Unit 5 Linear Equations and Inequalities

Students' Learning Outcomes

At the end of the unit, the students will be able to:

- Solve linear equations and inequalities with rational coefficients and represent the solution set on a real line.
- Solve two linear inequalities with two unknowns simultaneously.
- Interpret and identify regions in plane bounded by two linear inequalities in two unknowns.
- Find maximum and minimum values of a function using points in the feasible solution.

INTRODUCTION

Linear equations and inequalities are widely used in various fields to model and solve real-world problems. They help in understanding relationships between variables and making decisions. In this unit, our main goal will be to optimize (maximum or minimum) a quantity under consideration subject to certain constraint restrictions.

5.1 Linear Equation

An equation of the form $ax + b = 0$ where 'a' and 'b' are constants, $a \neq 0$ and 'x' is a variable, is called a linear equation in one variable. In linear equation, the highest power of the variable is always 1.

> **Remember!**
> $ax + b = 0$ and $a \neq 0$ is also called the general form of linear equation in one variable.

5.1.1 Solving a Linear Equation in One Variable

Solving a linear equation in one variable means finding the value of the variable that makes the equation true. To solve the equation, the goal is to isolate the variable on one side of the equation and determine its value.

Steps to Solve a Linear Equation in One Variable

Simplify Both Sides (if necessary)

- Combine like terms on each side of the equation.
- Simplify expressions, including distributing any multiplication over parentheses.

Isolate the Variable Term

- Move all terms containing the variable to one side of the equation and all constant terms numbers to the other side. We can do this by adding or subtracting terms from both sides of the equation.

Solve for the Variable

- Once the variable term is isolated, solve for the variable by dividing or multiplying both sides of the equation by the co-efficient of the variable.

Check Your Solution

- Substitute the solution into the original equation to ensure that solution is correct.

Example 1: Solve the following equations and represent their solutions on real line:

(i) $3x-5=7$ (ii) $\frac{x-2}{5}-\frac{x-4}{2}=2$

Solution **(i)** $3x-5=7$

$$3x-5+5=7+5$$
$$3x=12$$
$$\frac{3x}{x}=\frac{12}{3}$$
$$x=4$$

> **Remember!**
> A linear equation in one variable has only one solution.

Check: Substitute $x=4$ into the original equation

$$3(4)-5=7$$
$$12-5=7$$
$$7=7$$

So, $x=4$ is a solution because it makes the original equation true.

Representation of the solution on a number line:

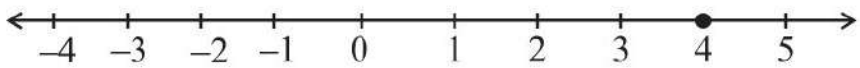


Fig. 5.1

(ii) $\frac{x-2}{5}-\frac{x-4}{2}=2$

$$\frac{2(x-2)-5(x-4)}{10}=2$$
$$\frac{2x-4-5x+20}{10}=2$$
$$\frac{-3x+16}{10}=2$$

> **Remember!**
> We check the solution after solving linear equation to ensure the accuracy of our work.

$$\frac{-3x+16}{10}\times 10 = 2\times 10$$

$$-3x+16 = 20$$

$$-3x+16-16 = 20-16$$

$$-3x = 4$$

$$x = -\frac{4}{3}$$

Check: Substitute $x=-\frac{4}{3}$ into the original equation

$$\frac{-\frac{4}{3}-2}{5}-\frac{-\frac{4}{3}-4}{2}=2$$

$$\Rightarrow \quad \frac{\frac{-4-6}{3}}{5}-\frac{\frac{-4-12}{3}}{2}=2$$

$$\Rightarrow \quad \frac{-10}{15}-\frac{-16}{6}=2$$

$$\Rightarrow \quad -\frac{2}{3}+\frac{8}{3}=2$$

$$\Rightarrow \quad \frac{-2+8}{3}=2$$

$$\Rightarrow \quad \frac{6}{3}=2$$

$$\Rightarrow \quad 2=2$$

So, $x=-\frac{4}{3}$ is the solution of given equation.

Representation of the solution on a number line:

Fig. 5.2

5.2 Linear Inequalities

Inequalities are expressed by the following four symbols:

$>$ (greater than), $<$ (less than), $\geq$ (greater than or equal to), $\leq$ (less than or equal to)

For example,

(i) $ax < b$ (ii) $ax + b \geq c$ (iii) $ax + by > c$ (iv) $ax + by \leq c$

are inequalities. Inequalities (i) and (ii) are in one variable while inequalities (iii) and (iv) are in two variables. The following operations will not affect the order (or sense) of inequality while changing it to simpler equivalent form:

(i) Adding or subtracting a constant to each side of it.

ii) Multiplying or dividing each side by a positive constant.

> **Do you know?**
> The order (or sense) of an inequality is changed by multiplying or dividing each side by a negative constant.

Example 2: Find solution of $\frac{2}{3}x - 1 < 0$ and also represent it on a real line.

