# Unit 4 Factorization and Algebraic Manipulation

## Students' Learning Outcomes

**At the end of the unit, the students will be able to:**

- Identify common factors, trinomial factoring, concretely, pictorially and symbolically.
- Factorize quadratic and cubic algebraic expressions:
  - $a^4 + a^2b^2 + b^4$ or $a^4 + b^4$
  - $x^2 + px + q$
  - $ax^2 + bx + c$
  - $(ax^2 + bx + c)(ax^2 + bx + d) + k$
  - $(x + a)(x + b)(x + c)(x + d) + k$
  - $(x + a)(x + b)(x + c)(x + d) + kx^2$
  - $a^3 + 3a^2b + 3ab^2 + b^3$
  - $a^3 - 3a^2b + 3ab^2 - b^3$
  - $a^3 \pm b^3$
- Find highest common factor and least common multiple of algebraic expressions and know relationship of LCM and HCF.
- Find square root of algebraic expression by factorization and division.
- Apply the concepts of factorization of quadratic and cubic algebraic expression to real-world problems (such as engineering, physics, and finance.)

## INTRODUCTION

Algebraic factorization is not just a mathematical technique limited to the classroom, it plays an important role in solving practical problems across various real-world scenarios. By breaking down complex algebraic expressions into simpler factors, we can make calculations more manageable and conceal important insights. Algebraic factorization has practical applications in finance, engineering science, business and daily life. This chapter will explore the techniques of algebraic factorization and demonstrate how these methods can be applied to real-world situations, making math a valuable asset in various aspects of life.

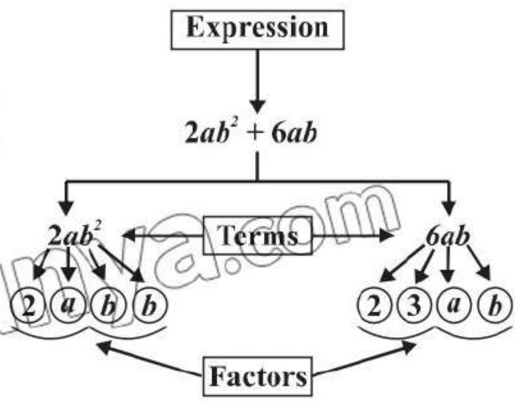
## 4.1 Identifying Common Factors and Trinomials Concretely, Pictorially and Symbolically

### 4.1.1 Common Factors

In algebra, a common factor is an expression that divides two or more expressions exactly. For example,

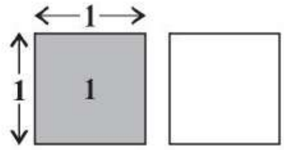
$$2x - 6 = 2(x - 3)$$

Here 2 is the common factor which exactly divides both terms $2x$ and 6.



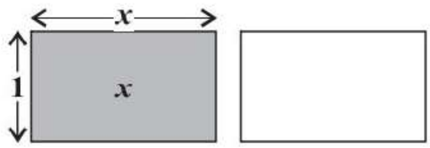
To represent trinomials concretely, we arrange unit tiles, rectangular tiles and the squared tiles into a rectangle. The factors of the trinomial are represented by the lengths of the sides of the rectangle.

### Unit Tiles



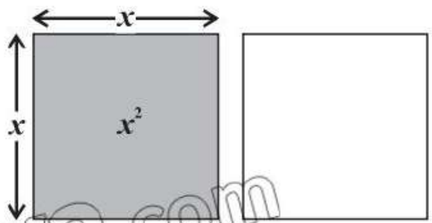

Here one grey unit tile represent 1 and one white unit tile represents –1. Both grey and white unit tiles form a zero pair.

### Rectangular Tiles




The grey rectangular tile represents $x$ and the white rectangular tile represents $-x$. Both grey and white rectangular tiles also form a zero pair.

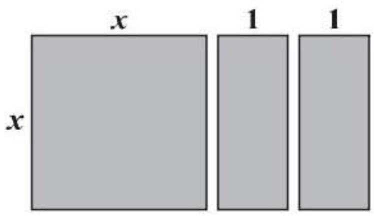
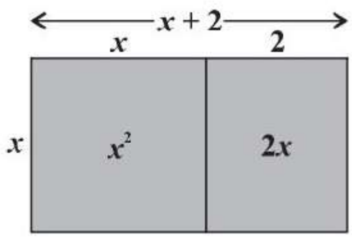
### Squared Tiles




The grey squared tile measure $x$ units on each side and it has an area of $x \times x = x^2$ units. This tile is labelled as $x^2$ tile. The white squared tile represents $-x^2$. Both grey and white squared tiles form a zero pair.

**Example 1:** Find common factor of $x^2 + 2x$ concretely, pictorially and symbolically

**Solution:** We arrange one $x^2$ tile and two $x$ tiles into a rectangle.

| Concretely | Pictorially | Symbolically |
|---|---|---|
|   |   | $x^2 + 2x = x(x + 2)$ |

### 4.1.2 Trinomial Factoring

Trinomial factoring is converting trinomial expression as a product of two binomial expressions. A trinomial is an expression with three terms and binomial is an expression with two terms.

