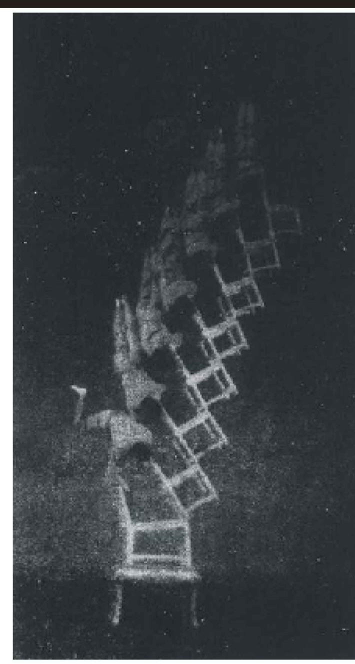
$\therefore$ $RP = QM + SP$

$$R_y = A_y + B_y \qquad \text{......... (ii)}$$

$$\vec{R}_y = R_y\,\hat{j} = (A_y + B_y)\,\hat{j}$$

Equation (ii) means that the sum of the magnitudes of y-components of two vectors is equal to the magnitude of y-component of the resultant.

Do You Know?



The Chinese acrobats in this incredible balancing act are in equilibrium.

Since $R_x\,\hat{i}$ and $R_y\,\hat{j}$ are rectangular components of the $\vec{R}$.

$$\vec{R} = R_x\,\hat{i} + R_y\,\hat{j} \qquad \text{......... (iii)}$$

Putting the values of R_x and R_y.

$$\therefore \quad \vec{R} = (A_x + B_x)\,\hat{i} + (A_y + B_y)\,\hat{j}$$

Its magnitude is

$$R = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2}$$

For direction

$$\tan\theta = \frac{R_y}{R_x}$$

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$

$$\theta = \tan^{-1}\left(\frac{A_y + B_y}{A_x + B_x}\right)$$

Similarly for any number of coplanar vectors $\vec{A}, \vec{B}, \vec{C}$ ....

$$R = \sqrt{(A_x + B_x + C_x + \ldots)^2 + (A_y + B_y + C_y + \ldots)^2}$$

$$\theta = \tan^{-1}\left(\frac{A_y + B_y + C_y + \ldots}{A_x + B_x + C_x + \ldots}\right)$$

Important Steps for Vector Addition by Rectangular Components

(ii) Find x-components , R_x of the resultant vector by adding the x–components of all the vectors.

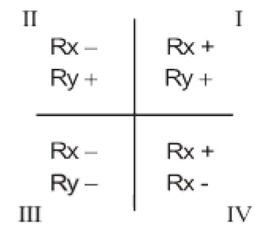
(iii) Find y-components , R_y of the resultant vector by adding the y–components of all the vectors.

(iv) Find magnitude of resultant vector $\vec{R}$ using

$$R = \sqrt{R_x^2 + R_y^2}$$

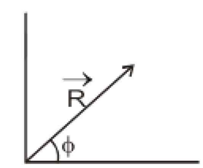
(v) Find the direction of resultant vector by using

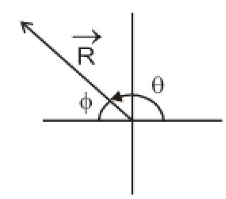
$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$



Where θ is the angle, which resultant vector makes with positive x-axis. Irrespective of sign of R_x and R_y determine the value of

$$\phi = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$



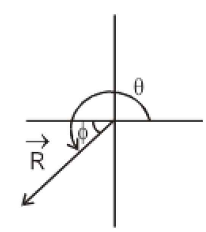


(i) If both R_x, R_y are + ve then $\vec{R}$ lies in first quadrant.

$\therefore \quad \theta = \phi$

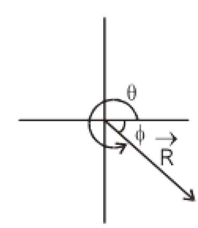
(ii) If R_x is –ve and R_y is +ve then $\vec{R}$ lies in second quadrant.

$\therefore \quad \theta = 180^o - \phi$



(iii) If R_x, R_y both are –ve then $\vec{R}$ lies in third quadrant.

$\therefore \quad \theta = 180^o + \phi$



(iv) If R_x is +ve and R_y is –ve then $\vec{R}$ lies in fourth quadrant.

$\therefore \quad \theta = 360^o - \phi$

Note: If θ is angle between $\vec{A}$ and $\vec{B}$, then their resultant is:

$$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

Point to Ponder

Why do you keep your legs far apart when you have to stand in the aisle of a bumpy-riding bus?

Ans. We should keep our legs apart so that when we bend, our centre of gravity will remain within live of action and we will remain in stable equilibrium.

PRODUCT OF TWO VECTORS

There are two types of vector multiplication.

(i) Scalar Product

(ii) Vector Product

Ans. SCALAR OR DOT-PRODUCT

When the product of two vectors results into a scalar quantity then the product is called scalar product. The scalar product between two vectors can be expressed as by putting a dot (•) between the vectors and can be written as $\vec{A} \cdot \vec{B}$. So this product is also called dot product.

Examples

(i) $W = \vec{F} \cdot \vec{d}$, where work is a scalar quantity because it is the dot product of two vectors force and displacement.

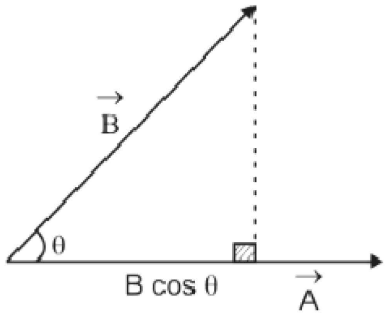
(ii) $P = \vec{F} \cdot \vec{V}$, where power is a vector because it is the dot product of two vectors force and velocity.