Solution:

$$\frac{2}{3}x - 1 < 0 \quad \text{...(i)}$$

$$\Rightarrow \quad \frac{2}{3}x < 1$$

$$\Rightarrow \quad 2x < 3$$

$$\Rightarrow \quad x < \frac{3}{2}$$

It means that all real numbers less than $\frac{3}{2}$ are in the solution of (i)

Thus, the interval $(-\infty, \frac{3}{2})$ or $-\infty < x < \frac{3}{2}$ is the solution of the given inequality which is shown in figure 5.3

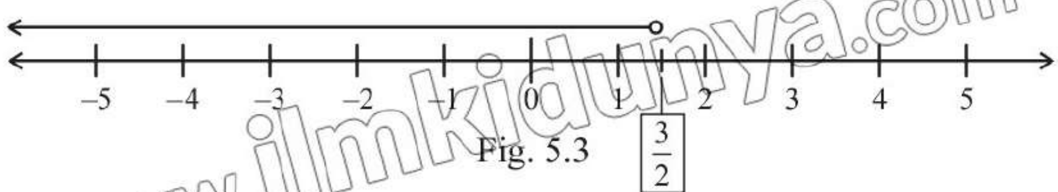
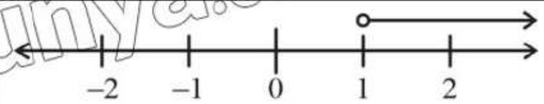
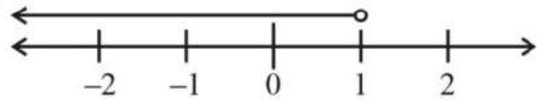
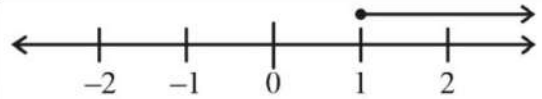
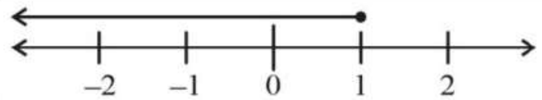


Fig. 5.3

We conclude that the solution of an inequality consists of all solutions of the inequality.

Following are the inequalities and their solutions on a real line:

Inequality	Solution	Representation on real line
$x > 1$	$(1, \infty)$ or $1 < x < \infty$	
$x < 1$	$(-\infty, 1)$ or $-\infty < x < 1$	
$x \geq 1$	$[1, \infty)$ or $1 \leq x < \infty$	
$x \leq 1$	$(-\infty, 1]$ or $-\infty < x \leq 1$	

5.2.1 Solution of a Linear Inequality in Two Variables

Generally, a linear inequality in two variables x and y can be one of the following forms:

$$ax + by < c; \quad ax + by > c; \quad ax + by \leq c; \quad ax + by \geq c$$

Where a, b, c are constants and a, b are not both zero.

We know that the graph of linear equation of the form $ax + by = c$ is a line which divides the plane into two disjoint regions as stated below:

(i) The set of ordered pairs (x, y) such that $ax + by < c$

(ii) The set of ordered pairs (x, y) such that $ax + by > c$

The regions (i) and (ii) are called **half planes** and the line $ax + by = c$ is called the boundary of each half plane.

Note that a **vertical line** divides the plane into **left and right half planes** while a **non-vertical line** divides the plane into **upper and lower half planes.**

A solution of a linear inequality in x and y is an ordered pair of numbers which satisfies the inequality.
For example, the ordered pair (1, 1) is a solution of the inequality $x + 2y < 6$ because $1 + 2(1) = 3 < 6$ which is true.

There are infinitely many ordered pairs that satisfy the inequality $x + 2y < 6$, so its graph will be a half plane.

Note that the linear equation $ax + by = c$ is called "**associated or corresponding equation**" of each of the above-mentioned inequalities.

Procedure for Graphing a linear Inequality in two Variables

(i) The corresponding equation of the inequality is first graphed by using 'dashes' if the inequality involves the symbols $>$ or $<$ and a solid line is drawn if the inequality involves the symbols $\geq$ or $\leq$.

(ii) A test point (not on the graph of the corresponding equation) is chosen which determines on which side of the boundary line the half plane line.

> **Do you know?**
>
> A test point is a point selected to determine which side of the boundary line represents the solution region for an inequality. Usually, we take origin (0,0) as a test point.
>
> - If the inequality holds true with the test point, the region containing this point is part of the solution.
> - If the inequality is false, the opposite region is the solution region.

Example 3: Solve the inequality $x + 2y < 6$.

