Unit 9 Similar Figures

Students' Learning Outcomes

At the end of the unit, the students will be able to:

- Identify similarity of polygons. Area and volume of similar figures.
- Solve problems using the relationship between areas of similar figures and volume of similar solids.
- Solve real life problems that involve the properties of regular polygons, triangles and parallelograms (such as building architectural structures, fencing, tiling, painting and carpeting a room).

INTRODUCTION

The concept of similarity dates back to ancient Greece, where Greek mathematicians, particularly Euclid, developed the fundamental principles of geometry. In his creative work, "The Elements", Euclid established the foundations of plane geometry, including the theory of similar triangles and polygons. Euclid's further work laid the groundwork for modern geometry and the concept of similarity remains central in many branches of mathematics, including trigonometry and algebra.

9.1 Similarity of Polygons

Similar figures have same shape but not necessarily of same size. Two polygons are similar if their corresponding angles are equal and the corresponding sides are proportional (i.e., the ratios of the lengths of corresponding sides are equal). This means that if two polygons are similar, one is a scaled version of the other. For example, all equilateral triangles are similar to each other because they have the same angles and the measure of the sides are proportional.

> **Remember!**
> Three or more than three-sided closed figure is called polygon.

9.1.1 Identification of Similar Triangles

(i) If two angles in one triangle are congruent to two corresponding angles in another triangle, the third angle in each triangle must be congruent. Since the angles are the same, the triangles are similar. Similarity symbol is '~'.

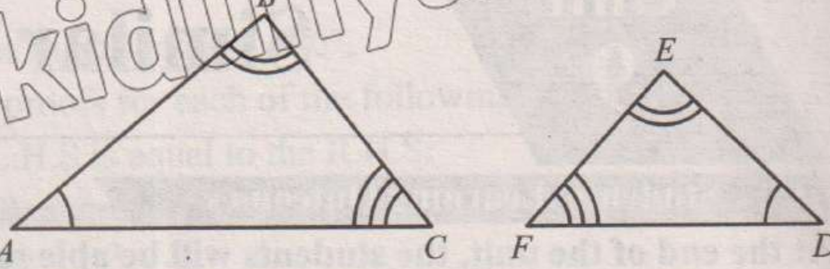
i.e., In the correspondence of the triangles *ABC* and *DEF*.

$m\angle A = m\angle D$

$m\angle B = m\angle E$

$m\angle C = m\angle F$

Hence, $\Delta ABC \sim \Delta DEF$

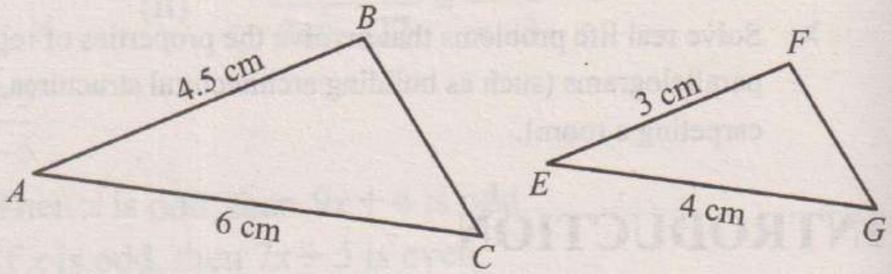


(ii) If the ratio of two corresponding sides and their included angle are equal, then the triangles are similar. In the correspondence of the triangles ABC and EFG, $m\angle ABC = m\angle EFG$ and the ratio of the corresponding sides are

$$\frac{m\overline{AB}}{m\overline{EF}} = \frac{4.5}{3} = \frac{3}{2}$$

and $\frac{m\overline{AC}}{m\overline{EG}} = \frac{6}{4} = \frac{3}{2}$ Hence

triangles ABC and EFG are similar.



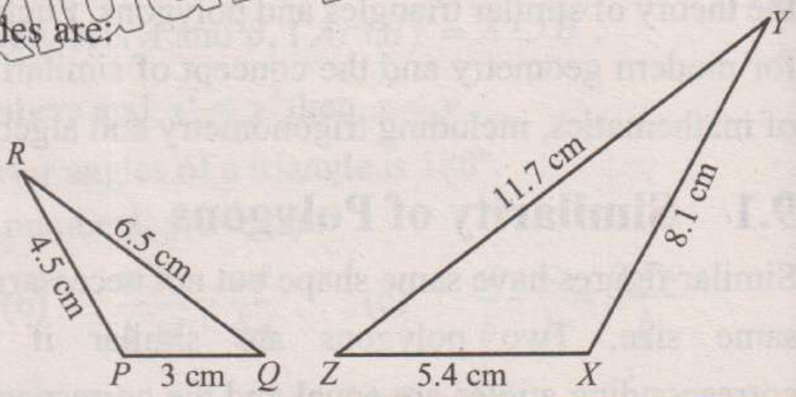
(iii) If the ratio of all the corresponding sides are equal, then the triangles are similar. In the corresponding of ΔPQR and ΔXZY,

the ratio of corresponding sides are:

$$\frac{m\overline{PQ}}{m\overline{XZ}} = \frac{m\overline{QR}}{m\overline{YZ}} = \frac{m\overline{PR}}{m\overline{XY}}$$

$$\frac{3}{5.4} = \frac{6.5}{11.7} = \frac{4.5}{8.1}$$

$$\frac{5}{9} = \frac{5}{9} = \frac{5}{9}$$



Hence, the ΔPQR and ΔXZY are similar.

Need to Know! Proportionality of sides means one side is k times of its corresponding side.

Example 1: If one pair of corresponding sides are parallel to each other, then the triangles so formed as shown in the figure are similar. i.e.,

In the figure, $\overline{AB}$ is parallel to $\overline{CD}$ and

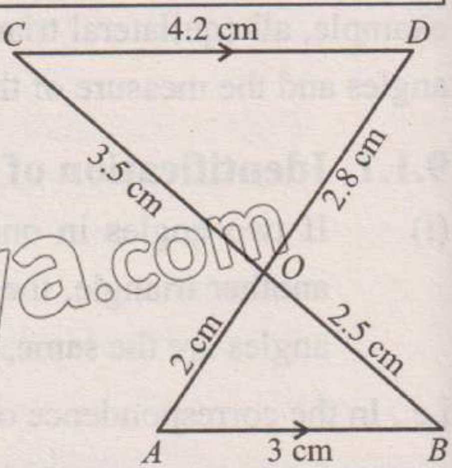
$m\angle AOB = m\angle DOC$ (Vertically opposite angles)

$m\angle A = m\angle D$ (Alternate angles of parallel lines)

$m\angle B = m\angle C$ (Alternate angles of parallel lines)

Since all three corresponding angles are equal,

so $\Delta OAB \sim \Delta ODC$



The ratio of corresponding sides are equal i.e.,

$$\frac{m\overline{OA}}{m\overline{OD}} = \frac{m\overline{AB}}{m\overline{DC}} = \frac{m\overline{OB}}{m\overline{OC}}$$

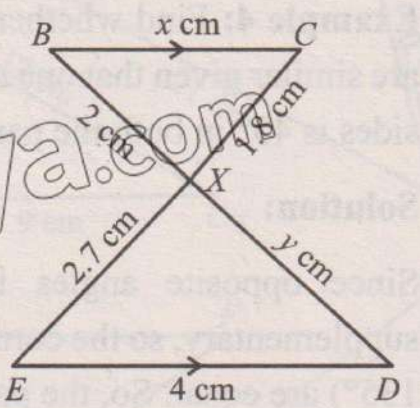
$$\frac{2}{2.8} = \frac{3}{4.2} = \frac{2.5}{3.5}$$

$$\frac{5}{7} = \frac{5}{7} = \frac{5}{7}$$

So, the triangles *OAB* and *ODC* are similar.

Example 2:
In the triangles *XBC* and *XDE*, find the value of *x* and *y*.

