Unit 10 Graphs of Functions

Students' Learning Outcomes

At the end of the unit, the students will be able to:

- Recall sketch graphs of linear functions (e.g. $y = ax + b$)
- Plot and interpret the graphs of quadratic, cubic, reciprocal and exponential functions.
  - Graph $y = ax^n$ where n is $+ve$ integer, $-ve$ integer, rational number for $x > 0$ and a is any real number.
  - Graph $y = ka^x$, where x is real $a > 1$.
- Discover exponential growth/decay of a practical phenomenon through its graph.
- Determine the gradients of curves by drawing tangents.
- Apply concepts of sketching and interpreting graphs to real-life problems (such as in tax payment, income and salary problems and cost and profit analysis)

INTRODUCTION

Graphs are powerful tools for visualizing and analyzing relationships between variables, making them essential in understanding various mathematical functions and their applications. In this unit, we explore the graphs of linear, quadratic, cubic, reciprocal and exponential functions. We will also examine how to determine the gradient of curves by drawing tangents. Finally, we will connect these concepts to real-life scenarios, learning how to sketch and interpret graphs to solve practical problems.

10.1 Functions and their Graphs

Functions are essential tools for representing real-world phenomena using mathematical concepts. A function can be expressed in various forms, including an equation, a graph, a numerical table or a verbal description. For example, the area of a circle depends on its radius.

In such cases, one variable y depends on another variable x. This relationship is expressed as:

$$y = f(x)$$

Here, f denotes the function, x is the independent variable (input) and y is the dependent variable (output) determined by the value of x.

10.1.1 Graph of Linear Functions

A linear function is a mathematical expression that represents a straight-line relationship between two variables. Its general form is $f(x) = mx + c$, where "m" is the slope or gradient of the line, indicating how steep it is and "c" is the y-intercept (the point where the line crosses the y-axis). It can also be written as $y = mx + c$.

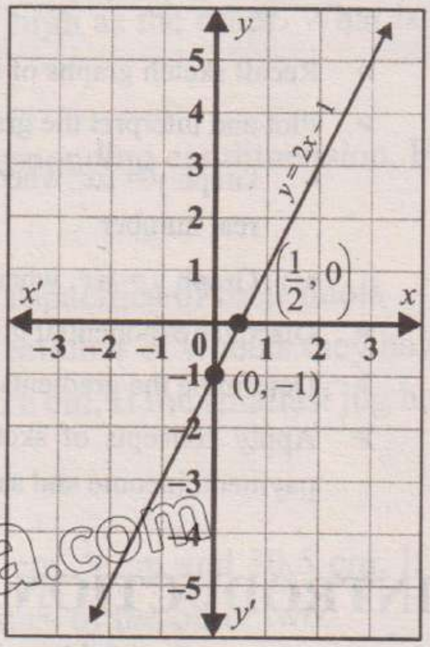
Example 1: Sketch the graph of $y = 2x - 1$.

Solution: To sketch the graph of linear function, we can find its x and y intercepts.

Put $x = 0$, we get $y = 5$. So $(0, -1)$ is the y-intercept.

Put $y = 0$, we get $x = \dfrac{1}{2}$. So $\left(\dfrac{1}{2}, 0\right)$ is the x-intercept.

The graph is a straight line that rises to the right because slope is positive.



10.1.2 Graph of Quadratic Functions

A quadratic function is a type of polynomial function that involves x^2 term. Its general form is:

$$y = ax^2 + bx + c$$

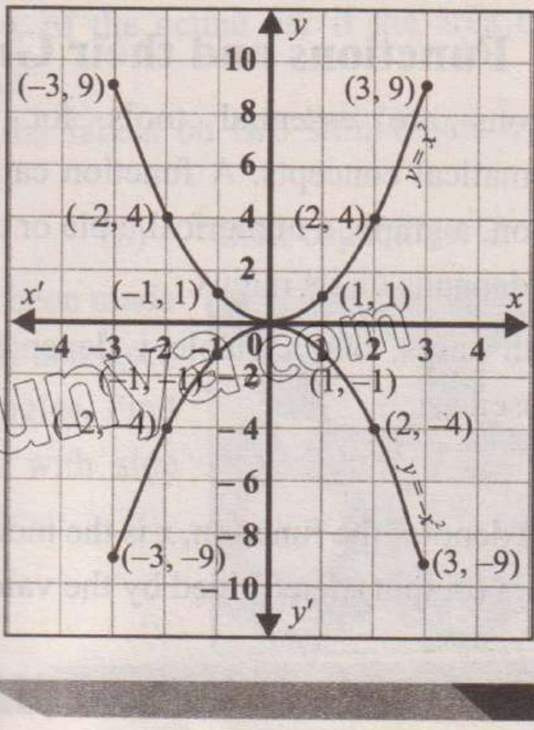
Where a, b, c are constants and $a \neq 0$.