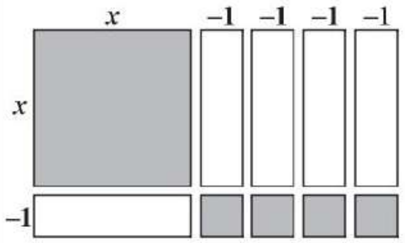
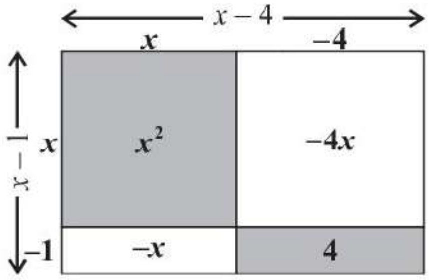
For example, $x^2 + 4x + 4$ and $3x^2 - x - 2$ are trinomials whereas $x + 2$ and $3x - 1$ are binomials.

**Teacher's Note**

Algebraic tiles of different sizes can easily be made with different coloured chart papers.

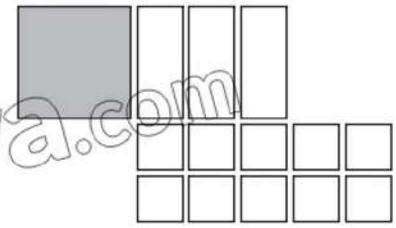
**Example 2:** Factorize $x^2 - 5x + 4$ concretely, pictorially and symbolically.

**Solution:**

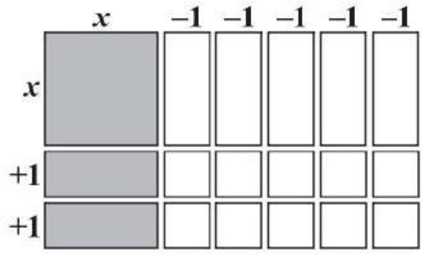
| Concretely | Pictorially | Symbolically |
|---|---|---|
| We arrange one $x^2$ tile, five $-x$ tiles and four unit tiles into a rectangle.   |   | $x^2 - 5x + 4$<br>$= (x - 1)(x - 4)$ |

**Example 3:** Factorize $x^2 - 3x - 10$ concretely, pictorially and symbolically.

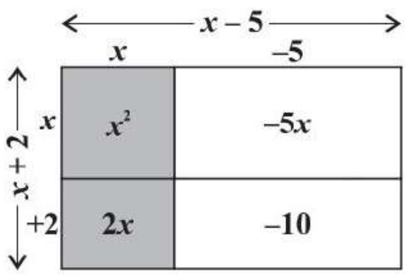
**Solution:**



Concretely we arrange one $x^2$ tile, three $-x$ tiles and ten $-1$ tiles into rectangle.




We see that there are not enough rectangular tiles to make a larger rectangle. To fix this issue, we add zero pair. Adding two $x$ tiles and two $-x$ tiles does not change the given expression because $2x - 2x = 0$.

| Pictorially | Symbolically |
|---|---|
|   | $x^2 - 3x - 10 = (x + 2)(x - 5)$ |

## 4.1.3 Factorizing Quadratic and Cubic Algebraic Expressions

**Type – I: Factorization of expression of the types $x^2 + px + q$ and $ax^2 + bx + c$**

The procedure is explained in the following examples to factorize the above type of expressions:

**Example 4:** Factorize: $x^2 + 9x + 14$

**Solution:** Two numbers whose product is +14 and their sum is 9 are +2, +7.

| Product of factors | Sum of factors |
|---|---|
| $14 \times 1 = 14$ | $14 + 1 = 15$ |
| $7 \times 2 = 14$ | $7 + 2 = 9$ |

So, $x^2 + 9x + 14$

$= x^2 + 2x + 7x + 14$

$= x(x + 2) + 7(x + 2)$

$= (x + 2)(x + 7)$

**Example 5:** Factorize: $x^2 - 11x + 24$

**Solution:** Two numbers whose product is +24 and their sum is –11 are –8, –3.

| Product of factors | Sum of factors |
|---|---|
| $24 \times 1 = 24$ | $24 + 1 = 25$ |
| $8 \times 3 = 24$ | $8 + 3 = 11$ |
| $(-8) \times (-3) = 24$ | $-8 - 3 = -11$ |
| $6 \times 4 = 24$ | $6 + 4 = 10$ |
| $12 \times 2 = 24$ | $12 + 2 = 14$ |

So, $x^2 - 11x + 24$

$= x^2 - 8x - 3x + 24$

$= x(x - 8) - 3(x - 8)$

$= (x - 8)(x - 3)$

**Example 6:** Factorize: $p^2 + 11p + 18$

**Solution:** $p^2 + 11p + 18$

$= p^2 + 9p + 2p + 18$ $\quad \because\ 9 + 2 = 11, 9 \times 2 = 18$

$= p(p + 9) + 2(p + 9)$

$= (p + 9)(p + 2)$

In all quadratic trinomials factorized so far, the coefficient of $x^2$ was 1. We will now consider cases where the coefficient of $x^2$ is not 1.

