Unit 6 Trigonometry

Students' Learning Outcomes

At the end of the unit, the students will be able to:

- Identify angles in standard positions expressed in degrees and radian.
- Apply Pythagoras theorem and the sine, the cosine and tangent ratios for acute angles of a right angle.
- Solve real life trigonometric problems in 2-D involving angles of elevation and depression
- Prove the trigonometric identities and apply them to draw different trigonometric relations.
- Solve real life problems involving trigonometric identities.

INTRODUCTION

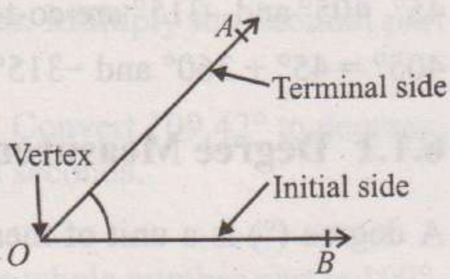
Trigonometry is a branch of mathematics that deals with the relationships between the angles and sides of a triangle, especially right-angled triangle. It plays a vital role in various fields such as physics, engineering, architecture and astronomy. The trigonometric concepts can solve problems involving angles and distances that appear in real-life situations such as calculating the height of buildings, distance between objects and angle measurements in navigation.

6.1 Identifying Angles in Standard Position (Degrees and Radians)

Brain teaser!

The plane geometry is the study of two dimensional figures. What is Euclidean geometry?

A plane figure which is formed by two rays sharing a common end point is called an angle. The two rays are known as the sides of the angle. The common end point is known as vertex. The amount of rotation or measure of opening between these rays is called an angle. $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are rays and angle is AOB. Written as $\angle AOB$ or $A\hat{O}B$.



The angle is said to be in standard position if:

(a) Its vertex is located at the origin of the coordinate plane.

(b) One of its rays (the initial side) lies along the positive x-axis.

Types of angles are:

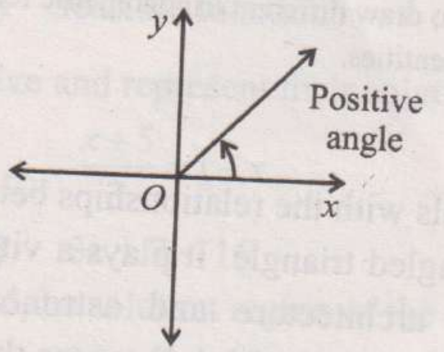
- Acute angle $0° < \theta < 90°$
- Obtuse angle $90° < \theta < 180°$
- Right angle $\theta = 90°$
- Straight angle $\theta = 180°$
- Reflex angle $180° < \theta < 360°$
- Full rotation $\theta = 360°$

(c) The other ray (the terminal side) determines the direction of the angle.

An angle is measured from the initial side to the terminal side. It is usually represented by Greek letters θ, α, β, γ etc.

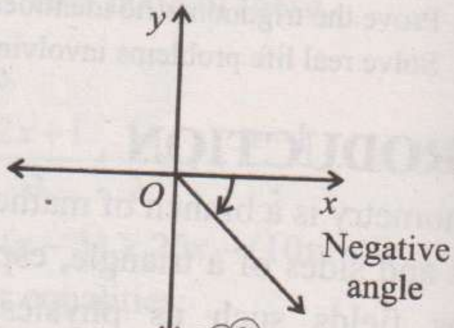
Positive angles

The angle will be positive if the terminal side is rotated counterclockwise from the initial side. The given angle is in 1^{st} quadrant



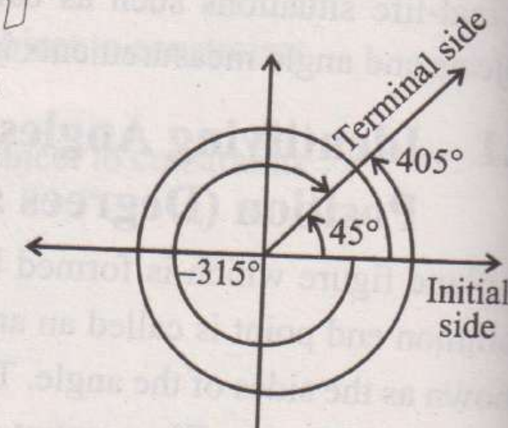
Negative angles

The angle will be negative if the terminal side is rotated clockwise from the initial side. The given angle is in 4^{th} quadrant



Co-Terminal Angles

Co-terminal angles are angles that share the same initial side and terminal side in standard position, but they may have different measures. These angles differ by a multiple of 360° or 2π rad. For example, 45°, 405° and −315° are co-terminal angles because $405° = 45° + 360°$ and $-315° = 45° - 360°$.



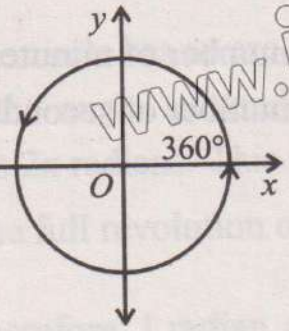
6.1.1 Degree Measurement

A degree (°) is a unit of measurement of angles. It represents $\frac{1}{360}$ of a full rotation around a point. In simpler terms, a degree is the measure of an angle, with a complete circle being 360°.

Why 360° Historically? The choice of 360° to divide a circle dates back to the **Babylonians**, who used a base-60 number system (sexagesimal system). They were among the first to formalize the concept of angle measurement, and 360 was chosen likely because it is a highly composite number (it can be divided by 2, 3, 4, 5, 6, 9, 10, 12, 15, and more), making calculations easier. This system persisted throughout ancient times and degrees became entrenched in various cultures and mathematical traditions.

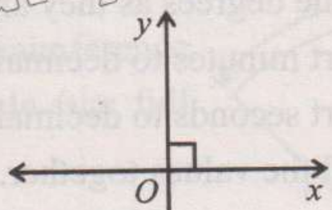
Full Circle

A full rotation around a central point forms an angle of 360°.



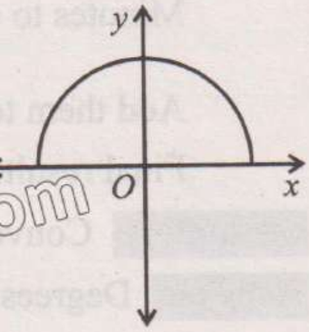
Right Angle

One-quarter of a full rotation, or a 90° angle, is called a right angle.



Half Circle

A straight angle, or half of a full rotation, measures 180°. The degree measure is further divided into minutes (′) and seconds (″).



$1° = 60'$ (60 minutes)

$1' = 60''$ (60 seconds)

$1° = 3600''$ (60×60 seconds)

6.1.2 Converting Degrees to Minutes and Seconds

To convert decimal degrees to degrees, minutes and seconds (DMS), follow the steps:

- Separate the whole number part (degrees) of the decimal.
- Multiply the decimal part by 60 to get the minutes.
- The whole number part of the result is the minutes. Multiply the decimal part of the minutes by 60 to get the seconds.

Example 1: Convert 73.12° to degrees, minutes, and seconds.

Solution:

Degrees: The whole number part is 73°.

Minutes: Take the decimal part (0.12) and multiply by 60: $0.12 \times 60 = 7.2$. The whole number part is 7, so it's 7 minutes.

Seconds: Now take the decimal part (0.2) and multiply by 60: $0.2 \times 60 = 12$. So, it's 12 seconds.

Final result: $73° \, 7' \, 12''$.

Example 2: Convert 109.42° to degrees, minutes, and seconds.

Solution:

Degrees: The whole number part is 109°.

Minutes: Take the decimal part (0.42) and multiply by 60: $0.42 \times 60 = 25.2$. The whole number part is 25, so it's 25 minutes.

Seconds: Now take the decimal part (0.2) and multiply by 60: $0.2 \times 60 = 12$. So, it's 12 seconds.

Final result: $109° \, 25' \, 12''$.

6.1.3 Converting from Degrees, Minutes and Seconds to Decimal Degrees

To convert from degrees, minutes and seconds (DMS) to decimal degrees, follow the steps:

- Keep the degrees as they are.
- Convert minutes to decimal degrees: Divide the number of minutes by 60.
- Convert seconds to decimal degrees: Divide the number of seconds by 3600.
- Add all the values together.

Example 3: Convert $45^\circ\ 45'\ 45''$ to decimal degrees.

Solution: Degrees: Keep 45.

Minutes to decimal: $\frac{45}{60} = 0.75$; Seconds to decimal: $\frac{45}{3600} = 0.0125$

Add them together: $45 + 0.75 + 0.0125 = 45.7625$

Final result: 45.7625°

Example 4: Convert $94^\circ\ 27'\ 54''$ to decimal degrees.