> **Keep in mind!**
> The graph of a quadratic function is always a **parabola**.
> - If $a > 0$, then the parabola opens upward like "◡".
> - If $a < 0$, then the parabola opens downward like "⌒".

Example 2: Plot the graphs of $y = x^2$ and $y = -x^2$ on the same diagram.

Solution: The following table shows several values of x and the given functions are evaluated at those values:

x	$y = x^2$	$y = -x^2$
−3	$(-3)^2 = 9$	−9
−2	$(-2)^2 = 4$	−4
−1	$(-1)^2 = 1$	−1
0	$(0)^2 = 0$	0
1	$(1)^2 = 1$	−1
2	$(2)^2 = 4$	−4
3	$(3)^2 = 9$	−9



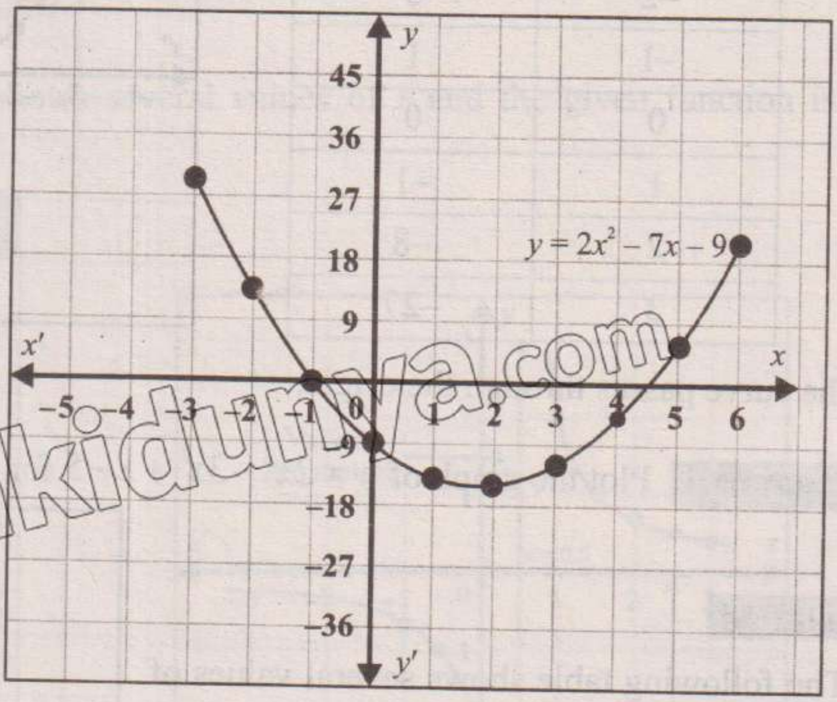
(i) Graph of $y = x^2$ represents parabola, passing through origin and opens upward.

(ii) Graph of $y = -x^2$ represents parabola, passing through origin and opens downward.

Example 3: Sketch the graph of $y = 2x^2 - 7x - 9$ for $-3 \le x \le 6$.

Solution: The values of x and y are given in the table and sketched in figure below:

x	y
–3	30
–2	13
–1	0
0	–9
1	–14
2	–15
3	–12
4	–5
5	6
6	21



Graph of $y = 2x^2 - 7x - 9$ represents parabola and opens upward. It intersects the y-axis at (0, –9) and x-axis at (–1, 0) and (4.5, 0).

10.1.3 Graph of Cubic Functions

A cubic function is a type of polynomial function of degree 3. Its standard form is:

$$y = ax^3 + bx^2 + cx + d$$

Where a, b, c, d are constants and $a \neq 0$.

Remember!

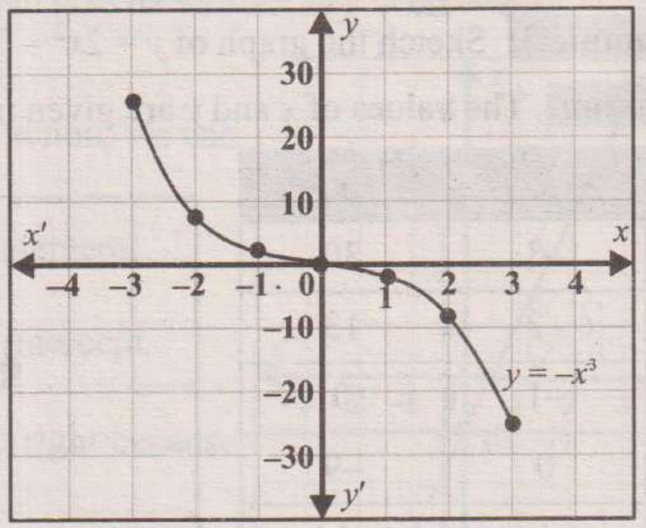
- The graph of a cubic function is a curve that can have at most two turning points.
- It has a general "S-shaped" appearance and depending on the coefficients, the shape may vary.
- Such functions are much more complicated and show more varied behaviour than linear and quadratic ones.