**Example 7:** Factorize: $2x^2 + 17x + 26$

**Solution:**

**Step – I:** Multiply the coefficient of $x^2$ with constant term. i.e.,

$$2 \times 26 = 52$$

**Step – II:** List all the factors of 52:

| | |
|---|---|
| 1, 52 | –1, – 52 |
| 2, 26 | –2, –26 |
| 4, 13 | –4, –13 |

**Remember!**

An expression having degree 2 is called a quadratic expression.

**Step – III:** Sum of factors equals middle term (17)

$1 + 52 = 53$ $\quad -1 - 52 = -53$

$2 + 26 = 28$ $\quad -2 - 26 = -28$

$4 + 13 = 17$ $\quad -4 - 13 = -17$

**Try Yourself!**

Factorize the following expressions:

(i) $x^2 + 7x - 18$

(ii) $t^2 - 5t - 24$

(iii) $6y^2 - y - 12$

**Step – IV:** Change the middle term in the given expression

$2x^2 + 17x + 26$

$= 2x^2 + 4x + 13x + 26$

**Step – V:** Take common from first two terms and last two terms

$= 2x(x + 2) + 13(x + 2)$

**Step – VI:** Again, take common from both terms

$= (x + 2)(2x + 13)$

**Example 8:** Factorize: $3x^2 - 4x - 4$

**Solution:** $3x^2 - 4x - 4$

$= 3x^2 + 2x - 6x - 4$ $\quad \because 2 \times (-6) = -12, +2 - 6 = -4$

$= x(3x + 2) - 2(3x + 2)$

$= (3x + 2)(x - 2)$

## EXERCISE 4.1

1. Factorize by identifying common factors.

(i) $6x + 12$ (ii) $15y^2 + 20y$ (iii) $-12x^2 - 3x$

(iv) $4a^2b + 8ab^2$ (v) $xy - 3x^2 + 2x$ (vi) $3a^2b - 9ab^2 + 15ab$

2. Factorize and represent pictorially:

(i) $5x + 15$ (ii) $x^2 + 4x + 3$ (iii) $x^2 + 6x + 8$

(iv) $x^2 + 4x + 4$

3. Factorize:

(i) $x^2 + x - 12$ (ii) $x^2 + 7x + 10$ (iii) $x^2 - 6x + 8$

(iv) $x^2 - x - 56$ (v) $x^2 - 10x - 24$ (vi) $y^2 + 4y - 12$

(vii) $y^2 + 13y + 36$ (viii) $x^2 - x - 2$

4. Factorize:

(i) $2x^2 + 7x + 3$ (ii) $2x^2 + 11x + 15$ (iii) $4x^2 + 13x + 3$

(iv) $3x^2 + 5x + 2$ (v) $3y^2 - 11y + 6$ (vi) $2y^2 - 5y + 2$

(vii) $4z^2 - 11z + 6$ (viii) $6 + 7x - 3x^2$

**Type – II: Factorization of the expression of the types $a^4 + a^2b^2 + b^4$ or $a^4 + b^4$**

Let's factorize the first expression

$$a^4 + a^2b^2 + b^4$$
$$= a^4 + b^4 + a^2b^2$$
$$= (a^2)^2 + (b^2)^2 + a^2b^2$$
$$= (a^2)^2 + (b^2)^2 + 2a^2b^2 - 2a^2b^2 + a^2b^2 \quad \text{(Adding and subtracting } 2a^2b^2\text{)}$$
$$= (a^2 + b^2)^2 - a^2b^2$$
$$= (a^2 + b^2)^2 - (ab)^2$$
$$= (a^2 + b^2 - ab)\,(a^2 + b^2 + ab)$$
$$= (a^2 - ab + b^2)\,(a^2 + ab + b^2)$$

**Remember!**

$a^2 - b^2 = (a - b)(a + b)$

$(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 = a^2 - 2ab + b^2$

**Example 9:** Factorize: $x^4 + x^2 + 25$

**Solution:**
$$x^4 + x^2 + 25$$
$$= x^4 + 25 + x^2$$
$$= (x^2)^2 + (5)^2 + 2(x^2)(5) - 2(x^2)(5) + x^2 \quad \text{(Adding and subtracting } 2(x^2)(5)\text{)}$$
$$= (x^2 + 5)^2 - 10x^2 + x^2$$
$$= (x^2 + 5)^2 - 9x^2$$
$$= (x^2 + 5)^2 - (3x)^2$$
$$= (x^2 + 5 - 3x)\,(x^2 + 5 + 3x)$$
$$= (x^2 - 3x + 5)\,(x^2 + 3x + 5)$$

**Activity**

- Prepare cards by writing several expressions.
- Divide students in small groups.
- Each group will draw a card and factorize the expression.
- The group which completes the most correct factorizations in a set time will win.