Solution: Degrees: Keep 94:

Minutes to decimal: $\frac{27}{60} = 0.45$; Seconds to decimal: $\frac{54}{3600} = 0.015$

Add them together: $94 + 0.45 + 0.015 = 94.465$

Final result: 94.465°

6.1.4 Circular Measure (Radian)

There is another system of angular measurement called circular system.

The radian, denoted by the symbol "rad", is the unit of angle in the International System of Units (SI) and is the standard unit of angular measure used in many areas of mathematics.

A radian is a unit of angular measure in mathematics, particularly in trigonometry. It is defined as, "the angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle" Unlike degrees, which are based on dividing a circle into 360 parts, the radian is inherently related to the circle's geometry and arc length.

Historical Background of the Radian

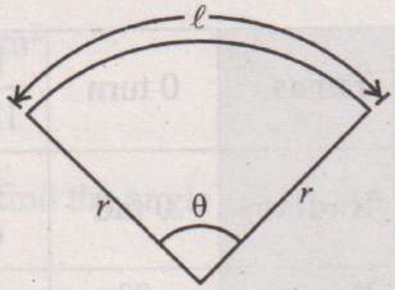
The concept of radian measure, was first formalized by mathematicians in the 18th century, but the principles behind it had been understood much earlier by Euclid and Archimedes.

The word "radian" comes from the radius of a circle, as the radian is fundamentally related to the ratio between the arc length and the radius.

The first known use of the term radian in the context of angular measurement was by Scottish mathematician James Thomson in 1873. His brother, William Thomson, also known as Lord Kelvin, was made a prominent physicist and both were influential in establishing radians as a standard unit.

If a circle of radius r, has an arc length equal to the radius of the circle, then the angle θ subtended by that arc is 1 radian:

$$\theta = \frac{r}{r} = 1 \text{ radian} \quad \left(\because \ \theta = \frac{\text{Arc length}}{\text{Radius}} = \frac{\ell}{r}\right)$$



A complete circle has an arc length equal to the circumference ($2\pi r$), so the angle subtended by the entire circle (the full rotation) is 2π radians. This means:

- One full revolution of a circle is 2π radians, or 360°.
- Therefore, 1 radian $= \dfrac{360^\circ}{2\pi} \approx 57.2958^\circ$ and $1^\circ = \dfrac{2\pi}{360} = 0.01745$ rad

Conversion between degrees and radians

Radians to degrees: $1 \text{ rad} = \dfrac{180}{\pi}$ degrees

Degrees to radians: $1^\circ = \dfrac{\pi}{180}$ rad

Example 5: Convert radians to degrees

(i) $\dfrac{5\pi}{3}$ rad (ii) $\dfrac{7\pi}{6}$ rad (iii) $\dfrac{11\pi}{6}$ rad (iv) 1.2 rad

Solution: (i) $\dfrac{5\pi}{3} \text{ rad} = \dfrac{5\pi}{3} \times \dfrac{180^\circ}{\pi} = 300^\circ$ $\left(1 \text{ rad} = \dfrac{180^\circ}{\pi}\right)$

(ii) $\dfrac{7\pi}{6} \text{ rad} = \dfrac{7\pi}{6} \times \dfrac{180^\circ}{\pi} = 210^\circ$

(iii) $\dfrac{11\pi}{6} \text{ rad} = \dfrac{11\pi}{6} \times \dfrac{180^\circ}{\pi} = 330^\circ$

(iv) $1.2 \text{ rad} = 1.2 \times \dfrac{180^\circ}{\pi} = 68.75^\circ$ $(\because \pi = 3.14159)$

Example 6: Convert degrees to radians

(i) 15° (ii) 75° (iii) 315° (iv) $15^\circ\, 15'$

Solution: (i) $15^\circ = 15 \times \dfrac{\pi}{180} = \dfrac{\pi}{12}$ rad or 0.262 rad

(ii) $75^\circ = 75 \times \dfrac{\pi}{180} = \dfrac{5\pi}{12}$ rad or 1.309 rad

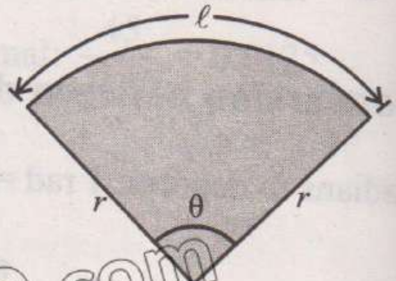
(iii) $315° = 315 \times \frac{\pi}{180} = \frac{7\pi}{4}$ rad or 5.498 rad

(iii) $15°\,15' = 15° + \left(\frac{15}{60}\right)° = 15.25° = 15.25 \times \frac{\pi}{180}$ rad $= 0.266$ rad

Turns	0 turn	$\frac{1}{12}$ turn	$\frac{1}{8}$ turn	$\frac{1}{6}$ turn	$\frac{1}{4}$ turn	$\frac{1}{2}$ turn	1 turn
Radians	0 rad	$\frac{\pi}{6}$ rad	$\frac{\pi}{4}$ rad	$\frac{\pi}{3}$ rad	$\frac{\pi}{2}$ rad	π rad	2π rad
Degrees	0°	30°	45°	60°	90°	180°	360°

Arc Length and Area of Sector

If r is radius and θ (rad) is the angle subtended by the arc of length 'ℓ', then



Arc length of sector $= \ell = r\theta$

and area of sector $= A = \frac{1}{2}r^2\theta$

Proof: We know that:

$$\ell = \frac{\theta}{360°} \times 2\pi r$$

$$= \frac{\theta}{2\pi} \times 2\pi r \quad (2\pi \text{ radians} = 360°)$$

$$= r\theta$$

Proof: We know that

$$A = \frac{\theta}{360°} \times \pi r^2$$

$$= \frac{\theta}{2\pi} \times \pi r^2 \quad (2\pi \text{ radians} = 360°)$$

$$= \frac{1}{2} r^2 \theta$$

Hence arc length, $\ell = r\theta$ and area of sector, $A = \frac{1}{2}r^2\theta$

Example 7: Find the arc length of a sector with radius $r = 10$ cm and central angle $\theta = 60°$.

Solution: Convert $\theta = 60°$ to radians: $\theta = 60 \times \frac{\pi}{180} = \frac{\pi}{3}$ radians.

$$\ell = r\theta = 10 \times \frac{\pi}{3} \approx 10.47 \text{ cm}$$

The arc length is approximately 10.47 cm

Example 8: Find the area of a sector with radius $r = 8$ cm and central angle $\theta = 45°$.

Solution: Convert $\theta = 45°$ to radians: $\theta = 45 \times \frac{\pi}{180} = \frac{\pi}{4}$ radians.

$$A = \frac{1}{2} r^2 \theta = \frac{1}{2} \times 8^2 \times \frac{\pi}{4} = 8\pi \text{ cm}^2 \approx 25.12 \text{ cm}^2.$$

The area of the sector is approximately 25.12 cm^2.

Example 9: If arc length of a sector of radius 5 cm is 11 cm, find the angle subtended by the arc in radians and degrees.

Solution: $r = 5$ cm ; $\ell = 11$ cm, ; $\theta = ?$

$\therefore \quad \ell = r\theta$

$$11 = 5\theta \quad \Rightarrow \theta = \frac{11}{5} = 2.2 \text{ rad}$$

$$\theta = 2.2 \times \frac{180°}{\pi} \approx 126.1°$$

Thus, the angle subtended by the arc in radians is 2.2 rad and degrees is 126.1°

EXERCISE 6.1

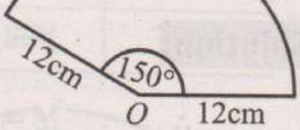
1. Find in which quadrant the following angles lie. Write a co-terminal angle for each:
   (i) 65° (ii) 135° (iii) −40° (iv) 210° (v) −150°
2. Convert the following into degrees, minutes, and seconds:
   (i) 123.456° (ii) 58.7891° (iii) 90.5678°
3. Convert the following into decimal degrees:
   (i) 65° 32' 15" (ii) 42° 18' 45" (iii) 78° 45' 36"
4. Convert the following into radians:
   (i) 36° (ii) 22.5° (iii) 67.5°
5. Convert the following into degrees:
   (i) $\frac{\pi}{16}$ rad (ii) $\frac{11\pi}{5}$ rad (iii) $\frac{7\pi}{6}$ rad
6. Find the arc length and area of a sector if:
   (i) $r = 6$ cm and central angle $\theta = \frac{\pi}{3}$ radians.
   (ii) $r = \frac{4.8}{\pi}$ cm and central angle $\theta = \frac{5\pi}{6}$ radians.

7. If the central angle of a sector is 60° and the radius of the circle is 12 cm, find the area of the sector and the percentage of the total area of the circle it represents.