Solution: The associated equation of the inequality

$$x + 2y < 6 \qquad \text{(i)}$$

is

$$x + 2y = 6 \qquad \text{(ii)}$$

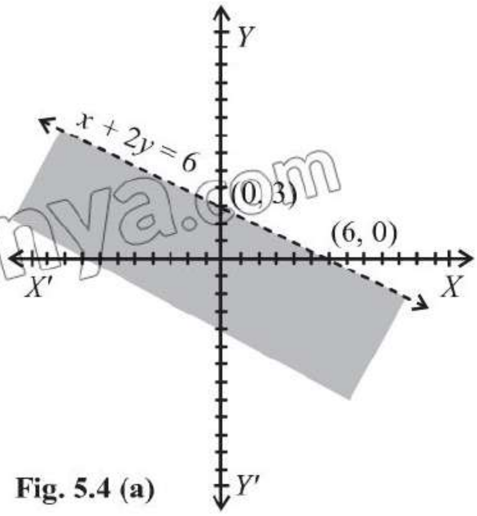


Fig. 5.4 (a)

The line (ii) intersects the x-axis and y-axis at (6, 0) and (0, 3) respectively. As no point of the line (ii) is a solution x of the inequality (i), so the graph of the line (ii) is shown by using dashes. We take O (0, 0) as a test point because it is not on the line (ii).

Substituting $x = 0$, $y = 0$ in the expression $x + 2y$ gives $0 - 2(0) = 0 < 6$. So, the point (0, 0) satisfies the inequality (i). Any other point below the line (ii) satisfies the inequality (i), that is all points in the half plane containing the point (0,0) satisfy the inequality (i).

Thus, the graph of the solution set of inequality (i) is a region on lies the origin-side of the line (ii), that is, the region below the line (ii). A portion of the open half plane below the line (ii) is shown as shaded region in figure 5.4(a)

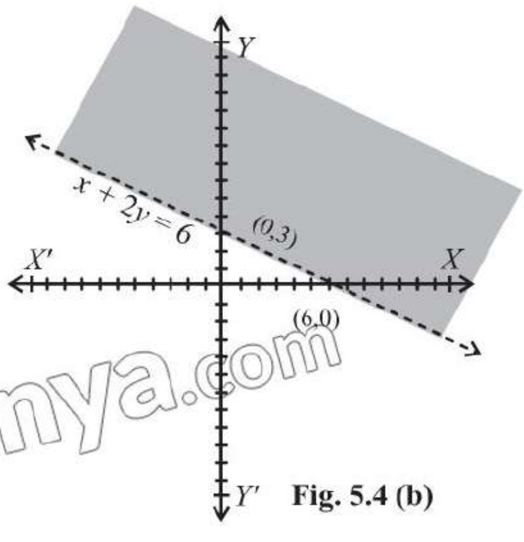


Fig. 5.4 (b)

Note:

All points above the dashed line satisfy the inequality $x + 2y > 6$ (iii)

A portion of the open half plane above the line (ii) is shown by shading in figure 5.4(b).

Note: 1. The graph of the inequality $x + 2y \leq 6$ ...(iv) The open half-plane below the line (ii) including the graph of the line (ii) is the graph of the inequality (iv). A portion of the graph of the inequality (iv) is shown by shading in fig. 5.4 (c).

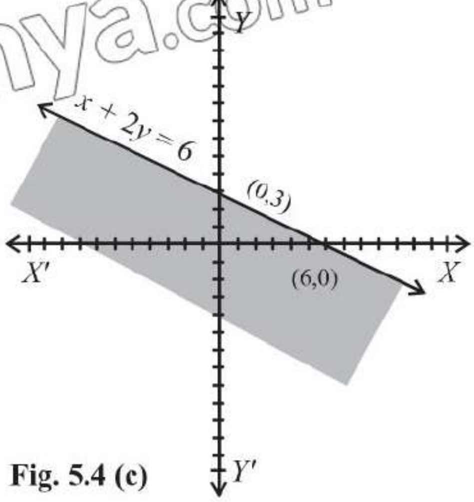


Fig. 5.4 (c)

Note: 2 All points on the line (ii) and above the line (ii) satisfy the inequality $x + 2y \geq 6$ .... (v). This means that the solution set of the inequality (v) consists of all points above the line (ii) and all points on the lines (ii). The graph of the inequality (v) is partially shown as shaded region in fig. 5.4 (d).

Note: 3 The graphs of $x + 2y \leq 6$ and $x + 2y \geq 6$ are closed half planes.

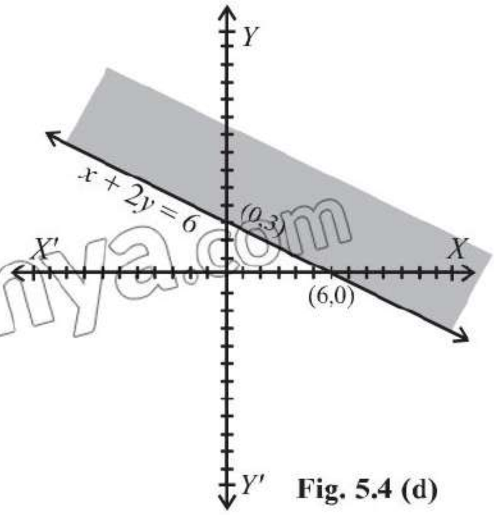


Fig. 5.4 (d)

Example 4: Solve the following linear inequalities in xy-plane:

(i) $2x \geq -3$

(ii) $y \leq 2$

Solution: (i) The inequality $2x \geq -3$ in xy-plane is considered as $2x + 0\,y \geq -3$ and its solution set consists of all point (x, y)

such that $x, y \in$ R and $x \geq -\frac{3}{2}$

The corresponding equation of the given inequality is $2x = -3$ …(i)

which is a vertical line (parallel to the y-axis) and its graph is drawn in figure 5.5(a).