**Example 10:** Factorize: $x^4 + y^4$

**Solution:**
$$x^4 + y^4$$
$$= (x^2)^2 + (y^2)^2$$
$$= (x^2)^2 + (y^2)^2 + 2(x^2)(y^2) - 2(x^2)(y^2) \quad \text{(Adding and subtracting } 2x^2y^2\text{)}$$
$$= (x^2 + y^2)^2 - (\sqrt{2}xy)^2$$
$$= (x^2 + y^2 - \sqrt{2}xy)\,(x^2 + y^2 + \sqrt{2}xy)$$
$$= (x^2 - \sqrt{2}xy + y^2)\,(x^2 + \sqrt{2}xy + y^2)$$

**Try Yourself!**

Factorize: (i) $64x^4y^4 + z^4$

(ii) $81x^4 + \dfrac{1}{81x^4} - 11$

**Example 11:** Factorize: $a^4 + 64$

**Solution:** $a^4 + 64$

$= (a^2)^2 + (8)^2$

$= (a^2)^2 + (8)^2 + 2(a^2)(8) - 2(a^2)(8)$ (Adding and subtracting $2(a^2)(8)$

$= (a^2 + 8)^2 - 16a^2$

$= (a^2 + 8)^2 - (4a)^2$

$= (a^2 + 8 - 4a)(a^2 + 8 + 4a)$

$= (a^2 - 4a + 8)(a^2 + 4a + 8)$

**Type – III: Factorization of the expression of the types**

- $(ax^2 + bx + c)(ax^2 + bx + d) + k$
- $(x + a)(x + b)(x + c)(x + d) + k$
- $(x + a)(x + b)(x + c)(x + d) + kx^2$

For explanation consider the following examples:

**Example 12:** Factorize: $(x^2 + 5x + 4)(x^2 + 5x + 6) - 3$

**Solution:** $(x^2 + 5x + 4)(x^2 + 5x + 6) - 3$

$= (y + 4)(y + 6) - 3$ ( Let $y = x^2 + 5x$ )

$= y^2 + 6y + 4y + 24 - 3$

$= y^2 + 10y + 21$

$= y^2 + 7y + 3y + 21$

$= y(y + 7) + 3(y + 7)$

$= (y + 7)(y + 3)$

$= (x^2 + 5x + 7)(x^2 + 5x + 3)$ $(\because y = x^2 + 5x)$

**Example 13:** Factorize: $(x + 2)(x + 3)(x + 4)(x + 5) - 15$

**Solution:** $(x + 2)(x + 3)(x + 4)(x + 5) - 15$

Re-arrange the given expression because $2 + 5 = 3 + 4$

$[(x + 2)(x + 5)][(x + 3)(x + 4)] - 15$

$= (x^2 + 5x + 2x + 10)(x^2 + 4x + 3x + 12) - 15$

$= (x^2 + 7x + 10)(x^2 + 7x + 12) - 15$

$= (y + 10)(y + 12) - 15$ ( Let $y = x^2 + 7x$ )

$= y^2 + 12y + 10y + 120 - 15$

$= y^2 + 22y + 105$

$= y^2 + 15y + 7y + 105$

$= y(y + 15) + 7(y + 15)$

$= (y + 15)(y + 7)$

$= (x^2 + 7x + 15)(x^2 + 7x + 7)$ $\quad (\because y = x^2 + 7x)$

**Example 14:** Factorize: $(x - 2)(x + 2)(x + 1)(x - 4) + 2x^2$

**Solution:** $(x - 2)(x + 2)(x + 1)(x - 4) + 2x^2$

$= [(x - 2)(x + 2)][(x + 1)(x - 4)] + 2x^2$ $\quad [\because (-2) \times 2 = 1 \times (-4)]$

$= (x^2 - 2^2)(x^2 - 4x + x - 4) + 2x^2$

$= (x^2 - 4)(x^2 - 3x - 4) + 2x^2$

$= y(y - 3x) + 2x^2$ $\quad$ ( Let $y = x^2 - 4$ )

$= y^2 - 3xy + 2x^2$

$= y^2 - 2xy - xy + 2x^2$

$= y(y - 2x) - x(y - 2x)$

$= (y - 2x)(y - x)$

$= (x^2 - 4 - 2x)(x^2 - 4 - x)$ $\quad (\because y = x^2 - 4)$

$= (x^2 - 2x - 4)(x^2 - x - 4)$

**Type – IV: Factorization of the expression of the types**

- $a^3 + 3a^2b + 3ab^2 + b^3$
- $a^3 - 3a^2b + 3ab^2 - b^3$

Factorization of such types of expressions is explained in the following examples:

**Example 15:** Factorize: $8x^3 + 60x^2 + 150x + 125$

**Solution:** $8x^3 + 60x^2 + 150x + 125$

$= (2x)^3 + 3(2x)^2(5) + 3(2x)(5)^2 + (5)^3$

$= (2x + 5)^3$

$= (2x + 5)(2x + 5)(2x + 5)$

**Remember!**
$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

**Example 16:** Factorize: $x^3 - 18x^2 + 108x - 216$

**Solution:** $x^3 - 18x^2 + 108x - 216$

$= (x)^3 - 3(x)^2(6) + 3(x)(6)^2 - (6)^3$

$= (x - 6)^3$

$= (x - 6)(x - 6)(x - 6)$

**Type – V: Factorization of the expression of the types $a^3 \pm b^3$**

The expression $a^3 + b^3$ is a sum of cubes and it can be factorized using the following identity:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

