Unit 8

# Logic



**Students' Learning Outcomes**

**At the end of the unit, the students will be able to:**

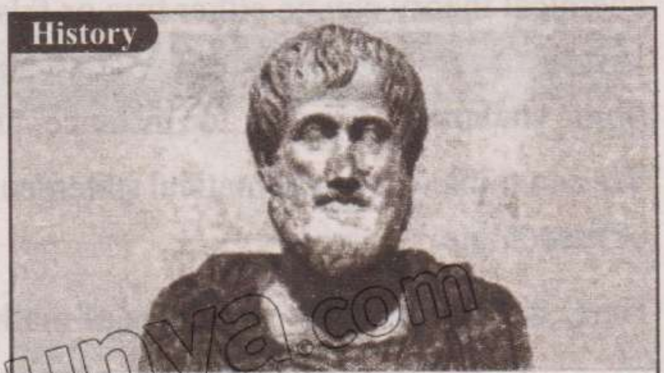
- Understand a mathematical statement and its proof
- Differentiate between an axiom, conjecture and theorem.
- Formulate simple deductive proofs [algebraic proofs that require showing the LHS to be equal to the RHS. e.g., showing $(x-3)^2+5=x^2-6x+14$]

## INTRODUCTION

Logic is a systematic method of reasoning that enables one to interpret the meanings of statements, examine their truth, and deduce new information from existing facts. Logic plays a key role in problem-solving and decision-making.

We generally use logic in our daily life while engaging in mathematics. For example, we often draw general conclusions from a limited number of observations or experiences. A person gets a penicillin injection once or twice and experiences a reaction soon afterward. He generalises that he is allergic to penicillin. This way of drawing conclusions is called **induction**. Inductive reasoning is helpful in natural sciences, where we must depend upon repeated experiments or observations. In fact greater part of our knowledge is based on induction. On many occasions, we have to adopt the opposite course. We have to conclude from accepted or well-known facts. We often consult lawyers or doctors

**History**



The history of logic began with **Aristotle**, who is considered the father of formal logic. He developed a system of deductive reasoning known as syllogistic logic, which became the foundation of logical thought. The **Stoics** followed, contributing to propositional logic and exploring paradoxes such as the Liar Paradox. During the medieval period, scholars like **Peter Abelard** and **William of Ockham** expanded Aristotle's work, introducing theories of semantics and consequences. In the **19th century**, logic advanced through the works of **George Boole**, who developed Boolean algebra, and **Gottlob Frege**, who formalized modern predicate logic. **Bertrand Russell** and **Alfred North Whitehead** attempted to reduce mathematics to logic in their seminal work, Principia Mathematica. The 20th century saw significant progress with **Kurt Gödel**, who introduced his incompleteness theorems, reshaping our understanding of mathematical logic(history-of-logic: http://individual.utoronto.ca/pking/miscellaneous/history-of-logic.pdf).

based on their good reputation. This way of reasoning i.e., drawing conclusions from premises believed to be true, is called **deduction**. One usual example of deduction is: All men are mortal. We are men. Therefore, we are also mortal. To study logic, we start with a statement.

## 8.1 Statement

A sentence or mathematical expression which may be true or false but not both is called a statement. This is correct so far as mathematics and other sciences are concerned. For instance, the statement $a = b$ can be either true or false. Similarly, any physical or chemical theory can be either true or false. However, in statistical or social sciences, it is sometimes impossible to divide all statements into two mutually exclusive classes. Some statements may be, for instance, undecided.

We can think of a mathematical statement as a unit of information that is either accurate or inaccurate.

Here, we discuss some examples of mathematical statements that are all true.

(i) For a non-zero real number $x$ and integers $m$ and $n$, we have: $x^m \cdot x^n = x^{m+n}$

(ii) The sum of the measures of the interior angles of a triangle is 180°

(iii) The circumference of a circle with radius $r$ is $2\pi r$

(iv) $Q \subseteq R$ (The set of rational numbers is a subset of the set of real numbers)

(v) $\frac{22}{7} \notin Q'$

(vi) The sum of two odd integers is an even integer

(vii) $x^2 - 5x + 6 = 0$, for $x = 2$ or $x = 3$

Further, we discuss some examples of mathematical statements that are all false.

