UNIT

08

Logic

Introduction

Logic is a systematic method of reasoning that enables one to interpret the meanings of statements, examine their truth, and deduce new information from existing facts. Logic plays a role in problem-solving and decision-making.

History

The history of logic began with Aristotle, who is considered the father of formal logic. He developed a system of deductive reasoning known as syllogistic logic, which became the foundation of logical thought. The Stories followed, contributing to propositional logic and exploring paradoxes such as the liar Paradox. During the medieval period, scholars like Peter Abelard and William of Ockham expanded Aristotle's work, introducing theories of semantics and consequences. In the 19th century, logic advanced through the works of George Boole, who developed Boolean algebra, and Gottlob Frege, who formalized modern predicate logic. Bertrand Russell and Alfred Noth Whitehead attempted to reduce mathematics to logic in their seminal work, Principia Mathematica. The 20th century saw significant progress with Kurt Godel, who introduced his incompleteness theorems, reshaping our understanding of mathematical logic (history-of-logic):

Statement

A sentence or Mathematical expression which may be true or false but not both is called a statement. For instance, the statement $a = b$ can be either true or false.

Here, we discuss some examples of mathematical statements that are all true.

(i) For a non-zero real number x and integers m and n, we have: $x^m . x^n = x^{m+n}$

(ii) The sum of the measures of interior angles of a triangle is 180°

(iii) The circumference of a circle with radius r is $2\pi r$.

(iv) $Q \subseteq R$ (Set of rational numbers is a subset of set of real numbers)

(v) $\frac{22}{7} \notin Q'$

(vi) The sum of two odd integers is an even integer.

(vii) The mathematical statement $x^2 - 5x + 6 = 0$, for $x = 2$ or $x = 3$.

Here, are some examples of mathematical statements that are all false.

(i) $3 + 4 = 8$

(ii) $Z \subseteq W$

(iii) All isosceles triangle are equilateral triangle

(iv) Between any two real numbers there is no real number.

(v) $\{1, 2, 3, 4\} \cap \{-1, -2, -3, -4\} = \{1, 2, 3, 4\}$

(vi) If a and b are the length and width of a rectangle then Area of a rectangle is $\frac{1}{2}(a \times b)$.

(vii) The sum of interior angle of an n-sided polygon is $(n-1) \times 180^\circ$

(viii) The sum of the interior angles of any quadrilateral is always 180°

(ix) The set of integers is a finite set.

Logical Operators

The statements will be denoted by the letters p, q etc. A brief list of the symbols which will be used is given below:

Symbols	How to be read	Symbolic expression	How to be read
$\sim$	not	$\sim p$	Not p, negation of p
$\wedge$	and	$p \wedge q$	p and q
$\vee$	or	$p \vee q$	p or q
$\rightarrow$	If . . . then, implies	$p \rightarrow q$	If p then q, p implies q
$\leftrightarrow$	Is equivalent to, if and only if	$p \leftrightarrow q$	p if and only if q, p is equivalent to q

Explanation of the use of the Symbols

Negation

If p is any statement its negation is denoted by $\sim p$, read 'not p'. It follows from this definition that if p is true, $\sim p$ is false and if p is false, $\sim p$ is true. The possible truth-values of p and $\sim p$ is given in table is called truth table where true value is denoted by T and false value is denoted by F.

Table 1

p	$\sim p$
T	F
F	T

Conjunction

Conjunction of two statements p and q is denoted symbolically as $p \wedge q$ (p and q). A conjunction is considered to be true only if both statements are true. So, the truth table of $p \wedge q$ is in table.

Table 2

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 1:

(i) Lahore is the capital of the Punjab and Quetta is the capital of Baloachistan. 09308001

(ii) $4 < 5 \wedge 8 > 10$ 09308002

(iii) $2 + 2 = 3 \wedge 6 + 6 = 10$ 09308003

Clearly conjunction (i) and (ii) are true whereas (iii) are false.

Disjunction

Disjunction of p and q is symbolically written $p \vee q$ (p or q). The disjunction $p \vee q$ is considered to be true when at least one of the statements is true. It is false when both of them are false.

The truth table is given as:

Table 3

P	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 2: 10 is a positive integer or 0 is a rational number. Find truth value of this disjunction. 09308004

Solution: Since the first statement is true, the disjunction is true.

Example 3: Triangle can have two right angles or Lahore is the capital of Sind. Find truth value of this disjunction. 09308005

Solution: Both the statements being false, the disjunction is false.

Implication or conditional

A compound statement of the form *if p* then q also written **p implies q** is called a **conditional** or an **implication**. p is called the **antecedent** or **hypothesis** and q is called the **consequent** or the **conclusion**.

A conditional is regarded as false only when the antecedent is true and consequent is false. In all other cases conditional is considered to be true. Its truth table is given in table.

p	q	$p\to q$
T	T	T
T	F	F
F	T	T
F	F	T

We attempt to clear the position with the help of an example. Consider the conditional:

If a person *A* lives at Lahore, then he lives in Pakistan.

If the antecedent is false i.e., *A* does not live in Lahore, all the same he may be living in Pakistan. We have no reason to say that he does not live in Pakistan.