8. Find the percentage of the area of sector subtending an angle $\frac{\pi}{8}$ radians.

9. A circular sector of radius $r = 12$ cm has an angle of 150°. This sector is cut out and then bent to form a cone. What is the slant height and the radius of the base of this cone?

Hint: Arc length of sector = circumference of cone.



6.2 Trigonometric Ratios

The functions that relate angles to side in a right-angled triangle are known as trigonometric functions (sine, cosine, tangent etc.) Their development is rooted in ancient geometry, blossomed through Indian and Islamic mathematics and became formalized in Europe during the Renaissance. Today, these functions are indispensable tools in both theoretical and applied sciences. Trigonometry has since been extensively used in various scientific disciplines such as physics (especially wave theory) engineering, and computer graphics.

History of Sine, Cosine and Tangent

Hipparchus of Nicaea (c. 190 - 120 BC) is considered the "father of trigonometry." He was the first to compile a trigonometric table for solving problems related to astronomy, using chord functions. Hipparchus divided a circle into 360 degrees and used this system for measuring angles.

In Islamic golden age, **Al-Battani** (c. 858 – 929 CE) was among the first to replace chord functions with the modern sine function and calculated tables of sines and tangents.

Al-Khwarizmi (c. 780–850 CE), known for his work in algebra, and Omar Khayyam (c. 1048–1131 CE) worked on spherical trigonometry, which has applications in astronomy.

Isaac Newton and **Gottfried Wilhelm Leibniz** (17th century) developed calculus, which further expanded the use of trigonometric functions beyond geometry into more abstract fields of mathematics.

Application of Trigonometric Ratios

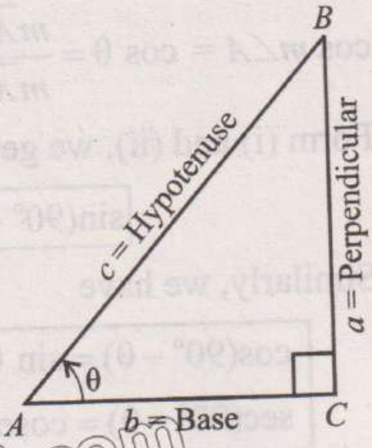
When we make use of a ruler or measuring tape to measure the thickness of a book, the length of a pencil, the height of a chair or table or dimensions of a classroom, we are making direct measurements.

In some cases, it is not possible to obtain direct measurements, because these are difficult and dangerous. For example, it is difficult to climb upon a flag pole to measure its height. To measure the height of a cliff is also difficult and dangerous.

These problems can be solved by indirect measurement with the help of trigonometry. For indirect measurements of distance or height it is very much useful. It also plays an important role in the field of surveying, navigation, engineering and many other branches of physical sciences. We make use of these concepts of trigonometry to solve many of the problems in these fields.

6.2.1 Trigonometric Ratios of an Acute Angle

The trigonometric ratios are applied to acute angle in a right-angled triangle, but the concepts extend to angles greater than 90° and are widely used in many areas of mathematics and science.



Let us consider a right-angled triangle *ACB* with respect to an angle θ (theta) = $m\angle CAB$ with $m\angle ACB = 90°$.

In the triangle *ACB*, the side *BC* is called perpendicular, which is opposite to an angle 'θ'.

The side *AC* is called the base and the side *AB* is called the hypotenuse. Let $m\overline{BC} = a$, $m\overline{AC} = b$ and $m\overline{AB} = c$.

For this right angled triangle *ACB*, the trigonometric ratios of an angle "θ" are defined as:

$$\sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{a}{c} \quad : \quad \operatorname{cosec}\theta = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{c}{a}$$

$$\cos\theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{b}{c} \quad : \quad \sec\theta = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{c}{b}$$

$$\tan\theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{a}{b} \quad : \quad \cot\theta = \frac{\text{Base}}{\text{Perpendicular}} = \frac{b}{a}$$

The six trigonometric ratios described with reference to a right-angled triangle *ACB* and are: sine (sin), cosine(cos), tangent(tan), cosecant (cosec or csc), secant (sec) and cotangent (cot).

We note that: $\tan\theta = \dfrac{a}{b}$

$$= \frac{a/c}{b/c} \quad \text{(Dividing by } c\text{)}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Similarly, $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Note:

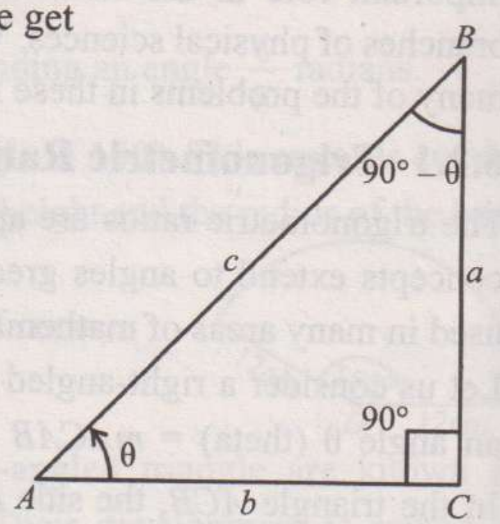
(i) $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

(ii) $\sec\theta = \dfrac{1}{\cos\theta}$

(iii) $\cot\theta = \dfrac{1}{\tan\theta}$

6.2.2 Trigonometric Ratios of Complementary Angles

We consider a right-angled triangle ACB, in which $m\angle A = \theta$, $m\angle C = 90^\circ$ then, $m\angle B = 90^\circ - \theta$. Using the trigonometric ratios of $\angle B$, we get



$$\sin m\angle B = \sin(90^\circ - \theta) = \frac{m\overline{AC}}{m\overline{AB}} = \frac{b}{c} \quad \ldots(i)$$

Using ratios of $\angle A$, we get

$$\cos m\angle A = \cos\theta = \frac{m\overline{AC}}{m\overline{AB}} = \frac{b}{c} \quad \ldots(ii)$$

Form (i) and (ii), we get,

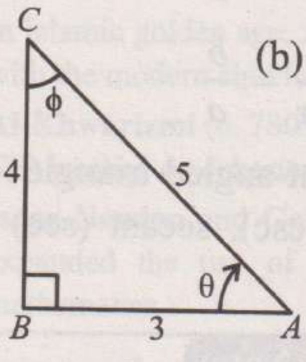
$$\boxed{\sin(90^\circ - \theta) = \cos\theta}$$

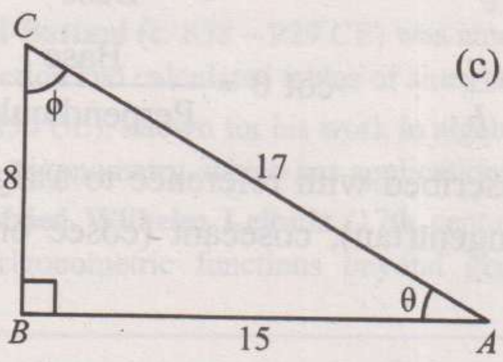
Similarly, we have

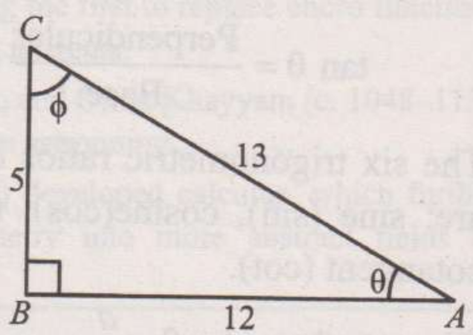
$\cos(90^\circ - \theta) = \sin\theta$;	$\tan(90^\circ - \theta) = \cot\theta$;	$\cot(90^\circ - \theta) = \tan\theta$
$\sec(90^\circ - \theta) = \operatorname{cosec}\theta$;	$\operatorname{cosec}(90^\circ - \theta) = \sec\theta$	

EXERCISE 6.2

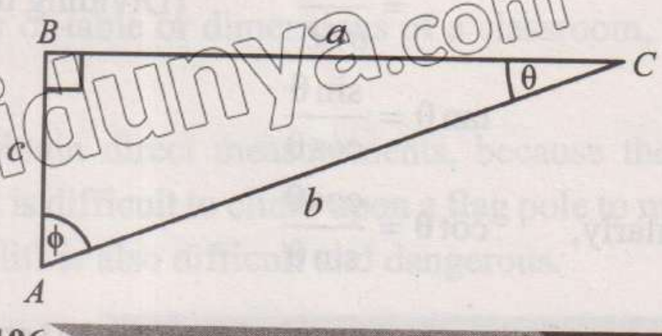
1. For each of the following right-angled triangles, find the trigonometric ratios:
   (i) $\sin\theta$ (ii) $\cos\theta$ (iii) $\tan\theta$ (iv) $\sec\theta$ (v) $\operatorname{cosec}\theta$
   (vi) $\cot\phi$ (vii) $\tan\phi$ (viii) $\operatorname{cosec}\phi$ (ix) $\sec\phi$ (x) $\cos\phi$