Example 4: Plot the graph of the following cubic function for $-3 \le x \le 3$:

$$y = -x^3$$

Solution: The following table shows several values of x and the given function is evaluated at those values:

x	$y = -x^3$
–3	27
–2	8
–1	1
0	0
1	–1
2	–8
3	–27



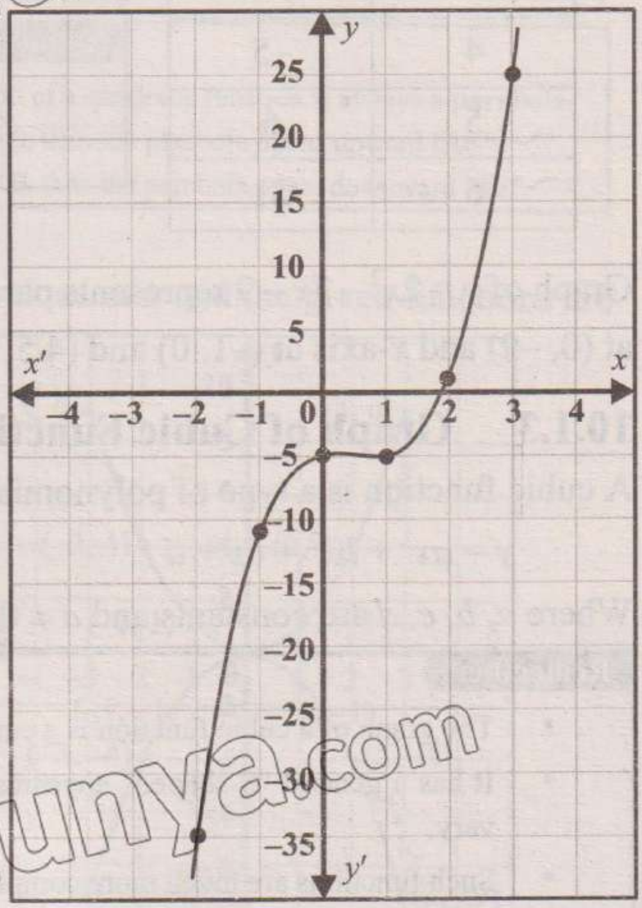
The curve passes through the origin.

Example 5: Plot the graph of $y = 2x^3 - 3x^2 + x - 5$ for $-2 \le x \le 3$.

Solution:

The following table shows several values of x and the given function is evaluated at those values:

x	y
–2	–35
–1	–11
0	–5
1	–5
2	1
3	25



The graph tells us that when $x = 0$, the function's value is -5.

10.1.4 Graph of Reciprocal Functions

A reciprocal function is a function of the form:

$$y = \frac{a}{x}$$

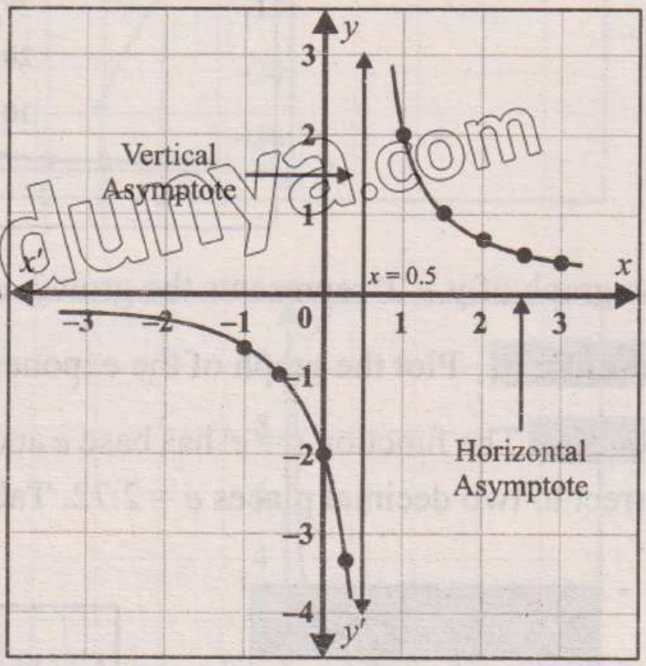
Where a is any real number and $x \neq 0$.

Example 6: Sketch the graph of the following reciprocal function:

$$y = \frac{1}{x - 0.5}, x \neq 0.5$$

Solution: The following table shows several values of x and the given function is evaluated at those values:

x	y
–1	–0.67
–0.5	–1
–0.2	–1.43
0	–2
0.2	–3.3
0.5	undefined
1	2
1.2	1.43
1.5	1
2	0.67
2.2	0.59
2.5	0.5
3	0.4



Remember!
An asymptote is a line that a graph approaches but never touches.