We cannot, therefore, say that the conditional is false. So we must regard it as true. Similarly, when both the antecedent and consequent of the conditional under consideration are false, then is no justification for quarrelling with the statement.

Biconditional $p\leftrightarrow q$

The statement $p\to q \wedge q\to p$ is shortly written $p \leftrightarrow q$ and is called the **biconditional** or **equivalence**. It is read ***p*** iff ***q*** (iff stands for "if and only if")

We draw up its truth table.

From the table it appears that

Table 5

p	q	$p\to q$	$q\to p$	$p\leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

$p\leftrightarrow q$ is true only when both statements p and q are true or both statements p and q are false.

Conditionals related with a given conditional.

Let p and q be the statements and $p\to q$ be a given conditional, then

(i) $q\to p$ is called the **converse** of $p\to q$;

(ii) $\sim p\to\sim q$ is called the **inverse** of $p\to q$;

(iii) $\sim q\to\sim p$ is called the **contrapositive** of $p\to q$.

The truth values of these new conditionals are given below in table.

Table 6

				Given conditional	Converse	Inverse	Contra positive
p	q	$\sim p$	$\sim q$	$p\to q$	$q\to p$	$\sim p\to\sim q$	$\sim q\to\sim p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

From the table it appears that

(i) Any conditional and its contrapositive are equivalent therefore any theorem may be proved by proving its contrapositive.

(ii) The converse and inverse are equivalent to each other.

Example 4: Prove that in any universe the empty set Φ is a subset of any set A. **09308006**

First Proof: Let U be the universal set consider the conditional:

$$\forall x \in U, x \in \Phi \rightarrow x \in A$$

(1) The antecedent of this conditional is false because no $x \in U$, is a member of φ.

Hence the conditional is true.

Second proof: The contrapositive of conditional (1) is

$$\forall x \in U, x \notin A \rightarrow x \notin \phi$$

(2). The consequent of this conditional is true. Therefore, the conditional is true.

Hence in any universe the empty set Φ is a subset of any set A.

Example 5: Construct the truth table of $[(p \rightarrow q) \wedge p]$ and $[(p \rightarrow q) \wedge p] \rightarrow q$ **0930800**

Solution: Desired truth table is given below:

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$[(p \rightarrow q) \wedge p] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Mathematical Proof

Suppose Fayyaz is a student in Grade 9. On day, he arrived home late due to heavy traffic in a city. His father, however, suspected that Fayyaz had not gone to school and instead spent the day elsewhere. To address his concerns, his father asked, "Tell me the truth, did you go to school today? Fayyaz responded, saying, "Yes, I did." Still doubtful, his father asked, "What proof do you have that you attended school? To satisfy his father's concern, Fayyaz says that my classmate Ahmad went to school with me and could confirm with him. But his father was still not convinced by his words. Now, how will he prove his father's claim that he went to school or not? To prove his father's claim, Fayyaz would need to present some evidence, like his attendance for that day, which was recorded in the school attendance register, or CCTV footage from the school to prove that he was indeed present that day.

Example 6: Prove the following mathematical statements. **09308008**

(a) If x is an odd integer, then x^2 is also an odd integers.

(b) The sum of two odd numbers is even number.

Proof (a): Let x be an odd integer. Then by definition of an odd integer, we can express x as:

$$x = 2k + 1 \text{ for some integer } k \in Z$$

$$\text{Now } x^2 = (2k+1)^2 = 4k^2 + 4k + 1$$

$$= 2(2k^2 + 2k) + 1$$

$$= 2m + 1, \text{ where } m = 2k^2 + 2k \in Z$$

$$\text{Thus, } x^2 = 2m + 1 \text{ for some } m \in Z$$

Therefore x^2 is an odd integers, by definition of an odd integers.

> **Note:**
> If x is odd, then x can be expressed in the form:
> $x = 2k + 1$ for some integer $k \in Z$

Proof (b): Let x and y be odd integers. Then by definition of an odd integer, we can express x and y as:

$x = 2k + 1$ and $y = 2n + 1$ for some integers k and n.

> **Note:**
> If x is an even integer, then x can be expressed in the form:
> $x = 2k$ for some integer $k \in Z$

Thus, $x + y = (2k+1) + (2n+1)$
$= 2k + 2n + 1 + 1$
$= 2(k + n + 1) = 2m$, where $k + n + 1 = m \in Z$

So, $x + y = 2m$ for some integer m.

Therefore $x + y$ is an even integer, by definition of an even integer.

Example 7: Prove that for any two non-empty sets A and B, $(A \cup B)' = A' \cap B'$.