Explanation

Consider $\vec{A}$ and $\vec{B}$ are the two vectors having θ angle between them, then the scalar product can be written as:

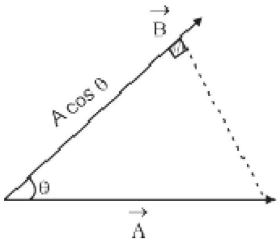
$$\vec{A} \cdot \vec{B} = AB\cos\theta$$

Now draw perpendicular from head of $\vec{B}$ on $\vec{A}$. Also draw a perpendicular from head of $\vec{A}$ on $\vec{B}$.



$$\vec{A} \cdot \vec{B} = AB\cos\theta$$
$$= A(B\cos\theta)$$
$$= \text{Magnitude of } \vec{A} \text{ times } \left(\text{Projection of } \vec{B} \text{ on } \vec{A}\right)$$

Similarly



$$\vec{B} \cdot \vec{A} = BA\cos\theta$$
$$= B(A\cos\theta)$$
$$= \text{Magnitude of } \vec{B} \text{ times } \left(\text{Projection of } \vec{A} \text{ on } \vec{B}\right)$$
$$= \text{Magnitude of } \vec{B}\left(\text{Magnitude of component of } \vec{A} \text{ in direction of } \vec{B}\right)$$

Characteristics of Scalar Product

Following are the characteristics of scalar product:

1. Scalar product is commutative (change of order of vectors has no effect) i.e., $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$.

Proof

$$\vec{A} \cdot \vec{B} = AB\cos\theta \quad \text{......... (1)}$$

$$\vec{B} \cdot \vec{A} = BA\cos\theta$$

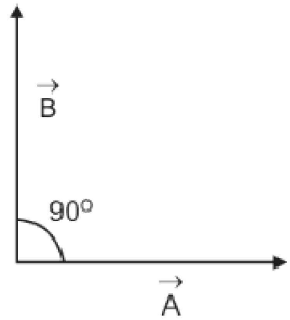
Since AB = BA

$$\therefore \quad \vec{B} \cdot \vec{A} = AB\cos\theta \quad \text{......... (2)}$$

From equations (1) and (2).

$$\therefore \quad \vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

2. The scalar product of two mutually perpendicular vector is zero i.e., $\vec{A} \cdot \vec{B} = 0$.

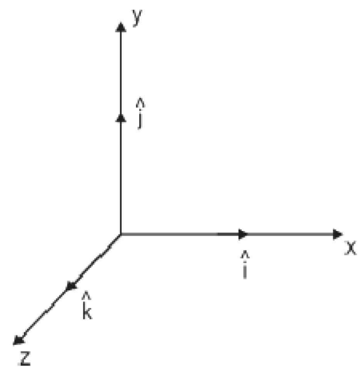


Now $\vec{A} \cdot \vec{B} = AB\cos\theta$

But $\theta = 90°$

$$\therefore \quad \vec{A} \cdot \vec{B} = AB\cos 90^{o}$$
$$= AB\,(0)$$
$$\vec{A} \cdot \vec{B} = 0$$

In case of unit vector



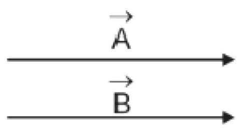
$$\hat{i} \cdot \hat{j} = 0$$
$$\hat{j} \cdot \hat{k} = 0$$
$$\hat{k} \cdot \hat{i} = 0$$

$$\boxed{\begin{aligned} \hat{i} \cdot \hat{j} &= |\hat{i}|\,|\hat{j}|\cos 90^{o} \\ &= 1 \times 1 \times 0 \\ &= 0 \end{aligned}}$$

3. The scalar product of two parallel vectors is equal to the product of their magnitudes i.e., $\vec{A} \cdot \vec{B} = AB$.

Now $\vec{A} \cdot \vec{B} = AB\cos\theta$

But $\theta = 0°$ because the vectors are parallel



As $\vec{A} \cdot \vec{B} = AB\cos\theta$

$$\therefore \quad \vec{A} \cdot \vec{B} = AB\cos 0^{o}$$
$$= AB \times 1$$
$$= AB$$

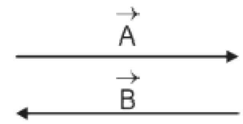
In the case of unit vectors

$$\hat{i}\cdot\hat{i} = 1,\ \hat{j}\cdot\hat{j} = 1,\ \hat{k}\cdot\hat{k} = 1.$$

$$\hat{i}\cdot\hat{i} = |\hat{i}||\hat{i}|\cos 0^o = 1 \times 1 \times 1 = 1$$

For anti–parallel vectors

i.e. $\theta = 180^o$



As $\vec{A}\cdot\vec{B} = AB\cos 180^o$

$= AB(-1)$

$\therefore\ \vec{A}\cdot\vec{B} = -AB$

4. The self scalar product of a $\vec{A}$ is equal to square of its magnitude i.e. $\vec{A}\cdot\vec{A} = A^2$.

As $\vec{A}\cdot\vec{B} = AB\cos\theta$

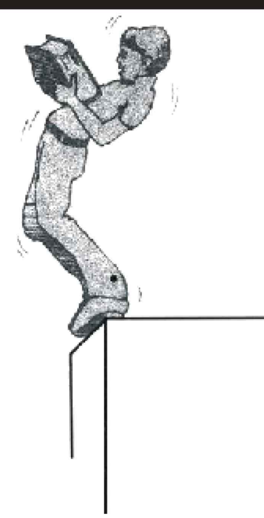
Put $\vec{B} = \vec{A}$

$\therefore\ \vec{A}\cdot\vec{A} = AA\cos 0^o$

$= A^2 \times 1$

$= A^2$

What Should You Do?