Thus, the graph of $2x \geq -3$ consists of boundary line and the open half-plane to the right of the line (i).

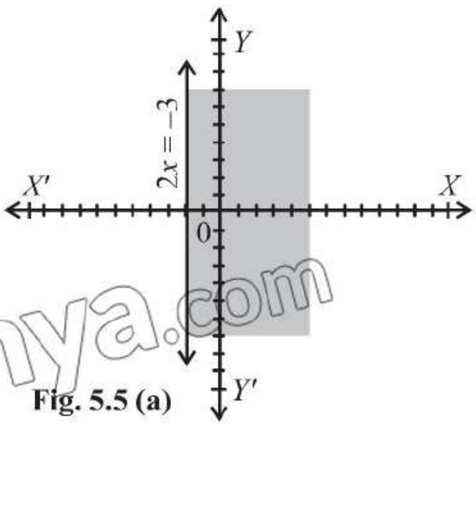


Fig. 5.5 (a)

(ii) The associated equation of the inequality $y \leq 2$ is $y = 2$ …(ii) which is a horizontal line (parallel to the x-axis) and its graph is shown in figure 5.5 (b). Here the solution set of the inequality $y < 2$ is the open half plane below the boundary line $y = 2$. Thus, the graph of $y \le 2$ consists of the boundary line and the closed half plane below it.

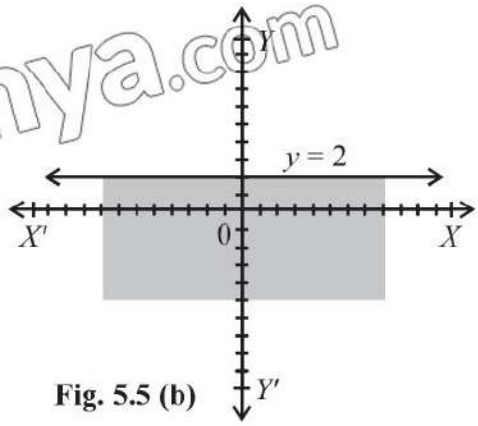


Fig. 5.5 (b)

5.2.2 Solution of Two Linear Inequalities in Two Variables

The graph of a system of linear inequalities consists of the set of all ordered pairs (x, y) in the xy-plane which simultaneously satisfies all the inequalities in the system. To find the graph of such a system, we draw the graph of each inequality in the system on the same coordinate axes and then take intersection of all the graphs. The common region so obtained is called the solution region for the system of inequalities.

Example 5: Find the solution region by drawing the graph of the system of inequalities

$$x - 2y \le 6$$
$$2x + y \ge 2$$

Solution: $x - 2y \le 6$ ...(i)

$2x + y \ge 2$ ...(ii)

The associated equation of (i) is

$x - 2y = 6$ ...(iii)

For x-intercept, put $y = 0$ in (iii), we get

$x - 2(0) = 6$

$x - 0 = 6$

$\Rightarrow$ $x = 6$, so the point is (6, 0)

For y-intercept, put $x = 0$ in (iii), we get

$0 - 2y = 6$

$\Rightarrow$ $-2y = 6$

$\Rightarrow$ $y = \dfrac{6}{-2} = -3$, so the point is (0, –3)

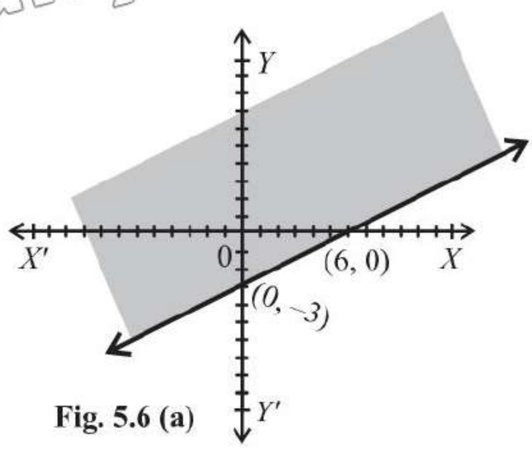


Fig. 5.6 (a)

The graph of the line $x - 2y = 6$ is drawn by joining the point (6, 0) and (0, –3). The point (0, 0) satisfies the inequality $x - 2y < 6$ because $0 - 2(0) = 0 < 6$. Thus, the graph of $x - 2y \le 6$ is the upper half-plane including the graph of the line $x - 2y = 6$. The closed half-plane is partially shown by shading in figure 5.6 (a).