(a)


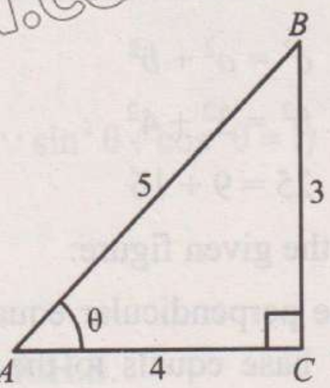
(b)


(c)


2. For the following right-angled triangle ABC find the trigonometric ratios for which $m\angle A = \phi$ and $m\angle C = \theta$
   (i) $\sin\theta$ (ii) $\cos\theta$
   (iii) $\tan\theta$ (iv) $\sin\phi$
   (v) $\cos\phi$ (vi) $\tan\phi$



3. Considering the adjoining triangle ABC, verify that:



(i) $\sin\theta \ \operatorname{cosec}\theta = 1$

(ii) $\cos\theta \ \sec\theta = 1$

(iii) $\tan\theta \ \cot\theta = 1$

4. Fill in the blanks.

(i) $\sin 30° = \sin(90° - 60°) =$ ______

(ii) $\cos 30° = \cos(90° - 60°) =$ ______

(iii) $\tan 30° = \tan(90° - 60°) =$ ______

(iv) $\tan 60° = \tan(90° - 30°) =$ ______

(v) $\sin 60° = \sin(90° - 30°) =$ ______

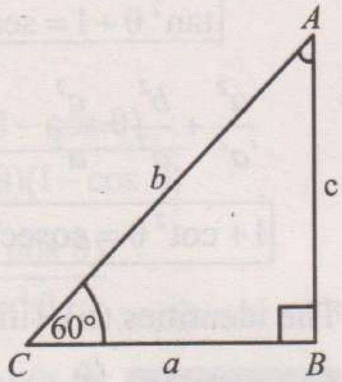
(vi) $\cos 60° = \cos(90° - 30°) =$ ______

(vii) $\sin 45° = \sin(90° - 45°) =$ ______

(viii) $\tan 45° = \tan(90° - 45°) =$ ______

(ix) $\cos 45° = \cos(90° - 45°) =$ ______

5. In a right angled triangle ABC, $m\angle B = 90°$ and C is an acute angle of 60°. Also $\sin m\angle A = \frac{a}{b}$, then find the following trigonometric ratios:



(i) $\frac{m\overline{BC}}{m\overline{AB}}$ (ii) $\cos 60°$

(iii) $\tan 60°$ (iv) $\operatorname{cosec}\frac{\pi}{3}$

(v) $\cot 60°$ (vi) $\sin 30°$

(vii) $\cos 30°$ (viii) $\tan\frac{\pi}{6}$

(ix) $\sec 30°$ (x) $\cot 30°$

6.3 Trigonometric Identities

Fundamental Trigonometric Identities

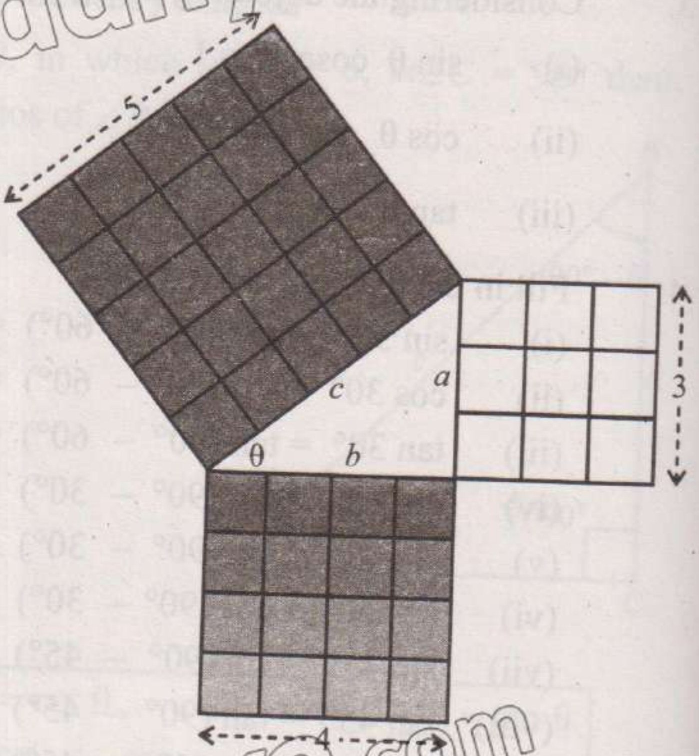
We shall consider some of the fundamental identities used in trigonometry. The key to these basic identities is the Pythagoras theorem in geometry.

"The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the other two sides".

$c^2 = a^2 + b^2$

$5^2 = 3^2 + 4^2$

$25 = 9 + 16$



In the given figure:

The perpendicular equals to the length 'a' base equals to the length 'b', and hypotenuse equals to the length 'c'.

By Pythagoras Theorem, we have

$$\boxed{a^2 + b^2 = c^2} \quad \ldots\text{(i)}$$

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{2} \quad \text{(Dividing by } c^2\text{)}$$

$$\boxed{\sin^2\theta + \cos^2\theta = 1} \ldots\text{(ii)}$$

$$a^2 + b^2 = c^2$$

$$\frac{a^2}{b^2} + \frac{b^2}{b^2} = \frac{c^2}{b^2}$$ Dividing equation (i) by b^2, we have

$$\boxed{\tan^2\theta + 1 = \sec^2\theta} \ldots\text{(iii)}$$

$$\frac{a^2}{a^2} + \frac{b^2}{a^2} = \frac{c^2}{a^2}$$ Dividing equation (i) by a^2, we have

$$\boxed{1 + \cot^2\theta = \text{cosec}^2\theta} \quad \ldots\text{(iv)}$$

The identities (ii), (iii) and (iv) are known as Pythagoras identities.

Example 10: Show that $(\sec^2\theta - 1)\cos^2\theta = \sin^2\theta$

Solution: L.H.S $= (\sec^2\theta - 1)\cos^2\theta$

$= \tan^2\theta \cdot \cos^2\theta \quad (\because 1 + \tan^2\theta = \sec^2\theta)$

$= \dfrac{\sin^2\theta}{\cos^2\theta} \cdot \cos^2\theta \quad \left(\because \tan\theta = \dfrac{\sin\theta}{\cos\theta}\right)$

$= \sin^2\theta = \text{R.H.S}$

Hence, $(\sec^2\theta - 1)\cos^2\theta = \sin^2\theta$

Example 11: Show that $\tan\theta + \cos\theta = \sec\theta \ \text{cosec}\ \theta$

Solution: L.H.S = tan θ + cot θ

$$=\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}$$

$$=\frac{\sin\theta\cdot\sin\theta+\cos\theta\cdot\cos\theta}{\cos\theta\cdot\sin\theta}$$

$$=\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}$$

$$=\frac{1}{\sin\theta\cos\theta}\quad(\because \sin^2\theta+\cos^2\theta=1)$$

$$=\frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta}$$

$$=\sec\theta\cdot\operatorname{cosec}\theta=\text{R.H.S.}$$

Hence, tan θ + cot θ = sec θ cosec θ

Example 12: Show that $\frac{1}{\operatorname{cosec}\theta-\cot\theta}-\frac{1}{\sin\theta}=\frac{1}{\sin\theta}-\frac{1}{\operatorname{cosec}\theta+\cot\theta}$