You are falling off the edge. What should you do to avoid falling?

Ans. In order to avoid falling we should bend in backward direction so that our centre of gravity will remain within line of action and we will remain In stable equilibrium.

5. Scalar product of two vectors $\vec{A}$ and $\vec{B}$ in terms of their rectangular components.

Let $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

Their scalar product can be written as

$$\vec{A}\cdot\vec{B} = \left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)\cdot\left(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\right)$$

$$= A_xB_x\,\hat{i}\cdot\hat{i} + A_xB_y\,\hat{i}\cdot\hat{j} + A_xB_z\,\hat{i}\cdot\hat{k}$$

$$+ A_yB_x\,\hat{j}\cdot\hat{i} + A_yB_y\,\hat{j}\cdot\hat{j} + A_yB_z\,\hat{j}\cdot\hat{k}$$

$$+ A_zB_x\,\hat{k}\cdot\hat{i} + A_zB_y\,\hat{k}\cdot\hat{j} + A_zB_z\,\hat{k}\cdot\hat{k}$$

$$= A_xB_x(1) + A_xB_y(0) + A_xB_z(0)$$

$$+ A_yB_x(0) + A_yB_y(1) + A_yB_z(0)$$

$$+ A_zB_x(0) + A_zB_y(0) + A_zB_z(1)$$

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

6. Scalar product holds distributive law, i.e.,

$$\vec{A}\cdot(\vec{B} + \vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$$

$$= \vec{A} \cdot mn\,\vec{B}$$

$$= m\vec{A} \cdot n\vec{B}$$

Q.9 Define vector product. Also explain the characteristic of vector product.

Ans. **VECTOR OR CROSS PRODUCT**

When the product of two vectors results into a vector quantity, then the product is called vector product. The vector product between two vectors can be expressed as by putting a cross (×) and can be written as $\vec{A} \times \vec{B}$. So this product is also called cross product.

Examples

(i) $\vec{\tau} = \vec{r} \times \vec{F}$ where torque is a vector because it is a cross product of moment arm and force.

(ii) $\vec{L} = \vec{r} \times \vec{p}$ where angular momentum is a vector because it is the cross-product of perpendicular distance and momentum.

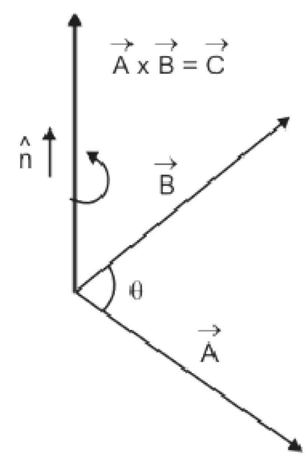
(iii) $\vec{F} = q\left(\vec{V} \times \vec{B}\right)$ where force is a vector because it is the cross-product of velocity and magnetic field.

Explanation

Consider $\vec{A}$ and $\vec{B}$ are the two vectors making an angle θ with each other, then their vector product is

$$\vec{A} \times \vec{B} = AB\sin\theta\,\hat{n}$$

Where $\hat{n}$ is a unit vector perpendicular to the plane containing $\vec{A}$ and $\vec{B}$.



$$\vec{A} \times \vec{B} = AB\sin\theta\,\hat{n}$$

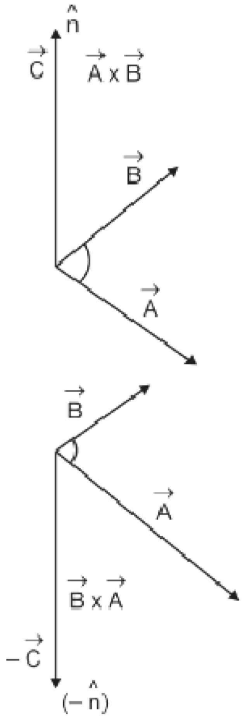
$$\left|\vec{A} \times \vec{B}\right| = AB\sin\theta\left|\hat{n}\right|$$

$$= AB\sin\theta \times 1$$

$$= AB\sin\theta$$

The direction of $\vec{A} \times \vec{B}$ is obtained by the right hand rule.

Right Hand Rule



Join the tails of two vectors $\vec{A}$ and $\vec{B}$ which determine the plane containing them. Rotate first vector in the plane, $(\vec{A})$ towards the second vector $(\vec{B})$ through the smaller of the two possible angles. Hold the right hand in such a way that thumb should be erect and fingers should be curl along the direction of rotation of $\vec{A}$, then thumb points toward the direction of $\vec{A} \times \vec{B}$.

Characteristics of Cross Product

Following are the characteristics of cross product:

1. Vector product is non-commutative i.e.

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

As by right hand-rule $\vec{A} \times \vec{B}$ is out of the paper and $\vec{B} \times \vec{A}$ is into the paper i.e., they have same magnitude but opposite in direction, therefore

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

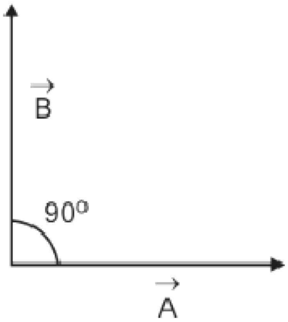
2. Cross product of two perpendicular vectors has maximum magnitude i.e.