(i) $3 + 4 = 8$

(ii) $Z \subseteq W$

(iii) All isosceles triangle are equilateral triangle

(iv) Between any two real numbers, there is no real number

(v) $\{1, 2, 3, 4\} \cap \{-1, -2, -3, -4\} = \{1, 2, 3, 4\}$

(vi) If $a$ and $b$ are the length and width of a rectangle, then the area of a rectangle is $\frac{1}{2}(a \times b)$.

(vii) The sum of interior angle of an $n$-sided polygon is $(n-1)\times 180^\circ$

(viii) The sum of the interior angles of any quadrilateral is always $180^\circ$.

(ix) The set of integers is finite.

The following section will discuss various standard methods for combining statements to create new statements.

## 8.1.1 Logical Operators

The letters $p$, $q$ etc., will use to donate the statements. A brief list of the symbols which will be used is given below:

| Symbols | How to be read | Symbolic expression | How to be read |
|---|---|---|---|
| ~ | Not | $\sim p$ | Not $p$, negation of $p$ |
| $\wedge$ | And | $p \wedge q$ | $p$ and $q$ |
| $\vee$ | Or | $p \vee q$ | $p$ or $q$ |
| $\rightarrow$ | If ... then, implies | $p \rightarrow q$ | If $p$ then $q$, $p$ implies $q$ |
| $\leftrightarrow$ | Is equivalent to, if and only if | $p \leftrightarrow q$ | $p$ if and only if $q$, $p$ is equivalent to $q$ |

## 8.1.2 Explanation of the Use of the Symbols

### 1. Negation

If $p$ is any statement, its negation is denoted by $\sim p$, read 'not $p$'. It follows from this definition that if $p$ is true, $\sim p$ is false, and if $p$ is false, $\sim p$ is true. The possible truth values of $p$ and $\sim p$ are given in table:1, which is called a truth table, where the true value is denoted by T and the false value is denoted by F.

**Table 1**

| $p$ | $\sim p$ |
|---|---|
| T | F |
| F | T |

### 2. Conjunction

The conjunction of two statements $p$ and $q$ is symbolically written as $p \wedge q$ ($p$ and $q$). A conjunction is considered to be true only if both statements are true. So, the truth table of $p \wedge q$ is given in Table: 2.

Table 2

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

**Example 1:** Whether the following statements are true or false.

(i) Lahore is the capital of the Punjab and Quetta is the capital of Balochistan.

(ii) $4 < 5 \wedge 8 < 10$ (iii) $2+2=3 \wedge 6+6=10$

**Solution:**

Clearly conjunctions (i) and (ii) are true whereas (iii) is false.

### 3. Disjunction

The disjunction of $p$ and $q$ is symbolically written as $p \vee q$ ($p$ or $q$). The disjunction $p \vee q$ is considered to be true when at least one of the statements is true. It is false when both of them are false. The truth table $p \vee q$ is given in Table: 3.

Table 3

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

**Example 2:** 10 is a positive integer or 0 is a rational number. Find truth value of this disjunction.

**Solution:** Since both statements are true, the disjunction is true.

**Example 3:** Triangle can have two right angles or Lahore is the capital of Sindh. Find the truth value of this disjunction.

**Solution:** Both statements are false, the disjunction is false.

### 4. Implication or conditional

A compound statement of the form if $p$ then $q$ ($p \rightarrow q$) also written as $p$ **implies** $q$ is called a **conditional** or an **implication**. $p$ is called the **antecedent** or **hypothesis** and $q$ is called the **consequent** or the **conclusion**.

A conditional is regarded as false only when the antecedent is true and the consequent is false. In all other cases conditional is considered to be true. So, the truth table of $p \rightarrow q$ is given in Table: 4.