Solution:

$$\text{L.H.S}=\frac{1}{\operatorname{cosec}\theta-\cot\theta}-\frac{1}{\sin\theta}$$

$$=\frac{1}{\frac{1}{\sin\theta}-\frac{\cos\theta}{\sin\theta}}-\frac{1}{\sin\theta}$$

$$=\frac{\sin\theta}{1-\cos\theta}-\frac{1}{\sin\theta}$$

$$=\frac{\sin\theta(1+\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}-\frac{1}{\sin\theta}$$

$$=\frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta}-\frac{1}{\sin\theta}$$

$$=\frac{\sin\theta(1+\cos\theta)}{\sin^2\theta}-\frac{1}{\sin\theta}$$

$$=\frac{1+\cos\theta}{\sin\theta}-\frac{1}{\sin\theta}$$

$$=\frac{1+\cos\theta-1}{\sin\theta}$$

$$=\frac{\cos\theta}{\sin\theta}=\cot\theta$$

$$\text{R.H.S}=\frac{1}{\sin\theta}-\frac{1}{\operatorname{cosec}\theta+\cot\theta}$$

$$=\frac{1}{\sin\theta}-\frac{1}{\frac{1}{\sin\theta}+\frac{\cos\theta}{\sin\theta}}$$

$$=\frac{1}{\sin\theta}-\frac{\sin\theta}{1+\cos\theta}$$

$$=\frac{1}{\sin\theta}-\frac{\sin\theta(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}$$

$$=\frac{1}{\sin\theta}-\frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta}$$

$$=\frac{1}{\sin\theta}-\frac{\sin\theta(1-\cos\theta)}{\sin^2\theta}$$

$$=\frac{1}{\sin\theta}-\frac{1-\cos\theta}{\sin\theta}$$

$$=\frac{1-1+\cos\theta}{\sin\theta}$$

$$=\frac{\cos\theta}{\sin\theta}=\cot\theta$$

Hence, L.H.S = R.H.S

Example 13: Show that $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$

Solution: $\text{L.H.S} = \sin^6\theta + \cos^6\theta$

$= (\sin^2\theta)^3 + (\cos^2\theta)^3$

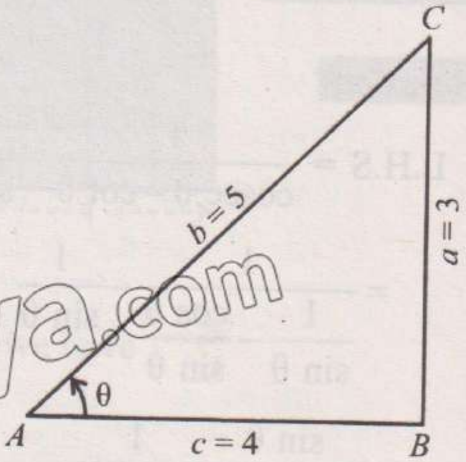
$= (\sin^2\theta + \cos^2\theta)(\sin^4\theta + \cos^4\theta - \sin^2\theta\cos^2\theta)$

$= (\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta - \sin^2\theta\cos^2\theta$

$= 1 - 3\sin^2\theta\cos^2\theta = \text{R.H.S}$

Hence, $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$

Example 14: If $\tan\theta = \dfrac{3}{4}$, find the remaining trigonometric ratios, when θ lies in first quadrant.



Solution: Given: $\tan\theta = \dfrac{3}{4} = \dfrac{a}{c}$,

Where, $a = 3, c = 4$

By Pythagoras theorem, we have

$$b^2 = a^2 + c^2$$

$$= 9 + 16 = 25$$

$$\Rightarrow \quad b = 5$$

Therefore, $\sin\theta = \dfrac{a}{b} = \dfrac{3}{5}$; $\text{coses}\,\theta = \dfrac{b}{a} = \dfrac{5}{3}$

$\cos\theta = \dfrac{c}{b} = \dfrac{4}{5}$; $\sec\theta = \dfrac{b}{c} = \dfrac{5}{4}$

$\cot\theta = \dfrac{c}{a} = \dfrac{4}{3}$

EXERCISE 6.3

1. If θ lies in first quadrant, find the remaining trigonometric ratios of θ.

(i) $\sin\theta = \dfrac{2}{3}$ (ii) $\cos\theta = \dfrac{3}{4}$ (iii) $\tan\theta = \dfrac{1}{2}$

(iv) $\sec\theta = 3$ (v) $\cot\theta = \dfrac{\sqrt{3}}{\sqrt{2}}$

Prove the following trigonometric identities:

2. $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

3. $\dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\tan\theta}$

4. $\dfrac{\sin\theta}{\operatorname{cosec}\theta} + \dfrac{\cos\theta}{\sec\theta} = 1$

5. $\cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

6. $\cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

7. $\dfrac{1-\sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1+\sin\theta}$

8. $(\sec\theta - \tan\theta)^2 = \dfrac{1-\sin\theta}{1+\sin\theta}$

9. $(\tan\theta + \cot\theta)^2 = \sec^2\theta\operatorname{cosec}^2\theta$

10. $\dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \tan\theta + \sec\theta$

11. $\sin^3\theta - \cos^3\theta = (\sin\theta - \cos\theta)(1 + \sin\theta\cos\theta)$

12. $\sin^6\theta - \cos^6\theta = (\sin^2\theta - \cos^2\theta)(1 - \sin^2\theta\cos^2\theta)$

6.4 Values of Trigonometric Ratios of Special Angles

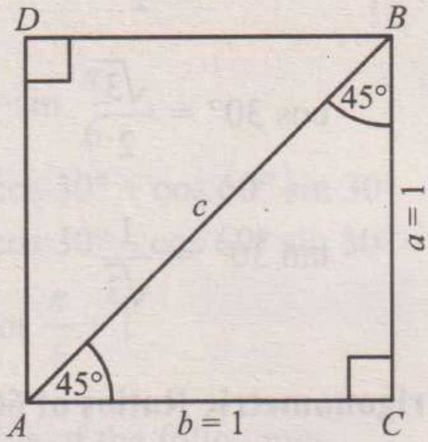
Trigonometric ratios of 45° $\left(\frac{\pi}{4}\text{ radian}\right)$:

Consider a square *ACBD* of side length 1 unit.

We know that the diagonals bisect the angles.

So, in the triangle *ABC*

$$m\angle A = m\angle B = 45° \text{ and } m\angle C = 90°.$$



Using Pythagoras theorem in ΔABC,

$$c^2 = a^2 + b^2$$
$$c^2 = 1 + 1$$
$$c^2 = 2 \quad \Rightarrow \quad c = \sqrt{2}$$

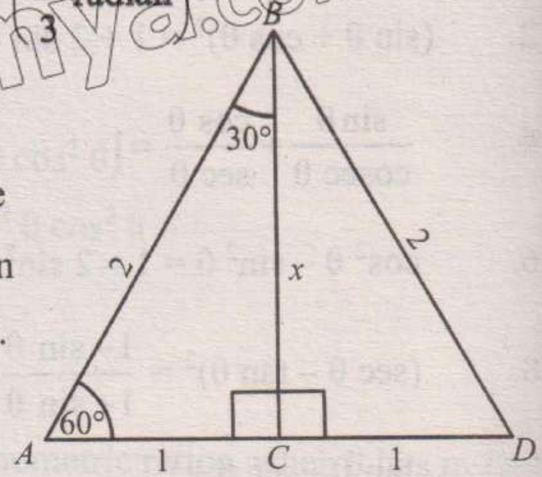
The trigonometric ratios are:

$\sin 45° = \dfrac{a}{c} = \dfrac{1}{\sqrt{2}}$; $\operatorname{cosec} 45° = \dfrac{c}{a} = \sqrt{2}$

$\cos 45° = \dfrac{b}{c} = \dfrac{1}{\sqrt{2}}$; $\sec 45° = \dfrac{c}{b} = \sqrt{2}$

$\tan 45° = \dfrac{a}{b} = 1$; $\cot 45° = \dfrac{b}{a} = 1$

Trigonometric Ratios of 30° $\left(\frac{\pi}{6}\text{radian}\right)$ and 60° $\left(\frac{\pi}{3}\text{radian}\right)$:



Consider an equilateral triangle ABD of side 2 units.

Draw a perpendicular bisector $\overline{BC}$ on $\overline{AD}$. The point C is the midpoint of $\overline{AD}$. So, $m\overline{AC} = m\overline{CD}$ in which $m\angle BAC = 60°, m\angle ABC = 30°, m\angle ACB = 90°$.

Let $m\overline{BC} = x$ units.