$$\vec{A} \times \vec{B} = AB\,\hat{n}$$

As $$\vec{A} \times \vec{B} = AB \sin \hat{n}$$

$$= AB \sin 90^\circ\,\hat{n}$$

$$= AB\,\hat{n}$$



In case of unit vector

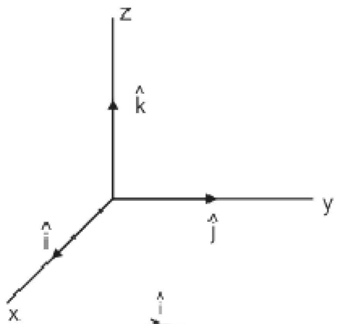
$$\hat{i} \times \hat{j} = \hat{k}$$

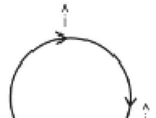
$$\hat{j} \times \hat{k} = \hat{i}$$

$$\hat{k} \times \hat{i} = \hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k}$$

$$\hat{k} \times \hat{j} = -\hat{i}$$





3. The cross product of two parallel vector is null vector i.e. $\vec{A} \times \vec{B} = \vec{0}$

As $\vec{A} \times \vec{B} = AB \sin\theta \hat{n}$

But $\theta = 0°$ because the vectors are parallel

As $\vec{A} \times \vec{B} = AB \sin\theta\ \hat{n}$

$\therefore \quad \vec{A} \times \vec{B} = AB \sin 0^o\ \hat{n}$

$= AB\,(0)\,\hat{n} = 0\,\hat{n}$

$\vec{A} \times \vec{B} = \vec{0}$

In case of unit vector

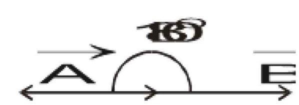
$\hat{i} \times \hat{i} = \vec{0}$

$\hat{j} \times \hat{j} = \vec{0}$

$\hat{k} \times \hat{k} = \vec{0}$

For anti-parallel vectors

$\theta = 180^o$



$\therefore \quad \vec{A} \times \vec{B} = AB \sin 180^o\ \hat{n}$

$= AB\,(0)\,\hat{n}$

$= 0\,\hat{n}$

$\vec{A} \times \vec{B} = \vec{0}$

4. Self vector product is equal to a null vector.

$\vec{A} \times \vec{A} = AA \sin 0^o\ \hat{n}$

$= A^2\,(0)\,\hat{n}$

$= 0\,\hat{n}$

$\vec{A} \times \vec{A} = \vec{0}$

5. Cross product of two vectors $\vec{A}$ and $\vec{B}$ in terms of their rectangular components.

$$\vec{A} \times \vec{B} = \left(A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}\right) \times \left(B_x\,\hat{i} + B_y\,\hat{j} + B_z\,\hat{k}\right)$$

$$= A_x B_x\,\hat{i} \times \hat{i} + A_x B_y\,\hat{i} \times \hat{j} + A_x B_z\,\hat{i} \times \hat{k}$$

$$+ A_y B_x\,\hat{j} \times \hat{i} + A_y B_y\,\hat{j} \times \hat{j} + A_y B_z\,\hat{j} \times \hat{k}$$

$$+ A_z B_x\,\hat{k} \times \hat{i} + A_z B_y\,\hat{k} \times \hat{j} + A_z B_z\,\hat{k} \times \hat{k}$$

$$= A_x B_x\,(0) + A_x B_y\left(\hat{k}\right) + A_x B_z\left(-\hat{j}\right)$$

$$+ A_y B_x\left(-\hat{k}\right) + A_y B_y\,(0) + A_y B_z\left(\hat{i}\right)$$

$$+ A_z B_x\left(\hat{j}\right) + A_z B_y\left(-\hat{i}\right) + A_z B_z\,(0)$$

$$= A_x B_y\,\hat{k} - A_x B_z\,\hat{j} - A_y B_x\,\hat{k} + A_y B_z\,\hat{i} + A_z B_x\,\hat{j} - A_z B_y\,\hat{i}$$

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\,\hat{i} + (A_z B_x - A_x B_z)\,\hat{j} + (A_x B_y - A_y B_x)\,\hat{k}$$

or

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec{A} \times \vec{B} = \hat{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \hat{j}\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \hat{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$= \hat{i}\,(A_yB_z - A_zB_y) - \hat{j}\,(A_xB_z - A_zB_x) + \hat{k}\,(A_xB_y - A_yB_x)$$

$$= \hat{i}\,(A_yB_z - A_zB_y) - \hat{j}\,[-1(A_zB_x - A_xB_z)] + \hat{k}\,(A_xB_y - A_yB_x)$$

$$= \hat{i}\,(A_yB_z - A_zB_y) + \hat{j}\,(A_zB_x - A_xB_z) + \hat{k}\,(A_xB_y - A_yB_x)$$

6. The magnitude of $\vec{A} \times \vec{B}$ is equal to the area of the parallelogram formed with $\vec{A}$ and $\vec{B}$ as two adjacent sides i.e.

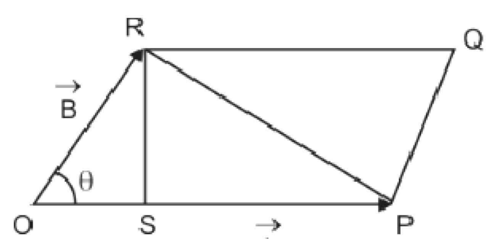
$$\left|\vec{A} \times \vec{B}\right| = \textbf{Area of parallelogram}$$

Solution

As $\vec{A} \times \vec{B} = AB \sin\theta\,\hat{n}$

$$\left|\vec{A} \times \vec{B}\right| = AB \sin\theta \quad \ldots\ldots\ldots \text{(i)}$$