10.1.5 Graph of Exponential Functions ($y = ka^x$ where x is real number, $a > 1$)

An exponential function is a mathematical function of the form:

$$y = ka^x$$

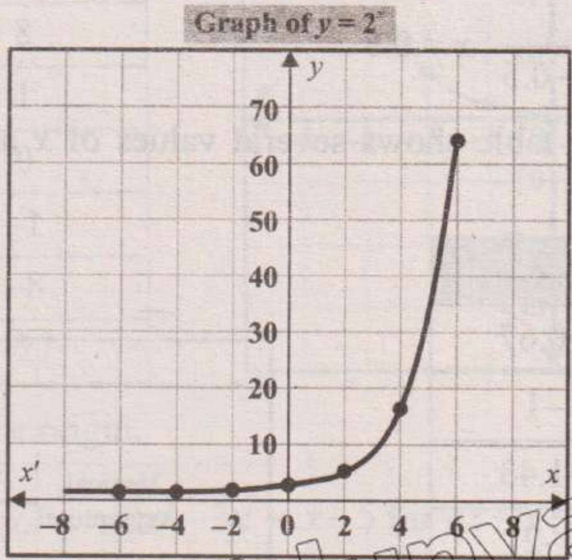
Where a, k are constants, x is variable and $a > 1$.

Example 7: Plot the graph of the exponential function $y = 2^x$ for $-6 \le x \le 6$.

Solution: The function $y = 2^x$ has base 2 and variable exponent x. Values of (x, y) are given in the table below:

x	−6	−4	−2	0	2	4	6
$y = 2^x$	0.02	0.06	0.25	1	4	16	64

Graph of the above points is given in the figure below:

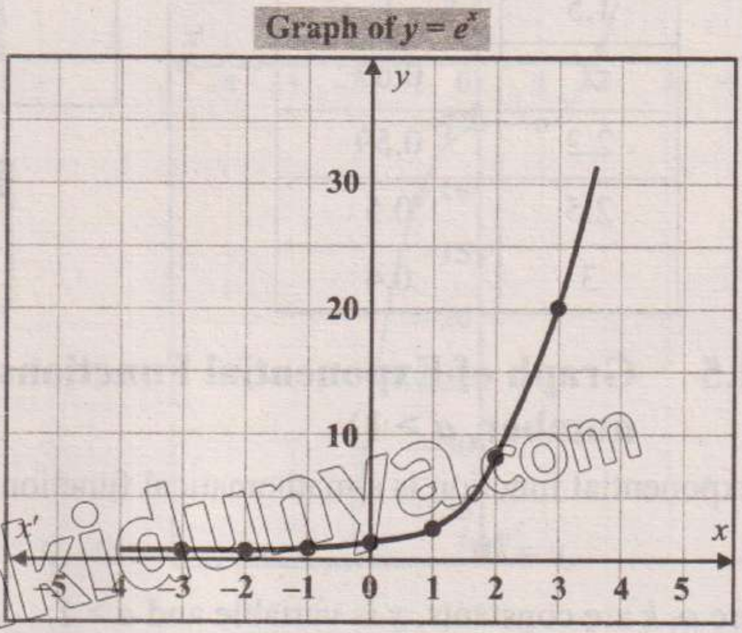


The graph of $y = 2^x$ represents the growth curve.

Example 8: Plot the graph of the exponential function, $y = e^x$.

Solution: The function $y = e^x$ has base e and variable power x. We know $e = 2.7182818$, correct to two decimal places $e = 2.72$. Table of x and y values is given below:

x	$y = e^x$
-3	0.05
-2	0.14
-1	0.37
0	1
1	2.72
2	7.40
3	20.09



10.1.6 Graphs of $y = ax^n$ (where n is +ve integer, –ve integer or rational number for $x > 0$ and a is any real number)

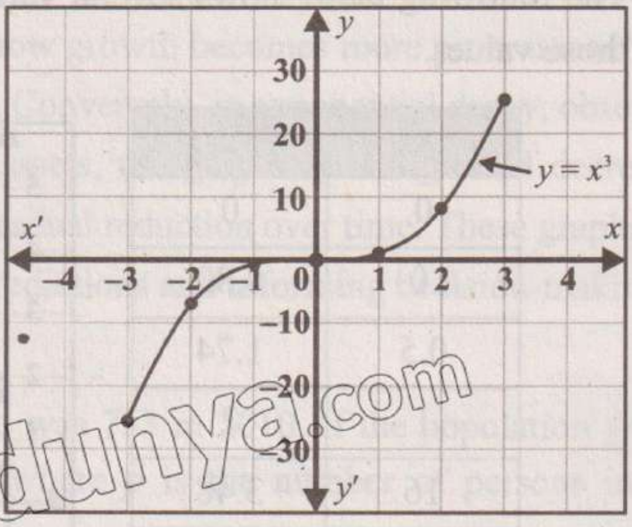
The graph of the function $y = ax^n$, where n is a positive integer, negative integer or rational number for $x > 0$ and a is any real number, exhibits distinct behaviours depending on the value of n. Following are the examples of these cases:

(i) When n is positive integer ($n = 3$)

Example 9: Plot the graph of $y = x^3$ for $-3 \le x \le 3$.