09308009

Proof: Let $x \in (A \cup B)'$

$\Rightarrow \quad x \notin (A \cup B)$

$\Rightarrow \quad x \notin A \text{ and } x \notin B$

$\Rightarrow \quad x \in A' \text{ and } x \in B'$

$\Rightarrow \quad x \in A' \cap B'$

But $x \in (A \cup B)'$ is an arbitrary element

Therefore $(A \cup B)' \subseteq A' \cap B'$ ....(1)

Now suppose that $y \in A' \cap B'$

$\Rightarrow \quad y \in A' \text{ and } y \in B'$

$\Rightarrow \quad y \notin A \text{ and } y \notin B$

$\Rightarrow \quad y \notin (A \cup B)$

$\Rightarrow \quad y \in (A \cup B)'$

Thus $A' \cap B' \subseteq (A \cup B)'$ ....(2)

From equations (1) and (2) we conclude that

$(A \cup B)' = A' \cap B'$

Hence proved

Note:

A set B is a subset of a set A if every element of set B is also an element of a set A. Mathematically, we write:

$B \subseteq A$ if $\forall x \in B \Rightarrow x \in A$

Theorem, Conjucture and Axion

Theorem: A theorem is a mathematical statement that has been proved true based on previously known facts. For example, the following statements are theorem

(i) Theorem: The sum of the interior angles of a quadrilateral is 360 degrees.

(ii) The Fundamental Theorem of Arithmetic: Every integer greater than 1 can be uniquely expressed as a product of prime numbers, up to the order of the factors.

(iii) Fermat's Last Theorem: There are no three positive integers $a, b, c, n \in N$ where $n > 2$ that satisfy the equation $a^n + b^n = c^n$.

Conjecture: A conjecture is a mathematical statement or hypothesis that is believed to be true based on observations, but has not yet been proved. In mathematics, conjectures often serve as hypotheses, and if a conjecture is proven to be true, it becomes a theorem. Conversely, if evidence is found that disproves it, the conjecture is shown to be false.

Conjecture. The Goldbach Conjecture states that:

Statement: Every even integer greater than 2 is a sum of two prime numbers.

Next, we are going to study the same statement which is known as axiom.

Axioms: A mathematical statement which we believe to be true without any evidence or requiring any proof. In other words, these statements are basic facts that form the starting point for further ideas and are based on everyday experiences. For example, the following are the statements of axioms.

Axiom: Through a given point, there pass infinitely many lines.

Euclid Axioms: A straight line can be drawn between any two points.

Peano Axioms: Every natural number has a successor, which is also a natural number.

Axiom of Extensionality: Two sets are equal if they have the same elements

Axiom of Power Set: For any set, there is a set of all its subsets.

Postulate: An idea that is suggested or accepted as a basic principle before a further idea is formed or developed from it.

We are going to prove a theorem.

Example 8: prove that $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$ where *a, b, c* and *d* are non-zero real numbers. 09308010

Solution:

$$\text{L.H.S} = \frac{a}{b}+\frac{c}{d}$$

$$= \frac{a}{b}\times 1+\frac{c}{d}\times 1 \quad \text{[Multiplicative identity]}$$

$$= \frac{a}{b}\times\left(d\times\frac{1}{d}\right)+\frac{c}{d}\times\left(b\times\frac{1}{b}\right)$$

[Multiplicative inverse]

$$= \frac{a}{b}\times\frac{d}{d}+\frac{c}{d}\times\frac{b}{b} \quad \because\ [a\times\frac{1}{b}=\frac{a}{b}]$$

$$= \frac{ad}{bd}+\frac{cb}{db}$$

[Rule of production of fraction $\frac{a}{b}.\frac{c}{d}=\frac{ac}{bd}$]

$$= \frac{ad}{bd}+\frac{bc}{bd}$$

[Commutative law of multiplication $ab=ba$]

$$= ad\times\frac{1}{bd}+bc\times\frac{1}{bd} \quad [a\times\frac{1}{b}=\frac{a}{b}]$$

$$= (ad+bc).\frac{1}{bd}$$

[Distributive property]

$$\text{L.H. S.} = \frac{(ad+bc)}{bd}$$

$$\text{L.H.S} = \text{R.H.S}$$

Thus, $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$

Deductive Proof

Deductive reasoning is a way of drawing conclusions from premises believed to be true.

If the premises are true, then the conclusion must also be true. For example: All humans need to breathe to live. Ahmad is a human. Therefore, Ahmad is also breathe to live.

Similarly in mathematics, deductive proof in algebraic expression is a technique to show the validity of mathematical statement through a logical reasoning based on known rules, theorem, axioms, or previously proven statements. Deductive reasoning are broadly used in algebra to validate different identities and for solving equations.

Example 9: Prove that:

$(x+1)^2 + 7= x^2+2x+8$

Deductive Proof: L.H.S 093080

$= (x+1)^2 + 7$

$= (x+1)\ (x+1)+ 7$

$= x(x+1)\ (x+1) + 7 \quad (\because x^m, x^n = x^{m+n})$

$= x.(x+1)+1.(x+1)+7$

($\because$ Right distribution la

($\because$ Left distribution la

$= x.x+x.1+1.x+1.1+7$

$= x^2+1.x+1.x+1+7$

($\because$ commutative law & $x^m.x^n = x^{m+n}$)

$= x^2+(1+1)x+8$ ($\because$ Left distribution law)

$= x^2+2x+8$

= R.H.S

$\Rightarrow$ L.H.S = R.H.S

Thus, $(x+)^2+7 = x^2+2x+8$. Hence proved

Example 10: Prove that $\frac{45x+15}{15} = 3x+1$ by justify each step. **09308012**