Consider ΔORS



$$\frac{RS}{OR} = \sin\theta$$

$$RS = OR \sin\theta$$

$$RS = B \sin\theta$$

Multiply and divide by 2

$$\left|\vec{A} \times \vec{B}\right| = \frac{2}{2}(OP)\,(RS)$$

$$= 2\left(\frac{1}{2}(\text{Base})\,(\text{Height})\right)$$

$$= 2\,[\text{Area of } \Delta OPR]$$

$$= \text{Area of } \Delta OPR + \text{Area of } \Delta PQR$$

$$\therefore \quad \left|\vec{A} \times \vec{B}\right| = \text{Area of parallelogram}$$

7. Vector product is distributive, i.e.,

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

8. Vector product is associative, i.e.,

$$mn(\vec{A} \times \vec{B}) = mn\,\vec{A} \times \vec{B}$$

$$= m\vec{A} \times n\vec{B}$$

$$= \vec{A} \times mn\,\vec{B}$$

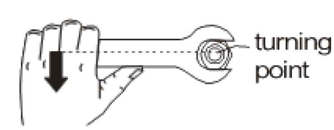
***Q.10 Explain torque. Also calculate the torque due to a free $\vec{F}$* acting on the rigid body.**

Ans. TORQUE (MOMENT OF FORCE)

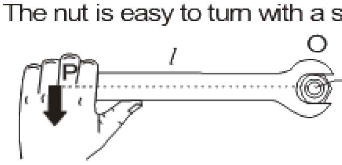
The turning effect of a force is called its torque and is equal to the product of Force and the perpendicular distance from its line of action to the pivot which is the point around which the body rotates. This perpendicular distance between line of action of force and pivot is called moment arm.

Magnitude of the Torque is represented by "τ" is given by

$$\tau = l\,F$$



The nut is easy to turn with a spanner.



It is easier still if the spanner has a long handle.

Torque depends upon two factors:

(i) Force

(ii) Moment arm

When the line of action of the applied force passed through the pivot, then

Moment arm $l = 0$

$$\therefore \quad \tau = 0\,(F)$$

$$\tau = 0$$

SI unit of torque is "Nm".

Dimensions

$$[\tau] = Nm$$
$$= kg\ m / s^2 \times m$$
$$= kg\ m^2 / s^2$$
$$= ML^2 / T^2$$
$$= [ML^2\ T^{-2}]$$

$F = ma$
$N = kg\ m/s^2$

Torque Due to a Force $\vec{F}$ Acting on a Rigid Body

Under the action of a force, if distance between the points of the body remains same such a body is called rigid body.

Let the force $\vec{F}$ acts on a rigid body at point P whose position vector relative to pivot O is $\vec{r}$. The force $\vec{F}$ can be resolved into two rectangular components i.e., $F \sin\theta$ perpendicular to $\vec{r}$ and $F \cos\theta$ in the direction of $\vec{r}$ as shown in Fig. (i). The torque due to $F \cos\theta$ is zero as its line of action of force passes through pivot O. Therefore torque due to $\vec{F}$ is equal to the torque due to $F \sin\theta$. As

$$\tau = \text{Moment arm} \times F$$
$$\tau = r F \sin\theta$$

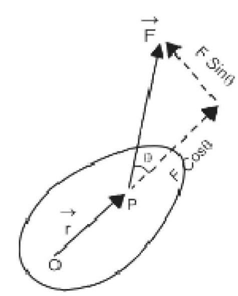


Fig. (i)

Alternatively the moment arm "l" is equal to the magnitude of the component of $\vec{r}$ perpendicular to the line of action $\vec{F}$ as shown in Fig. (ii).

As $\tau = l F$
$$\tau = r \sin\theta\ F$$
$$\tau = r F \sin\theta$$

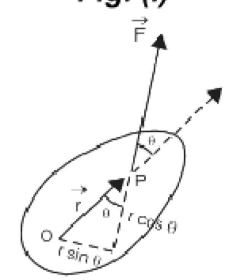


Fig. (ii)

Where θ is the angle between $\vec{r}$ and $\vec{F}$. From equations (1) and (2) torque can also be define as

Definition

The vector product of position vector $\vec{r}$ and the force $\vec{F}$ is also called torque.

$$\therefore \quad \vec{\tau} = \vec{r} \times \vec{F}$$

$$\vec{\tau} = r F \sin\theta \, \hat{n}$$

where $r F \sin\theta$ is the magnitude of torque. The direction of torque represented by $\hat{n}$ is perpendicular to the plane containing $\vec{r}$ and $\vec{F}$ given by right hand rule.

Note: *Torque is a vector quantity.*

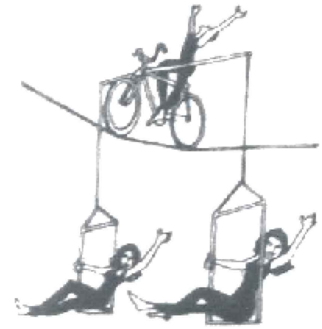
If $\vec{r}$ and $\vec{F}$ are in same direction, then torque will be minimum (0).

If $\vec{r}$ and $\vec{F}$ are in opposite direction, then torque will be minimum (0).

If $\vec{r}$ and $\vec{F}$ are perpendicular to each other, then torque will be maximum (rF).

Just as force determines the linear acceleration produced in a body, the torque acting on a body determines its angular acceleration. Torque is the analogous of force for rotational motion. If the body is at rest or rotating with uniform angular velocity, the angular acceleration will be zero. In this case the torque acting on the body will be zero.

Point to Ponder



Do you think the rider in the above figure is really in danger? What if people below were removed?

Ans. **There is no danger for the rider in the above figure because he is in stable equilibrium and no change if the people below were removed.**

Can You Do?

Stand with one arm and the side of one foot pressed against a wall. Can you raise the other leg side ways? If not, then why not?

Ans. **Yes.**

Q.11 Define equilibrium. Also state the conditions of equilibrium.