Table 4

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

We attempt to clear the position with the help of an example. Consider the conditional:

If person $A$ lives in Lahore, then he lives in Pakistan.

If the antecedent is false, i.e., $A$ does not live in Lahore, he may still be living in Pakistan. We have no reason to say that he does not live in Pakistan.

We cannot, therefore, say that the conditional is false. So we must regard it as true. Similarly, when both the antecedent and consequent of the conditional under consideration are false, then is no justification for quarrelling with the statement.

## 5. Biconditional $p \leftrightarrow q$

The statement $p \to q \wedge q \to p$ is shortly written as $p \leftrightarrow q$ and is called the **biconditional** or **equivalence**. It is read $p$ iff $q$ (iff stands for "if and only if")

We draw up its truth table.

**Table 5**

| $p$ | $q$ | $p \to q$ | $q \to p$ | $p \leftrightarrow q$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | T | F |
| F | T | T | F | F |
| F | F | T | T | T |

From the Table 5 it appears that $p \leftrightarrow q$ is true only when both statements $p$ and $q$ are true or both statements $p$ and $q$ are false.

## 6. Conditionals related with a given conditional.

Let $p$ and $q$ be the statements and $p \to q$ be a given conditional, then

(i) $q \to p$ is called the **converse** of $p \to q$;

(ii) $\sim p \to \sim q$ is called the **inverse** of $p \to q$;

(iii) $\sim q \to \sim p$ is called the **contrapositive** of $p \to q$.

The truth values of these new conditionals are given below in Table 6.

**Table 6**

| | | | | Given conditional | Converse | Inverse | Contrapositive |
|---|---|---|---|---|---|---|---|
| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \to q$ | $q \to p$ | $\sim p \to \sim q$ | $\sim q \to \sim p$ |
| T | T | F | F | T | T | T | T |
| T | F | F | T | F | T | T | F |
| F | T | T | F | T | F | F | T |
| F | F | T | T | T | T | T | T |

From the table 6, it appears that

(i) Any conditional and its contrapositive are equivalent; therefore, any theorem may be proved by proving its contrapositive.

(ii) The converse and inverse are equivalent to each other.

**Example 4:** Prove that in any universal set, the empty set $\phi$ is a subset of any set $A$.

**Solation:** Let $U$ be the universal set. Consider the conditional:

$$\forall x \in U, x \in \phi \rightarrow x \in A \quad \ldots(i)$$

The antecedent of this conditional is false because no $x \in U$, is a member of $\phi$.
Hence, the conditional is true.

**Example 5:** Construct the truth table of $[(p \rightarrow q) \wedge p]$ and $[(p \rightarrow q) \wedge p] \rightarrow q$

**Solution:**

The desired truth Table 7 is given below:

**Table 7**

| $p$ | $q$ | $p \rightarrow q$ | $(p \rightarrow q) \wedge p$ | $[(p \rightarrow q) \wedge p] \rightarrow q$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | F | T |
| F | T | T | F | T |
| F | F | T | F | T |

## 8.1.3 Mathematical Proof

Suppose Fayyaz is a student in Grade 9. One day, he arrived home late due to heavy traffic in a city. His father, however, suspected that Fayyaz had not gone to school and instead spent the day elsewhere. To address his concerns, his father asked, "Tell me the truth, did you go to school today? Fayyaz responded, saying, "Yes, I did." Still doubtful, his father asked, "What proof do you have that you attended school? To satisfy his father's concern, Fayaz says that my classmate Ahmad went to school with me and could confirm with him. But his father was still not convinced by his words. Now, how will he prove his father's claim that he went to school or not? To prove his father's claim, Fayyaz would need to present some evidence, like his attendance for that day, which was recorded in the school attendance register, or CCTV footage from the school to prove that he was indeed present that day.