Solution: Since $\overline{BC}$ is parallel to $\overline{ED}$, so the triangles *XBC* and *XDE* are similar, so, the ratio of the corresponding sides are:



$$\frac{m\overline{XB}}{m\overline{XD}} = \frac{m\overline{BC}}{m\overline{DE}} = \frac{m\overline{XC}}{m\overline{XE}}$$

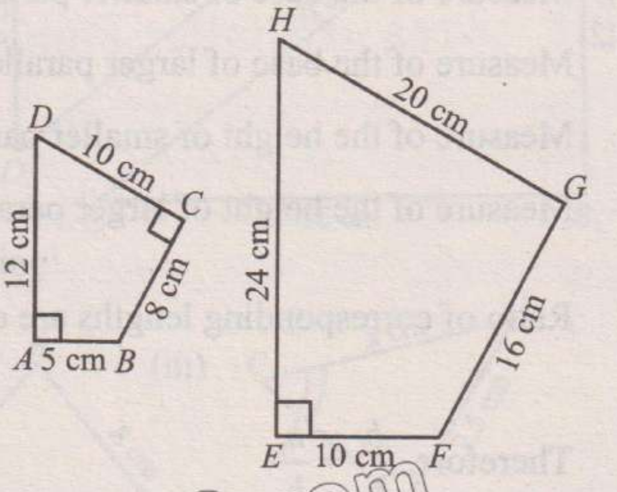
$$\frac{2}{y} = \frac{x}{4} = \frac{1.8}{2.7}$$

$$\frac{x}{4} = \frac{1.8}{2.7} \Rightarrow x = \frac{1.8}{2.7} \times 4 = 2.67\text{ cm}$$

$$\frac{2}{y} = \frac{1.8}{2.7} \Rightarrow y = \frac{2.7}{1.8} \times 2 = 3\text{ cm}$$

9.1.2 Similarity of Quadrilaterals

Example 3: The Quadrilateral *ABCD* has side lengths $m\overline{AB}$ = 5cm, $m\overline{BC}$ = 8, $m\overline{CD}$ = 10 cm, $m\overline{AD}$ = 12cm, and its angles are $m\angle A = 90°$, $m\angle B$ = 120° and $m\angle C = 90°$. Quadrilateral *EFGH* has side lengths $m\overline{EF}$ = 10 cm, $m\overline{FG}$ = 16 cm, $m\overline{GH}$ = 20cm, $m\overline{EH}$ = 24 cm and its angles are $m\angle E = 90°$, $m\angle F = 120°$ and $m\angle H = 60°$. Prove that the quadrilateral *ABCD* is similar to the quadrilateral *EFGH*. (Diagrams are not drawn to scale).



Solution: We see that in the quadrilateral *ABCD*:

$$m\angle D = 360° - (90° + 120° + 90°) = 60°.$$

In the quadrilateral *EFGH*, $m\angle G = 360° - (90° + 120° + 60°) = 90°$.

Now, check if the corresponding angles of the quadrilaterals are congruent:

$m\angle A = m\angle E = 90°$, $m\angle B = m\angle F = 120°$, $m\angle C = m\angle G = 90°$ and $m\angle D = m\angle H = 60°$.

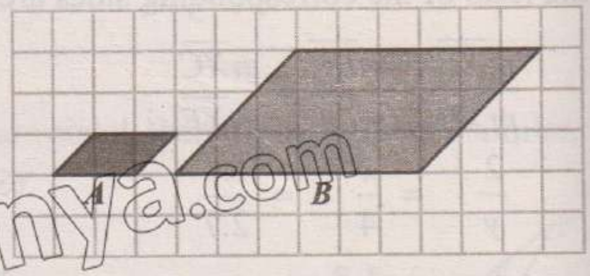
Next, check the ratios of the corresponding sides:

Ratio of $\overline{AB}$ to $\overline{EF}$: $\frac{m\overline{AB}}{m\overline{EF}} = \frac{5}{10} = \frac{1}{2}$, Ratio of $\overline{BC}$ to $\overline{FG}$: $\frac{m\overline{BC}}{m\overline{FG}} = \frac{8}{16} = \frac{1}{2}$

Ratio of $\overline{CD}$ to $\overline{GH}$: $\frac{m\overline{CD}}{m\overline{GH}} = \frac{10}{20} = \frac{1}{2}$ Ratio of $\overline{AD}$ to $\overline{EH}$: $\frac{m\overline{AD}}{m\overline{EH}} = \frac{12}{24} = \frac{1}{2}$

Since the corresponding angles are congruent and the corresponding sides are proportional (with a ratio of $\frac{1}{2}$), so the quadrilateral *ABCD* is similar to the quadrilateral *EFGH*.

Example 4: Find whether the parallelograms are similar given that one of the angle between sides is 45° in both the parallelograms.



Solution:

Since opposite angles in a parallelogram are equal and adjacent angles are supplementary, so the corresponding angles in both parallelograms (45°, 135°, 45°, and 135°) are equal. So, the parallelograms are similar.

Measure of the base of smaller parallelogram, $b_1 = 2$units

Measure of the base of larger parallelogram, $b_2 = 6$ units.

Measure of the height of smaller parallelogram, $h_1 = 1$ unit

Measure of the height of larger parallelogram, $h_2 = 3$ units.

Ratio of corresponding lengths are equal. i.e., $\frac{b_1}{b_2} = \frac{2}{6} = \frac{1}{3}$ and $\frac{h_1}{h_2} = \frac{1}{3}$

Therefore, $\frac{b_1}{b_2} = \frac{h_1}{h_2}$

Example 5 The perimeter of a regular octagon is 48 cm. Another octagon has sides that are 1.2 times the sides of the first octagon. What is the length of side of the second octagon?

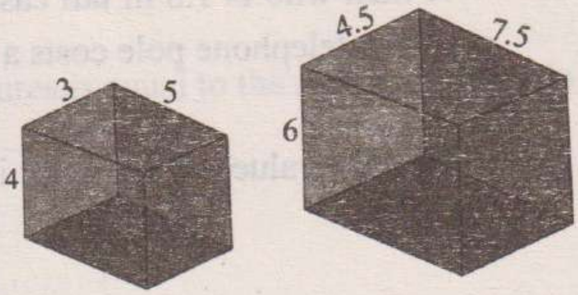
Solution: Perimeter of first regular octagon = 48 cm

Side length of first regular octagon $= \frac{48}{8} = 6$ cm.

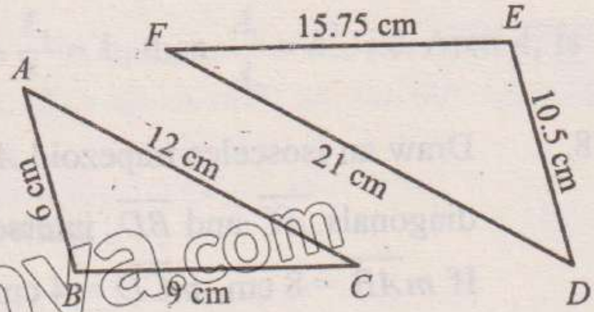
Side length of second regular octagon $= 6 \times 1.2 = 7.2$ cm.

EXERCISE 9.1

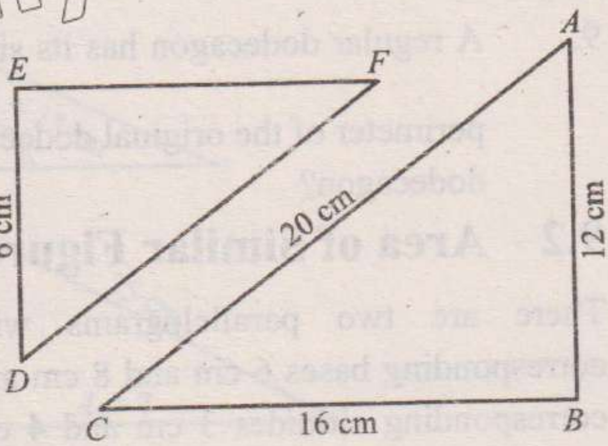
1. Find whether the solids are similar. All lengths are in cm.



2. In triangle ABC, the sides are given as $m\overline{AB} = 6$ cm, $m\overline{BC} = 9$ cm and $m\overline{CA} = 12$ cm. In triangle DEF, the sides are given as $m\overline{DE} = 10.5$ cm, $m\overline{EF} = 15.75$ cm, and $m\overline{FD} = 21$ cm. Prove that the triangles are similar.