Using Pythagoras theorem in the ΔABC.

$$2^2 = 1^2 + x^2$$

$$x^2 = 4 - 1 \Rightarrow x^2 = 3 \Rightarrow x = \sqrt{3}\ (m\overline{BC} = \sqrt{3}\text{ units})$$

Trigonometric ratios of 30° $\left(\frac{\pi}{6}\text{radian}\right)$:

In the triangle ABC with $m\angle ABC = 30°$

$\sin 30° = \frac{1}{2}$; $\text{cosec } 30° = 2$

$\cos 30° = \frac{\sqrt{3}}{2}$; $\sec 30° = \frac{2}{\sqrt{3}}$

$\tan 30° = \frac{1}{\sqrt{3}}$; $\cot 30° = \sqrt{3}$

Trigonometric Ratios of 60° $\left(\frac{\pi}{3}\text{radian}\right)$:

In right angled triangle ABC, with $m\angle A = 60°$

$\sin 60° = \frac{\sqrt{3}}{2}$; $\cos 60° = \frac{1}{2}$; $\tan 60° = \sqrt{3}$

$\text{cosec } 60° = \frac{2}{\sqrt{3}}$; $\sec 60° = 2$; $\cot 60° = \frac{1}{\sqrt{3}}$

These results in the form of a table can be written as:

θ	0°	$30° = \frac{\pi}{6}$	$45° = \frac{\pi}{4}$	$60° = \frac{\pi}{3}$	$90° = \frac{\pi}{2}$
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

EXERCISE 6.4

1. Find the value of the following trigonometric ratios without using the calculator.

(i) sin 30° (ii) cos 30° (iii) $\tan\frac{\pi}{6}$ (iv) tan 60°

(v) sec 60° (vi) $\cos\frac{\pi}{3}$ (vii) cot 60° (viii) sin 60°

(ix) sec 30° (x) cosec 30° (xi) sin 45° (xii) $\cos\frac{\pi}{4}$

2. Evaluate:

(i) 2 sin 60° cos60° (ii) $2\cos\frac{\pi}{6}\sin\frac{\pi}{6}$

(iii) 2 sin 45° + 2cos 45° (iv) sin 60° cos 30° + cos 60° sin 30°

(v) cos 60° cos 30° – sin 60° sin 30° (vi) sin 60° cos 30° – cos 60° sin 30°

(vii) cos 60° cos 30° + sin 60° sin 30° (viii) $\tan\frac{\pi}{6}\cot\frac{\pi}{6}+1$

3. If $\sin\frac{\pi}{4}$ and $\cos\frac{\pi}{4}$ equal to $\frac{1}{\sqrt{2}}$ each, then find the value of the followings:

(i) 2 sin 45° – 2 cos 45° (ii) 3 cos 45° + 4 sin 45°

(iii) 5 cos 45° – 3 sin 45°

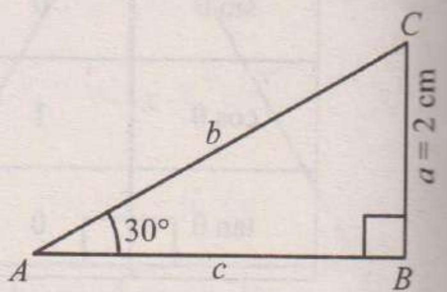
6.5 Solution of a Triangle

We know that there are three sides and three angles in a triangle. Out of these six elements, if we know three of them including at least one side, then we can find the measures of the remaining elements. Finding the measures of the remaining elements is called the solution of a triangle Here we learn the solution of a right angled triangle only.

Case I: When measures of one side and one angle are given.

Example 15: Solve triangle ABC, in which $m\angle B = 90°$, $m\angle A = 30°$, $a = 2$ cm

Solution:



We are required to find b, c and $m\angle C$.

Now $m\angle C = m\angle B - m\angle A$

$= 90° - 30°$

$= 60°$ ...(i)

$$\frac{a}{b} = \sin 30°$$

$$\Rightarrow \quad \frac{2}{b} = \sin 30° \qquad (\because a = 2)$$

$$\Rightarrow \quad \frac{2}{b} = \frac{1}{2} \qquad \left(\because \sin 30° = \frac{1}{2}\right)$$

$$\Rightarrow \quad b = 4\,\text{cm} \qquad \text{...(ii)}$$

and $\frac{a}{c} = \tan 30°$

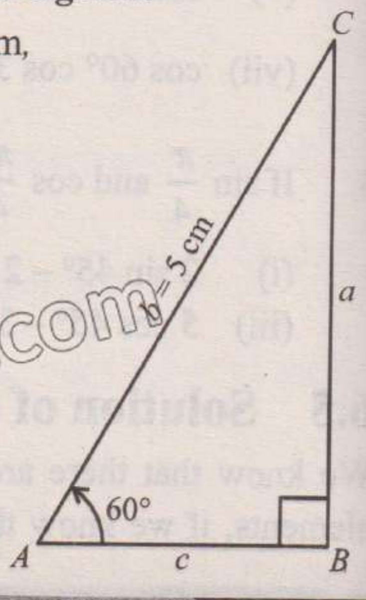
$$\Rightarrow \quad \frac{2}{c} = \frac{1}{\sqrt{3}} \qquad \left(\because a = 2, \tan 30° = \frac{1}{\sqrt{3}}\right)$$

thus $c = 2\sqrt{3}$ cm ...(iii)

(i), (ii) and (iii) are the required results.

Case II: When measure of the hypotenuse and an angle are given.

Example 16: Solve triangle ABC, when $m\angle A = 60°$, $b = 5$ cm, $m\angle B = 90°$



Soiution: We are required to find a, c and $m\angle C$

$m\angle A = 60°$

$m\angle B = 90°$

$m\angle C = m\angle B - m\angle A$

$= 90° - 60°$

$= 30°$ ...(i)

Now $\frac{a}{b} = \sin 60°$

$\frac{a}{5} = \frac{\sqrt{3}}{2}$ $\left(\because b = 5, \sin 60° = \frac{\sqrt{3}}{2}\right)$

$\Rightarrow$ $a = \frac{5\sqrt{3}}{2}$

$\Rightarrow$ $a = 4.33$ cm ...(ii)

and $\frac{c}{b} = \cos 60°$

$\frac{c}{5} = \frac{1}{2}$ $\left(\because b = 5, \cos 60° = \frac{1}{2}\right)$

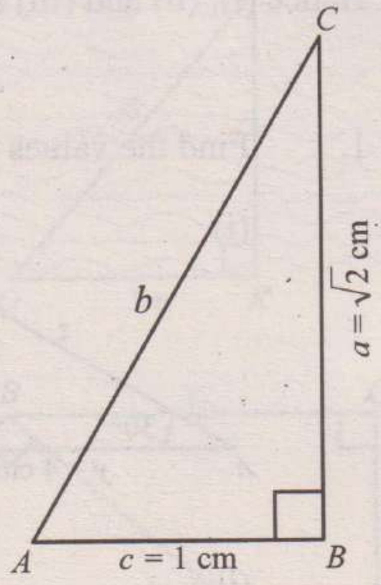
$\Rightarrow$ $c = \frac{5}{2}$

$\Rightarrow$ $c = 2.5$ cm ...(iii)

(i), (ii) and (iii) are the required results.

Case III: When measure of two sides are given.

Example 17: Solve triangle ABC, when $a = \sqrt{2}$ cm, $c = 1$ cm and $m\angle B = 90°$



Solution: We are required to find b, $m\angle A$, $m\angle C$.

By Pythagoras theorem, we have

$b^2 = c^2 + a^2$

or $b^2 = (1)^2 + (\sqrt{2})^2$

or $b^2 = 1 + 2$

or $b^2 = 3$

or $b = \sqrt{3}$ cm ...(i)

Now $\sin m\angle A = \frac{a}{b} = \frac{\sqrt{2}}{\sqrt{3}} \Rightarrow m\angle A = \sin^{-1}\sqrt{\frac{2}{3}} = 54.7°$

$\Rightarrow$ $m\angle A = 54.7°$ ...(ii)

and $m\angle C = m\angle B - m\angle A$

$= 90° - 54.7°$

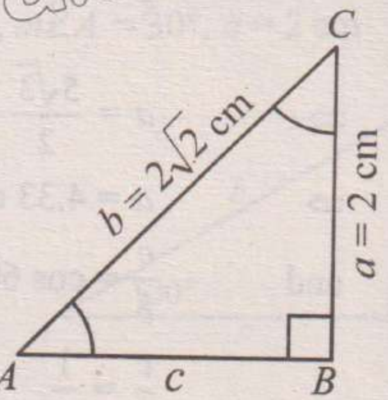
$= 35.3°$ ...(iii)

(i), (ii) and (iii) are the required results.

Case IV: When measure of one side and hypotenuse are given.

Example 18: Solve tringle ABC, when $a = 2\text{cm}$, $b = 2\sqrt{2}$ cm and $m\angle B = 90^\circ$

Solution: We are required to find $m\angle A$, $m\angle C$ and c.



By Pythagoras theorem, we have

$$b^2 = a^2 + c^2$$

or $\quad c^2 = b^2 - a^2$

$$= (2\sqrt{2})^2 - (2)^2$$

$$= 8 - 4 = 4$$

or $\quad c = 2\text{cm} \quad \ldots(\text{i})$

Now $\quad \dfrac{c}{b} = \cos m\angle A$

or $\quad \dfrac{c}{b} = \cos m\angle A = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \quad m\angle A = 45^\circ \quad \ldots(\text{ii})$

Thus, $\quad m\angle C = m\angle B - m\angle A$

$$= 90^\circ - 45^\circ$$

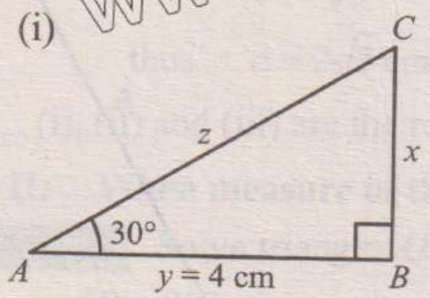
$$= 45^\circ \ldots(\text{iii})$$

Hence (i), (ii) and (iii) are the required results.