Consider another situation, you have bought a mobile phone with a warranty of about one year. After using the mobile phone for a few days, your mobile phone breaks down, so you take it to the mobile company or service provider. The customer support representative will ask you for proof if you want to claim your mobile phone's warranty. To claim the warranty on the mobile phone, you must present the warranty card as documented proof to the customer service representative. Generally, we have to prove and disprove many claims and statements in our daily routine. In mathematics, proofs provides the evidence that a statement is correct, demonstrating a logical sequence of steps that lead to the final conclusion.

**Example 6:** Prove the following mathematical statements.

(a) If $x$ is an odd integer, then $x^2$ is also an odd integer

(b) The sum of two odd numbers is an even number

**Solution:**

(a) Let $x$ be an odd integer. Then by definition of an odd integer, we can express $x$ as:

$x = 2k + 1$ for some $k \in Z$

Now $x^2 = (2k + 1)^2 = 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

$= 2m + 1$, where $m = 2k^2 + 2k \in Z$

Thus, $x^2 = 2m + 1$ for some $m \in Z$

Therefore, $x^2$ is an odd integer, by definition of an odd integer.

> **Note:** If $x$ is odd, then $x$ can be expressed in the form: $x = 2k+1$ for some $k \in Z$

(b) Let $x$ and $y$ be odd integers. Then by definition of an odd integer, we can express $x$ and $y$ as:

$x = 2k + 1$ and $y = 2n + 1$ for some $k$ and $n \in Z$.

> **Note:** If $x$ is an even integer, then $x$ can be expressed in the form: $x = 2k$ for some $k \in Z$

Thus, $x + y = (2k + 1) + (2n + 1)$

$= 2k + 2n + 1 + 1$

$= 2(k + n + 1) = 2m$, where $k + n + 1 = m \in Z$

So, $x + y = 2m$ for some $m \in Z$.

Therefore, $x + y$ is an even integer, by definition of an even integer.

**Example 7:** Prove that for any two non-empty sets $A$ and $B$, $(A\cup B)' = A'\cap B'$.

**Proof:** Let $x\in(A\cup B)'$

$\Rightarrow \quad x\notin(A\cup B)$

$\Rightarrow \quad x\notin A \text{ and } x\notin B$

$\Rightarrow \quad x\in A' \text{ and } x\in B'$

$\Rightarrow \quad x\in A'\cap B'$

But $x\in(A\cup B)'$ is an arbitrary element

Therefore, $(A\cup B)' \subseteq A'\cap B'$ … (i)

Now, suppose that $y\in A'\cap B'$

$\Rightarrow \quad y\in A' \text{ and } y\in B'$

$\Rightarrow \quad y\notin A \text{ and } y\notin B$

$\Rightarrow \quad y\notin(A\cup B)$

$\Rightarrow \quad y\in(A\cup B)'$

Thus $A'\cap B' \subseteq (A\cup B)'$ …(ii)

From equations (i) and (ii) we conclude that

$(A\cup B)' = A'\cap B'$, hence proved.

> **Note:**
> $A$ set $B$ is a subset of a set $A$ if every element of set $B$ is also an element of a set $A$.
> Mathematically, we write it as:
> $B\subseteq A$ if $\forall x\in B \Rightarrow x\in A$

## 8.1.4 Theorem, Conjecture and Axiom

In previous sections, we have explored mathematical statements and their corresponding proofs. We will now move on to a more advanced concept known as theorems. A **theorem** is a mathematical statement that has been proved true based on previously known facts. For example, the following statements are theorem:

(i) **Theorem:** The sum of the interior angles of a quadrilateral is 360 degrees.

(ii) **The Fundamental Theorem of Arithmetic:** Every integer greater than 1 can be uniquely expressed as a product of prime numbers up to the order of the factors.

(iii) **Fermat's Last Theorem**: There are no three positive integers $a$, $b$, $c$, which satisfy the equation $a^n + b^n = c^n$, where $n\in N$ and $n > 2$

One of the famous theorems was named after the 17th-century French mathematician Pierre Fermat. Let's examine Fermat's Last Theorem for specific values of $n$ and see how they apply. For $n = 2$, the statement simplifies to $a^2 + b^2 = c^2$ which does have solutions. This is the well-known Pythagorean theorem. For instance, $3^2 + 4^2 = 5^2$ holds true because $9 + 16 = 25$.