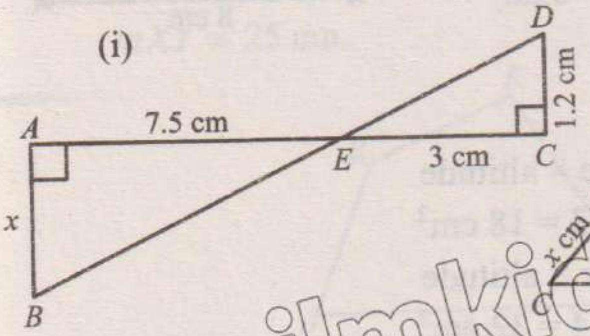


3. In the given figure, $\Delta ABC \sim \Delta DEF$, $m\overline{AB} = 12$ cm, $m\overline{AC} = 20$ cm and $m\overline{BC} = 16$ cm. In ΔDEF, $m\overline{DE} = 6$ cm. Find $m\overline{DF}$ and $m\overline{EF}$

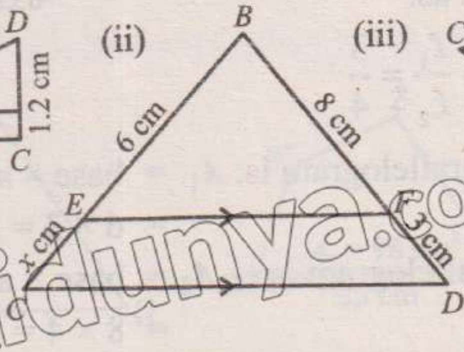


4. Find the value of x in each of the following:

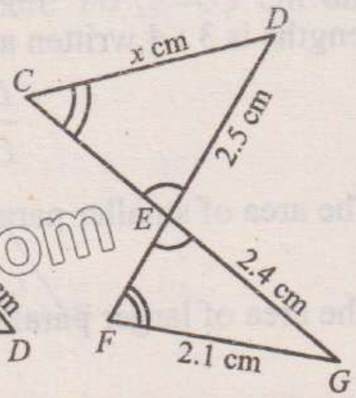
(i)



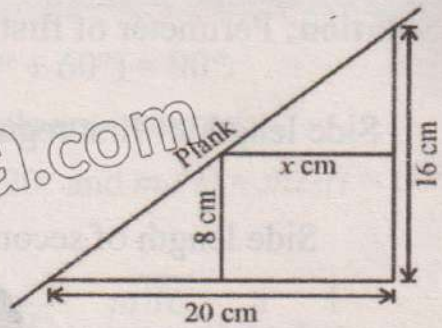
(ii)



(iii)

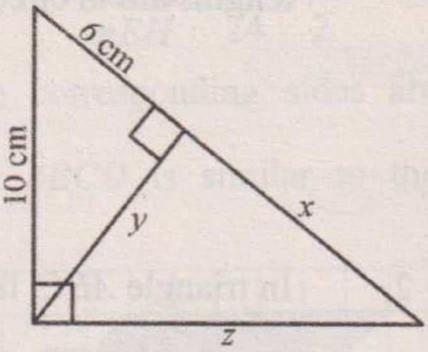


5. A plank is placed straight upstairs that 20 cm wide and 16 cm deep. A rectangular box of height 8 cm and width x cm is placed on a stair under the plank. Find the value of x.



6. A man who is 1.8 m tall casts a shadow of a 0.76 m in length. If at the same time a telephone pole costs a 3 m shadow, find the height of the pole.

7. Find the values of x, y and z in the given figure.

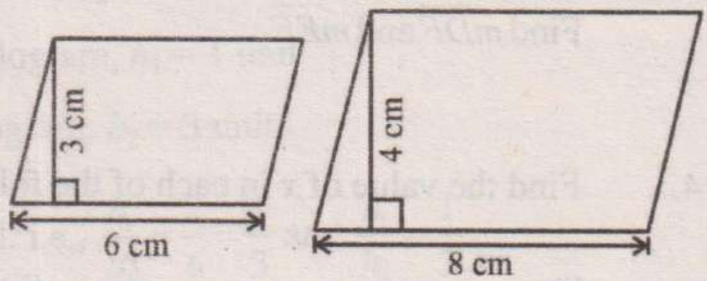


8. Draw an isosceles trapezoid $ABCD$ where $\overline{AB} \parallel \overline{CD}$ and $m\overline{AB} > m\overline{CD}$. Draw diagonals $\overline{AC}$ and $\overline{BD}$, intersecting at E. Prove that $\triangle ABE$ is similar to $\triangle CDE$. If $m\overline{AB} = 8$ cm, $m\overline{CD} = 4$ cm, and $m\overline{AE} = 3$ cm, find the length of $\overline{CE}$.

9. A regular dodecagon has its side lengths decreased by a factor of $\frac{1}{\sqrt{2}}$. If the perimeter of the original dodecagon is 72 cm. What is the side length of scaled dodecagon?

9.2 Area of Similar Figures

There are two parallelograms with corresponding bases 6 cm and 8 cm and corresponding altitudes 3 cm and 4 cm respectively. The ratio between their lengths is 3 : 4 written as:



$$\frac{\ell_1}{\ell_2} = \frac{3}{4}$$

The area of smaller parallelogram is: A_1 = base × altitude
$= 6 \times 3 = 18 \text{ cm}^2$

The area of larger parallelogram is: A_2 = base × altitude
$= 8 \times 4 = 32 \text{ cm}^2$

The ratio of their areas is: $\dfrac{A_1}{A_2} = \dfrac{9}{16} = \left(\dfrac{3}{4}\right)^2$

$$\frac{A_1}{A_2} = \left(\frac{\ell_1}{\ell_2}\right)^2$$

Where A_1 and A_2 are areas and ℓ_1 and ℓ_2 are any two corresponding lengths of similar figures.

Hence the ratio of the areas of any two similar figures is equal to the square of the ratio of any two corresponding lengths of the figures.

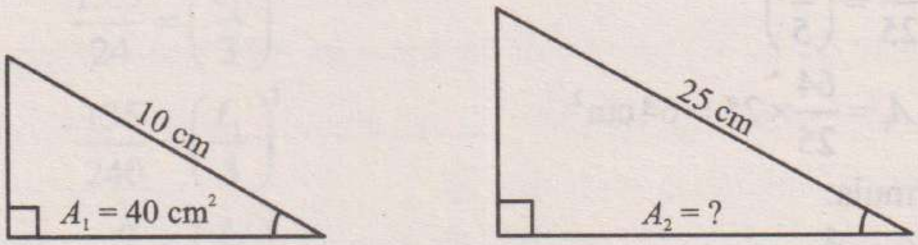
$$\frac{A_1}{A_2} = \left(\frac{\ell_1}{\ell_2}\right)^2$$

Since each length is k times of the other, we take $\dfrac{\ell_1}{\ell_2} = k$, then $\dfrac{A_1}{A_2} = k^2$. i.e. Area A_1 is k^2 times the area A_2. k is called scale factor.

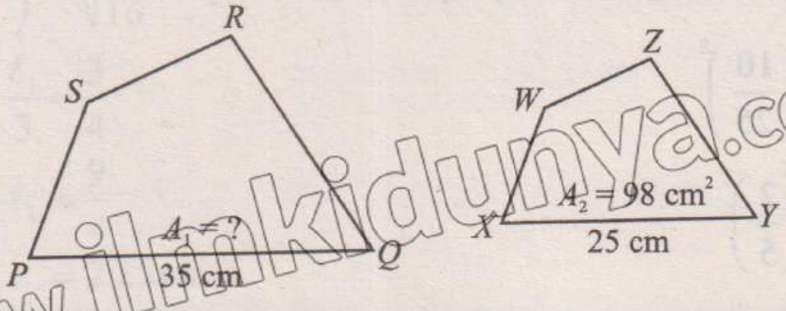
Example 6: Find the unknown value in the following:

(i)

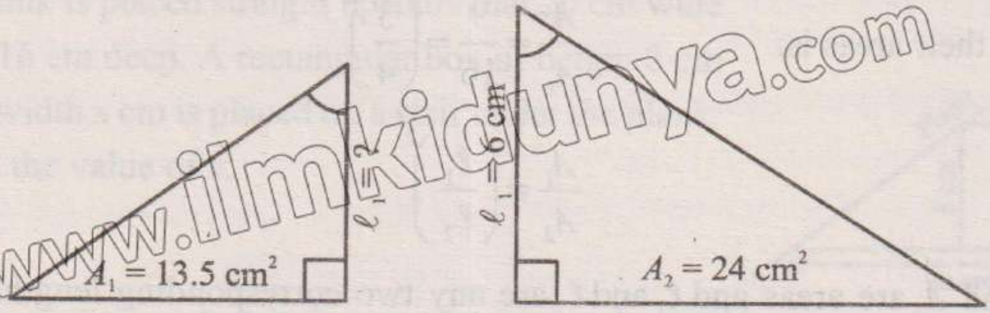
(ii)



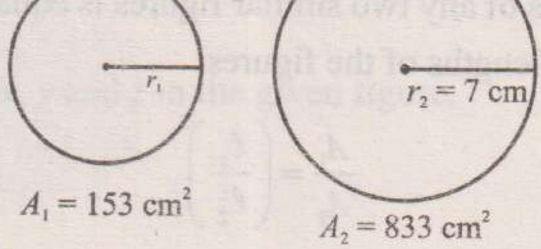
(iii) The quadrilaterals *PQRS* and *XYZW* are similar where $m\overline{PQ} = 35$ cm and $m\overline{XY} = 25$ cm.