Solution: The table shows several values of x and the given function is evaluated at those values:

x	$y = x^3$
–3	–27
–2	–8
–1	–1
0	0
1	1
2	8
3	27



The curve passes through the origin.

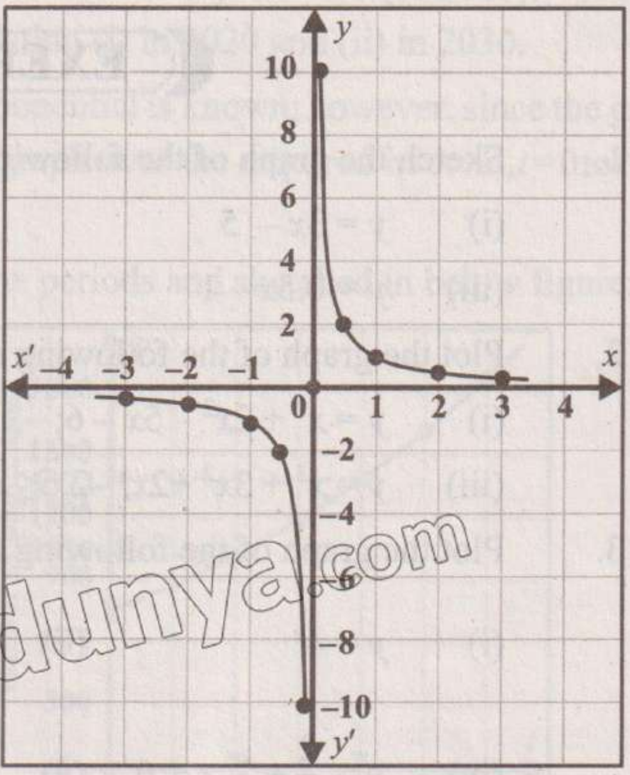
(ii) When n is negative integer ($n = -1$)

Example 10: Plot the graph of $y = x^{-1}$

Solution: $y = x^{-1} = \frac{1}{x}$

The following table shows several values of x and the given function is evaluated at those values:

x	$y = \frac{1}{x}$
–3	–0.3
–2	–0.5
–1	–1
–0.5	–2
–0.1	–10
0.1	10
0.5	2
1	1
2	0.5
3	0.3



The above graph consists of two branches, one in the first quadrant and the other in the third quadrant. Both branches approach but never touch the x-axis or the y-axis.

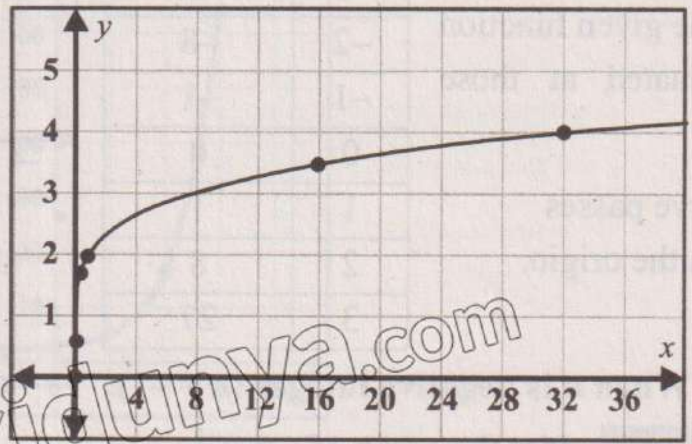
(iii) When n is rational number $\left(n=\frac{1}{5}\right)$

Example 11: Plot the graph of $y=2x^{\frac{1}{5}}$.

Solution: $y=2x^{\frac{1}{5}}$

The following table shows several values of x and the given function is evaluated at those values.

x	y
0	0
0.01	0.80
0.5	1.74
1	2
16	3.48
32	4



EXERCISE 10.1

1. Sketch the graph of the following linear functions:
   (i) $y=3x-5$ (ii) $y=-2x+8$
   (iii) $y=0.5x-1$
2. Plot the graph of the following quadratic and cubic functions:
   (i) $y=x^3+2x^2-5x-6;\ -3.5\leq x\leq 2.5$ (ii) $y=x^2+x-2$
   (iii) $y=x^3+3x^2+2x;\ -2.5\leq x\leq 0.5$ (iv) $y=5x^2-2x-3$
3. Plot the graph of the following functions:
   (i) $y=4^x$ (ii) $y=5^{-x}$ (iii) $y=\frac{1}{x-3}\ \ x\neq 3$
   (iv) $y=\frac{2}{x}+3, x\neq 0$ (v) $y=x^{\frac{1}{2}}$ (vi) $y=3x^{\frac{1}{3}}$
   (vii) $y=2x^{-2}$

10.2 Exponential Growth / Decay of a Practical Phenomenon through its Graph

Exponential growth and decay are widely observed in real-world phenomenon and their graphical representations offer critical insights into these processes. In exponential growth, such as population expansion, compound interest in finance or the spread of infectious diseases, the graph starts slowly but accelerates rapidly as time progresses. The curve increases steeply, showcasing how growth becomes more pronounced with time due to constant proportional changes. Conversely, in exponential decay, observed in cooling of objects or depreciation of assets, the graph starts high and decreases sharply before levelling off, indicating a gradual reduction over time. These graphs are essential for interpreting trends, making predictions and informing decision-making in diverse fields.