Ans. **EQUILIBRIUM**

If a body, under the action of a number of forces, is at rest or moving with uniform velocity, it is said to be in equilibrium.

1. **Static Equilibrium**

   If a body is at rest, it is said to be in Static Equilibrium. For example book lying on the table.

2. **Dynamic Equilibrium**

   If a body is moving with uniform velocity, or rotating with uniform angular velocity, it is said to be in Dynamic Equilibrium. For example jumping of paratrooper.

First Condition of Equilibrium

The vector sum of all the forces acting on a body must be equal to zero.

$$\Sigma \vec{F} = \vec{0}$$

In case of coplanar forces, the resultant force $\vec{F}_x$ is equal to sum of x–directed forces acting on the body. Therefore

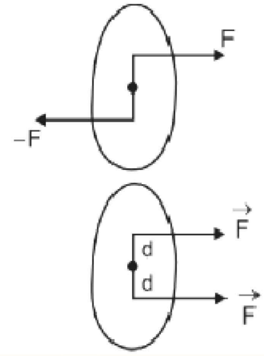
$$\therefore \quad \Sigma \vec{F}_x = \vec{0}$$

Similarly for y–directed forces, the resultant $\vec{F}_y$ should be zero. Therefore

$$\therefore \quad \Sigma \vec{F}_y = \vec{0}$$

Second Condition of Equilibrium

Let two equal and opposite forces are acting on body as shown in figure.



Although, the first condition of equilibrium is satisfied, yet it may rotate having clockwise turning effect.

Thus for a body in equilibrium, the vector sum of all the torques acting on it about arbitrary axis should be zero. This is known as second condition of equilibrium. Mathematically

$$\Sigma \vec{\tau} = \vec{0}$$

Requirements for a body to be in complete equilibrium are

(i) $\Sigma \vec{F} = \vec{0}$

i.e. $\Sigma \vec{F}_x = \vec{0}$

$\Sigma \vec{F}_y = \vec{0}$

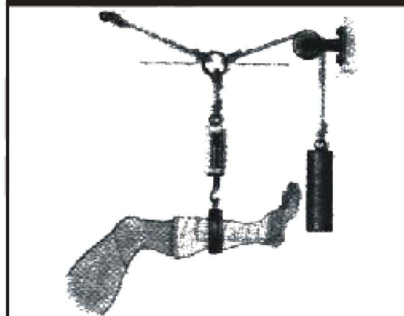
and $\Sigma \vec{\tau} = \vec{0}$

When first condition is satisfied, there is no linear acceleration and body will be in translation equilibrium (linear equilibrium).

When second condition is satisfied that is no angular acceleration and the body will be in rotational a equilibrium.

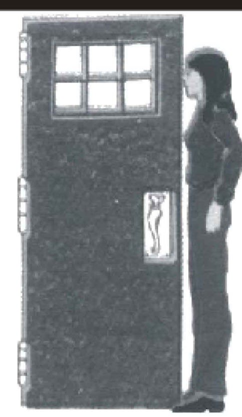
For a body to be in complete equilibrium, both conditions should be satisfied, i.e., both linear acceleration and angular acceleration should be zero.

Interesting Application



A concurrent force system in equilibrium. The tension applied can be adjusted as desired.

Can You Do?



With your nose touching the end of the door, put your feet astride the door and try to rise up on your toes.

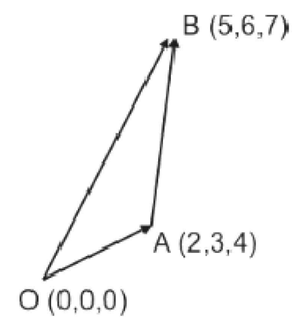
SOLVED EXAMPLES

EXAMPLE 2.1

The position of two aeroplanes at any instant are represented by points A (2, 3, 4) and B (5, 6, 7) from an origin 'O' in km as shown in the figure.

(a) What are their position vectors.

(b) Calculate the distance between two aeroplanes.



Data

Position of two acroplanes are

A(2, 3, 4) and B(5, 6, 7)

To Find

Distance between two aeroplanes AB = ?

SOLUTION

Position vector of aeroplane A is

$$\overrightarrow{OA} = 2\hat{i} + 3\hat{j} + 4\hat{k}$$

and that of B is

$$\overrightarrow{OB} = (5, 6, 7) - (0, 0, 0)$$

$$= 5\hat{i} + 6\hat{j} + 7\hat{k}$$

By head to tail rule

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

$$\therefore \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= 5\hat{i} + 6\hat{j} + 7\hat{k} - 2\hat{i} - 3\hat{j} - 4\hat{k}$$

$$= 3\hat{i} + 3\hat{j} + 3\hat{k}$$

$\therefore$ Distance between two aeroplanes is

$$AB = \sqrt{(3)^2 + (3)^2 + (3)^2}$$

$$= \sqrt{9 + 9 + 9}$$

$$= \sqrt{27}$$

Distance between two aeroplanes = AB = 5.2 km

EXAMPLE 2.2

Two forces of magnitudes 10N and 20N act on a body in directions making angles 30^o and 60^o with x-axis respectively. Find the resultant force.

Data

First force = F_1 = 10 N

Second force = F_2 = 20 N

First angle = θ_1 = 30^o

Second angle = θ_2 = 60^o

To Find

Resultant force = $\vec{F}$ = ?