(iv)



(v)



Solution: (i) Since two pairs of corresponding angles are equal i.e., triangles are similar. We use the formula for ratio of areas of similar figures.

$$\frac{A_1}{A_2} = \left(\frac{\ell_1}{\ell_2}\right)^2$$

Here $\ell_1 = 2.4$ cm, $\ell_2 = 1.5$ cm, $A_2 = 25\ cm^2$, $A_1 = ?$

$$\frac{A_1}{25} = \left(\frac{2.4}{1.5}\right)^2$$

$$\frac{A_1}{25} = \left(\frac{8}{5}\right)^2$$

$$A_1 = \frac{64}{25} \times 25 = 64\,cm^2$$

(ii) Apply formula:

$$\frac{A_1}{A_2} = \left(\frac{\ell_1}{\ell_2}\right)^2$$

Here $\ell_1 = 10$ cm, $\ell_2 = 25$ cm, $A_1 = 40\ cm^2$, $A_2 = ?$

$$\frac{40}{A_2} = \left(\frac{10}{25}\right)^2$$

$$\frac{40}{A_2} = \left(\frac{2}{5}\right)^2$$

$$\frac{40}{A_2} = \frac{4}{25}$$

$$A_2 = 40 \times \frac{25}{4} = 250\,\text{cm}^2$$

(iii) It is given that the quadrilateral *PQRS* is similar to quadrilateral *XYZW*.

$$\frac{A_1}{A_2} = \left(\frac{\ell_1}{\ell_2}\right)^2$$

Here $\ell_1 = 35$ cm, $\ell_2 = 25$ cm, $A_1 = ?$, $A_2 = 98$ cm^2

$$\frac{A_1}{98} = \left(\frac{35}{25}\right)^2$$

$$\frac{A_1}{98} = \left(\frac{7}{5}\right)^2$$

$$A_1 = \frac{49}{25} \times 98 = 192.08\,\text{cm}^2$$

(iv) Since two pairs of corresponding angles in both triangles are equal, so triangles are similar.

$$\therefore \quad \frac{A_1}{A_2} = \left(\frac{\ell_1}{\ell_2}\right)^2$$

Here $\ell_1 = ?$, $\ell_2 = 3$ cm, $A_1 = 13.5$ cm^2, $A_2 = 24$ cm^2

$$\frac{13.5}{24} = \left(\frac{\ell_1}{3}\right)^2$$

$$\frac{135}{240} = \left(\frac{\ell_1}{3}\right)^2$$

$$\frac{9}{16} = \left(\frac{\ell_1}{3}\right)^2$$

$$\sqrt{\left(\frac{\ell_1}{3}\right)^2} = \sqrt{\frac{9}{16}} \quad \text{(Taking square root)}$$

$$\frac{\ell_1}{3} = \frac{3}{4}$$

$$\ell_1 = \frac{9}{4}$$

$$= 2.25\,\text{cm}$$

(v) For similar spheres

$$\frac{A_1}{A_2}=\left(\frac{\ell_1}{\ell_2}\right)^2$$

Here $r_1 = ?$, $r_2 = 7$ cm , $A_1 = 153$ cm^2, $A_2 = 833$ cm^2

$$\frac{153}{833}=\left(\frac{r_1}{7}\right)^2$$

$$\frac{9}{49}=\left(\frac{r_1}{7}\right)^2$$

$$\sqrt{\left(\frac{r_1}{7}\right)^2}=\sqrt{\frac{9}{49}} \quad \text{(Taking square root)}$$

$$\frac{r_1}{7}=\frac{3}{7} \quad \Rightarrow \ r_1 = 3\,\text{cm}$$

Example 7: Two polygons are similar with a ratio of corresponding sides being $\frac{3}{5}$. If the area of the smaller polygon is 54 cm², find the area of the larger polygon.

Solution: The ratio of the areas of two similar polygons is the square of the ratio of corresponding sides. So, $\frac{\text{Area of larger polygon}}{\text{Area of smaller polygon}}=\left(\frac{5}{3}\right)^2=\frac{25}{9}$

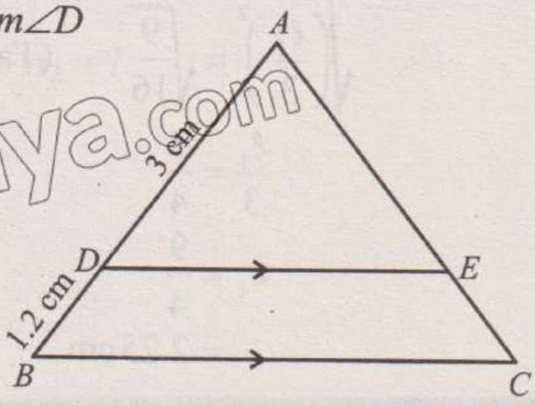
Therefore, Area of larger polygon $=\frac{25}{9}\times 54 = 150$ cm^2

Example 8 Given that $\overline{BC} \parallel \overline{DE}$, prove that the triangles ABC and ADE are similar.

(i) If $m\overline{AB} = 3$ cm and $m\overline{BD} = 1.2$ cm, find the ratio of area of ΔABC to the area of ΔADE.

(ii) If area of ΔADE is 125 cm², find the area of ΔABC and area of trapezium $BCED$.

Solution: Since $m\angle A = m\angle A$ (common), $m\angle B = m\angle D$ and $m\angle C = m\angle E$ (Corresponding angles of parallel lines $\overline{BC}$ and $\overline{DE}$). Hence ΔABC is similar to ΔADE.



(i) Ratio of sides $=\frac{m\overline{AB}}{m\overline{AD}}=\frac{3+1.2}{3}=\frac{4.2}{3}=\frac{7}{5}$

$$\therefore \quad \frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta ADE} = \left(\frac{\ell_1}{\ell_2}\right)^2 = \left(\frac{7}{5}\right)^2 = \frac{49}{25}$$

(ii) Area of $\Delta ADE = 125\ \text{cm}^2$

$$\frac{\text{Area of } \Delta ABC}{125} = \frac{49}{25}$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{49}{\cancel{25}} \times \cancel{125}^{5} = 245\ \text{cm}^2$$

Area of trapezium $BCED$ = Area of ΔABC – Area of ΔADE

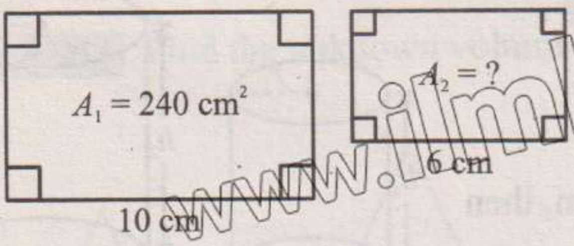
$= 245 - 125 = 120\ \text{cm}^2$

EXERCISE 9.2

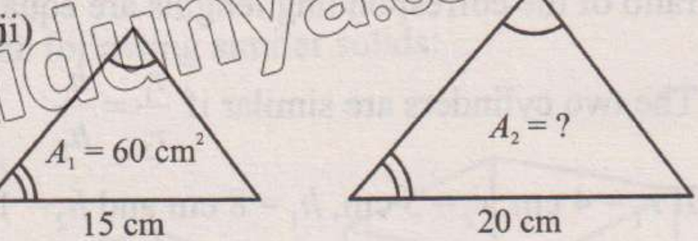
1. Find the ratio of the areas of similar figures if the ratio of their corresponding lengths are: (i) 1:3 (ii) 3:4 (iii) 2:7 (iv) 8:9 (v) 6:5

2. Find the unknowns in the following figures:

(i)



(ii)