Example 12: The population of a village was 753 in 2010. If the population grows according to the equation $p = 753e^{0.03t}$, where p is the number of persons in the population at time t,

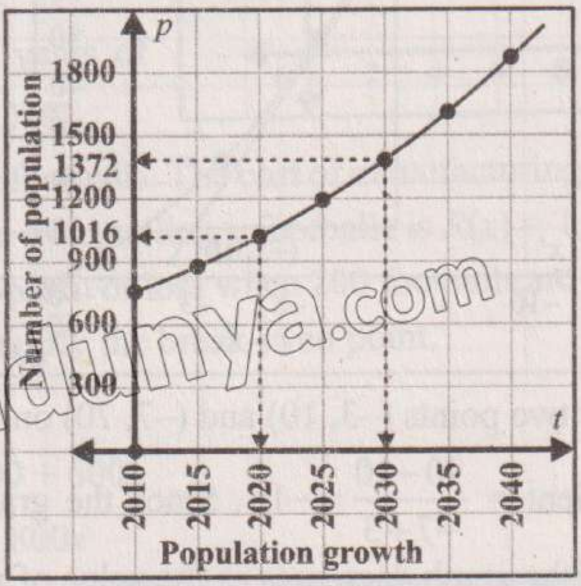
(a) Graph the population equation for $t = 0$ (in 2010) to $t = 30$ (in 2040).

(b) From the graph, estimate the population (i) in 2020 and (ii) in 2030.

Solution: (a) The general shape of the exponential is known; however, since the graph is being used for estimations, an accurate graph over the required interval, $t=0$ to $t=30$, is required.

Calculate a table of values for different time periods and sketched in below figure:

t	p
0	753
5	874.9
10	1016.4
15	1180.9
20	1372.1
25	1594.1
30	1852.1



(b) From graph,

(i) In 2020 ($t = 10$) the population is 1016 persons.

(ii) In 2030 ($t = 20$) the population is 1372 persons.

10.2.1 Gradients of Curves by Drawing Tangents

The gradient or slope of a graph at any point is equal to the gradient of the tangent to the curve at that point. Remember that a tangent is a line that just touches a curve only at one point (and doesn't cross it).

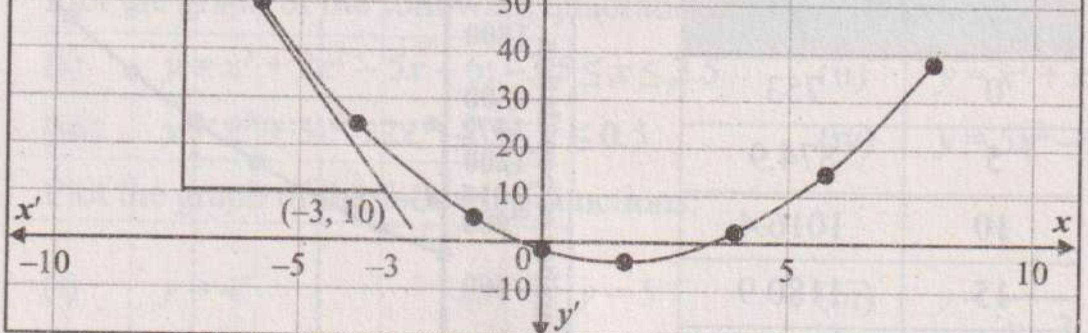
The gradient between two points is defined as:

$$\text{Gradient} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 13: Sketch the graph of $y = x^2 - 3x - 2$ for values of x from -8 to 8, draw a tangent line at $x = -6$ and determine the gradient.

Solution: Calculate the y-values for given values of x. The results are given in the table and sketched in below figure:

x	-8	-6	-4	-2	0	2	4	6	8
y	86	52	26	8	-2	-4	2	16	38



Consider two points $(-3, 10)$ and $(-7, 70)$ on the tangent line.

So, gradient $= \dfrac{70-10}{-7+3} = -15$. Since the gradient is negative, this indicates that the height of the graph decreases as the value of x increases.

10.2.2 Applications of Graph in Real-Life

Applying concepts of sketching and interpreting graphs to real-life problems enables individuals to visualize and analyse complex relationships, make informed decisions and optimize solutions. In tax payment scenarios, graphing concepts help identify optimal income levels, tax brackets, and liability. In income and salary problems, graphing facilitates analysis of compensation packages and income growth. By sketching salary against experience, patterns or anomalies in compensation structures become apparent. In cost and profit analysis, graphing enables businesses to visualize cost-profit relationships, determine break-even points, and optimize production levels.