The expression $a^3 - b^3$ is a difference of cubes and it can be factorized using the following identity:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

**Example 17:** Factorize: $8x^3 + 27$

**Solution:** $8x^3 + 27$

$= (2x)^3 + (3)^3$

$= (2x + 3)[(2x)^2 - (2x)(3) + (3)^2]$

$= (2x + 3)(4x^2 - 6x + 9)$

**Example 18:** Factorize: $x^3 - 27y^3$

**Solution:** $x^3 - 27y^3$

$= (x)^3 - (3y)^3$

$= (x - 3y)[(x)^2 + (x)(3y) + (3y)^2]$

$= (x - 3y)(x^2 + 3xy + 9y^2)$

> **Do you know?**
> $(a + b)^2 \neq a^2 + b^2$
> $(a - b)^2 \neq a^2 - b^2$
> $(a + b)^3 \neq a^3 + b^3$
> $(a - b)^3 \neq a^3 - b^3$

## EXERCISE 4.2

1. Factorize each of the following expressions:

   (i) $4x^4 + 81y^4$ (ii) $a^4 + 64b^4$ (iii) $x^4 + 4x^2 + 16$

   (iv) $x^4 - 14x^2 + 1$ (v) $x^4 - 30x^2y^2 + 9y^4$ (vi) $x^4 + 11x^2y^2 + y^4$

2. Factorize each of the following expressions:

   (i) $(x + 1)(x + 2)(x + 3)(x + 4) + 1$ (ii) $(x + 2)(x - 7)(x - 4)(x - 1) + 17$

   (iii) $(2x^2 + 7x + 3)(2x^2 + 7x + 5) + 1$ (iv) $(3x^2 + 5x + 3)(3x^2 + 5x + 5) - 3$

   (v) $(x + 1)(x + 2)(x + 3)(x + 6) - 3x^2$ (vi) $(x + 1)(x - 1)(x + 2)(x - 2) + 13x^2$

3. Factorize:

   (i) $8x^3 + 12x^2 + 6x + 1$ (ii) $27a^3 + 108a^2b + 144ab^2 + 64b^3$

   (iii) $x^3 + 48x^2y + 108xy^2 + 216y^3$ (iv) $8x^3 - 125y^3 + 150xy^2 - 60x^2y$

4. Factorize:

   (i) $125a^3 - 1$ (ii) $64x^3 + 125$ (iii) $x^6 - 27$

   (iv) $1000a^3 + 1$ (v) $343x^3 + 216$ (vi) $27 - 512y^3$

# 4.3 Highest Common Factor (HCF) and Least Common Multiple (LCM) of Algebraic Expressions

## 4.3.1 Highest Common Factor (HCF)

The HCF of two or more algebraic expressions refers to the greatest algebraic expression which divides them without leaving a remainder.

We can find HCF of given expressions by the following two methods:

(a) By factorization (b) By division

### (a) HCF by Factorization Method

**Example 19:** Find the HCF of $6x^2y$, $9xy^2$

**Solution:**

$$6x^2y = 2 \times 3 \times x \times x \times y$$
$$9xy^2 = 3 \times 3 \times x \times y \times y$$

$\therefore$ HCF $= 3 \times x \times y$ (Product of common factors)

$= 3xy$

**Example 20:** Find the HCF by factorization method $x^2 - 27$, $x^2 + 6x - 27$, $x^2 - 9$

**Solution:**

$$x^3 - 27 = x^3 - 3^3$$
$$= (x-3)[(x)^2 + (3)(x) + (3)^2]$$
$$= (x-3)(x^2 + 3x + 9)$$
$$x^2 + 6x - 27 = x^2 + 9x - 3x - 27$$
$$= x(x+9) - 3(x+9)$$
$$= (x+9)(x-3)$$
$$x^2 - 9 = x^2 - 3^2$$
$$= (x-3)(x+3)$$

Hence, HCF $= x - 3$

### (b) HCF by Division Method

**Example 21:** Find HCF of $6x^3 - 17x^2 - 5x + 6$ and $6x^3 - 5x^2 - 3x + 2$ by using division method.

**Solution:**

$$\begin{array}{r|l} & \quad 1 \\ \hline 6x^3 - 17x^3 - 5x + 6 & 6x^2 - \ 5x^2 - 3x + 2 \\ & \underline{-6x^3 \mp 17x^2 \mp 5x \pm 6} \\ & \qquad 12x^2 + 2x - 4 \end{array}$$

Here, $12x^2 + 2x - 4 = 2(6x^2 + x - 2)$

2 is not common in both the given polynomials, so we ignore it and consider only $6x^2 + x - 2$.

$$\begin{array}{r|l} & \quad\quad\quad\quad x - 3 \\ \hline 6x^2 + x - 2 & 6x^2 - 17x^2 - 5x + 6 \\ & \underline{\mp 6x^3 \pm x^2 \mp 2x} \\ & \quad\quad -18x^2 - 3x + 6 \\ & \quad\quad \underline{\mp 18x^3 \mp 3x \pm 6} \\ & \quad\quad\quad\quad 0 \end{array}$$

Hence, HCF $= 6x^2 + x - 2$

## 4.3.2 Least Common Multiple (LCM)

The LCM of two or more algebraic expressions is the smallest expression that is divisible by each of the given expressions.