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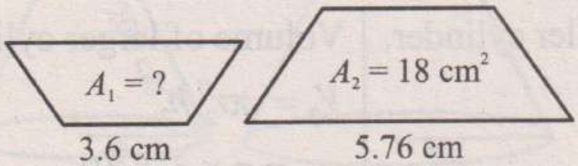

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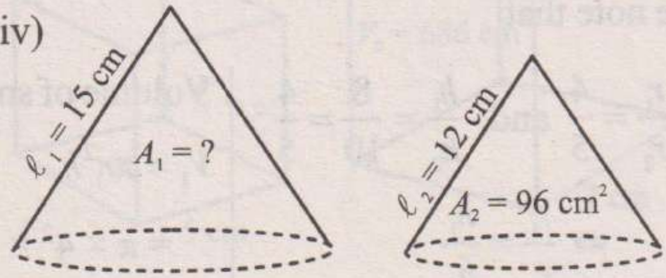
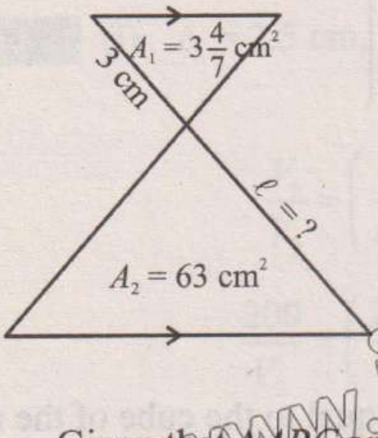
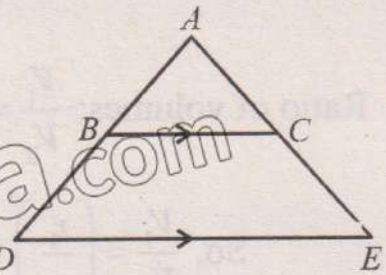


(v)



3. Given that area of ΔABC = 36 cm² and $m\overline{AB} = 6$ cm, $m\overline{BD} = 4$ cm. Find
(a) the area of ΔADE
(b) the area of trapezium $BCED$

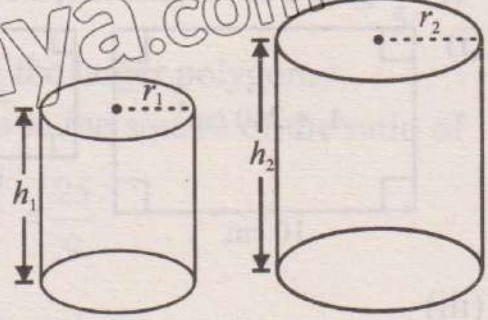


4. Given that ΔABC and ΔDEF are similar, with a scale factor of $k = 3$. If the area of ΔABC is 50 cm², find the area of triangle ΔDEF?

5. Quadrilaterals *ABCD* and *EFGH* are similar, with a scale factor of $k = \frac{1}{4}$. If the area of quadrilateral *ABCD* is 64 cm^2, find the area of quadrilateral *EFGH*.
6. The areas of two similar triangles are 16 cm^2 and 25 cm^2. What is the ratio of a pair of corresponding sides?
7. The areas of two similar triangles are 144 cm^2 and 81 cm^2. If the base of the large triangle is 30 cm, find the corresponding base of the smaller triangle.
8. A regular heptagon is inscribed in a larger regular heptagon and each side of the larger heptagon is 1.7 times the side of the smaller heptagon. If the area of the smaller heptagon is known to be 100 cm^2, find the area of the larger heptagon.

9.3 Volume of Similar Solids

Two solids are said to be similar if they have same shape but possibly different sizes. Two solids are similar if lengths of the corresponding sides are proportional i.e., the ratio of the corresponding lengths are equal. e.g.,



The two cylinders are similar if $\frac{r_1}{r_2} = \frac{h_1}{h_2}$

If $r_1 = 4$ cm, $r_2 = 5$ cm, $h_1 = 8$ cm and $h_2 = 10$ cm, then we note that:

$\frac{r_1}{r_2} = \frac{4}{5}$ and $\frac{h_1}{h_2} = \frac{8}{10} = \frac{4}{5}$.

$\Rightarrow \frac{r_1}{r_2} = \frac{h_1}{h_2}$

Volume of smaller cylinder,

$$V_1 = \pi r_1^2 h_1$$
$$= \pi \times 4^2 \times 8$$
$$= 128\pi \text{ cm}^2$$

Volume of larger cylinder,

$$V_2 = \pi r_2^2 h_2$$
$$= \pi \times 5^2 \times 10$$
$$= 250\pi \text{ cm}^2$$

Ratio of volumes: $\frac{V_1}{V_2} = \frac{128\pi}{250\pi} = \frac{64}{125} = \left(\frac{4}{5}\right)^3$

So, $\frac{V_1}{V_2} = \left(\frac{r_1}{r_2}\right)^3$ or $\frac{V_1}{V_2} = \left(\frac{h_1}{h_2}\right)^3$

Hence the ratio of the volume of any two similar solids is equal to the cube of the ratio of any two corresponding lengths of the solids.

$$\frac{V_1}{V_2} = \left(\frac{\ell_1}{\ell_2}\right)^3$$

Since each length is k times of the other, we take $\frac{\ell_1}{\ell_2} = k$, then $\frac{V_1}{V_2} = k^3$. i.e., Volume V_1 is k^3 times the volume V_2 and k is called scale factor.

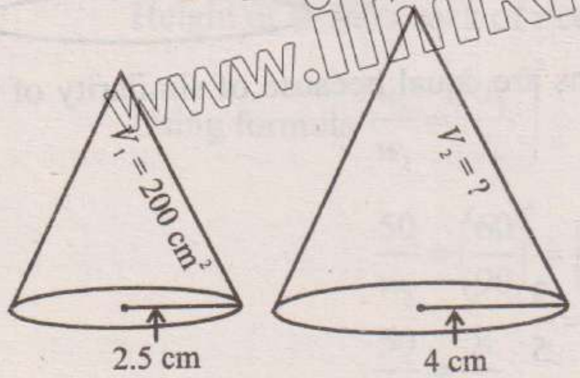
Since mass of a substance is proportional to its volume, the ratio of the mass of two similar solids is equal of to the ratio of their volumes. If the masses of two similar solids are w_1 and w_2 and volumes are V_1 and V_2, then

$$\frac{V_1}{V_2} = \frac{w_1}{w_2}$$

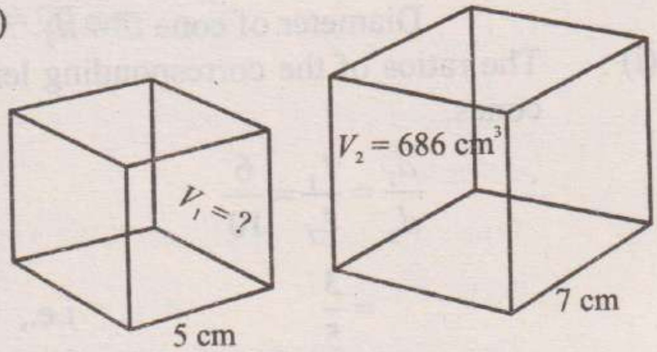
Therefore, $\frac{w_1}{w_2} = \left(\frac{\ell_1}{\ell_2}\right)^3$

Example 9: Find the unknown volume in the following similar solids:

(i)



(ii)



Solution: (i) $\ell_1 = 2.5$ cm, $\ell_2 = 4$ cm

$$\frac{V_1}{V_2} = \left(\frac{\ell_1}{\ell_2}\right)^3$$

$$\frac{200}{V_2} = \left(\frac{2.5}{4}\right)^3$$

$$\frac{200}{V_2} = \left(\frac{5}{8}\right)^3$$

$$V_2 = 200 \times \frac{512}{125}$$

$$V_2 = 819.2 \text{ cm}^3$$

(ii) Using formula $\frac{V_1}{V_2} = \left(\frac{\ell_1}{\ell_2}\right)^3$

$$\frac{V_1}{686} = \left(\frac{5}{7}\right)^3 \qquad \left[\begin{array}{l}\ell_1 = 5 \text{ cm}, \ell_2 = 7 \text{ cm} \\ V_1 = ?, V_2 = 686 \text{ cm}^3\end{array}\right]$$

$$\frac{V_1}{686} = \frac{125}{343}$$

$$V_1 = \frac{125}{343} \times 686$$

$$= 250 \text{ cm}^3$$

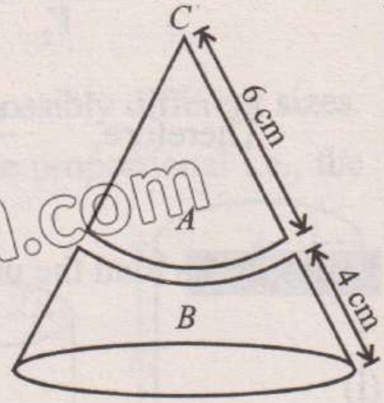
Example 10: A solid cone C is cut into two pieces A and B with sloping edges 6 cm and 4 cm. Find the ratio of:

(i) the diameters of the bases of the cones A and C.