Deductive Proof: L.H.S $= \frac{45x+15}{15}$

$= \frac{1}{15} \times (45x+15)$

$= \frac{1}{15} \times (45x+15) \quad \left(\because \frac{a}{b} = \frac{1}{b} \times a\right)$

$= \frac{1}{15} \times (15 \times 3x + 15 \times 1)$

($\because$ Multiplicative identity)

$= \frac{1}{15} \times 15(3x+1)$ ($\because$ Distributive law)

$= \left(\frac{1}{15} \times 15\right).(3x+1)$ ($\because$ Associative law)

$= 1.(3x+1)$ ($\because$ Multiplicative inverse)

$= 3x + 1 = \text{R.H.S}$ ($\because$ Multiplicative identity)

Thus, $\frac{45x+15}{15} = 3x + 1$ hence proved.

Review Exercise 8

Q.1 Choose the correct option.

(i) Which of the following expressions is: **09308013** often related to inductive reasoning?

(a) Based on repeated experiments
(b) If and only if statements
(c) Statement is proven by a theorem°
(d) Based on general principles

(ii) Which of the following sentences describe deductive reasoning? **09308014**

(a) General conclusions from a limited number of observations.
(b) Based on repeated experiments
(c) Based on units of information that are accurate
(d) Draw conclusion from well-known facts

(iii) Which one of the following statements is true? **09308015**

(a) The set of integers in finite
(b) The sum of the interior angles of any quadrilateral is always 180°
(c) $\frac{22}{7} \notin Q'$
(d) All isosceles triangles are equilateral triangles.

(iv) Which of the following statements is the best to represent the negation of the statement "The stove is burning"? **09308016**

(a) The stove is not burning.
(b) The stove is dim
(c) The stove is turned to low heat
(d) It is both burning and not burning

(v) The conjunction of two statements p and q is the true when: **09308017**

(a) Both p and q are false
(b) Both p and q are true
(c) Only q is true
(d) Only p is true

(vi) A conditional is regarded as false only when: **09308018**

(a) Antecedent is true and consequent is false
(b) Consequent is true and antecedent is false
(c) Antecedent is true only
(d) Consequent is false only.

(vii) Contrapositive of q → p is: **09308019**

(a) q → ~p (b) q → ~p
(c) ~p → ~q (d) ~q → ~p

(viii) The statement "Every integer greater than 2 is a sum of two prime numbers" is: **09308020**

(a) theorem (b) conjecture
(c) axiom (d) postulates

(ix) The statement "A straight line can be drawn between any two points" is: **09308021**

(a) theorem (b) conjecture
(c) axiom (d) logic

(x) The statement "The sum of the interio angle of a triangle is 180°" is: 09308022
(a) converse (b) theorem
(c) axiom (d) conditional

Answers Key

i	**a**	ii	**d**	iii	**c**	iv	**a**	v	**b**
vi	**a**	vii	**c**	viii	**b**	ix	**c**	x	**b**

Multiple Choice Questions (Additional)

1. Who is considered Father of formal logic? 09308023
(a) Aristotle (b) Alfred Noth
(c) Bertrand Russell (d) Kurt Godel

2. The conjunction of two statements p and q is denoted by: 09308024
(a) $p \wedge q$ (b) $p \vee q$
(c) $\sim p \wedge \sim q$ (d) $\sim p \vee \sim q$

3. The disjunction of two statements p and q is denoted by: 09308025
(a) $p \wedge q$ (b) $p \vee q$
(c) $\sim p \wedge \sim q$ (d) $\sim p \vee \sim q$

4. The conjunction of negations of two statements p and q is denoted by: 09308026
(a) $p \wedge q$ (b) $p \vee q$
(c) $\sim p \wedge \sim q$ (d) $\sim p \vee \sim q$

5. The disjunction of negation of two statements p and q is denoted by: 09308027
(a) $p \wedge q$ (b) $p \vee q$
(c) $\sim p \wedge \sim q$ (d) $\sim p \vee \sim q$

6. The negation of statement p is denoted by: 09308028
(a) $\wedge p$ (b) $\vee p$
(c) $\sim p$ (d) $-p$

7. The conjunction $p \wedge q$ is True when p and q are: 09308029
(a) T,T (b) T,F
(c) F,T (d) F,F

8. The disjunction $p \vee q$ is False when p and q are: 09308030
(a) T,T (b) T,F
(c) F,T (d) F,F

9. Any conditional and it------are equivalent. 09308031
(a) negation (b) contrapositive
(c) converse (d) Inverse

10. Which of the following is the odd number for $k \in N$? 09308032
(a) $k+1$ (b) $2k$
(c) $2k+1$ (d) $2k+2$

11. If a=b, b=c then a=c is an example of : 09308033
(a) axiom (b) postulate
(c) theorem (d) proof

12. The statement that has been proved true based on previously known facts is: 09308034
(a) axiom (b) postulate
(c) theorem (d) proof

Answer Key

1	**a**	2	**a**	3	**b**	4	**c**	5	**d**	6	**c**	7	**a**	8	**d**	9	**b**	10	**c**
11	**a**	12	**c**																

Q.2 Write the converse, inverse and contrapositive of the following conditionals:

(i) $\sim p \rightarrow q$ 09308035

Solution:

$\sim p \rightarrow q$

Converse: $q \rightarrow \sim p$

Inverse: $p \rightarrow \sim q$

Contra positive: $\sim q \rightarrow p$

(ii) $q \to p$ 09308036

Solution:

$q \to p$

Converse: $p \to q$

Inverse: $\sim q \to \sim p$

Contra positive: $\sim p \to \sim q$

(iii) $\sim p \to \sim q$ 09308037

Solution

$\sim p \to \sim q$

Converse: $\sim q \to \sim p$

Inverse: $p \to q$

Contra positive: $q \to p$

(iv) $\sim q \to \sim p$ 09308038

Solution:

Converse: $\sim p \to \sim q$

Inverse: $q \to p$

Contra positive: $p \to q$

Q.3 Write the truth table of the following

(i) $\sim(p \vee q) \vee (\sim q)$ 09308039

Solution:

p	q	$p \vee q$	$\sim(p \vee q)$	$\sim q$	$\sim(p \vee p) \vee (\sim q)$
T	T	T	F	F	F
T	F	T	F	T	T
F	T	T	F	F	F
F	F	F	T	T	T

(ii) $\sim(\sim q \vee \sim p)$ 09308040

Solution:

p	q	$\sim p$	$\sim q$	$(\sim q \vee \sim p)$	$\sim(\sim q \vee \sim p)$
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

(iii) $(q \vee p) \leftrightarrow (p \wedge q)$ 09308041

Solution:

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \leftrightarrow (p \sim p)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

Q.4 Differentiate between a Mathematical Statement and its proof and provide two examples. 09308042

Solution

A sentence or mathematical expression which may be true or false but not both is called a statement. This is correct so far as mathematics and other sciences are concerned. For instance, the statement $a = b$ can be either true or false.

We can think of a mathematical statement as a unit of information that is either accurate or inaccurate.

Here, we discuss some examples of mathematical statements are:

(i) For a non-zero real number x and integers m and n, we have: $x^m.x^n = x^{m+n}$.

(ii) The sum of the measures of the interior angles of a triangle is 180°

(iii) The circumference of a circle with radius r is $2\pi r$.

(iv) $Q \subseteq R$ (The set of rational numbers is a subset of the set of real numbers).

Mathematical Proof

A proof is a step-by-step logical explanation to establish the truth of a mathematiczal statement. It is a detailed and logical procedure which explains how the statement is true. The proof involves axioms, postulates theorems, and logical deductions.

Example

Proof involves the properties of parallel lines and alternate interior angles to show that the sum of the three angles is $180°$.

(ii) Proof involves the properties of sum of angles on a point on the a line is $180°$.

Q.5 What is the difference between an axiom and a theorem? Provide examples of each.

Solution:

A theorem is a mathematical statement that has been proved true based on previously known facts. For example, the following statements are theorem:

(i) Theorem: The sum of the interior angles of a quadrilateral is 360 degrees.

(ii) The fundamental Theorem of **Arithmetic:** Every integer greater than 1 can be uniquely expressed as a product of prime numbers up to the order of the factors:

An Axiom is a mathematical statement that we believe to be true without any evidence or requiring any proof. For example, the following are the statements of axioms.

Axiom: Through a given point, infinitely many lines can pass.

Euclid Axioms: A straight line can be drawn between any two points.

Q.6 What is the importance of logical reasoning in mathematical proofs? Give an example to illustrate your point. 09308043

Logic is a systematic method of reasoning that enables one to interpret the meanings of statements, examine their truth, and deduce new information from existing facts. Logic plays a role in problem-solving and decision-making.

We generally use logic in our daily life while engaging in mathematics. For example, we often draw general conclusion from a limited number of observations or experiences. A person gets a penicillin injection once or twice and experiences a reaction soon afterward. He generalizes that he is allergic to penicillin. This way of drawing conclusions is called induction. Inductive reasoning is helpful in natural sciences, where we must depend upon repeated experiments or observations. In fact greater part of our knowledge is based on induction.

(ii) Statement: "The sum of two odd numbers is always even." 09308044

Solution:

"The sum of two odd numbers is always even."

Proof: If m and n be any two numbers then 2m and 2n are even numbers. [multiple of 2]

$2m + 1$ and $2n + 1$ are odd numbers [even+1 = odd]

Sum: $(2n+1) + (2n+1)$

$= 2m+2n+2$

$= 2(m+n+1)$

Being multiple of 2 the sum of even numbers.

Q.7 Indicate each of the following whether it is an axiom, conjecture**, or** theorem**, and explain your reasoning.**

(i) "There is exactly one straight line through any two points." 09308045

Solution:

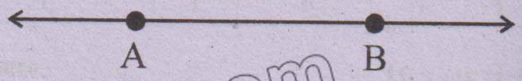
"Through any two points, there is exactly one straight line."

The statement is an Axiom.

Explanation

Take two points A and B in a plane.

Draw a line, passing through these points.



Now, try to pass any other lines through the same points A and B. you will get the same straight line even after trying several times, so it is our common observation that through any two points, there is exactly one straight line". The statement is self evidence and there is no need to prove it mathematically.