To find the LCM by factorization, we use the formula.

$$\text{LCM} = \text{Common factors} \times \text{Non-common factors}$$

**Example 22:** Find the LCM of $4x^2y$, $8x^3y^2$.

**Solution:** $4x^2y = 2 \times 2 \times x \times x \times y$

$8x^3y^2 = 2 \times 2 \times 2 \times x \times x \times x \times y \times y$

Common factors $= 2 \times 2 \times x \times x \times y = 4x^2y$

Non-common factors $= 2 \times x \times y = 2xy$

LCM = Common factors × Non-common factors

$= 4x^2y \times 2xy = 8x^3y^2$

**Example 23:** Find the LCM of the polynomials $x^2 - 3x + 2$, $x^2 - 1$ and $x^2 - 5x + 4$.

**Solution:** As $x^2 - 3x + 2 = x^2 - 2x - x + 2$

$= x(x - 2) - 1(x - 2)$

$= (x - 2)(x - 1)$

And $x^2 - 1 = (x - 1)(x + 1)$

$x^2 - 5x + 4 = x^2 - 4x - x + 4$

$= x(x - 4) - 1(x - 4)$

$= (x - 4)(x - 1)$

Common factors $= x - 1$

Non-common factors $= (x + 1)(x - 2)(x - 4)$

LCM = Common factors × Non-common factors

$= (x-1) \times (x+1)(x-2)(x-4)$

$= (x-1)(x+1)(x-2)(x-4)$

### 4.3.3 Relationship Between LCM and HCF

The relationship between LCM and HCF can be expressed as follows:

$$\text{LCM} \times \text{HCF} = p(x) \times q(x)$$

Where, $p(x) = 1^{st}$ polynomial

$q(x) = 2^{nd}$ polynomial

**Example 24:** LCM and HCF of two polynomials are $x^3 - 10x^2 + 11x + 70$ and $x - 7$. If one of the polynomials is $x^2 - 12x + 35$, find the other polynomial.

**Solution:** Given that: LCM $= x^3 - 10x^2 + 11x + 70$

HCF $= x - 7$

$p(x) = x^2 - 12x + 35$

As we know that: $q(x) = \dfrac{\text{LCM} \times \text{HCF}}{p(x)}$

$$= \frac{(x^3 - 10x^2 + 11x + 70)(x-7)}{x^2 - 12x + 35}$$

$$\begin{array}{r|l} & x + 2 \\ \hline x^2 - 12x + 35 & x^3 - 10x^2 + 11x + 70 \\ & \underset{-}{x^3} \underset{+}{\mp} 12x^2 \underset{-}{\pm} 35x \\ \hline & 2x^2 - 24x + 70 \\ & \underset{-}{2x^2} \underset{+}{\mp} 24x \underset{-}{\pm} 70 \\ \hline & 0 \end{array}$$

So, $q(x) = (x+2)(x-7)$

$= x^2 - 7x + 2x - 14$

$= x^2 - 5x - 14$

**Example 25:** The LCM of $x^2y + xy^2$ and $x^2 + xy$ is $xy(x+y)$. Find the HCF.

**Solution:** Given that: LCM $= xy(x+y)$

HCF = ?

$1^{st}$ polynomial $= x^2y + xy^2$

$2^{nd}$ polynomial $= x^2 + xy$

As we know that: LCM × HCF = Product of two polynomials

$$\text{HCF} = \frac{\text{Product of two polynomials}}{\text{LCM}}$$

$$= \frac{(x^2y + xy^2)(x^2 + xy)}{xy(x+y)}$$

$$= \frac{xy(x+y)x(x+y)}{xy(x+y)}$$

$$= x(x+y)$$

## EXERCISE 4.3

1. Find HCF by factorization method.
   (i) $21x^2y, 35xy^2$ (ii) $4x^2 - 9y^2, 2x^2 - 3xy$
   (iii) $x^3 - 1, x^2 + x + 1$ (iv) $a^3 + 2a^2 - 3a, 2a^3 + 5a^2 - 3a$
   (v) $t^2 + 3t - 4, t^2 + 5t + 4, t^2 - 1$ (vi) $x^2 + 15x + 56, x^2 + 5x - 24, x^2 + 8x$
2. Find HCF of the following expressions by using division method.
   (i) $27x^3 + 9x^2 - 3x - 9,\ 3x - 2$ (ii) $x^3 - 9x^2 + 21x - 15,\ x^2 - 4x + 3$
   (iii) $2x^3 + 2x^2 + 2x + 2,\ 6x^3 + 12x^2 + 6x + 12$
   (iv) $2x^3 - 4x^2 + 6x,\ x^3 - 2x,\ 3x^2 - 6x$
3. Find LCM of the following expressions by using prime factorization method.
   (i) $2a^2b,\ 4ab^2,\ 6ab$ (ii) $x^2 + x,\ x^3 + x^2$
   (iii) $a^2 - 4a + 4, a^2 - 2a$ (iv) $x^4 - 16,\ x^3 - 4x$
   (v) $16 - 4x^2,\ x^2 + x - 6,\ 4 - x^2$
4. The HCF of two polynomials is $y - 7$ and their LCM is $y^3 - 10y^2 + 11y + 70$. If one of the polynomials is $y^2 - 5y - 14$, find the other.
5. The LCM and HCF of two polynomial $p(x)$ and $q(x)$ are $36x^3(x + a)(x^3 - a^3)$ and $x^2(x - a)$ respectively. If $p(x) = 4x^2(x^2 - a^2)$, find $q(x)$.
6. The HCF and LCM of two polynomials is $(x + a)$ and $12x^2(x + a)(x^2 - a^2)$ respectively. Find the product of the two polynomials.