Now, let's examine the statement for $n = 3$. The statement becomes $a^3 + b^3 = c^3$.

After centuries of searching, no such integer solution has been found, and Wiles' proof confirmed that no such numbers exist. For example, $3^3 + 4^3 \neq 5^3$ because $91 \neq 125$.

Fermat claimed he could prove this theorem but noted that the margin of his book was too small for such a meaningful explanation. Despite his assertion, many mathematicians found it challenging to prove the theorem for centuries. The theorem remained unproven for over 350 years and became one of the most famous problems in mathematics. In 1993, Andrew Wiles from Princeton University announced a proof after working on it for over seven years, spanning hundreds of pages. This illustrates that some factual statements are not immediately evident.

**Conjecture:** A **conjecture** is a mathematical statement or hypothesis that is believed to be true based on observations but has not yet been proved. In mathematics, conjectures often serve as hypotheses, and if a conjecture is proven to be true, it becomes a theorem. Conversely, if evidence is found that disproves it, thc conjecture is shown to be false. Here, is another well-known statement that has gained enough recognition to be named. First proposed in the 18th century by the German mathematician Christian Goldbach, it is known as the Goldbach Conjecture. The Goldbach Conjecture states that:

**Statement:** Every even integer greater than 2 is a sum of two prime numbers.

We must agree that the conjecture is either true or false. It appears to be true based on empirical evidence, as many even numbers greater than 2 can indeed be written as the sum of two prime numbers: for example, $4 = 2 + 2$, $6 = 3 + 3$, $12 = 5 + 7$, among others. However, this does not preclude the possibility that some large even number may exist that cannot be expressed as the sum of two primes. The conjecture would be proven false if such a number is found. Despite extensive efforts since Goldbach first posed the problem over 260 years ago, no proof has been found to determine whether the conjecture is true or false. Nevertheless, conjecture is a valid mathematical statement, as it must be either true or false.

In mathematics, we frequently encounter situations where it is necessary to determine the truth of a given statement without proving it. Next, we will study the same statement, which is known as axiom.

An **axiom** is a mathematical statement that we believe to be true without any evidence or requiring any proof. In other words, these statements are basic facts that form the starting point for further ideas and are based on everyday experiences. Moreover, there is no evidence contradicting these statements. For example, the following are the statements of axioms.

**Axiom:** Through a given point, infinitely many lines can pass.

**Euclid Axioms:** A straight line can be drawn between any two points.

**Peano Axioms:** Every natural number has a successor, which is also a natural number.

**Axiom of Extensionality**: Two sets are equal if they have the same elements.

**Axiom of Power Set**: Any set has a set of all its subsets.

Considering the above example, we will find that there is no need to prove these statements. For example, our intuition recognizes that infinitely many lines can pass through a point, so there is no need to prove it.

Axioms are sometimes referred to as postulates. Both Axioms and Postulates describe statements that are accepted as true without requiring proof. However, postulates are associated explicitly with geometry, while axioms can pertain to broader mathematical contexts.

Next, we are going to prove the statement of a theorem.

**Example 8:** Prove that $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$ where *a, b, c* and *d* are non-zero real numbers.

**Solution:** $\text{L.H.S} = \frac{a}{b}+\frac{c}{d}=\frac{a}{b}\times 1+\frac{c}{d}\times 1$ ($\because$ Multiplicative identity)

$=\frac{a}{b}\times\left(d\times\frac{1}{d}\right)+\frac{c}{d}\times\left(b\times\frac{1}{b}\right)$ ($\because$ Multiplicative inverse)

$=\frac{a}{b}\times\frac{d}{d}+\frac{c}{d}\times\frac{b}{b}$ $\left(\because a\times\frac{1}{b}=\frac{a}{b}\right)$