(ii) "Every even number greater than 2 can be written as the sum of two prime numbers." **09308046**

Solution:

"Every even number greater than 2 can be written as the sum of two prime numbers."

The statement is conjecture.

Explanation

Take even numbers greater than2. Like, 4, 6, 8, 10, 12….

$4 = 2+2$, $6 = 3+3$, $8 = 3+5$, $10 = 3+7$, $12 = 5+7$

On the base of empirical evidence, it appears to be true but this does not preclude the possibility that cannot be expressed as the sum of two primes. Since, there is need to prove the statement mathematically, so we can say that statement is not proved yet mathematical. So it is a conjecture.

(iii) "The sum of the angles in a triangle is 180 degrees." **09308047**

Solution:

"The sum of the angles in a triangle is 180 degrees."

The statement is a theorem.

Explanation

The statement is the well known fact of geometry which has been proved mathematical. So it is a geometrical theorem.

Q.8 Formulate Simple Deductive Proofs

For each of the following algebraic expressions, prove that the LHS is equal to the RHS:

(i) Prove that $(x-4)^2+9=x^2-8x+25$ **09308048**

Solution:

$(x-4)^2+9=x^2-8x+25$

Proof

L.H.S $= (x-4)^2+9$

expanding by identity $(a-b)^2=a^2-2ab+b^2$

$$= (x)^2-2(x)(4)+(4)^2+9$$
$$= x^2-8x+16+9$$
$$= x^2-8x+25$$

L.H.S = R.H.S

Hence proved

(ii) Prove that $(x+1)^2-(x-1)^2=4x$ **09308049**

Solution:

$(x+1)^2-(x-1)^2=4x$

L.H.S $= (x+1)^2-(x-1)^2$

Using algebraic identities.

$(a+b)^2=a^2+2ab+b^2$

$(a-b)^2=a^2-2ab+b^2$

$$= [(x)^2+(1)^2+2(x)(1)]-[(x)^2+(1)^2-2(x)(1)]$$
$$= (x^2+1+2x)-(x^2+1-2x)$$
$$= \cancel{x^2}+\cancel{1}+2x-\cancel{x^2}-\cancel{1}+2x$$
$$= 2x+2x$$
$$= 4x$$

L.H.S = R.H.S. Hence $(x+1)^2-(x-1)^2=4x$

(iii) Prove that $(x+5)^2-(x-5)^2=20x$ **09308050**

Solution:

$(x+5)^2-(x-5)^2=20x$

L.H.S $= (x+5)^2-(x-5)^2$

$$= [(x)^2+(5)^2+2(x)(5)]-[(x)^2+(5)^2-2(x)(5)]$$
$$= (x^2+25+10x)-(x^2+25-10x)$$
$$= x^2+25+10x-x^2-25+10x$$
$$= 10x+10x$$

L.H.S $=20x$

L.H.S = R.H.S

Hence, $(x+5)^2-(x-5)^2=20x$

Q.9 Prove the following by justifying each step.

(i) $\dfrac{4+16x}{4}=1+4x$ **09308051**

Solution:

$$\text{L.H.S} = \frac{4+16x}{4}=\frac{1}{4}(4+16x)\quad(\because \frac{a}{b}=a\times\frac{1}{b})$$

$$= \frac{1}{4}(1\times4+4\times4x)\ (\because \text{Multiplicative Identity})$$

$$= \frac{1}{4}\times 4\,(1+4x)\qquad(\because \text{Distributive law})$$

$$= 1(1+4x)\qquad(\because \text{Multiplicative inverse})$$

= 1+4x (∵ Multiplicative identity)

L.H.S = R.H.S

Thus $\frac{4+16x}{4}$ = 1+4x , hence proved.

(ii) $\frac{6x^2+18x}{3x^2-9}=\frac{2x}{x-3}$

09308052

Solution:

$\frac{6x^2+18x}{3x^2-9}=\frac{2x}{x-3}$

L.H.S = $\frac{6x^2+18x}{3x^2-3}$

= $\frac{6\times x^2+6\times 3x}{3\times x^2-3\times 1}$ Factorization

= $\frac{6x(x+3)}{3(x^2-9)}$ (∵ Distributive law)

= $\frac{1}{3}\times\frac{3\times 2x(\times+3)}{x^2-3^2}$ "Factors of 6x"

= $\frac{1\times 2x(x+3)}{(x+3)(x-3)}$ (Multiplicative inverse)

∵ $a^2-b^2=(a+b)(a-b)$

= $\frac{2x(x+3)}{(x+3)(x-3)}$ (Multiplicative identity)

= $\frac{1(x+3)}{(x+3)}\times\frac{(2x)}{x-3}$

= $\frac{1}{(x+3)}\times(x+3)\times\frac{(2x)}{(x-3)}$ Multiplicative identity

= $\frac{1\times(2x)}{x-3}$ Multiplicative inverse

= $\frac{2x}{x-3}$ Multiplicative inverse

L.H.S = R.H.S

Thus $\frac{6x^2+18x}{3(x^2-9)}=\frac{2x}{x+3}$, hence proved

(iii) $\frac{x^2+7x+10}{x^2-3x-10}=\frac{x+5}{x-5}$ 09308053 **(correction)**

Solution:

$\frac{x^2+7x+10}{x^2-3x-10}$

L.H.S = $\frac{x^2+2x+5x+10}{x^2-5x+2x-10}$

[midterm breaking]

= $\frac{x(x+2)+5(x+2)}{x(x-5)+2(x-5)}$ (∵ Distributive law)

= $\frac{(x+2)(x+5)}{(x-5)(x+2)}$ (∵ Distributive law)

= $\frac{(x+2)(x+5)}{(x+2)(x-5)}$ (Commutative law)

= $\frac{1}{(x+2)}\times(x+2)\times\frac{(x+5)}{(x-5)}$ $\left(\because \frac{a}{b}=a\times\frac{1}{b}\right)$

= $\frac{(x+5)}{(x-5)}$ (Multiplicative inverse)

L.H.S = R.H.S Hence proved.

Q.10 Suppose *x* is an integer. If *x* is odd then 9*x* + 4 is odd. 09308054

Solution:

Proof:

Given expression = 9x+4

If an integer x is odd then it can be written as:

x = 2k+1 where k ∈ Z

Put x = 2k+1in given expression

9x +4 = 9(2k+1)+4

= 18k+9+4

= 18k+13

= 18k+12+1

= 2(9k+6)+1

Being multiple of 2 the term 2(9k+6) is clearly even.

9x +4= Even + 1

9x +4=Odd [Even +1=Odd]

Hence it is proved that if an integer x is odd the 9x+4 is an odd.

Q.11 Suppose *x* is an integer. If *x* is odd then 7*x* + 5 is even. 09308055

Solution:

Proof:

Given expression = 7x+5

If an integer x is odd then it can be written as:

$x = 2k+1$ where $k \in Z$

Put x=2k+1in given expression

$7x+5=7(2k+1)+5$

$= \quad 14k+7+5$

$= \quad 14k+12$

$= \quad 2(7k+6)$

$7x+5$ = Even (Multiple of 2)

Since k is an integer therefore 7k+6 is also an integer and being a multiple of 2 the result 2(7k+6) is even.

Hence it is proved that if an integer x is odd the 7x+5 is an Even.

Q.12 Prove the following statement:

(a) If x is an odd integer then show that $x^2 - 4x + 6$ is odd. **09308056**

Proof:

Given expression = $x^2 - 4x + 6$

If an integer x is odd then it can be written as:

$x=2k+1$ where $k \in Z$

Put x=2k+1in given expression

$$\begin{aligned} x^2 - 4x + 6 &= (2k+1)^2 - 4(2k+1)+6 \\ &= (2k)^2+(1)^2+2(2k)(1)-8k-4+6 \\ &= 4k^2+1 +4k -8k+2 \\ &= 4k^2 -4k+2+1 \\ &= 2(2k^2 -2k+1)+1 \end{aligned}$$

Being multiple of 2 the expression

$2(2k^2 - 2k+1)$ is an even for any integer k.

$x^2 - 4x + 6$= Even +1

Since adding 1 to even number forms Odd number so,

$x^2 - 4x + 6$= Odd

Hence proved if x is an odd integer then $x^2 - 4x + 6$ is odd.

(b) If x is an even integer then show that $x^2 + 2x + 4$ is even. **09308057**

Proof:

Given expression = $x^2 + 2x + 4$

If an integer x is odd then it can be written as:

$x=2k$ where $k \in Z$

Put x=2k in given expression

$$\begin{aligned} x^2 + 2x + 4 &= (2k)^2+2(2k)+4 \\ &= 4k^2+4k+4 \\ &= 4k^2+4k+4 \\ &= 2(2k^2+2k+2) \end{aligned}$$

Being multiple of 2 the expression $2(2k^2+2k+2)$ is an even for any integer k.

$x^2 + 2x + 4$= Even

Hence proved if x is an even integer then $x^2 + 2x + 4$ is even.

Q.13 Prove that for any two non-empty set A and B, $(A \cap B)' = A' \cup B'$. **09308058**

Solution:

Proof:

Let $x \in (A \cap B)'$

$\Rightarrow x \notin (A \cap B)$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \in A'$ or $x \in B'$

$\Rightarrow x \in (A' \cup B')$

But $x \in (A \cap B)'$ is an arbitrary element.

Therefore $(A \cap B)' \subseteq (A' \cup B')$ …….(i)

Let $y \in (A' \cup B')$

$\Rightarrow y \in A'$ or $y \in B'$

$\Rightarrow y \notin A$ or $y \notin B$

$\Rightarrow y \notin (A \cap B)$

$\Rightarrow y \in (A \cap B)'$

$\Rightarrow (A' \cup B') \subseteq (A \cap B)'$ ………(ii)

From equation (i) and (ii) we conclude that $(A \cap B)' = A' \cup B'$.