(ii) the area of the bases of the cones A and C.

(iii) the volumes of the cones A and C.

(iv) If volume of cone A is 72 cm^3, find the volume of solid B.



Solution: Let diameter of cone $A = d_1$

Diameter of cone $C = d_2$

(i) The ratios of the corresponding lengths are equal because of similarity of the cones.

$$\therefore \quad \frac{d_1}{d_2} = \frac{\ell_1}{\ell_2} = \frac{6}{10}$$

$$= \frac{3}{5} \qquad \text{i.e., } \frac{\ell_1}{\ell_2} = \frac{3}{5}$$

(ii) $$\frac{\text{Area of cone } A}{\text{Area of cone } C} = \left(\frac{\ell_1}{\ell_2}\right)^2$$

$$= \left(\frac{3}{5}\right)^2 = \frac{9}{25}$$

(iii) $$\frac{\text{Volume of cone } A}{\text{Volume of cone } C} = \left(\frac{\ell_1}{\ell_2}\right)^3$$

$$= \left(\frac{3}{5}\right)^3 = \frac{27}{125}$$

(iv) V_1 = Volume of cone A = 72 cm^3

V_2 = Volume of cone C = ?

$$\therefore \quad \frac{V_1}{V_2} = \left(\frac{\ell_1}{\ell_2}\right)^3$$

$$\frac{72}{V_2} = \frac{27}{125}$$

$$V_2 = \frac{72 \times 125}{27} = 333\frac{1}{3} \text{ cm}^3$$

Volume of solid B = Volume of cone C – Volume of cone A

$$= 333\frac{1}{3} - 72 = 261\frac{1}{3} \text{ cm}^3$$

Example 11: The mass of sack of rice is 50 kg and height 60 cm. Find the mass of the similar sack of rice with height of 90 cm.

Solution: Mass of the smaller sack of rice $w_1 = 50$ kg

Height of smaller sack of rice $h_1 = 60$ cm

Mass of larger sack of rice $w_2 = ?$

Height of smaller sack of rice $h_2 = 90$ cm

Using formula $\frac{w_1}{w_2} = \left(\frac{h_1}{h_2}\right)^3$

$$\frac{50}{w_2} = \left(\frac{60}{90}\right)^3 = \left(\frac{2}{3}\right)^3$$

$$\frac{50}{w_2} = \frac{8}{27}$$

$$w_2 = \frac{27 \times 50}{8} = 168.75 \text{ kg}$$

Example 12: The ratio of the corresponding lengths of two similar cylindrical cans is 3 : 2.

(i) The larger cylindrical can has surface area of 67.5 square metres. Find the surface area of the smaller cylindrieal can.

(ii) The smaller cylindrical can has a volume of 132 cubic metres. Find the volume of larger tin can.

Solution: (i) Surface area of larger can = A_1 = 67.5 m^2

Surface area of smaller can = A_2 = ?

Ratio of corresponding lengths is $\frac{\ell_1}{\ell_2} = \frac{3}{2}$

Using formula for areas of the similar figures:

$$\frac{A_1}{A_2} = \left(\frac{\ell_1}{\ell_2}\right)^2$$

$$\frac{67.5}{A_2} = \left(\frac{3}{2}\right)^2 \Rightarrow A_2 = 67.5 \times \frac{4}{9} = 30\ m^2$$

(ii) Volume of smaller can $= V_2 = 132\ m^3$

Volume of larger can $= V_1 = ?$

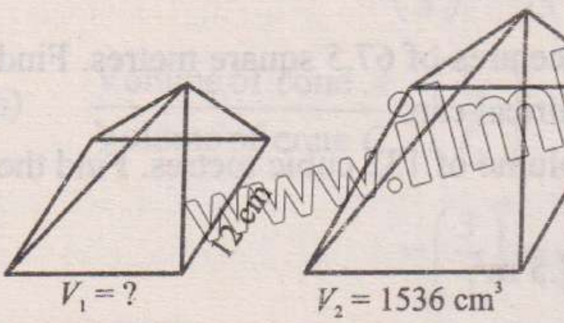
Using formula for volume of similar figures: $\frac{V_1}{V_2} = \left(\frac{\ell_1}{\ell_2}\right)^3$

$$\frac{V_1}{132} = \left(\frac{3}{2}\right)^3 \Rightarrow V_1 = 132 \times \frac{27}{8} = 445.5\ m^3$$

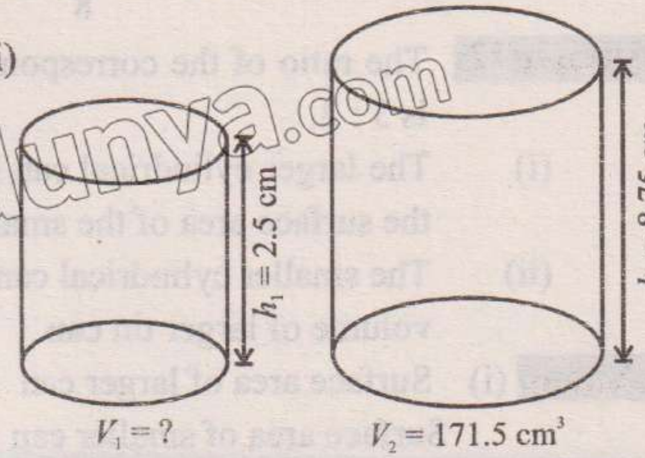
EXERCISE 9.3

1. The radii of two spheres are in the ratio 3 : 4. What is the ratio of their volumes?
2. Two regular tetrahedrons have volumes in the ratio 8 : 27. What is the ratio of their sides?
3. Two right cones have volumes in the ratio 64 : 125. What is the ratio of:
   (i) their heights (ii) their base areas?
4. Find the missing value in the following similar solids.

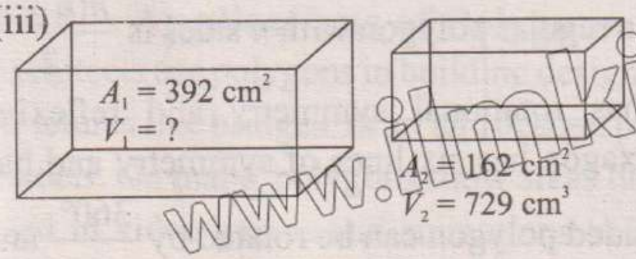
(i)



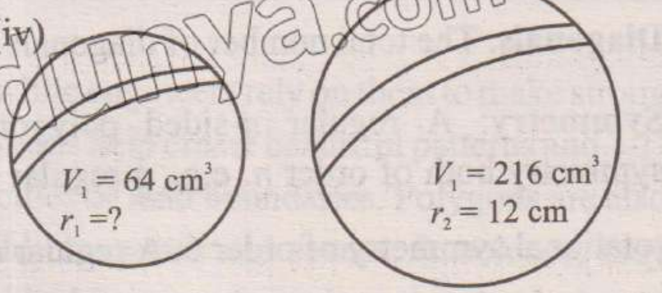
(ii)



(iii)



(iv)



5. The ratio of the corresponding lengths of two similar canonical cans is 3 : 2.
   (i) The larger canonical can have surface area of 96 m^2. Find the surface area of the smaller canonical can.
   (ii) The smaller canonical can have a volume of 240 m^3. Find the volume of larger canonical can.
6. The ratio of the heights of two similar cylindrical water tanks is 5 : 3.
   (i) If the surface area of the larger tank is 250 square metres, find the surface area of the smaller tank.
   (ii) If the volume of the smaller tank is 270 cubic metres, find the volume of the larger tank.

9.4 Geometrical Properties of Polygon and their Applications

9.4.1 Geometrical Properties of Regular Polygon

A regular polygon has all sides and all angles equal. Some of the common regular polygons are equilateral triangles, squares, regular pentagons, regular hexagons, etc.