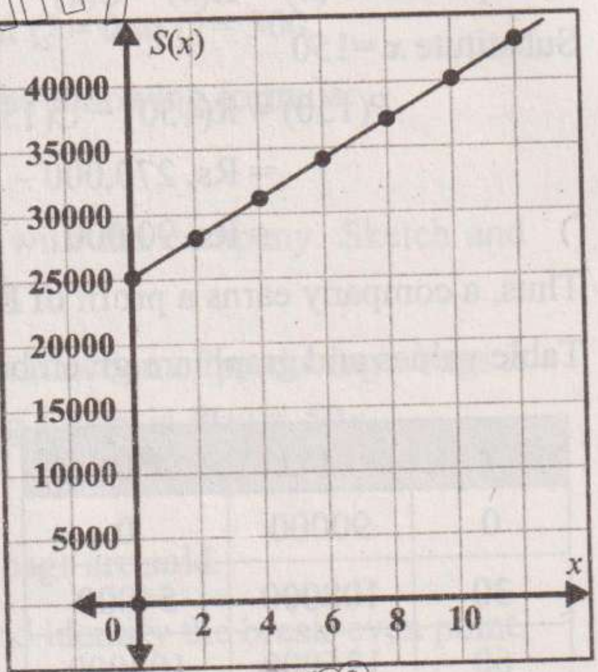
Example 14: Majid's salary $S(x)$ in rupees is based on the following formula:

$$S(x) = 25000+1500x\ ,$$

where x is the number of years he worked. Sketch and interpret the graph of salary function for $0 \leq x \leq 10$.

Solution: Table values and graph are given below:

x	$S(x)$
0	25000
2	28000
4	31000
6	34000
8	37000
10	40000



Majid's salary increases linearly with years of service and rises by Rs. 1500 for every year.

Example 15: A company manufactures footballs. The cost of manufacturing x footballs is $C(x) = 90,000 + 600x$. The revenue from selling x footballs is $R(x) = 1,800x$. Find the break-even point and determine the profit or loss when 200 footballs are sold. Draw the graphs of both the functions and identify the break-even point.

Solution: Given that

Cost function: $C(x) = 90,000 + 600x$

Revenue function: $R(x) = 1,800x$

The break-even point occurs when $R(x) = C(x)$

$$1800x = 90000 + 600x$$

$$1200x = 90000$$

$$\Rightarrow x = \frac{90000}{1200}$$

$$x = 75$$

So, at the break-even point, 75 footballs are produced or sold.

Next, we find the profit for 150 footballs

When $x = 150$, revenue:

$R(150) = 1,800(150)$

$= \text{Rs. } 270,000$

and $C(150) = 90,000 + 600(150)$

$= \text{Rs. } 180,000$

Now profit: $P(x) = R(x) - C(x)$

Substitute $x = 150$

$P(150) = R(150) - C(150)$

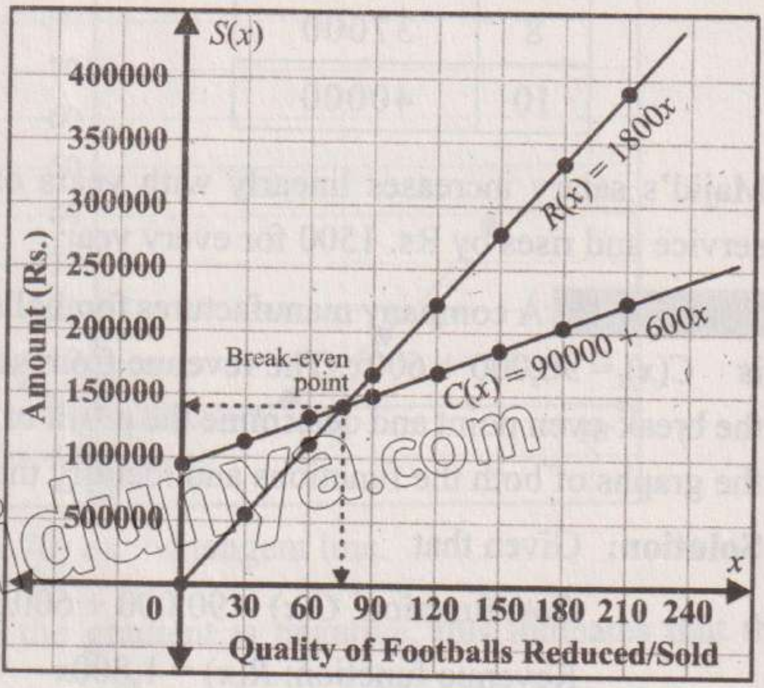
$= \text{Rs. } 270,000 - \text{Rs. } 180,000$

$= \text{Rs. } 90,000$

Thus, a company earns a profit of Rs. 90,000 when selling 150 footballs.