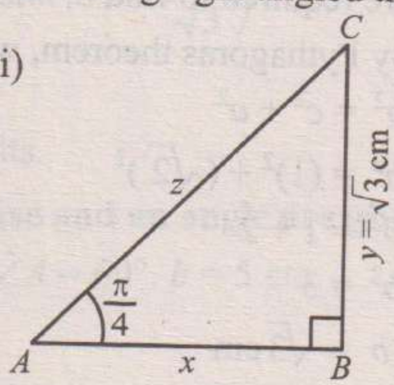
EXERCISE 6.5

1. Find the values of x, y and z from the following right angled triangles.

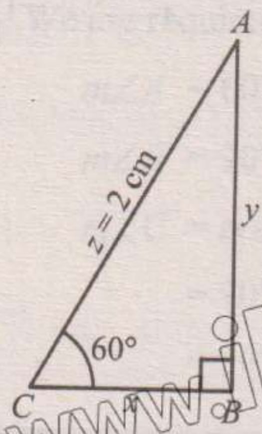
(i)



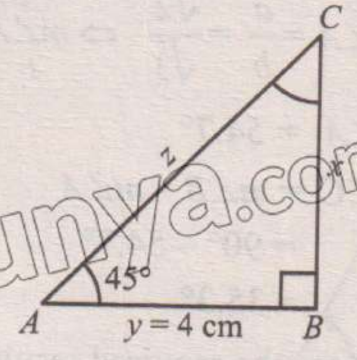
(ii)



(iii)

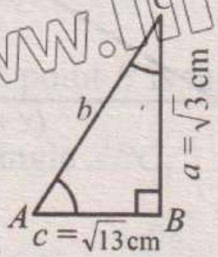


(iv)

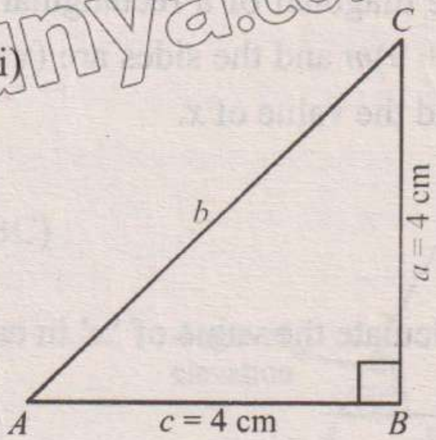


2. Find the unknown side and angles of the following triangles.

(i)



(ii)



3. Each side of a square field is 60 m long. Find the lengths of the diagonals of the field.

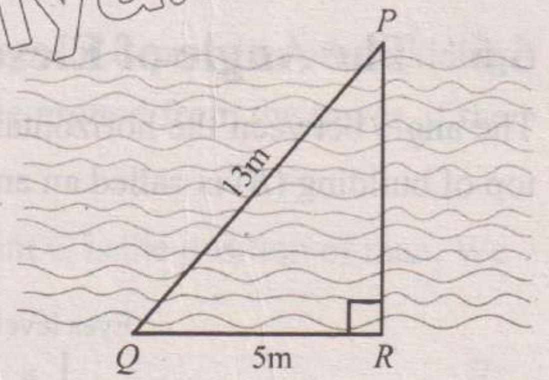
4. Solve the following triangles when $m\angle B = 90°$:

(i) $m\angle C = 60°$, $c = 3\sqrt{3}$ cm (ii) $m\angle C = 45°$, $a = 8$ cm

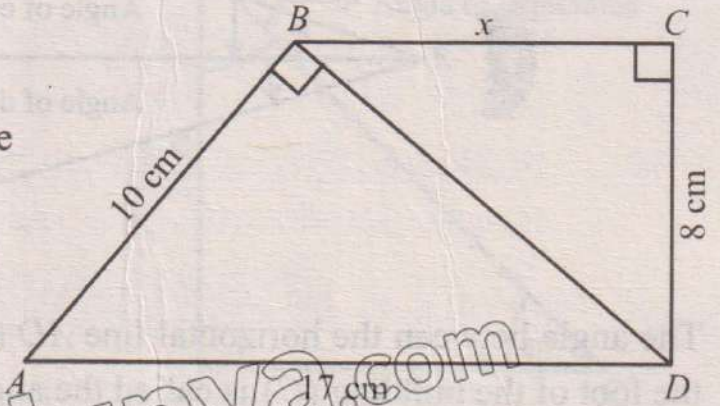
(iii) $a = 12$ cm, $c = 6$ cm (iv) $m\angle A = 60°$, $c = 4$ cm

(v) $m\angle A = 30°$, $c = 4$ cm (vi) $b = 10$ cm, $a = 6$ cm

5. Let Q and R be the two points on the same bank of a canal. The point P is placed on the other bank straight to point R. Find the width of the canal and the angle PQR in radians.

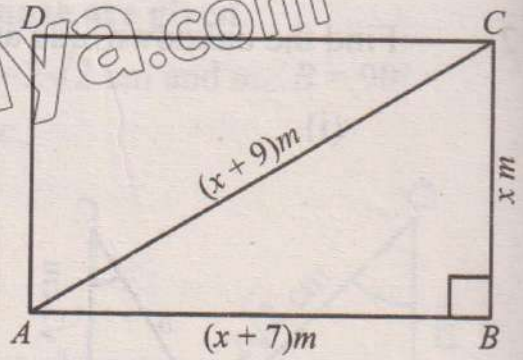


6. Calculate the length x in the adjoining figure.

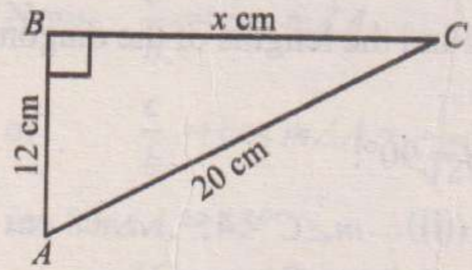


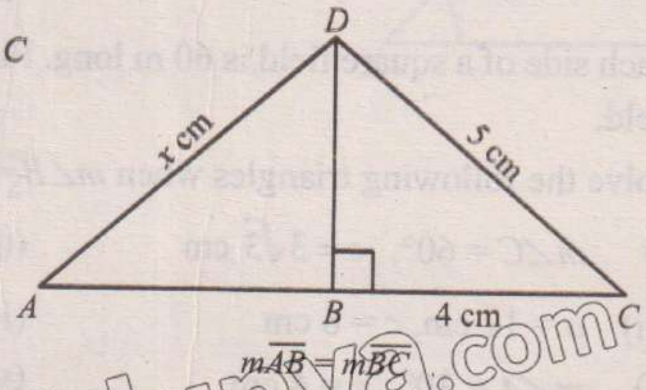
7. If the ladder is placed along the wall such that the foot of the ladder is 2 m away from the wall. If the length of the ladder is 8 m, find the height of the wall.

8. The diagonal of a rectangular field $ABCD$ is $(x + 9)m$ and the sides are $(x + 7)$m and x m. Find the value of x.



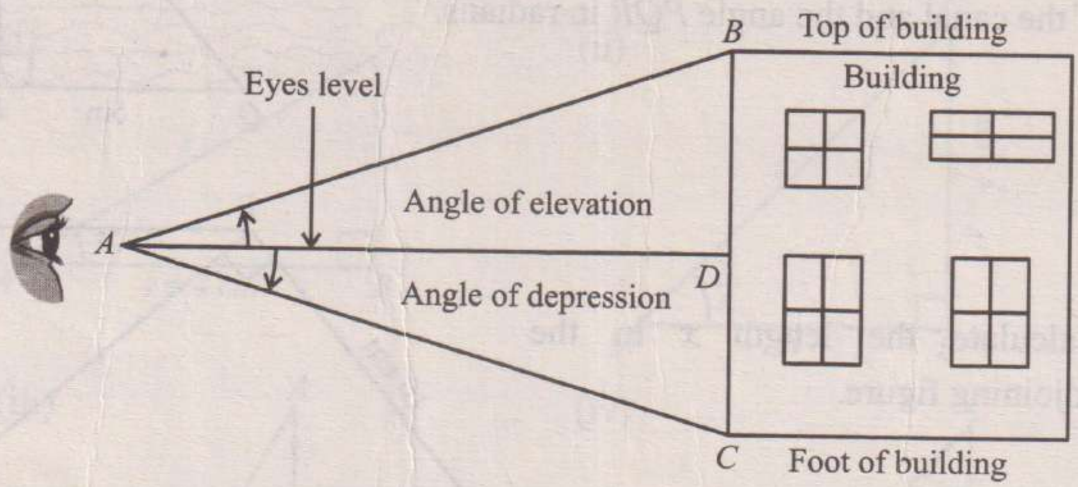
9. Calculate the value of 'x' in each case.