## 4.4 Square Root of an Algebraic Expression

The square root of an algebraic expression refers to a value that, when multiplied by itself, gives the original expression. Just like finding the square root of a number, taking the square root of an algebraic expression involves determining what expression, when squared, results in the given expression.

For example, square root of $4a^2$ is $\pm 2a$ because $2a \times 2a = 4a^2$ and $(-2a) \times (-2a) = 4a^2$

There are following two methods for finding the square root of an algebraic expression:

(a) By factorization method (b) By division method

**(a) Square Root by Factorization Method**

**Example 26:** Find the square root of the expression $36x^4 - 36x^2 + 9$

**Solution:**

$$\begin{aligned} & 36x^4 - 36x^2 + 9 \\ &= 9(4x^4 - 4x^2 + 1) \\ &= 9[(2x^2)^2 - 2(2x^2)(1) + (1)^2] \\ &= 3^2(2x^2 - 1)^2 \end{aligned}$$

Taking square root on both sides

$$\begin{aligned} \sqrt{36x^4 - 36x^2 + 9} &= \sqrt{3^2(2x^2-1)^2} \\ &= \sqrt{3^2} \cdot \sqrt{(2x^2-1)^2} \\ &= \pm 3(2x^2 - 1) \end{aligned}$$

**(b) Square Root by Division Method**

When the degree of the polynomial is higher, division method in finding the square root is very useful.

**Example 27:** Find the square root of the polynomial $x^4 - 12x^3 + 42x^2 - 36x + 9$.

**Solution:** Multiply $x^2$ by $x^2$ to get $x^4$

Multiply the quotient ($x^2$) by 2, so we get $2x^2$. By dividing $-12x^3$ by $2x^2$, we get $-6x$. By continuing in this way, we get the remainder.

Hence, square root of $x^4 - 12x^3 + 42x^2 - 36x + 9$ is $\pm(x^2 - 6x + 3)$

$$\begin{array}{r|l} & \quad x^2 - 6x + 3 \\ \hline & x^4 - 12x^3 + 42x^2 - 36x + 9 \\ x^2 & -x^4 \\ \hline & -12x^3 + 42x^2 \\ 2x^2 - 6x & \mp 12x^3 \pm 36x^2 \\ \hline & 6x^2 - 36x + 9 \\ 2x^2 - 12x + 3 & -6x^2 \mp 36x \pm 9 \\ \hline & 0 \end{array}$$

## 4.4.1 Real World Problems of Factorization

In this section, we will apply the concept of factorization of quadratic and cubic algebraic expressions to real world problems such as engineering, physics and finance.

**Example 28:** Cost function for producing a part is modeled by:

$$C(x) = 5x^2 - 25x + 30$$

Where $x$ is the width of the component and $C(x)$ is the cost. Find the value of $x$ where $C(x)$ is minimum.

**Solution:**
$$C(x) = 5x^2 - 25x + 30$$
$$= 5(x^2 - 5x + 6)$$
$$= 5(x^2 - 2x - 3x + 6)$$
$$= 5[x(x - 2) - 3(x - 2)]$$
$$= 5(x - 2)(x - 3)$$

Thus, the minimum cost occurs when $x = 2$ or $x = 3$.

**Example 29:** The potential energy $U(x)$ of a particle moving in a cubic potential is expressed as:
$$U(x) = x^3 - 6x^2 + 12x - 8$$
Factorize the expression to find the points where the energy in minimized.

**Solution:**
$$U(x) = x^3 - 6x^2 + 12x - 8$$
$$= (x)^3 - 3(x)^2(2) + 3(x)(2)^2 - (2)^3$$
$$= (x - 2)^3$$
$$= (x - 2)(x - 2)(x - 2)$$

The factorized form of the potential energy function shows that the energy is minimized at $x = 2$.

**Example 30:** A company's profit $P(x)$ is modeled by the quadratic equation:
$$P(x) = -5x^2 + 50x - 120$$
Where $x$ represents the number of units produced and $P(x)$ represents the profit in dollars. Find how many units should be produced to maximize profit.