Sum of Interior Angles: The formula for sum of interior angles of n-sided polygon is $(n-2)\times180°$.

Interior Angle: For a regular n-sided polygon:

$$\text{Size of each Interior Angle} = \frac{(n-2)\times180°}{n}$$

For instance, a regular hexagon has $n = 6$, so each interior angle is

$$\frac{(6-2)\times180°}{6} = \frac{720°}{6} = 120°$$

Exterior Angle: The sum of all exterior angles of any polygon is always 360° regardless of the number of sides. The exterior angle of a regular n-sided polygon is:

$$\text{Exterior Angle} = \frac{360°}{n}$$

The interior and exterior angles are supplementary at a vertex i.e.,

Interior + exterior angle = 180°

Diagonals: The total number of diagonals in a regular polygon with n sides is $\frac{n(n-3)}{2}$

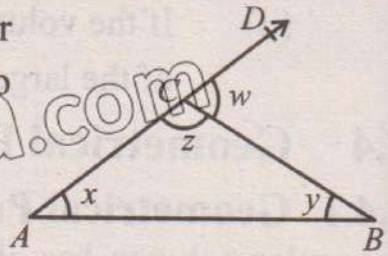
Symmetry: A regular n-sided polygon has rotational symmetry and reflexive symmetry both of order n. e.g., a regular hexagon has six lines of symmetry and has rotational symmetry of order 6. A regular n-sided polygon can be rotated by $\frac{360°}{n}$ and will look the same.

9.4.2 Geometrical Properties of Triangle

A triangle is a polygon with three sides and three angles. Triangles come in various types based on side length and angle measure.

Angle sum: The sum of the interior angles in any triangle is always 180°. In equilateral triangle, all sides are equal, and each angle is 60°. It has three lines of symmetry and rotational symmetry of order 3. In isosceles triangle, two sides are equal, and the angles opposite to the equal sides are also equal. It has one line of symmetry.

Exterior angle of a triangle: The measure of an exterior angle in a triangle is equal to sum of the measures of two opposite interior angles i.e.,



In ΔABC, $m\angle A + m\angle B = m\angle BCD$

i.e., $x + y = w$

9.4.3 Geometrical Properties of Parallelogram

A parallelogram is a quadrilateral whose opposite sides are parallel and equal in length and opposite angles are equal. Its adjacent angles are supplementary. The diagonals of a parallelogram bisect each other (they cross each other at the midpoint). They are not equal in length.

Recall:
Rectangle: All angles are 90° and diagonals are equal.
Rhombus: All sides are equal, and diagonals bisect each other at right angles.
Square: All sides are equal, all angles are 90° and diagonals are equal and bisect each other at right angles.

Example 13: Find the measure of each interior angle of a regular pentagon.

Solution: Interior angle $= \frac{(n-2)\times 180°}{n}$

$$= \frac{(5-2)\times 180°}{5} = \frac{540°}{5} = 108°$$

Each exterior angle is: $\frac{360°}{5} = 72°$

9.4.4 Applications of Polygons

Architects use polygons in building designs, while engineers rely on them to make strong structures like bridges. In art and design, polygons help create beautiful patterns and 3-D models. On maps, polygons show areas like cities or land boundaries. Polygons are also used in video games and animations to build characters and scenes. In science, they appear in molecular shapes, natural patterns like honeycombs and even in the design of telescope mirrors. Their simple, versatile shapes make polygons essential in many fields.

Tessellation

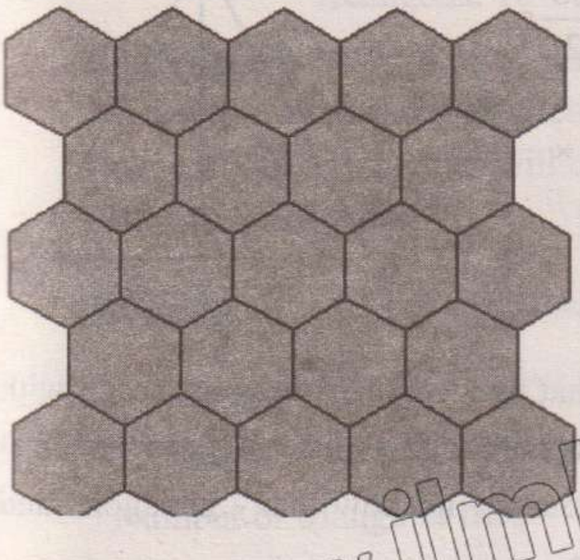
A tessellation is a pattern of shapes that fit together perfectly, without any gaps or overlaps, covering a plane. These shapes can be repeated infinitely to create a repeating pattern. Tessellations can be created using a single shape or a combination of shapes. They can be regular or irregular and they can exhibit various symmetries and patterns.

> **Remember:**
> Equilateral triangles can tessellate perfectly because the internal angle of each equilateral triangle is 60°, and six of these triangles meet at a point to form a 360° angle, allowing them to fill space seamlessly. Squares can tessellate perfectly because each square has an internal angle of 90° and four squares meet at a point to form a 360° angle.

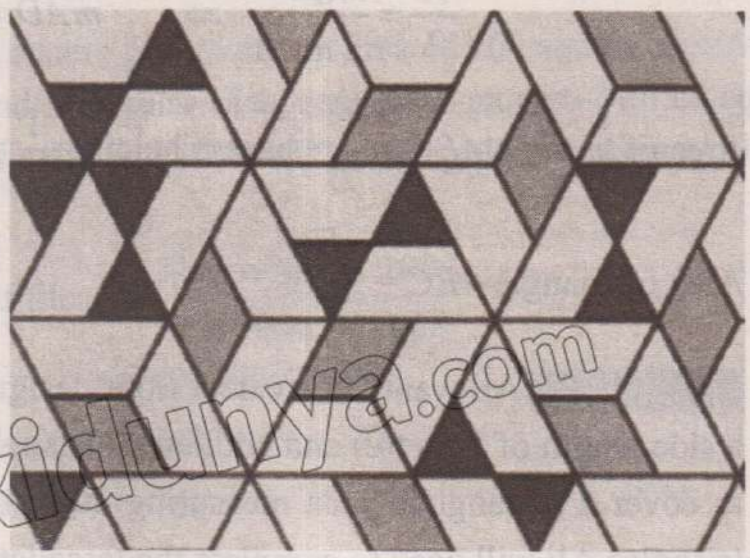
Only three regular polygons can tessellate the plane on their own: equilateral triangles, squares, and regular hexagons. They have symmetries. Hexagons (interior angle 120°) can tessellate perfectly because three hexagons meet at each vertex to form a 360° angle with no space creating a natural look inspired by honeycombs.

> **Remember:**
> Regular pentagons and other polygons with angles that don't add up to 360° at each vertex cannot form gap-free patterns. i.e., Tessellation is not possible.

Regular Tessellation



Irregular Tessellation



Example 14: A tessellation is created using a combination of regular pentagons and decagons. Find the sum of the angles at a vertex where a pentagon and a decagon meet.

Solution

Interior angle of regular decagon $= \frac{(n-2)\times 180^\circ}{n}$

$$= \frac{(10-2)\times 180^\circ}{10} = \frac{1440^\circ}{10} = 144^\circ$$

Interior angle of regular pentagon = 108°

Sum of angles = 144° + 108° = 252°. Since, angle sum ≠ 360°. Tessellation cannot be done.

Example 15: A parallelogram-shaped room has a base of 10 metres and a height of 8m. Babar wants to carpet the room using rolls that cover 20 m² each. How many rolls of carpet do he need?

Solution: The area of the parallelogram = A = base × height = 10 × 8 = 80 m²

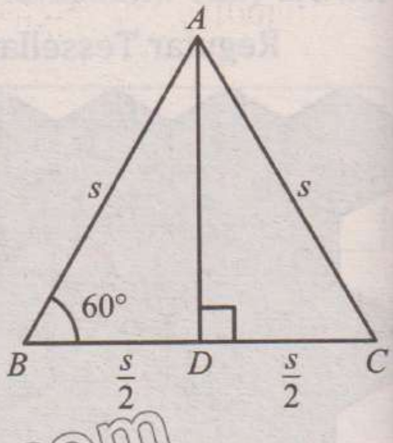
Number of rolls needed: $\frac{80}{20} = 4$ rolls

Example 16: Find the area of the equilateral triangle ABC of side length s.