The associated equation of (ii) is

$2x + y = 2 \qquad \ldots(\text{iv})$

For x-intercept, put $y = 0$ in (iv) , we get

$2x + 0 = 2$

$\Rightarrow \quad 2x = 2$

$\Rightarrow \quad x = 1$, so the point is (1, 0)

For y-intercept, put $x = 0$ in (iv), we get

$2(0) + y = 2$

$\Rightarrow \quad y = 2$, so the point is (0, 2)

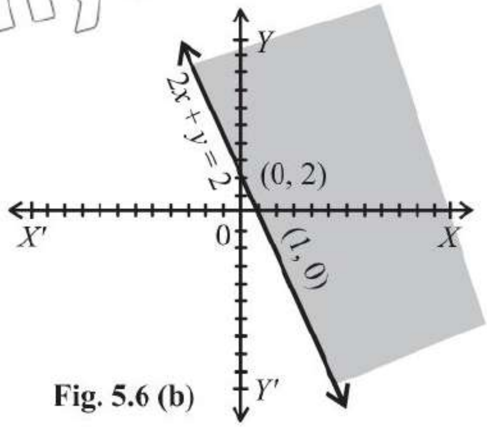


Fig. 5.6 (b)

We draw the graph of the line $2x + y = 2$ joining the points (1, 0) and (0, 2). The point (0, 0) does not satisfy the inequality $2x + y > 2$ because $2(0) + 0 = 0 \ngtr 2$. Thus, the graph of the inequality $2x + y \geq 2$ is the closed half-plane not on the origin-side of the line $2x + y = 2$ and partially shown by shading in figure 5.6 (b).

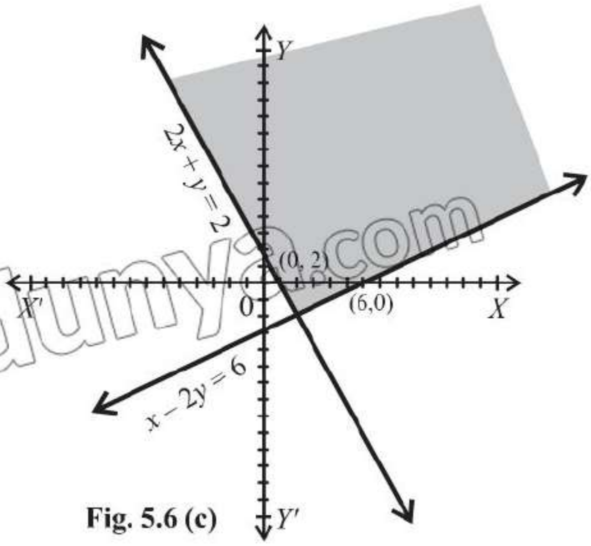


Fig. 5.6 (c)

The solution region of the given system of inequalities is the intersection of the graphs indicated in figures 5.6 (a) and 5.6 (b) is shown as shaded region in figure 5.6 (c).

EXERCISE 5.1

1. Solve and represent the solution on a real line.

(i) $12x + 30 = -6$ (ii) $\frac{x}{3} + 6 = -12$ (iii) $\frac{x}{2} - \frac{3x}{4} = \frac{1}{12}$

(iv) $2 = 7(2x + 4) + 12x$ (v) $\frac{2x-1}{3} - \frac{3}{4}x = \frac{5}{6}$ (vi) $\frac{-5x}{10} = 9 - \frac{10}{5}x$

2. Solve each inequality and represent the solution on a real line.

(i) $x - 6 \leq -2$ (ii) $-9 > -16 + x$ (iii) $3 + 2x \geq 3$

(iv) $6(x + 10) \leq 0$ (v) $\frac{5}{3}x - \frac{3}{4} < \frac{-1}{12}$ (vi) $\frac{1}{4}x - \frac{1}{2} \leq -1 + \frac{1}{2}x$

3. Shade the solution region for the following linear inequalities in xy-plane:

(i) $2x + y \le 6$ (ii) $3x + 7y \ge 21$ (iii) $3x - 2y \ge 6$

(iv) $5x - 4y \le 20$ (v) $2x + 1 \ge 0$ (vi) $3y - 4 \le 0$

4. Indicate the solution region of the following linear inequalities by shading:

(i) $2x - 3y \le 6$, $2x + 3y \le 12$ (ii) $x + y \ge 5$, $-y + x \le 1$ (iii) $3x + 7y \ge 21$, $x - y \le 2$

(iv) $4x - 3y \le 12$, $x \ge -\frac{3}{2}$ (v) $3x + 7y \ge 21$, $y \le 4$ (vi) $5x + 7y \le 35$, $x - 2y \le 2$

5.3 Feasible Solution

While tackling a certain problem from everyday life each linear inequality concerning the problem is named as **problem constraint**. The system of linear inequalities involved in the problem concerned is called **problem constraints**. The variables used in the system of linear inequalities relating to the problems of everyday life are non-negative and are called **non-negative constraints**. These non-negative constraints play an important role for taking decision. So, these variables are called **decision variables**. A region which is restricted to the first quadrant is referred to as a **feasible region** for the set of given constraints. Each point of the feasible region is called a **feasible solution** of the system of linear inequalities (or for the set of a given constraints).