6.6 The Angle of Elevation and the Angle of Depression

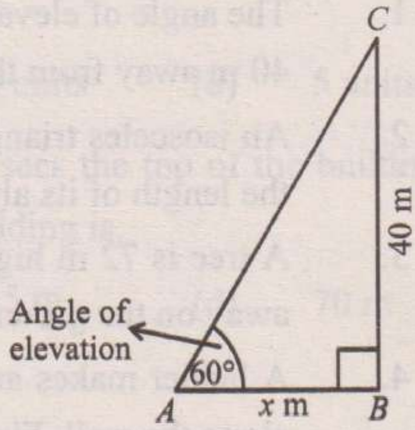
The angle between the horizontal line AD (eye level) and a line from the eye A to the top of building (B) is called an angle of elevation.



The angle between the horizontal line AD (eye level) and the line from the eye 'A' to the foot of the building (C) is called the angle of depression.

Example 19: The angle of elevation of the top of a pole 40 m high is 60° when seen from a point on the ground level. Find the distance of the point from the foot of the pole.

Solution: In the triangle ABC, we have



$$m\overline{BC} = 40 \text{ m}$$

$$m\angle A = 60^\circ$$

Let $m\overline{AB} = x$ (the point B is the foot of the pole BC)

In right angled triangle ABC,

$$\tan 60^\circ = \frac{m\overline{BC}}{m\overline{AB}}$$

$$\sqrt{3} = \frac{40}{x}$$

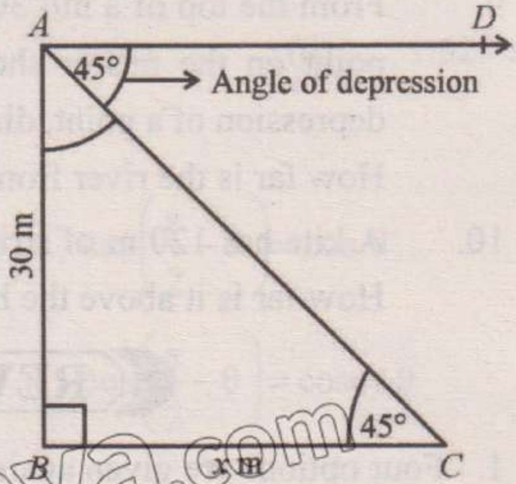
$$\Rightarrow \quad x = \frac{40}{\sqrt{3}}$$

$$\Rightarrow \quad x = 23.09 \text{ m}$$

Hence, distance of the point from the foot of the pole = 23.09 m

Example 20: From the top of a lookout tower, the angle of depression of a building has on the ground level of 45°. How far is a man on the ground from the tower, if the height of the tower is 30 m?

Solution: In the triangle ABC, AB is the tower and point C is the position of man. We have



$$m\overline{AB} = 30 \text{ m}$$

$$m\angle CAD = m\angle C = 45^\circ$$

$$m\overline{BC} = x\text{m} = ?$$

Let x be the base of right angled triangle ABC,

$$\tan 45^\circ = \frac{m\overline{AB}}{m\overline{BC}}$$

$$\Rightarrow \quad 1 = \frac{30}{x}$$

$$\Rightarrow \quad x = 30 \text{ m}$$

Hence, man is 30 *m* far from the tower.

EXERCISE 6.6

1. The angle of elevation of the top of a flag post from a point on the ground level 40 m away from the flag post is 60°. Find the height of the post.
2. An isosceles triangle has a vertical angle of 120° and a base 10 cm long. Find the length of its altitude.
3. A tree is 72 m high. Find the angle of elevation of its top from a point 100 m away on the ground level.
4. A ladder makes an angle of 60° with the ground and reaches a height of 10m along the wall. Find the length of the ladder.
5. A light house tower is 150 m high from the sea level. The angle of depression from the top of the tower to a ship is 60°. Find the distance between the ship and the tower.
6. Measure of an angle of elevation of the top of a pole is 15° from a point on the ground, in walking 100 m towards the pole the measure of angle is found to be 30°. Find the height of the pole.
7. Find the measure of an angle of elevation of the Sun, if a tower 300 m high casts a shadow 450 m long.
8. Measure of angle of elevation of the top of a cliff is 25°, on walking 100 metres towards the cliff, measure of angle of elevation of the top is 45°. Find the height of the cliff.
9. From the top of a hill 300 m high, the measure of the angle of depression of a point on the nearer shore of the river is 70° and measure of the angle of depression of a point, directly across the river is 50°. Find the width of the river How far is the river from the foot of the hill?
10. A kite has 120 m of string attached to it when at an angle of elevation of 50°. How far is it above the hand holding it? (Assume that the string is stretched.

REVIEW EXERCISE 6

1. Four options are given against each statement. Encircle the correct one.
   (i) The value of $\tan^{-1} 2$ in radians is:

   (a) $\frac{\pi}{2}$ (b) $\frac{3\pi}{2}$ (c) 1.11π (d) 1.11

(ii) In a right triangle, the hypotenuse is 13 units and one of the angles is $\theta = 30°$. The length of the opposite side is:

(a) 6.5 units (b) 7.5 units (c) 6 units (d) 5 units

(iii) A person standing 50 m away from a building sees the top of the building at an angle of elevation of 45°. Height of the building is:

(a) 50 m (b) 25 m (c) 35 m (d) 70 m

(iv) $\sec^2\theta - \tan^2\theta =$ ________.

(a) $\sin^2\theta$ (b) 1 (c) $\cos^2\theta$ (d) $\cot^2\theta$

(v) If $\sin\theta = \frac{3}{5}$ and θ is an acute angle, $\cos^2\theta =$ ________.

(a) $\frac{7}{25}$ (b) $\frac{24}{25}$ (c) $\frac{16}{25}$ (d) $\frac{4}{25}$

(vi) $\frac{5\pi}{24}$ rad = ________ degrees.

(a) 30° (b) 37.5° (c) 45° (d) 52.5°

(vii) 292.5° = ________ rad.

(a) $\frac{17\pi}{6}$ (b) $\frac{17\pi}{4}$ (c) 1.6π (d) 1.625π

(viii) Which of the following is a valid identity?

(a) $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ (b) $\cos\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

(c) $\cos\left(\frac{\pi}{2} - \theta\right) = \sec\theta$ (d) $\cos\left(\frac{\pi}{2} - \theta\right) = \operatorname{cosec}\theta$

(ix) $\sin 60° =$ ________.

(a) 1 (b) $\frac{1}{2}$ (c) $\sqrt{(3)^2}$ (d) $\frac{\sqrt{3}}{2}$

(x) $\cos^2 100\pi + \sin^2 100\pi =$ ________.

(a) 1 (b) 2 (c) 3 (d) 4

2. Convert the given angles from:

(a) degrees to radians giving answer in terms of π.

(i) $255°$ (ii) $75° 45'$ (iii) $142.5°$

(b) radians to degrees giving answer in degrees and minutes.

(i) $\dfrac{17\pi}{24}$ (ii) $\dfrac{7\pi}{12}$ (iii) $\dfrac{11\pi}{16}$

3. Prove the following trigonometric identities:

(i) $\dfrac{\sin\theta}{1-\cos\theta} = \dfrac{1+\cos\theta}{\sin\theta}$

(ii) $\sin\theta(\text{cosec}\,\theta - \sin\theta) = \dfrac{1}{\sec^2\theta}$

(iii) $\dfrac{\text{cosec}\,\theta - \sec\theta}{\text{cosec}\,\theta + \sec\theta} = \dfrac{1-\tan\theta}{1+\tan\theta}$

(iv) $\tan\theta + \cot\theta = \dfrac{1}{\sin\theta\cos\theta}$

(v) $\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta} + \dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta} = \dfrac{2}{1-2\sin^2\theta}$

(vi) $\dfrac{1+\cos\theta}{1-\cos\theta} = (\text{cosec}\,\theta + \cot\theta)^2$

4. If $\tan\theta = \dfrac{3}{\sqrt{2}}$ then find the remaining trigonometric ratios when θ lies in first quadrant.

5. From a point on the ground, the angle of elevation to the top of a 30 m high building is 28°. How far is the point from the base of the building?

6. A ladder leaning against a wall forms an angle of 65° with the ground. If the ladder is 10 m long, how high does it reach on the wall?