**Solution:**
$$P(x) = -5x^2 + 50x - 120$$
$$= -5(x^2 - 10x + 24)$$
$$= -5[x^2 - 4x - 6x + 24]$$
$$= -5[x(x - 4) - 6(x - 4)]$$
$$= -5(x - 4)(x - 6)$$

We can see that profit will be 0 when $x = 4$ or $x = 6$. As coefficients of $x^2$ is negative, the maximum profit occurs at the midpoint between 4 and 6.

Which is: $x = \dfrac{4+6}{2} = \dfrac{10}{2} = 5$

Thus, the company should produce 5 units to maximize profit.

## EXERCISE 4.4

1. Find the square root of the following polynomials by factorization method:
   (i) $x^2 - 8x + 16$ (ii) $9x^2 + 12x + 4$
   (iii) $36a^2 + 84a + 49$ (iv) $64y^2 - 32y + 4$

(v) $200t^2 - 120t + 18$ (vi) $40x^2 + 120x + 90$

2. Find the square root of the following polynomials by division method:
   (i) $4x^4 - 28x^3 + 37x^2 + 42x + 9$
   (ii) $121x^4 - 198x^3 - 183x^2 + 216x + 144$
   (iii) $x^4 - 10x^3y + 27x^2y^2 - 10xy^3 + y^4$
   (iv) $4x^4 - 12x^3 + 37x^2 - 42x + 49$

3. An investor's return $R(x)$ in rupees after investing $x$ thousand rupees is given by quadratic expression:
$$R(x) = -x^2 + 6x - 8$$
Factorize the expression and find the investment levels that result in zero return.

4. A company's profit $P(x)$ in rupees from selling $x$ units of a product is modeled by the cubic expression:
$$P(x) = x^3 - 15x^2 + 75x - 125$$
Find the break-even point(s), where the profit is zero.

5. The potential energy $V(x)$ in an electric field varies as a cubic function of distance $x$, given by:
$$V(x) = 2x^3 - 6x^2 + 4x$$
Determine where the potential energy is zero.

6. In structural engineering, the deflection $Y(x)$ of a beam is given by:
$$Y(x) = 2x^2 - 8x + 6$$
This equation gives the vertical deflection at any point $x$ along the beam. Find the points of zero deflection.

## REVIEW EXERCISE 4

1. Four options are given against each statement. Encircle the correct option.
   i. The factorization of $12x + 36$ is:
      (a) $12(x + 3)$ (b) $12(3x)$ (c) $12(3x + 1)$ (d) $x(12 + 36x)$
   ii. The factors of $4x^2 - 12y + 9$ are:
      (a) $(2x + 3)^2$ (b) $(2x - 3)^2$
      (c) $(2x - 3)(2x + 3)$ (d) $(2 + 3x)(2 - 3x)^2$
   iii. The HCF of $a^3b^3$ and $ab^2$ is:
      (a) $a^3b^3$ (b) $ab^2$ (c) $a^4b^5$ (d) $a^2b$

iv. The LCM of $16x^2$, $4x$ and $30xy$ is:

(a) $480x^3y$ (b) $240xy$ (c) $240x^2y$ (d) $120x^4y$

v. Product of LCM and HCF = ________ of two polynomials.

(a) sum (b) difference (c) product (d) quotient

vi. The square root of $x^2 - 6x + 9$ is:

(a) $\pm(x-3)$ (b) $\pm(x+3)$ (c) $x-3$ (d) $x+3$

vii. The LCM of $(a-b)^2$ and $(a-b)^4$ is:

(a) $(a-b)^2$ (b) $(a-b)^3$ (c) $(a-b)^4$ (d) $(a-b)^6$

viii. Factorization of $x^3 + 3x^2 + 3x + 1$ is:

(a) $(x+1)^3$ (b) $(x-1)^3$

(c) $(x+1)(x^2+x+1)$ (d) $(x-1)(x^2-x+1)$

ix. Cubic polynomial has degree:

(a) 1 (b) 2 (c) 3 (d) 4

x. One of the factors of $x^3 - 27$ is:

(a) $x-3$ (b) $x+3$ (c) $x^2-3x+9$ (d) Both $a$ and $c$

2. Factorize the following expressions:

(i) $4x^3 + 18x^2 - 12x$ (ii) $x^3 + 64y^3$

(iii) $x^3y^3 - 8$ (iv) $-x^2 - 23x - 60$

(v) $2x^2 + 7x + 3$ (vi) $x^4 + 64$

(vii) $x^4 + 2x^2 + 9$ (viii) $(x+3)(x+4)(x+5)(x+6) - 360$

(ix) $(x^2 + 6x + 3)(x^2 + 6x - 9) + 36$

3. Find LCM and HCF by prime factorization method:

(i) $4x^3 + 12x^2,\ 8x^2 + 16x$ (ii) $x^3 + 3x^2 - 4x,\ x^2 - x - 6$

(iii) $x^2 + 8x + 16,\ x^2 - 16$ (iv) $x^3 - 9x,\ x^2 - 4x + 3$

4. Find square root by factorization and division method of the expression $16x^4 + 8x^2 + 1$.

5. Huria is analyzing the total cost of her loan, modeled by the expression $C(x) = x^2 - 8x + 15$, where $x$ represents the number of years. What is the optimal repayment period for Huria's loan?