$=\frac{ad}{bd}+\frac{cb}{db}$ $\left(\because \text{Rule of production of fraction } \frac{a}{b}.\frac{c}{d}=\frac{ac}{bd}\right)$

$$= \frac{ad}{bd} + \frac{bc}{bd} \quad (\because \text{ Commutative law of multiplication } ab = ba)$$

$$= ad \times \frac{1}{bd} + bc \times \frac{1}{bd} \quad \left(\because \quad a \times \frac{1}{b} = \frac{a}{b}\right)$$

$$= (ad + bc).\frac{1}{bd} \quad (\because \text{ Distributive property})$$

$$= \frac{(ad + bc)}{bd} = \text{R.H.S}$$

$\Rightarrow$ L.H.S = R.H.S

Thus, $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

Hence proved.

### 8.1.5 Deductive Proof

As discussed earlier, deductive reasoning is a way of drawing conclusions from premises believed to be true. If the premises are true, then the conclusion must also be true. For example: All human beings need to breathe to live. Ahmad is a human. Therefore, Ahmad is also breathing to live.

Similarly, in mathematics, deductive proof in an algebraic expression is a technique to show the validity of a mathematical statement through logical reasoning based on known rules, theorems, axioms, or previously proven statements. Deductive reasoning is broadly used in algebra to validate identities and solve equations.

**Example 9:** Prove that: $(x + 1)^2 + 7 = x^2 + 2x + 8$

**Solution:** **Proof:** L.H.S $= (x + 1)^2 + 7$

$= (x + 1)(x + 1) + 7 \quad (\because x^m . x^n = x^{m+n})$

$= x.(x + 1) + 1.(x + 1) + 7 \quad (\because$ Right distributive law)

$= x \cdot x + x \cdot 1 + 1 \cdot x + 1 \cdot 1 + 7 \quad (\because$ Right distributive law)

$= x^2 + 1 \cdot x + 1 \cdot x + 1 + 7 \quad (\because$ Commutative law & $x^m . x^n = x^{m+n})$

$= x^2 + (1 + 1)x + 8 \quad (\because$ Left distributive law)

$= x^2 + 2x + 8$ = R.H.S

$\Rightarrow$ L.H.S = R.H.S

Thus, $(x + 1)^2 + 7 = x^2 + 2x + 8$. Hence proved

**Example 10:** Prove that $\frac{45x + 15}{15} = 3x + 1$ by justifying each step.

**Solution:** **Proof:** $\text{L.H.S} = \dfrac{45x+15}{15}$

$= \dfrac{1}{15}\times(45x+15)$ $\left(\because \dfrac{a}{b} = \dfrac{1}{b}\times a\right)$

$= \dfrac{1}{15}\times(15\times 3x + 15\times 1)$ (∵ Multiplicative Identity)

$= \dfrac{1}{15}\times 15(3x+1)$ (∵ Distributive Law)

$= \left(\dfrac{1}{15}\times 15\right)\cdot(3x+1)$ (∵ Associative Law)

$= 1\cdot(3x+1)$ (∵ Multiplicative Inverse)

$= 3x+1 = \text{R.H.S}$ (∵ Multiplicative Identity)

$\Rightarrow$ L.H.S = R.H.S

Thus, $\dfrac{45x+15}{15} = 3x+1$ hence proved.

## EXERCISE 8

1. Four options are given against each statement. Encircle the correct option.

(i) Which of the following expressions is often related to inductive reasoning?
   (a) based on repeated experiments (b) if and only if statements
   (c) Statement is proven by a theorem (d) based on general principles

(ii) Which of the following sentences describe deductive reasoning?
   (a) general conclusions from a limited number of observations
   (b) based on repeated experiments
   (c) based on units of information that are accurate
   (d) draw conclusion from well-known facts

(iii) Which one of the following statements is true?
   (a) The set of integers is finite
   (b) The sum of the interior angles of any quadrilateral is always $180^\circ$
   (c) $\dfrac{22}{7} \notin Q'$
   (d) All isosceles triangles are equilateral triangles

(iv) Which of the following statements is the best to represent the negation of the statement "The stove is burning"?
   (a) the stove is not burning.