Table values and graph are given below:

x	$C(x)$	$R(x)$
0	90000	0
30	108000	54000
60	126000	108000
90	144000	162000
120	162000	216000
150	180000	27000
180	198000	324000
210	216000	378000



EXERCISE 10.2

1. Plot the graph of $y = 2x^2 - 4x + 3$ from -1 to 3. Draw tangent at (2, 3) and find the gradient.

2. Plot the graph of $y = 3x^2 + x + 1$ and draw tangent at (1, 5). Also find gradient of the tangent line at this point.

3. The strength of students in a school was 1000 in 2016. If the strength decay according to the equation $S = 1000\ e^{-t}$, where S is the number of students at time t.

   (a) Graph the given equation for $t = 0$ (in 2016) to $t = 9$ (*in* 2025).

   (b) From the graph, estimate the student's strength in 2019 and in 2023.

4. The demand and supply functions for a product are given by the equations $P_d = 400 - 5Q$, $P_s = 3Q + 24$:

   Plot the graph of each function over the interval $Q = 0$ to $Q = 300$.

5. Shahid's salary S(x) in rupees is based on the following formula:

   $$S(x) = 45000 + 4500x,$$

   where x is the number of years he has been with the company. Sketch and interpret the graph of salary function for $0 \leq x \leq 5$.

6. A company manufactures school bags. The cost function of producing x bags is $C(x) = 1200 + 20x$ and the revenue from selling x bags is $R(x) = 50x$.

   (a) Find the break-even point.

   (b) Determine the profit or loss when 250 bags are sold.

   (c) Plot the graphs of both the functions and identify the break-even point.

7. A newspaper agency fixed cost of Rs. 70 per edition and marginal printing and distribution costs of Rs. 40 per copy. Profit function is $p(x) = 10\,x - 70$, where x is the number of newspapers. Plot the graph and find profit for 500 newspapers.

8. Ali manufactures expensive shirts for sale to a school. Its cost (in rupees) for x shirts is $C(x) = 1500 + 10x + 0.2x^2$, $0 \leq x \leq 150$. Plot the graph and find the cost of 200 shirts.

REVIEW EXERCISE 10

1. Four options are given against each statement. Encircle the correct option.

(i) $x = 5$ represents:

(a) x-axis (b) y-axis
(c) line || to x-axis (d) line || to y-axis

(ii) Slope of the line $y = 5x + 3$ is:

(a) 3 (b) −3 (c) 5 (d) −5

(iii) The y- intercepts of $y = -2x - 1$ is:

(a) −2 (b) 2
(c) −1 (d) 1

(iv) The graph of $y = x^3$, cuts the x-axis at:

(a) $x = 0$ (b) $x = 1$ (c) $x = -1$ (d) $x = 2$

(v) The graph of 3^x represents:

(a) growth (b) decay (c) both (a) and (b) (d) a line

(vi) The graph of $y = -x^2 + 5$ opens:

(a) upward (b) downward (c) left side (d) right side

(vii) The graph of $y = x^2 - 9$ opens:

(a) upward (b) downward (c) left side (d) right side

(viii) $y = 5^x$ is ____________ function.

(a) linear (b) quadratic (c) cubic (d) exponential

(ix) Reciprocal function is:

(a) $y = 7^x$ (b) $y = \dfrac{2}{x}$ (c) $y = 2x^2$ (d) $y = 5x^3$

(x) $y = -3x^3 + 7$ is ____________ function.

(a) exponential (b) cubic (c) linear (d) reciprocal

2. Plot the graph of the following functions:

(i) $y = 3^{-x}$ for x from −2 to 4 (ii) $y = \dfrac{2}{x}, x \neq 0$

3. Sales for a new magazine are expected to grow according to the equation: $S = 200000\,(1 - e^{-0.05t})$, where t is given in weeks.

   (a) Plot graph of sales for the first 50 weeks.

   (b) Calculate the number of magazines sold, when $t = 5$ and $t = 35$.

4. Plot the graph of following for x from –5 to 5:

   (i) $y = x^2 - 3$ (ii) $y = 15 - x^2$

5. Plot the graph of $y = \frac{1}{2}\,(x + 4)(x - 1)(x - 3)$ from –5 to 4.

6. The supply and demand functions for a particular market are given by the equations:

   $P_s = Q^2 + 5$ and $P_d = Q^2 - 10Q$, where P represents price and Q represents quantity,

   Sketch the graph of each function over the interval $Q = -20$ to $Q = 20$.

7. A television manufacturer company make 40 inches LEDs. The cost of manufacturing x LEDs is $C(x) = 60{,}000 + 250x$ and the revenue from selling x LEDs is $R(x) = 1200x$. Find the break-even point and find the profit or loss when 100 LEDs are sold. Identify the break-even point graphically.