Hence proved

Alternate Method:

$(A \cap B)' = A' \cup B'$

The logical form of the theorem is:

$\sim(p \wedge q) = \sim p \vee \sim q$

p	q	$\sim p$	$\sim q$	$(p \wedge q)$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

From the last two columns of the truth table we observe that $\sim(p \wedge q) = \sim p \vee \sim q$

Hence, $(A \cap B)' = A' \cup B'$

Q.14 If x and y are positive real numbers and $x^2 < y^2$ then $x < y$. **09308059**

Solution:

$x^2 < y^2$ then $x < y$

$x^2 < y^2$

Given that x > 0, y > 0 square is defined for positive numbers.

$\Rightarrow \sqrt{x^2} < \sqrt{y^2}$

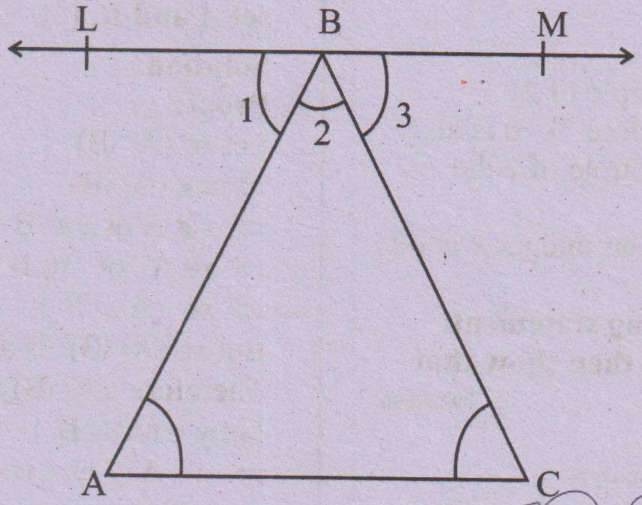
$\Rightarrow x < x$

Q.15 The sum of the interior angle of a triangle is 180° 09308060

Solution:

triangle is 180°

Given: A ΔABC



To prove

$m\angle A + m\angle B + m\angle C = 180°$

Construction

Passing through vertex B, draw $\overleftrightarrow{LM} \parallel \overline{AC}$.

Proof

$m\angle 1 + m\angle 2 + m\angle 3 = 180°$ ......(i)	Sum of st. Line angles
$m\angle 1 + m\angle A$ ......(ii)	Alternate angles are equal.
$m\angle 2 = m\angle ABC$ ......(iii)	From figure
$m\angle 3 = m\angle C$ ......(iv)	Alternate angles are equal.
Now (i) can be written as	
$m\angle A + m\angle ABC + m\angle C = 180°$	From (i), (ii), (iii) and (iv)
$m\angle A + m\angle B + m\angle C = 180°$	$m\angle ABC = m\angle B$

Hence, the sum of the interior angles of a a triangle is 180°.

Q.16 If a, b and c are non-zero real numbers prove that: 09308061

(a) $\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$

Solution:

$\frac{a}{b} = \frac{c}{d}$

Multiplying sides by bd

$\frac{a}{b}(bd) = \frac{c}{d}(bd)$

By commutative property bd = db

$\frac{a.1}{b}(bd) = \frac{c.1}{d}(db)$

By associative property

$a\left(\frac{1}{b}.b\right)d = c\left(\frac{1}{d}.d\right)b$

$a(1)\,d = c(1)b$ (Multiplicative inverse)

$ad = cb$ Multiplicative identity

$ad = bc$ (∵ $cb = bc$)

Again,

$ad = bc$

Multiplying B.S by $\left(\frac{1}{b}.\frac{1}{d}\right)$

$$ad\left(\frac{1}{b}.\frac{1}{d}\right) = bc\left(\frac{1}{b}.\frac{1}{d}\right)$$

By associative property.

$$a.\frac{1}{b}\times d.\frac{1}{d} = b.\frac{1}{b}\times c.\frac{1}{d} =$$

$\frac{a}{b}\times 1 = 1\times\frac{c}{d}$ (Multiplicative inverse)

$\frac{a}{b} = \frac{c}{d}$ (Multiplicative identity)

(b) $\frac{a}{b}.\frac{c}{d} = \frac{ac}{bd}$ **09308062**

Solution:

$$\frac{a}{b}.\frac{c}{d} = \frac{ac}{bd}$$

$$\text{L.H.S} = \frac{a}{b}.\frac{c}{d}$$

$= \left(a\times\frac{1}{b}\right)\times\left(c\times\frac{1}{d}\right)$ (Multiplicative rule)

$= (a\times c)\times\left(\frac{1}{b}\times\frac{1}{d}\right)$ (Associative property)

$= ac\times\frac{1}{bd}$

$= \frac{ac}{bd}$ (Multiplication rule)

L.H.S = R.H.S

(c) $\frac{a}{b}+\frac{c}{b} = \frac{a+c}{b}$ **09308063**

Solution:

$$\frac{a}{b}+\frac{c}{b} = \frac{a+c}{b}$$

$\text{L.H.S} = \frac{a}{b}+\frac{c}{b}$

$\text{L.H.S} = a\times\frac{1}{b}\times c\times\frac{1}{b}$ (Multiplicative rule)

$\text{L.H.S} = (a+c)\times\frac{1}{b}$ (Right distributive property)

$\text{L.H.S} = \frac{a+c}{b}$ (Multiplication rule)

L.H.S = R.H.S