Example 6: Shade the feasible region and find the corner points for the following system of inequalities (or subject to the following constraints).

$$x - y \le 3$$

$$x + 2y \le 6, \quad x \ge 0, \quad y \ge 0$$

Solution: The associated equations for the inequalities

$x - y \le 3$ … (i) and $x + 2y \le 6$ … (ii)

are $x - y = 3$ … (1) and $x + 2y = 6$ … (2)

As the point (3, 0) and (0,–3) are on the line (1), so the graph of $x - y = 3$ is drawn by joining the points (3, 0) and (0,–3) by solid line.

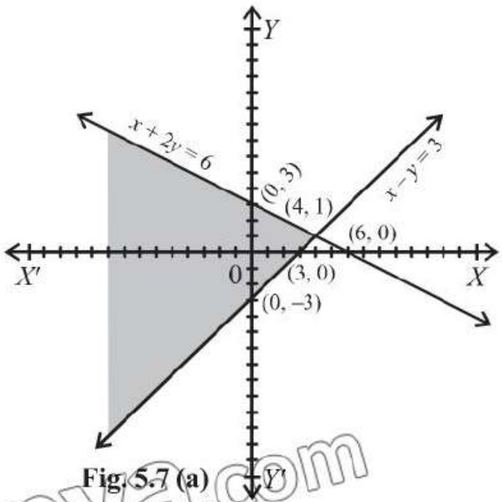


Fig. 5.7 (a)

Similarly, line (2) is graphed by joining the points (6, 0) and (0, 3) by solid line. For $x = 0$ and $y = 0$, we have;

$0 - 0 = 0 < 3$ and $0 + 2(0) = 0 < 6$

So, both the closed half-planes are on the origin sides of the lines (1) and (2). The intersection of these closed half-planes is partially displayed as shaded region in fig. 5.7(a).

The graph of $y \geq 0$, will be the closed upper half plane. The intersection of graph shown in figure 5.7(a) and closed upper half plane is partially displayed as shaded region in figure 5.7 (b).

The graph of $x \geq 0$ will be closed right half plane. The intersection of the graph shown in fig. 5.7(a) and closed right half plane is graphed in fig. 5.7 (c).

Finally, the graph of the given system of linear inequalities is displayed in figure 5.7 (d) which is the feasible region for the given system of linear inequalities. The points (0, 0), (3, 0), (4, 1) and (0, 3) are corner points of the feasible region.

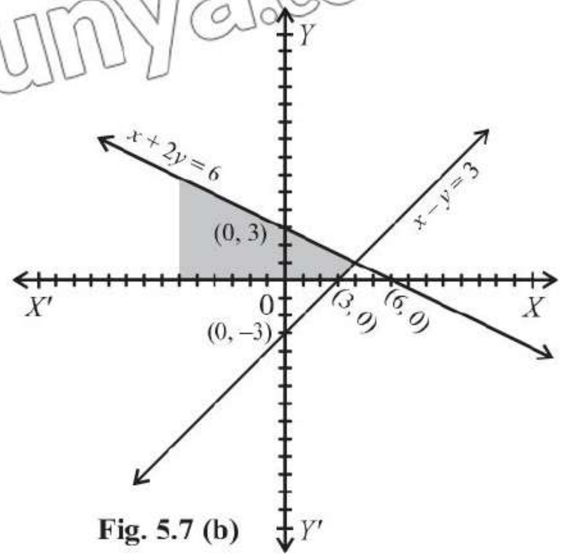


Fig. 5.7 (b)

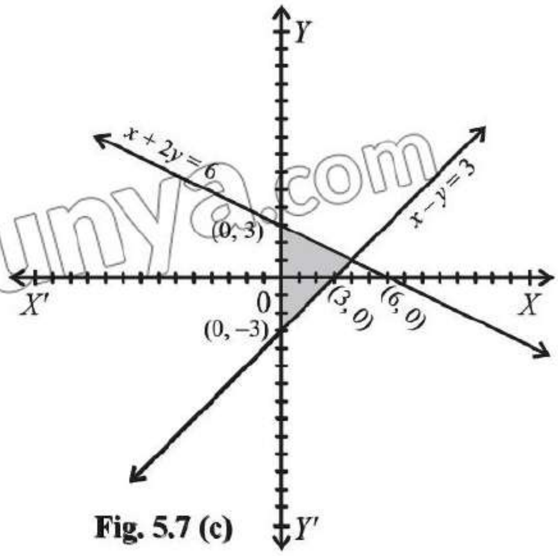


Fig. 5.7 (c)

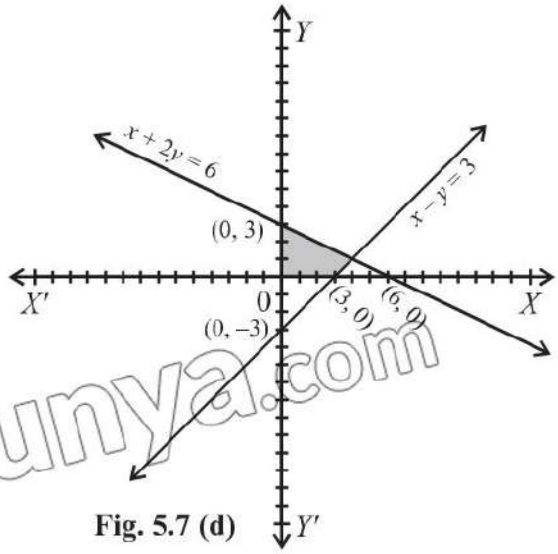


Fig. 5.7 (d)

Remember!