SOLUTION

Resolving F_1 and F_2 into its components

$$F_{1x} = F_1 \cos 30^o$$
$$= 10 \times 0.866$$
$$= 8.66 \text{ N}$$

$$F_{1y} = F_1 \sin 30^o$$
$$= 10 \times 0.5$$
$$= 5 \text{ N}$$

$$F_{2x} = F_2 \cos 60^o$$
$$= 20 \times 0.5$$
$$= 10 \text{ N}$$

$$F_{2y} = F_2 \sin 60^o$$
$$= 20 \times 0.866$$
$$= 17.32 \text{ N}$$

Now $F_x = F_{1x} + F_{2x}$

$$= 8.66 + 10$$
$$= 18.66 \text{ N}$$

Also $F_y = F_{1y} + F_{2y}$

$$= 5 + 17.32$$
$$= 22.32 \text{ N}$$

$$= \sqrt{(18.66)^2 + (22.32)^2}$$

$$= \sqrt{348.19 + 498.18}$$

$$= \sqrt{84637}$$

$$F = 29 \text{ N}$$

For direction using

$$\phi = \tan^{-1} \frac{F_y}{F_x}$$

Putting values

$$\phi = \tan^{-1} \frac{22.32}{18.66}$$

$$= \tan^{-1} 1.196$$

$$\phi = 50^o$$

Since F_x and F_y are positive hence resultant lies in first quadrant.

$$\therefore \quad \theta = \phi$$

$$\therefore \quad \theta = 50^o$$

Result

Resultant force = F = 29 N

EXAMPLE 2.3

Find the angle between two forces of equal magnitude when the magnitude of their resultant is also equal to the magnitude of either of these forces.

Data

Angle between two forces F_1 and F_2 $\theta = ?$

Such that

$$F_1 = F_2 = F$$

Also $R = F$

SOLUTION

Using $R = \sqrt{F_1^2 + F_2^2 + 2 F_1 F_2 \cos\theta}$

$$\because \quad F_1 = F_2 = R = F$$

$$\therefore \quad F = \sqrt{F^2 + F^2 + 2 F F \cos\theta}$$

$$F = \sqrt{2 F^2 + 2 F^2 \cos\theta}$$

$$F^2 = 2F^2(1 + \cos\theta)$$

$$1 = 2(1 + \cos\theta)$$

$$\frac{1}{2} = 1 + \cos\theta$$

$$\cos\theta = \frac{1}{2} - 1$$

$$= -\frac{1}{2}$$

$$\theta = \cos^{-1}\left(-\frac{1}{2}\right)$$

$$\theta = 120^{\circ}$$

EXAMPLE 2.4

A force $F = 2\hat{i} + 3\hat{j}$ units, has its point of application moved from point A(1, 3) to the point B(5, 7). Find the work done.

Data

$$\vec{F} = 2\hat{i} + 3\hat{j}$$

Distance covered from

A (1, 3) to B (5, 7)

To Find

Work done = W = ?

SOLUTION

Using $\vec{d} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}$

$$= (5 - 1)\hat{i} + (7 - 3)\hat{j}$$

$$\vec{d} = 4\hat{i} + 4\hat{j}$$

Now using

$$W = \vec{F} \cdot \vec{d}$$

$$\therefore W = \left(2\hat{i} + 3\hat{j}\right) \cdot \left(4\hat{i} + 4\hat{j}\right)$$

Result

Work done = W = 20 Joule

EXAMPLE 2.5

Find the projection of vector $\vec{A} = 2\hat{i} - 8\hat{j} + \hat{k}$ in the direction of vector $\vec{B} = 3\hat{i} - 4\hat{j} - 12\hat{k}$.

Data

$$\vec{A} = 2\hat{i} - 8\hat{j} + \hat{k}$$

$$\vec{B} = 3\hat{i} - 4\hat{j} - 12\hat{k}$$

To Find

Projection of $\vec{A}$ along $\vec{B}$ = $A\cos\theta$ = ?

SOLUTION

Using $\vec{A} \cdot \vec{B} = AB\cos\theta$

$$\therefore \quad A\cos\theta = \frac{\vec{A} \cdot \vec{B}}{B}$$

$$= \frac{(2\hat{i} - 8\hat{j} + \hat{k}) \cdot (3\hat{i} - 4\hat{j} - 12\hat{k})}{\sqrt{(3)^2 + (-4)^2 + (-12)^2}}$$

$$= \frac{6 + 32 - 12}{\sqrt{9 + 16 + 144}}$$

$$= \frac{26}{\sqrt{169}}$$

$$= \frac{26}{13}$$

$$= 2$$

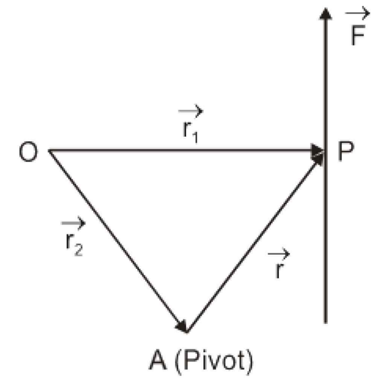
Result

Projection of $\vec{A}$ along $\vec{B}$ = $A\cos\theta$ = 2

EXAMPLE 2.6

The line of action of a force $\vec{F}$ passes through a point P of a body whose position vector in metre is $\hat{i} - 2\hat{j} + \hat{k}$. If $\vec{F} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ (in Newton) determine the torque about the point 'A' whose position vector (in metre) is $2\hat{i} + \hat{j} + \hat{k}$.