Solution: Draw perpendicular from A to side BC at point D. In the right angled triangle ABD:

Using trigonometric ratios: $\sin 60^\circ = \frac{\text{Perpendicular}}{\text{Hypotenuse}}$

$$\frac{\sqrt{3}}{2} = \frac{m\overline{AD}}{s} \quad \Rightarrow \quad m\overline{AD} = \frac{\sqrt{3}}{2}s$$



Area of triangle $ABC = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times s \times \frac{\sqrt{3}}{2}s$

Area of triangle $ABC = \frac{\sqrt{3}}{4}s^2$

Example 17: Ali wants to create a floor design that uses regular hexagons (each with a side length of 1 metre) and equilateral triangles (each with a side length of 1 metre) to cover a rectangular area measuring 10 m by 5 m. Find how many hexagons and triangles Ali will need to complete the tessellation.

Solution: To find the area of an equilateral triangle with side length s, we can use the formula:

$$\text{Area of a triangle} = \frac{\sqrt{3}}{4} \cdot s^2$$

Multiply by 6 (since there are 6 triangles)

$$\text{Area of a hexagon} = \frac{6\sqrt{3}}{4} \cdot s^2 = \frac{3\sqrt{3}}{2} \cdot s^2$$

$$\text{Area of a hexagon} = \frac{3\sqrt{3}}{2} \times s^2 \approx \frac{3\sqrt{3}}{2} \times (1\text{ m})^2 \approx 2.598\text{ m}^2$$

$$\text{Area of an equilateral triangle} = \frac{\sqrt{3}}{4} \times s^2 \approx \frac{\sqrt{3}}{4} \times (1\text{ m})^2 \approx 0.433\text{ m}^2$$

$$\text{Area of the rectangular floor} = 10\text{ m} \times 5\text{ m}$$

$$= 50\text{ m}^2$$

Determine the arrangement: Assume a pattern where one hexagon is surrounded by 6 triangles. The area covered by one hexagon and the 6 surrounding triangles:

Total area covered by 1 hexagon and 6 triangles

$= 2.598\text{ m}^2 + 6 \times 0.433\text{ m}^2 \approx 2.598\text{ m}^2 + 2.598\text{ m}^2 = 5.196\text{ m}^2$

Calculate the total number of hexagons and triangles needed:

$$\text{Number of sets} = \frac{50\text{ m}^2}{5.196\text{ m}^2} \approx 9.62\text{ sets}$$

Rounding up, you can fit 10 sets of the pattern. Therefore, we need:

- Hexagons: 10
- Triangles: $10 \times 6 = 60$

Example 18: Falak plans to tile a square patio with an area of 100 square metres. He decides to use both square tiles and triangular tiles, each with an area of 0.25 square metres. If 60% of the tiles will be square and 40% will be triangular, how many tiles of each shape are needed?

Solution:

$$\text{Total number of tiles} = \frac{\text{Patio Area}}{\text{Tile Area}} = \frac{100}{0.25}$$

$$= 400\text{ tiles}$$

$$\text{Number of square tiles} = 400 \times 0.6 = 240$$

$$\text{Number of triangular tiles} = 400 \times 0.4 = 160$$

EXERCISE 9.4

1. (i) What is the sum of the interior angles of a decagon (10-sided polygon)?
   (ii) Calculate the measure of each interior angle of a regular hexagon.
   (iii) What is each exterior angle of a regular pentagon?
   (iv) If the sum of the interior angles of a polygon is 1260°, how many sides does the polygon have?
2. In a parallelogram $ABCD$, $m\overline{AB} = 10$ cm, $m\overline{AD} = 6$ cm and $m\angle BAD = 45°$. Calculate the area of $ABCD$.
3. In a parallelogram $ABCD$ if $m\angle DAB = 70°$, find the measures of all other angles in the parallelogram.
4. A shape is created by cutting a square in half diagonally and then attaching a right-angled triangle to the hypotenuse of each half. Explain why this shape can tessellate and calculate the interior angle of the new shape.
5. A tessellation is created by repeatedly reflecting a basic shape. The basic shape is a right-angled triangle with sides of length 3, 4, and 5 units. Find: The minimum number of reflections needed to create a tessellation that covers a square with an area of 3600 square units.
6. A tessellation is created using regular hexagons. Each hexagon has a side length of 5 cm. Find the total area of the tessellation if it consists of 25 hexagons and total perimeter of the outer edge of the tessellation, assuming it's a perfect hexagon.
7. A rectangular floor is 12 m by 15 m. How many square tiles, each 1 m by 1 m, are needed to cover the floor?
8. A rectangular wall is 10 m tall and 120 m wide. How many gallons of paint are needed to cover the wall, if one gallon covers 35 m^2?
9. A rectangular wall has a length of 10 m and a width of 4 meters. If 1 litre of paint covers 7 m^2, how many liters of paint are needed to cover the wall?
10. A window has a trapezoidal shape with parallel sides of 3 m and 1.5 m and a height of 2 m. Find the area of the window.

REVIEW EXERCISE 9

1. Four options are given against each statement. Encircle the correct one.

(i) If two polygons are similar, then:
(a) their corresponding angles are equal.
(b) their areas are equal.
(c) their volumes are equal.
(d) their corresponding sides are equal.

(ii) The ratio of the areas of two similar polygons is:
(a) equal to the ratio of their perimeters.
(b) equal to the square of the ratio of their corresponding sides.
(c) equal to the cube of the ratio of their corresponding sides.
(d) equal to the sum of their corresponding sides.

(iii) If the volume of two similar solids is 125 cm^3 and 27 cm^3, the ratio of their corresponding heights is -----------.
(a) 3:5 (b) 5:3 (c) 25:9 (d) 9:25

(iv) The exterior angle of regular pentagon is:
(a) 40° (b) 45° (c) 60° (d) 72°

(v) A parallelogram has an area of 64 cm^2 and a similar parallelogram has an area of 144 cm^2. If a side of the smaller parallelogram is 8 cm, the corresponding side of the larger parallelogram is:
(a) 10 cm (b) 12 cm (c) 18 cm (d) 16 cm

(vi) The total number of diagonals in a polygon with 9 sides is:
(a) 18 (b) 21 (c) 25 (d) 27

(vii) Two spheres are similar, and their radii are in the ratio 4:5. If the surface area of the larger sphere is 500π cm^2, what is the surface area of the smaller sphere?
(a) 256π cm^2 (b) 320π cm^2 (c) 400π cm^2 (d) 405π cm^2

(viii) A regular polygon has an exterior angle of 30°. How many diagonals does the Polygon have?
(a) 54 (b) 90 (c) 72 (d) 108

(ix) In a regular hexagon, the ratio of the length of a diagonal to the side length is:
(a) $\sqrt{3} : 1$ (b) $2 : 1$ (c) $3 : 2$ (d) $2 : 3$

(x) A regular polygon has an interior angle of 165°. How many sides does it have?

(a) 15 (b) 16 (c) 20 (d) 24

2. If the sum of the interior angles of a polygon is 1080°, how many sides does the polygon has?

3. Two similar bottles are such that one is twice as high as the other. What is the ratio of their surface areas and their capacities?

4. Each dimension of a model car is $\frac{1}{10}$ of the corresponding car dimension. Find the ratio of:

(a) the areas of their windscreens (b) the capacities of their boots
(c) the widths of the cars (d) the number of wheels they have.

5. Three similar jugs have heights 8 cm, 12 cm and 16 cm. If the smallest jug holds $\frac{1}{2}$ litre, find the capacities of the other two.

6. Three similar drinking glasses have heights 7.5 cm, 9 cm and 10.5 cm. If the tallest glass holds 343 millilitres, find the capacities of the other two.

7. A toy manufacturer produces model cars which are similar in every way to the actual cars. If the ratio of the door area of the model to the door area of the car is 1 cm to 2500 cm, find:

(a) the ratio of their lengths
(b) the ratio of the capacities of their petrol tanks
(c) the width of the model, if the actual car is 150 cm wide
(d) the area of the rear window of the actual car if the area of the rear window of the model is 3 cm^2.

8. The ratio of the areas of two similar labels on two similar jars of coffee is 144 : 169. Find the ratio of

(a) the heights of the two jars (b) their capacities.

9. A tessellation of tiles on a floor has been made using a repeating pattern of a regular hexagon, six squares and six equilateral triangles. Find the total area of a single pattern with side length $\frac{1}{2}$ metre of each polygon.