A point of a solution region where two of its boundary lines intersect, is called a **corner point** or **vertex** of the solution region.

5.3.2 Maximum and Minimum Values of a Function in the Feasible Solution

A function which is to be maximized or minimized is called an **objective function**. Note that there are infinitely many feasible solutions in the feasible region. The feasible solution which maximizes or minimizes the objective function is called the **optimal solution**.

Procedure for determining optimal solution

(i) Graph the solution set of linear inequality constraints to determine feasible region.

(ii) Find the corner points of the feasible region.

(iii) Evaluate the objective function at each corner point to find the optimal solution.

Example 7: Find the maximum and minimum values of the function defined as:

$f(x,y) = 2x + 3y$

subject to the constraints;

$x - y \leq 2$

$x + y \leq 4$

$x \geq 0, y \geq 0$

Solution: $x - y \leq 2$ ...(i)

$x + y \leq 4$ ...(ii)

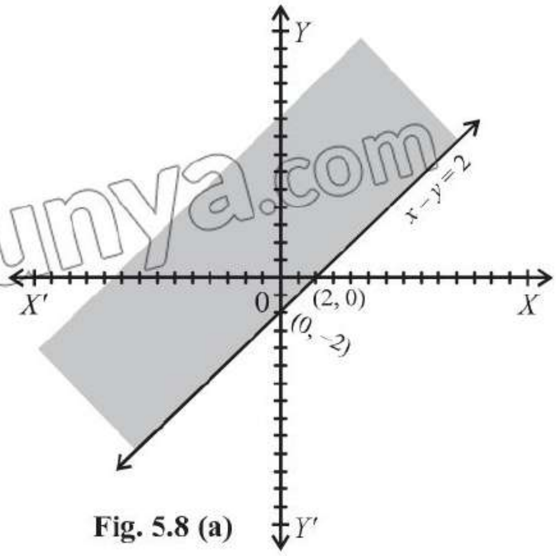


Fig. 5.8 (a)

The associated equation of (i) is

$x - y = 2$

x-intercept and y-intercept of $x - y = 2$ are (2, 0) and (0, –2) respectively. The graph of the line x – y = 2 is drawn by joining the points (2, 0) and (0, –2). The point (0,0) satisfies the inequality $x - y \leq 2$ because $0 - 0 = 0 < 2$. Thus, the graph of $x - y \leq 2$ is the upper half-plane including the graph of the line $x - y = 2$. The closed half-plane is partially shown by shading in figure 5.8(a).

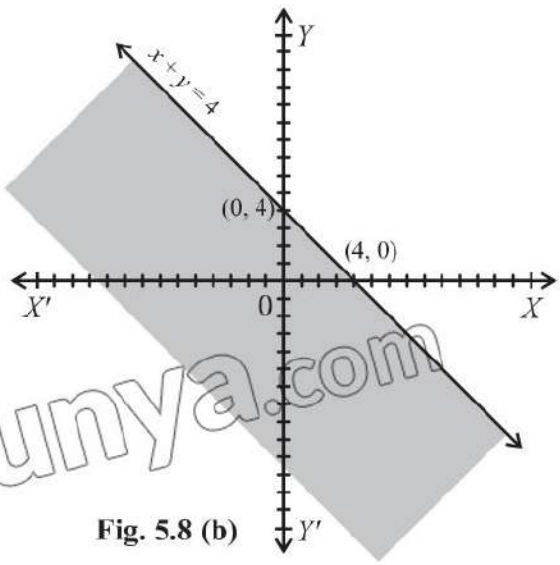


Fig. 5.8 (b)

The associated equation of (ii) is $x + y = 4$

x-intercept and y-intercept of $x + y = 4$ are (4, 0) and (0, 4). The graph of the line $x + y = 4$ is drawn by joining the points (4,0) and (0,4). The point (0, 0) satisfies the inequality $x + y \leq 4$. The closed half-plane is partially shown by shading in figure 5.8 (b).

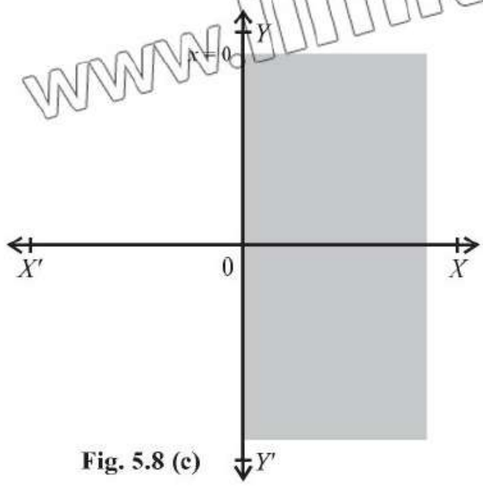


Fig. 5.8 (c)