(b) the stove is dim
(c) the stove is turned to low heat
(d) it is both burning and not burning.

(v) The conjunction of two statements $p$ and $q$ is true when:
(a) both $p$ and $q$ are false. (b) both $p$ and $q$ are true.
(c) only $q$ is true. (d) only $p$ is true

(vi) A conditional is regarded as false only when:
(a) antecedent is true and consequent is false.
(b) consequent is true and antecedent is false.
(c) antecedent is true only.
(d) consequent is false only.

(vii) Contrapositive of $q \rightarrow p$ is
(a) $q \rightarrow \sim p$ (b) $\sim q \rightarrow p$ (c) $\sim p \rightarrow \sim q$ (d) $\sim q \rightarrow \sim p$

(viii) The statement "Every integer greater than 2 is a sum of two prime numbers" is:
(a) theorem (b) conjecture (c) axiom (d) postulates

(ix) The statement "A straight line can be drawn between any two points" is :
(a) theorem (b) conjecture (c) axiom (d) logic

(x) The statement "The sum of the interior angle of a triangle is 180°" is:
(a) converse (b) theorem (c) axiom (d) conditional

2. Write the converse, inverse and contrapositive of the following conditionals:
(i) $\sim p \rightarrow q$ (ii) $q \rightarrow p$ (iii) $\sim p \rightarrow \sim q$ (iv) $\sim q \rightarrow \sim p$

3. Write the truth table of the following
(i) $\sim(p \vee q) \vee (\sim q)$ (ii) $\sim(\sim q \vee \sim p)$ (iii) $(p \vee q) \leftrightarrow (p \wedge q)$

4. Differentiate between a mathematical statement and its proof. Given two examples.

5. What is the difference between an axiom and a theorem? Give examples of each.

6. What is the importance of logical reasoning in mathematical proofs? Give an example to illustrate your point.

7. Indicate whether it is an axiom, conjecture or theorem and explain your reasoning.
(i) There is exactly one straight line through any two points.
(ii) Every even number greater than 2 can be written as the sum of two prime numbers.

(iii) The sum of the angles in a triangle is $180^\circ$.

8. Formulate simple deductive proofs for each of the following algebraic expressions, prove that the L.H.S is equal to the R.H.S:
   (i) prove that $(x-4)^2+9=x^2-8x+25$
   (ii) prove that $(x+1)^2-(x-1)^2=4x$
   (iii) prove that $(x+5)^2-(x-5)^2=20x$
9. Prove the following by justifying each step:
   (i) $\dfrac{4+16x}{4}=1+4x$ (ii) $\dfrac{6x^2+18x}{3x^2-27}=\dfrac{2x}{x-3}$
   (iii) $\dfrac{x^2+7x+10}{x^2-3x-10}=\dfrac{x+5}{x-5}$
10. Suppose $x$ is an integer. Then $x$ is odd, then $9x+4$ is odd.
11. Suppose $x$ is an integer. If $x$ is odd, then $7x+5$ is even.
12. Prove the following statements
    (a) If $x$ is an odd integer, then show that it $x^2-4x+6$is odd.
    (b) If $x$ is an even integer then show that $x^2+2x+4$ is even.
13. Prove that for any two non-empty sets $A$ and $B$, $(A\cap B)'=A'\cup B'$.
14. If $x$ and $y$ are positive real numbers and $x^2<y^2$ then $x<y$.
15. Prove that the sum of the interior angles of a triangle is $180^\circ$.
16. If $a$, $b$ and $c$ are non-zero real numbers, prove that:
    (a) $\dfrac{a}{b}=\dfrac{c}{d}\Leftrightarrow ad=bc$ (b) $\dfrac{a}{b}.\dfrac{c}{d}=\dfrac{ac}{bd}$ (c) $\dfrac{a}{b}+\dfrac{c}{b}=\dfrac{a+c}{b}$