Data



$$\vec{r}_1 = \hat{i} - 2\hat{j} + \hat{k}$$

$$\vec{F} = 2\hat{i} - 3\hat{j} + 4\hat{k} \text{ (N)}$$

To Find

$$\vec{\tau} \text{ about A} = ?$$

SOLUTION

$$\vec{r}_2 = 2\hat{i} + \hat{j} + \hat{k}$$

By head to tail rule

$$\vec{r}_1 = \vec{r}_2 + \vec{r}$$

$$\therefore \quad \vec{r} = \vec{r}_1 - \vec{r}_2$$

Putting values

$$\vec{r} = (\hat{i} - 2\hat{j} + \hat{k}) - (2\hat{i} + \hat{j} + \hat{k})$$

$$= \hat{i} - 2\hat{j} + \hat{k} - 2\hat{i} - \hat{j} - \hat{k}$$

$$= -\hat{i} - 3\hat{j}$$

Now using

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\vec{\tau} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -3 & 0 \\ 2 & -3 & 4 \end{vmatrix}$$

$$= \hat{i}(-12 + 0) - \hat{j}(-4 - 0) + \hat{k}(3 + 6)$$

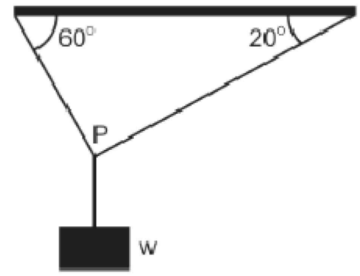
$$= -12\hat{i} + 4\hat{j} + 9\hat{k} \text{ (Nm)}$$

Result

$$\vec{\tau} \text{ about A} = -12\hat{i} + 4\hat{j} + 9\hat{k} \text{ (Nm)}$$

EXAMPLE 2.7

A load is suspended by two cords as shown in the figure. Determine the maximum load that can be suspended at P if maximum breaking stress of the cord used is 50N.



Data

Force = F = 50 N

To Find

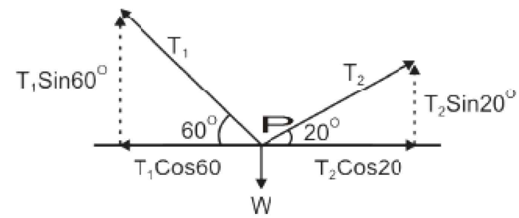
Maximum load = W = ?

SOLUTION

Resolving T_1 and T_2 into its components

Using 1st condition of equilibrium

Now using



$\Sigma F_y = 0$ and $\Sigma F_x = 0$

$T_1 \sin 60^o + T_2 \sin 20^o - W = 0$

$50\ (0.866) + 26.6\ (0.34) = W$

$W = 52\ N$

$T_2 \cos 20° - T_1 \cos 60° = 0$

$T_1 = 1.88\ T_2$

Since $T_1 > T_2$

$\therefore$ T_1 has maximum stress.

$T_1 = 50\ N$

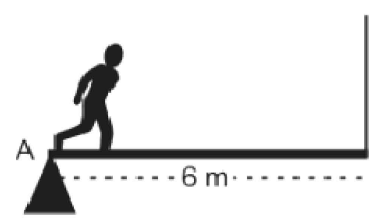
then $T_2 = 26.6\ N$

Result

Maximum load = W = 52 N

EXAMPLE 2.8

A uniform beam of 200N is supported horizontally as shown. If the breaking stress of the rope is 400N how far can the man of weight 400N walk from point A on the beam as shown in figure?



Data

Weight of beam = 200 N

Breaking stress of rope = 400N

Distance of man from point A = d = ?

Weight of man = 400N

To Find

Distance = d = ?

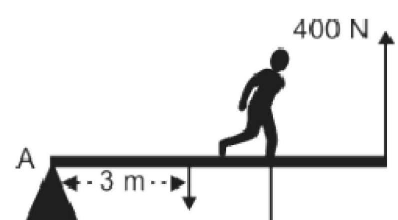
SOLUTION

Taking point A as pivot

Using $\Sigma \tau = 0$

$400 \times 6 - 400 \times d - 200 \times 3 = 0$

$2400 - 400\ d - 600 = 0$



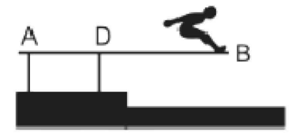
Result

Distance = d = 4.5 m

EXAMPLE 2.9

A body weighing 300N is standing at the edge of a uniform diving board 4.0m in length. The weight of the board is 200N. Find the forces exerted by pedestals on the board.

Data



Weight of boy = 300N

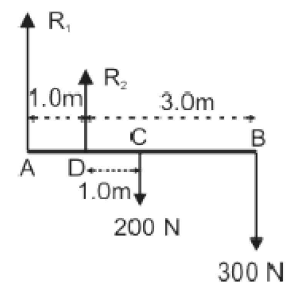
Length of board = 4m

Weight of board = 200N

To Find

Forces exerted by pedestals on board = ?

SOLUTION



Let R_1 and R_2 are the reaction forces exerted by the pedestals on the board. A little consideration will show that R_1 is in the wrong direction because the board must be actually pressed down in order to keep it in equilibrium. We will see that this assumption will be automatically corrected by calculations.

Now applying condition of equilibrium

$$\Sigma F_y = 0$$

$$R_1 + R_2 = 200 + 300$$

$$R_1 + R_2 = 500N \quad \ldots\ldots\ldots (1)$$

Now applying $\Sigma \tau = 0$ about point D.

$$-R_1 \times AD - 300 \times DB - 200 \times DC = 0$$

$$-R_1 \times 1 - 300 \times 3 - 200 \times 1 = 6$$

$$-R_1 - 900 - 200 = 0$$

$$-R_1 = 1100$$

$$R_1 = -1100 \text{ N}$$

Negative sign shows R_1 is directed downward.

⇒ Putting value of R_1 in equation (1).

$$-1100 + R_2 = 500$$

$$R_2 = 500 + 1100$$

$$= 1600 \text{ N}$$

$$\boxed{R_1 = -1100 \text{ N} = 1.1 \text{ KN}}$$

Force exerted by pedestals on board $= R_1 = 1.1$ KN

$= R_2 = 1.6$ KN

The negative sign tR_1 shows that it is directed downward.