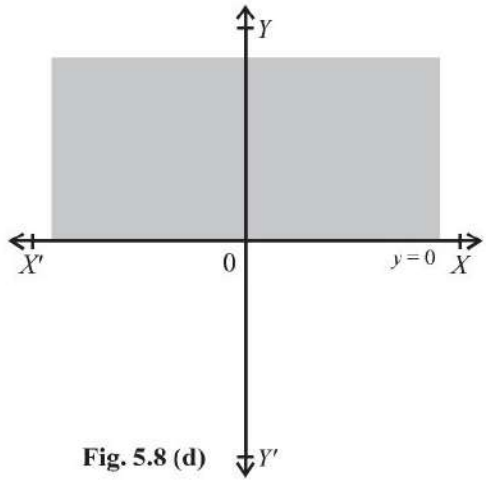


Fig. 5.8 (d)

The graph of $x \geq 0$ and $y \geq 0$ is shown by shading in figures 5.8 (c) and 5.8 (d) respectively.

The feasible region of the given system of inequalities is the intersection of the graphs indicated in figures 5.8 (a), 5.8 (b), 5.8 (c) and 5.8 (d) and is shown as shaded region $ABCD$ in figure 5.8 (e).

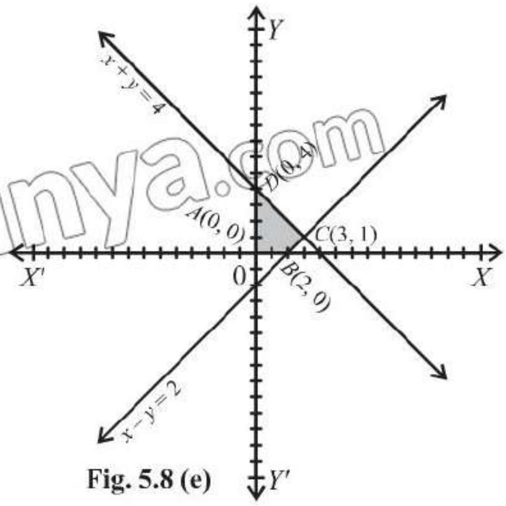


Fig. 5.8 (e)

Corner points of the feasible region are (0, 0), (2, 0), (3, 1) and (0, 4). Now, we find values of $f(x, y) = 2x + 3y$ at the corner points.

$$f(0, 0) = 2(0) + 3(0) = 0$$
$$f(2, 0) = 2(2) + 3(0) = 4$$
$$f(3, 1) = 2(3) + 3(1) = 9$$
$$f(0, 4) = 2(0) + 3(4) = 12$$

Thus, the minimum value of f is 0 at the corner point (0, 0) and maximum value of f is 12 at corner point (0,4).

EXERCISE 5.2

1. Maximize $f(x, y) = 2x + 5y$; subject to the constraints

   $2y - x \leq 8$; $x - y \leq 4$; $x \geq 0$; $y \geq 0$

2. Maximize $f(x, y) = x + 3y$; subject to the constraints

   $2x + 5y \leq 30$; $5x + 4y \leq 20$; $x \geq 0$; $y \geq 0$

3. Maximize $z = 2x + 3y$; subject to the constraints:

$$2x + y \leq 4 \quad ; \quad 4x - y \leq 4 \quad ; \quad x \geq 0; \quad y \geq 0$$

4. Minimize $z = 2x + y$; subject to the constraints:

$$x + y \geq 3 \quad ; \quad 7x + 5y \leq 35 \quad ; \quad x \geq 0; \quad y \geq 0$$

5. Maximize the function defined as; $f(x, y) = 2x + 3y$ subject to the constraints:

$$2x + y \leq 8 \quad ; \quad x + 2y \leq 14 \quad ; \quad x \geq 0; \quad y \geq 0$$

6. Find minimum and maximum values of $z = 3x + y$; subject to the constraints:

$$3x + 5y \geq 15 \quad ; \quad x + 6y \geq 9 \quad ; \quad x \geq 0; \quad y \geq 0$$

REVIEW EXERCISE 5

1. Four options are given against each statement. Encircle the correct one.

i. In the following, linear equation is:
   (a) $5x > 7$ (b) $4x - 2 < 1$
   (c) $2x + 1 = 1$ (d) $4 = 1 + 3$

ii. Solution of $5x - 10 = 10$ is:
   (a) 0 (b) 50
   (c) 4 (d) -4

iii. If $7x + 4 < 6x + 6$, then x belongs to the interval
   (a) $(2, \infty)$ (b) $[2, \infty)$
   (c) $(-\infty, 2)$ (d) $(-\infty, 2]$

iv. A vertical line divides the plane into
   (a) left half plane (b) right half plane
   (c) full plane (d) two half planes

v. The linear equation formed out of the linear inequality is called
   (a) linear equation (b) associated equation
   (c) quadratic equal (d) none of these

vi. $3x + 4 < 0$ is:
   (a) equation (b) inequality
   (c) not inequality (d) identity

vii. Corner point is also called:
   (a) code (b) vertex
   (c) curve (d) region